Mock Final

(1)

(a) $(1-x^3)\dfrac{dy}{dx} = 3x^2y$

$\Rightarrow \displaystyle\int \frac{dy}{y} = \int \frac{3x^2\,dx}{1-x^3}$

$f(x) = 1-x^3$

$f'(x) = -3x^2$

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$\Rightarrow \ln|y| = -\ln|1-x^3| + c$

$\Rightarrow \ln|y| + \ln|1-x^3| = c$

$\Rightarrow \ln|(y)(1-x^3)| = c$ $\quad (\because \ln|a| + \ln|b| = \ln|a.b|)$

$\Rightarrow y(1-x^3) = e^c = c_1$

$\Rightarrow \boxed{y(1-x^3) = c_1}$ Answer

(b) $t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0$

$\Rightarrow y' + \dfrac{4t^2}{t^3}y = \dfrac{e^{-t}}{t^3} \quad \Rightarrow y' + \underbrace{\dfrac{4}{t}}_{p(t)}y = \underbrace{e^{-t}\cdot t^{-3}}_{g(t)}$

$u(t) = e^{\int p(t)dt} = e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = e^{\ln|t|^4} = t^4$

$\Rightarrow y(t) = \dfrac{1}{u(t)}\left[\int u(t)g(t)dt + c\right]$

$\Rightarrow y(t) = \dfrac{1}{t^4}\left[\int e^{-t}t^{-3}t^4\,dt + c\right]$

$= \dfrac{1}{t^4}\left[\int e^{-t}t\,dt + c\right] = \boxed{\dfrac{1}{t^4}\left[-e^{-t}(t+1) + c\right]}$

Answer

2

(c) $\dfrac{dy}{dx} = \dfrac{xy^2 - x - y^2 + 1}{xy - 3y + 2x - 6}$

$= \dfrac{x(y^2-1) - 1(y^2-1)}{y(x-3) + 2(x-3)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(x-1)(y^2-1)}{(y+2)(x-3)}$

$\Rightarrow \displaystyle\int \frac{(y+2)\,dy}{(y^2-1)} = \int \left(\frac{(x-1)}{(x-3)}\right) dx$

$\Rightarrow \displaystyle\int \frac{y}{y^2-1}\,dy + 2\int \frac{dy}{y^2-1} = \int \left(\frac{x-3+2}{x-3}\right) dx$

$\Rightarrow \displaystyle\frac{1}{2}\int \frac{2y}{y^2-1}\,dy + \frac{2}{2}\int \left[\frac{1}{y-1} - \frac{1}{y+1}\right] dy = \int \frac{x-3}{x-3}\,dx + 2\int \frac{dx}{x-3}$

Left hand side

$$\Rightarrow \underbrace{\frac{1}{2}\int \frac{2y}{y^2-1}\,dy}_{\int \frac{f'(x)}{f(x)}dx} + \int \frac{dy}{y-1} - \int \frac{dy}{y+1}$$

$$\Rightarrow \frac{1}{2}\ln|y^2-1| + \ln|y-1| - \ln|y+1| \quad \text{——— (A)}$$

Right hand side

$$\Rightarrow \int 1\,dx + 2\int \frac{dx}{x-3}$$

$$= x + 2\ln|x-3| + c \quad \text{——— (B)}$$

From (A) & (B),

$$\ln\left|\frac{(y^2-1)^{1/2}\cdot(y-1)}{y+1}\right| = x + 2\ln|x-3| + c$$

$$\Rightarrow \boxed{\ln\left|\frac{(y^2-1)^{3/2}}{y+1}\right| = x + 2\ln|x-3| + c}$$

(e) $(1+e^x)\frac{dy}{dx} + e^x y = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{e^x}{1+e^x}\, y = 0$$

$$\Rightarrow \frac{dy}{dx} = -y\left(\frac{e^x}{1+e^x}\right)$$

$$\Rightarrow \int \frac{dy}{y} = -\int \left(\frac{e^x}{1+e^x}\right) dx$$

$f(x) = e^x + 1$

$\Rightarrow f'(x) = e^x$

$$\Rightarrow \ln|y| = -\ln|1+e^x| + C$$

$$\Rightarrow \ln|y| + \ln|1+e^x| = C$$

$$\Rightarrow \ln|y(1+e^x)| = C$$

$$\Rightarrow y(1+e^x) = e^C = C_1$$

$$\Rightarrow \boxed{y(1+e^x) = C_1}$$

(f) $\cos x \frac{dy}{dx} + y \sin x = 1$

$$\Rightarrow \frac{dy}{dx} + \frac{\sin x}{\cos x}\, y = \frac{1}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} + \underbrace{(\tan(x))}_{p(x)}\, y = \left.\frac{1}{\cos x}\right\} \leftarrow g(x)$$

