LECTURE NOTES FOR MA3203 RING THEORY

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Contents

1. Quivers

1.1. Quivers, vertices, arrows and paths.

Definition 1.1. A *quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph,

$$\Gamma_0 = \{\text{vertices}\}(= \{1, 2, \ldots, n\}).$$
$$\Gamma_1 = \{\text{arrows}\}.$$

We always assume that Γ_0 and Γ_1 are finite sets.

Example 1.2. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, $\Gamma_0 = \{1, 2\}$ and $\Gamma_1 = \{\alpha\}$.

Example 1.3. $\Gamma\colon\ \begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$, $\Gamma_0 = \{1\}$ and $\Gamma_1 = \{\alpha\}$.

Example 1.4. $\Gamma\colon$
$$\begin{tikzcd} 1 \arrow[rr, bend left=10, "\alpha"] \arrow[rr, bend right=10, "\beta"'] & & 2 \arrow[loop right, "\gamma"] \arrow[dl, bend right, "\delta"'] \\ & 3 \arrow[ul, "\theta"] \arrow[ur, bend right, "\epsilon"'] & \end{tikzcd}$$
, $\Gamma_0 = \{1, 2, 3\}$ and $\Gamma_1 = \{\alpha, \beta, \gamma, \delta, \epsilon, \theta\}$.

Have maps: $s, e\colon \Gamma_1 \to \Gamma_0$

$$\mathfrak{s}(\alpha) = \text{the vertex where } \alpha \in \Gamma_1 \text{ starts},$$
$$\mathfrak{e}(\alpha) = \text{the vertex where } \alpha \in \Gamma_1 \text{ ends}.$$

Definition 1.5. $\Gamma = (\Gamma_0, \Gamma_1)$ quiver. A *path* in Γ is either

(i) an ordered sequence of arrows $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$, where
$$\mathfrak{e}(\alpha_t) = \mathfrak{s}(\alpha_{t+1})$$
for $t = 1, 2, \ldots, n-1$ (*non-trivial path*) or

(ii) e_i for each i in Γ_0 (*trivial path*).

In addition,

$$\mathfrak{s}(p) = \mathfrak{s}(\alpha_1) \qquad \mathfrak{s}(e_i) = i$$
$$\mathfrak{e}(p) = \mathfrak{e}(\alpha_n) \qquad \mathfrak{e}(e_i) = i$$

Example 1.6. Γ:
$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\gamma"] & 3 \\ & 4 & \end{tikzcd}$$

Paths:

(i) $\alpha, \beta, \gamma, \beta\alpha, \gamma\alpha$.
(ii) e_1, e_2, e_3, e_4.

Example 1.7. Γ: $1 \circlearrowright \alpha$.

Paths:

(i) $\alpha, \alpha^2 = \alpha\alpha, \alpha^3 = \alpha\alpha\alpha, \ldots$.
(ii) e_1.

1.2. **Path algebras.** Given $\Gamma = (\Gamma_0, \Gamma_1)$, a quiver, and k a field.

The *path algebra* $k\Gamma$: $k\Gamma$ is the vector space with all the paths in Γ as a basis.

The elements in $k\Gamma$:
$$a_1p_1 + a_2p_2 + \cdots + a_tp_t$$
where $a_i \in k$ and p_i are paths in Γ.

Example 1.8. Continueing Example 1.6:

$$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5\alpha + a_6\beta + a_7\gamma + a_8\beta\alpha + a_9\gamma\alpha$$
$$y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5\alpha + b_6\beta + b_7\gamma + b_8\beta\alpha + b_9\gamma\alpha$$

$$x + y = (a_1 + b_1)e_1 + (a_2 + b_2)e_2 + (a_3 + b_3)e_3 + (a_4 + b_4)e_4 + (a_5 + b_5)\alpha$$
$$+ (a_6 + b_6)\beta + (a_7 + b_7)\gamma + (a_8 + b_8)\beta\alpha + (a_9 + b_9(\gamma\alpha$$

p, q paths in Γ:

(1) p, q both non-trivial
$$p \cdot q = \begin{cases} pq, & \text{if } \mathfrak{e}(q) = \mathfrak{s}(p) \\ 0, & \text{otherwise} \end{cases}$$

(2) p non-trivial, q trivial, $q = e_i$
$$p \cdot q = \begin{cases} p, & \text{if } \mathfrak{s}(p) = i = \mathfrak{e}(q) \\ 0, & \text{otherwise} \end{cases}$$
$$q \cdot p = \begin{cases} p, & \text{if } \mathfrak{e}(p) = i = \mathfrak{s}(q) \\ 0, & \text{otherwise} \end{cases}$$

(3) $p = e_i$, $q = e_j$ (both trivial)
$$p \cdot q = \begin{cases} e_i, & \text{if } \mathfrak{e}(q) = j = i = \mathfrak{s}(p) \\ 0, & \text{otherwise} \end{cases}$$

This is extended distributively to an operator on $k\Gamma$ (see [1, page 50]).