$$\Rightarrow \mu(x) = e^{\int p(x)dx}$$

$$= e^{\int \tan(x)\,dx}$$

$$\Rightarrow \int \tan(x)\,dx = -\ln(\cos x)$$

$$\Rightarrow \mu(x) = e^{-\ln(\cos x)dx}$$

$$= (\cos(x))^{-1}$$

$$\Rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\,dx + C\right]$$

$$= \cos(x)\left[\int \frac{dx}{(\cos x)^2} + C\right]$$

$$= \cos(x)\left[\int \sec^2 x\, dx + C\right]$$

$$\boxed{y(x) = \cos(x)\left[\tan x + C\right]}$$

2

(a) $\frac{dx}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & -4 \end{pmatrix} x \quad \Big| \; x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 0 & -4-\lambda \end{vmatrix} = (-1-\lambda)(-4-\lambda)$$
$$= (\lambda+1)(\lambda+4)$$

$\det(A-\lambda I) = 0$

$\Rightarrow \lambda = -1, -4$

$\Rightarrow$ $\boxed{\text{Eigen values : } \lambda_1 = -1 \;\; \Big| \; \lambda_2 = -4}$

For $\lambda_1 = -1$

$$\Rightarrow A - \lambda_1 I = \begin{bmatrix} -1-(-1) & 1 \\ 0 & -4-(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$$

$\Rightarrow (A - \lambda_1 I) V_1 = 0$

$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow v_2 = 0 \; ; \; v_1 = c$ (any number)

$= 1$ (assumption)

$\Rightarrow V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda_2 = -4$

$$\Rightarrow A - \lambda_2 I = \begin{bmatrix} -1-(-4) & 1 \\ 0 & -4-(-4) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow (A - \lambda_2 I) V_2 = 0$

$\Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow 3v_1 + v_2 = 0$

$\Rightarrow v_1 = -\frac{1}{3} v_2$

$\Rightarrow v_2 = 3 \quad \& \; v_1 = -1$ (assumption)

$\Rightarrow \quad v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\Rightarrow \boxed{\text{Eigenvectors}: \; v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; ; \; v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$

$\Rightarrow$ General Solution: $\quad c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$$\boxed{x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}} \text{ Answer}$$

$\Rightarrow \quad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\Rightarrow @ \, t = 0 \quad \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} c_1 - c_2 &= 2 \\ 3c_2 &= 3 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 3 \\ c_2 &= 1 \end{aligned}$$

$$\Rightarrow \boxed{x(t) = 3e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-4t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}} \text{ Answer}$$

(b) $\frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x \quad \Big| \quad x(0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} \Rightarrow (1-\lambda)^2 - 4$

$\det(A-\lambda I) = 0 \Rightarrow (\lambda - 1)^2 - 4 = 0 \Rightarrow (\lambda-1)^2 = 4$

$\Rightarrow (\lambda - 1) = \pm 2$

$\Rightarrow \lambda = 3, -1$

$\Rightarrow \boxed{\text{Eigenvalues: } \lambda_1 = -1 \ , \lambda_2 = 3}$

For $\lambda_1 = -1$

$A - \lambda_1 I = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow (A - \lambda_1 I) V_1 = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 2v_1 + v_2 = 0$

$\Rightarrow v_1 = -2v_2 \Rightarrow v_1 = 1$ and $v_2 = -2$

$\Rightarrow V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda_2 = 3$,

$A - \lambda_2 I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow (A - \lambda_2 I) V_2 = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2v_1 + v_2 = 0$

$\Rightarrow v_1 = \frac{v_2}{2} \Rightarrow v_2 = 2$ and $v_1 = 1$

$\Rightarrow V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow \boxed{\text{Eigenvectors: } \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{V_1} \text{ and } \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{V_2}}$

General Solution : $c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$

$$\Rightarrow \boxed{c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

ANSWER

Constants : $X(0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

@ $t=0$

$$\Rightarrow \begin{bmatrix} 4 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 4 \\ -2c_1 + 2c_2 &= 6 \end{aligned}\right\} \quad \begin{aligned} c_1 + c_2 &= 4 \\ -c_1 + c_2 &= 3 \end{aligned}$$

$$2c_2 = 7$$

$$\boxed{\begin{aligned} c_2 &= 3.5 \\ c_1 &= 0.5 \end{aligned}}$$

$$\Rightarrow \boxed{x(t) = 0.5 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 3.5 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

ANSWER

(C) $\frac{dx}{dt} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$

$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = 0$ $\qquad = (3-\lambda)(-1-\lambda) + 8$

$\Rightarrow \lambda^2 - 2\lambda + 5 = 0$ $\qquad = (\lambda - 3)(\lambda + 1) + 8$

$\Rightarrow \lambda = 1 \pm 2i$ $\qquad = \lambda^2 - 2\lambda + 5$

$\Rightarrow \boxed{\lambda_1 = 1 + 2i \,,\, \lambda_2 = 1 - 2i}$

For λ_1, $\quad A - \lambda_1 I = \begin{bmatrix} 3-1+2i & -2 \\ 4 & -1-1-2i \end{bmatrix} = \begin{bmatrix} 2+2i & -2 \\ 4 & -2-2i \end{bmatrix}$

$\Rightarrow (A - \lambda_1 I) v_1 = 0$

$\Rightarrow \begin{bmatrix} 2+2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{aligned} (2+2i) v_1 - 2 v_2 &= 0 \\ 4 v_1 + (-2-2i) v_2 &= 0 \end{aligned}\right\}$ Solving this,

$v_1 = 1$

$v_2 = 1 - i$

$\Rightarrow \boxed{V_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \text{ and so } v_2 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}}$

For a system with complex eigenvalues, the general solution is given by

$$X = C_1 U(t) + C_2 W(t)$$

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

For eigen vectors,

$$(A - \lambda_1 I) v_1 = 0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda_1 & -2 \\ 4 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (3-\lambda_1) v_1 - 2 v_2 = 0$$

$$\Rightarrow (3-\lambda_1) v_1 = 2 v_2$$

$$\Rightarrow v_1 = \frac{2 v_2}{3-\lambda_1}$$

$\lambda_1 = 1 + 2i$

$$\Rightarrow v_1 = \frac{2}{3-(1+2i)} v_2$$

$$= \frac{2}{2-2i} v_2 = \frac{1}{1-i} v_2$$

$$\Rightarrow v_1 = \frac{1}{1-i} v_2$$

$$\Rightarrow v_2 = (1-i) v_1$$

$\Rightarrow v_1 = 1$ (assumption)

and $v_2 = 1 - i$

$\Rightarrow V_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$ and it follows that

$$V_2 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

where

$$u(t) = e^{\mu t}(a\cos\nu t - b\sin\nu t)$$

$$w(t) = e^{\mu t}(a\sin\nu t + b\cos\nu t)$$

where the eigen vector $\mathbf{v}_1 = a + ib$ | $\lambda_1 = \mu + i\nu = 1 + 2i \Rightarrow \boxed{\begin{matrix}\mu = 1 \\ \nu = 2\end{matrix}}$

Here

$$v_1 = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = a + ib$$

$$= \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{a} \pm i \underbrace{\begin{bmatrix} 0 \\ +1 \end{bmatrix}}_{b}$$

$$\Rightarrow u(t) = e^{t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ +1 \end{bmatrix}\sin(2t)\right)$$

$$= e^{t}\left(\begin{bmatrix} \cos(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix}\right)$$

$$\Rightarrow w(t) = e^{t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ +1 \end{bmatrix}\cos(2t)\right)$$

$$= e^{t}\left(\begin{bmatrix} \sin(2t) \\ \sin(2t) + \cos(2t) \end{bmatrix}\right)$$

$$\Rightarrow \boxed{X = c_1 e^{t}\begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + c_2 e^{t}\begin{bmatrix} \sin(2t) \\ \sin(2t) + \cos(2t) \end{bmatrix}}$$

ANSWER

3

(a) $y'' - 2y' - 3y = 0 \quad \Big| \; y(0) = 1, \; y'(0) = 0$

Let $x_1 = y$ and $x_2 = y'$

$\Rightarrow x_1' = x_2$

and $x_2' = 2x_2 + 3x_1$

$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \frac{dx}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ eigen values of A: $\det(A - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} -\lambda & 1 \\ 3 & -\lambda + 2 \end{bmatrix} = 0$$

$$\Rightarrow \lambda = -1, 3$$

$\Rightarrow$ eigen vectors of A

For $\lambda_1 = -1$, $(A - \lambda_1 I) v_1 = \begin{bmatrix} +1 & 1 \\ 3 & +3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -v_1 + v_2 = 0 \Rightarrow v_1 = -1$ (assumption)