Example 1.9. Γ: $1 \xrightarrow{\alpha} 2$, k field.

Elements in $k\Gamma$: $a_1e_1 + a_2e_2 + a_3\alpha = y$.

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

$$\begin{aligned}(e_1+e_2)\cdot y &= (e_1+e_2)(a_1e_1+a_2e_2+a_3\alpha)\\ &= a_1e_1^2 + a_2\underbrace{e_1e_2}_{=0} + a_3\underbrace{e_1\alpha}_{=0} + a_1\underbrace{e_2e_1}_{=0} + a_2e_2^2 + a_3e_2\alpha\\ &= a_1e_1 + a_2e_2 + a_3\alpha = y\end{aligned}$$

Similary $y \cdot (e_1 + e_2) = y$. Hence, $e_1 + e_2$ acts like 1 in $k\Gamma$.

Basis for $k\Gamma$: $\{e_1, e_2, \alpha\}$, $\dim_k k\Gamma = 3$.

Example 1.10. Γ: $1 \circlearrowright \alpha$, and k a field.

$k\Gamma$ has basis: $\{e_1, \alpha, \alpha^2, \alpha^3, \ldots\}$, that is, $\dim_k k\Gamma = \infty$.

Elemenets in $k\Gamma$: $a_0e_1 + a_1\alpha + a\alpha^2 + \cdots + a_t\alpha^t$, with a_i in k and $t \geqslant 0$.

Notes

(1) In general, $\{e_i\}_{i\epsilon\Gamma}$ are orthogonal idempotents in $k\Gamma$, ie.

$$\begin{cases} e_i^2 = e_i \\ e_ie_j = 0 \text{ for i}\neq\text{j} \end{cases}$$

(2) Suppose $\Gamma_0 = \{1, 2, ..., n\}$. Then $e_1 + e_2 + ... + e_n$ acts like 1 in $k\Gamma$. Enough to show that $p = (e_1 + e_2 + ... + e_n)p = p(e_1 + e_2 + ... + e_n)$ for any path p. Suppose that $\mathfrak{s}(p) = i$ and $\mathfrak{e}(p) = j$ Then

$$(e_1 + e_2 + ... + e_n)p = e_1p + e_2p + ... + e_jp + ... + e_np = e_jp \overset{\text{def}}{=} p$$

$$p(e_1 + e_2 + ... + e_n) = pe_1 + pe_2 + ... + pe_i + ... + pe_n = e_jp \overset{\text{def}}{=} p$$

(3) $\implies e_1 + e_2 + ... + e_n = 1_{k\Gamma} =$ identity in $k\Gamma$

Can show: $k\Gamma$ is a k-algebra with $e_1+e_2+...+e_n$ as an identity (see [1, page 50])

recall: Λ ring, k field

Definition 1.11. Λ is a k-algebra, if Λ is a vector space over k ($k \times \Lambda \longrightarrow \Lambda$, Λ is a module over k, $\alpha \in k, \lambda \in \Lambda, \alpha \cdot \lambda$) and $\alpha(\lambda \cdot \lambda') = (\alpha \cdot \lambda) \cdot \lambda' = \lambda(\alpha \cdot \lambda')$ $\forall \alpha \in k, \forall \lambda, \lambda' \in \Lambda$

Equivalent: Λ is a k-algebra, if $\exists \phi$: $k \to \Lambda$ a ring homomorphism such that $\operatorname{Im} \phi \subseteq Z(\Lambda) = \{z \in \Lambda | z\lambda = \lambda z, \forall \lambda \in \Lambda\}$ ($\iff \exists R \subseteq \Lambda$ subring such that $R \simeq k$ with $R \subseteq Z(\Lambda)$)

$\phi(a) = a \cdot 1_\Lambda$ For $k\Gamma$ the ring homomorphisme ϕ: $k \to k\Gamma$ is given by $\phi(a) = ae_1 + ae_2 + ... + ae_n$

Exercises:

(1) $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field.
Find a k-algebra isomorphisme

$$\psi\colon k\Gamma \to \begin{pmatrix} \mathrm{k} & 0 \\ \mathrm{k} & \mathrm{k} \end{pmatrix}$$

(2) $\Gamma\colon\ 1 \circlearrowright \alpha$. k field.
Show that $k\Gamma \simeq k[x]$ as k-algebra's.