$\Rightarrow v_2 = 1$

$$\Rightarrow \boxed{V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

Similarly for $\lambda_2 = 3$ $(A - \lambda_2 I) V_2 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \boxed{V_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$\Rightarrow$ General Solution:

$$\boxed{X(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$\Rightarrow$ Initial condition $\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ @ $t=0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\Rightarrow$

$$c_2 - c_1 = 1$$
$$3c_2 + c_1 = 0$$

$$4c_2 = 1 \Rightarrow \boxed{\begin{array}{l} c_2 = 1/4 \\ \Rightarrow c_1 = -3/4 \end{array}}$$

$$\Rightarrow \boxed{X(t) = -\frac{3}{4} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{4} e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

ANSWER

(b) $y'' + 4y = 0$; $y(0) = 1$, $y'(0) = -2$

Let $\left.\begin{array}{l} y = x_1 \\ y' = x_2 \end{array}\right\} \Rightarrow \begin{array}{l} x_1' = x_2 \\ \text{and } x_2' = -4x_1 \end{array}$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigen values of A :

$$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda^2 + 4 = 0$$

$$\Rightarrow \boxed{\lambda = \pm 2i}$$

$\lambda = u + iv$

$\boxed{\begin{array}{l} u = 0 \\ v = 2 \end{array}}$

$\Rightarrow \lambda_1 = 2i$ and

$\lambda_2 = -2i$

For eigen vectors,

$\lambda_1 = 2i$

$$(A - \lambda_1 I)v_1 = 0 \Rightarrow \begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2i\,v_1 + v_2 = 0 \Rightarrow v_2 = +2i\,v_1$

and $-4v_1 - 2i\,v_2 = 0$

$$\Rightarrow v_1 = \begin{bmatrix} -0.5i \\ 1 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} -0.5i \\ 1 \end{bmatrix} = a \pm ib \quad \text{and} \quad V_2 = \begin{bmatrix} 0.5i \\ 1 \end{bmatrix}$$ (15)

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

For complex eigen values

$$X = C_1 U(t) + C_2 W(t)$$

$$U(t) = e^{ut}(a\cos vt - b\sin vt)$$

$$= e^{0t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} +0.5 \\ 0 \end{bmatrix}\sin(2t)\right)$$

$$= \left(\begin{bmatrix} -0.5\sin(2t) \\ \cos(2t) \end{bmatrix}\right)$$

$$W(t) = e^{ut}(a\sin vt + b\cos vt)$$

$$= e^{0t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} +0.5 \\ 0 \end{bmatrix}\cos(2t)\right)$$

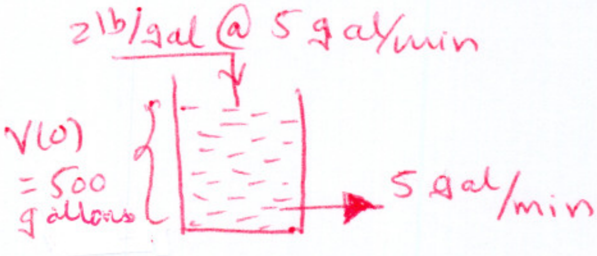
$$= \left(\begin{bmatrix} +0.5\cos 2t \\ \sin(2t) \end{bmatrix}\right)$$

$$\Rightarrow X = C_1 \begin{bmatrix} -0.5\sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} +0.5\cos(2t) \\ \sin(2t) \end{bmatrix}$$

Initial Conditions: $X(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

@ $t=0$ $$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} +0.5 \\ 0 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} C_1 = -2 \\ C_2 = +2 \end{matrix}}$$ ANSWER

4



A(t) : amount of salt

$A(0) = 0$ (∵ pure water)

$$\frac{dA}{dt} = \text{inrate amount} - \text{out rate amount}$$

$$= 2\frac{\text{lb}}{\text{gal}} \times 5\frac{\text{gal}}{\text{min}} - \frac{A(t)\,\text{lb}}{500\text{ gal}} \times \frac{5\text{gal}}{\text{min}}$$

Answer (a)

$$\boxed{\frac{dA}{dt} = 10 - \frac{A}{100}} \Rightarrow \boxed{\frac{dA}{dt} + \frac{A}{100} = 10}$$

$$\Rightarrow \mu(t) = e^{\int \frac{1}{100}dt}$$

$$= e^{t/100}$$

$$\Rightarrow A(t) = \frac{1}{\mu(t)}\left[\int \mu(t) g(t) dt + c\right]$$

$$= \frac{1}{e^{t/100}}\left[\int e^{t/100}\, 10\, dt + c\right]$$

$$= e^{-t/100}\left[10\int e^{t/100} dt + c\right]$$

$$= e^{-t/100}\left[10 \cdot \frac{e^{t/100}}{1/100} + c\right]$$

$$A(t) = 1000 + ce^{-t/100}$$

$A(0) = 0 \Rightarrow 1000 + C = 0$

$\Rightarrow C = -1000$

$\Rightarrow \boxed{A(t) = 1000 - 1000e^{-t/100}}$ → Answer (b)

after 5 minutes

$\Rightarrow A(5) = 1000 - 1000e^{-5/100}$

$\boxed{A(5) = 48.7 \text{ pounds}}$

5

$$\boxed{\frac{dT}{dt} = k(T - T_A)}$$

$T(0) = 70°F$

$T_A = 10°F$

$T(1/2) = 50°F$

$$\int \frac{dT}{T - T_A} = k\int dt$$

$$\ln|T - T_A| = kt + c$$

$$T - T_A = e^{kt+c} = ce^{kt}$$

$$\boxed{T = T_A + ce^{kt}}$$

@ $t=0 \Rightarrow T(0) = 70°F$

$$\Rightarrow 70 = T_A + c$$

$$\Rightarrow c = 70 - T_A = 70 - 10 = 60$$

$$\Rightarrow \boxed{c = 60}$$

$$\Rightarrow \boxed{T(t) = 10 + 60e^{kt}}$$

@ $t = 1/2 \Rightarrow T(1/2) = 50$

$$\Rightarrow 50 = 10 + 60e^{k(1/2)}$$

$$\Rightarrow 40 = 60e^{k(1/2)}$$

$$\Rightarrow \frac{2}{3} = e^{k(1/2)}$$

$$\Rightarrow \frac{k}{2} = -0.405$$

$$\Rightarrow \boxed{k = -0.81}$$

$$\Rightarrow \boxed{T(t) = 10 + 60e^{(-0.81)t}}$$

→ Answer (b)

@ $t = 1$,

$$T(1) = 10 + 60e^{-0.81}$$

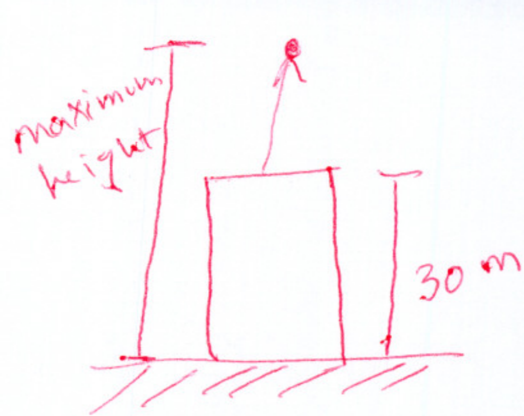
$$\boxed{T(1) = 36.67}$$ → Answer (c)

when $T = 15 \Rightarrow t = ?$

$$\Rightarrow 15 = 10 + 60e^{-(0.81)t}$$ → Answer (d)

$$\Rightarrow \boxed{t = 3.067}$$ minutes

6



$m = 0.25$ kg

$V(0) = 20$ m/sec

$\downarrow$ direction = positive

$ma = -mg$

$\cancel{m}\frac{dv}{dt} = \cancel{m}g \Rightarrow \boxed{\frac{dv}{dt} = -g = -9.8}$ → Answer (a)

$\Rightarrow \int dv = -9.8 \int dt$

$\Rightarrow V(t) = -9.8t + C$

@ $t = 0 \Rightarrow V(0) = 20$

$\Rightarrow 20 = C$

$\Rightarrow V(t) = -9.8t + 20$

$\boxed{V(t) = 20 - 9.8t}$ → Answer (b)

$X(t) = \int V(t)\,dt$

$= \int (20 - 9.8t)\,dt$

$= 20t - 9.8\frac{t^2}{2} + C$

$\boxed{X(t) = 20t - 4.9t^2 + C}$

@ $t = 0$, $X(t) = 0$

$\Rightarrow X(t) = 20t - 4.9t^2$

@ max height $t = 2.04$ seconds

$\Rightarrow \boxed{X(2.04) = 20.40\text{ m}}$

@ max height

$V(t) = 0$

$\Rightarrow 20 - 9.8(t) = 0$

$\Rightarrow t = \frac{20}{9.8} = 2.04$ seconds.

$\Rightarrow \boxed{t = 2.04\text{ seconds}}$ → Answer (c)

Maximum height = 20.40 + 30

= 50.4 meters.

⑦ Spring mass System

The differential equation is

$$m u'' + r u' + k u = 0$$

where u = position of the weight

$\Rightarrow u(0) = 0$

$u'(0) = 0.25$ ft/sec

Substituting all the values,

$\Rightarrow$

$$2u'' + 4u' + 2u = 0$$

$$\Rightarrow \boxed{u'' + 2u' + u = 0} \longrightarrow \text{Answer}$$

$$\boxed{\begin{array}{l} u(0) = 0 \\ u'(0) = 0.25 \end{array}}$$

for spring constant

$W = 4$ lb

$x = 2$ ft

$\Rightarrow k = \dfrac{W}{x} = \dfrac{4}{2} = 2\,\dfrac{\text{lb}}{\text{ft}}$

New Weight = 64 lbs

$W = mg$

$m = \dfrac{W}{g} = \dfrac{64}{32} = 2$ slugs

$r = 4\,\dfrac{\text{lb}}{\text{ft/sec}}$

8 Spring mass System

$\Rightarrow mu'' + \gamma u' + ku = 0$

u = position of the mass

$u(0) = 0$

$u'(0) = 0.5$ ft/sec

$m = 1/2$ slug

$k = 6$ lb/ft

for damping constant,

$\gamma = \frac{F}{v}$

Given, $\boxed{F = 2v}$

$\Rightarrow \gamma = \frac{2v}{v} = 2$

Substituting the values,

$\Rightarrow$

$\frac{1}{2}u'' + 2u' + 6u = 0$

$\Rightarrow \boxed{u'' + 4u' + 12u = 0}$

9

A 30-year old lady has \$100,000 of savings in an account that pays 5% annual interest, compounded continually. She hopes to have \$4 million in this account when she is 70, at which time she will retire. Suppose she deposits 30% of her earnings every year into the saving account. How much money does she need to earn in order to reach the goal? How much money would she be able to withdraw every year from her account after age 70, given that she wants the account to last until she is 100?

Solution :- 30 year old lady,

initial amount $S_0 = \$100,000$

Rate $r = 5\% = 0.05$

At Age 70 $\rightarrow$ $S = \$4,000,000$

$\therefore$ 30 ——— 70 (age)

$t = 70 - 30 = 40 \Rightarrow S(40) = \$4000,000$

$K = 30\%$ of pay

$\boxed{K = 0.3 \times \text{Pay}}$

$$S(t) = S_0 e^{rt} + \left(\frac{K}{r}\right)(e^{rt} - 1)$$

$$\therefore S(40) = S_0 e^{r(40)} + \left(\frac{K}{r}\right)(e^{r(40)} - 1)$$

$$\therefore 4\,000\,000 = (100\,000)e^{(0.05)(40)} + \left(\frac{K}{0.05}\right)(e^{(0.05)(40)} - 1)$$

$$\therefore \frac{K}{0.05}(e^2 - 1) = 3,261,094.39$$

$\therefore \boxed{K = 25,520.9 \text{ dollars}}$ $\Rightarrow$ $\boxed{\text{Pay} = \frac{K}{0.3} = \$85,069.80}$

ANSWER

ONCE RETIRED

age $70 \longrightarrow 100$

Here $t = 100 - 70 = 30$

$S_0 = S(70)$
$= 4$ million

At age 100
$S(30) = 0$ (zero).
$r = 0.05$

$$S(t) = S_0 e^{rt} + \left(\frac{k}{r}\right)(e^{rt} - 1)$$

Money withdrawn $\Rightarrow$ k -ve

$$\therefore S(30) = S_0 e^{r(30)} + \left(\frac{-k}{r}\right)(e^{r(30)} - 1)$$

$$\therefore 0 = (4\,000\,000)\, e^{(0.05)(30)} - \frac{k}{(0.05)}(e^{(0.05)(30)} - 1)$$

$$\therefore -17{,}926{,}756.28 = -\frac{k}{0.05}(3.4816)$$

$$\therefore \boxed{k = \$257{,}443.38}$$ ANSWER.