Definition 1.12. A non-trival path p in Γ is an oriented cycle if

$$\mathfrak{e}(p) = \mathfrak{s}(p)$$

wrong path!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

Example 1.13. $\Gamma\colon\ 1 \circlearrowright \alpha$

Cycles: $\alpha, \alpha^3, \gamma\beta\alpha, \beta\alpha^10\gamma, \ldots\ \dim_k k\Gamma = \infty$

Proposition 1.14. $\Gamma = (\Gamma_0, \Gamma_1)$ *quiver, k field.* $\mathit{dim}_k k\Gamma < \infty \iff \Gamma$ *has no oriented cycles.*

Proof. Exercise □

Proposition 1.15. *Assume that* $\Gamma = (\Gamma_0, \Gamma_1)$ *has no oriented cycles.* $k\Gamma$ *is semisimpel* $\iff \Gamma_1 = \emptyset$

Proof. proposition 1.14 $\implies \dim_k k\Gamma < \infty \implies k\Gamma$ is a left artinian ring.
$k\Gamma$ semisimpel $\iff$ no non-zero nilpotent left ideals in $k\Gamma$
$\implies$: Assume that $\Gamma_1 \neq \emptyset$. Let α_1 be an arrow in Γ. Want to find a vertex where at least one arrow ends and no arrow starts. if

$$\mathfrak{e}(\alpha_1)$$

is such a vertex, we are done. If not, there is an arrow α_2 starting in

$$\mathfrak{e}(\alpha_1)$$

. If also

$$\mathfrak{e}(\alpha_2)$$

is not as above, we continue. Since Γ has no oriented cycles and Γ is finite, we must end up in a vertex v, where arrows only end and no arrows starts. Say, $\alpha = \alpha_t$ is an arrow ending in v. Then consider $k\Gamma\alpha = k\alpha$ Since $(a_1\alpha)(a_2\alpha = (a_1a_2)\underbrace{(\alpha\alpha)}_{=0} = 0$

$\implies (k\Gamma\alpha)^2 = (0)$ and $k\Gamma\alpha \neq (0)$
$\implies k\Gamma$ is not semisimpel.

$\impliedby$: assume that $\Gamma_1 = \emptyset$. then Γ 1 2 $\cdots$ n (n vertice)
Basis for $k\Gamma\colon \{e_1, e_2, \cdots, e_n\}$. Elements in $k\Gamma\colon a_1e_1 + a_2e_2 + \cdots + a_ne_n$ with $a_i \in k$.
Have a ring homomorphisme. $\psi\colon \underbrace{k \times \cdots \times k}_{n} \to k\Gamma$
given by $\psi(a_1, a_2, \cdots, a_n) = a_1e_1 + a_2e_2 + \cdots a_ne_n$(check this!). Show that ψ is an isomorphisme. Therefore $k\Gamma$ is semisimpel, since $k\Gamma$ is isomorphic to a finite product of full matrix rings over divisjon rings. □

$k\Gamma$ is not always semisimpel, but some factor of $k\Gamma$ is.

Proposition 1.16. $\Gamma = (\Gamma_0, \Gamma_1)$ *quiver, k field. Let* $J = \{$*all linear combinations of non-trivial paths*$\}$ *Then* J *is an ideal in* $k\Gamma$ *and* $k\Gamma/J \simeq \underbrace{k \times \cdots \times k}_{|\Gamma_0|}$, *-semisimpel*

Proof. "proof"
Define $\psi\colon k\Gamma \to \underbrace{k \times \cdots \times}_{|\Gamma_0|=n} = k^n$
$\psi(a_1e_1 + a_2e_2 + \cdots + a_ne_n +$ linear combinations of non-trivial paths$) = (a_1, a_2, \cdots, a_n)$

<u>Check:</u>

(1) ψ is well-defined
(2) ψ homomorphism of rings
(3) $\ker\psi = J$

$\implies k\Gamma/J \simeq \mathrm{Im}\psi = k^n$ □

1.3. Modules.

Example 1.17. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field What is a module over $k\Gamma$? Let M be a left $k\Gamma$-module. <u>Recall:</u> $1_{k\Gamma} = e_1 + e_2$,

$$e_ie_j = \begin{cases} e_i^2 = e_i \\ e_ie_j = 0 \text{ for i} \neq \text{j} \end{cases}$$

<u>Claim:</u> $M = e_1M \oplus e_2M$ as vector space over k.

<u>Proof:</u>

$$\begin{aligned} &m = 1_{k\Gamma} * m = (e_1 + e_2)m = e_1m + e_2m \in e_1M + e_2M \\ \implies &M \subseteq e_1M + e_2M \subseteq M \implies M = e_1M + e_2M \end{aligned}$$

Let $m \in e_1M \cap e_2M$, i.e $m = e_1m' = e_2m''$

$$\begin{aligned} e_1m &= e_1(e_1m') = (e_1e_1)m' = e_1m' = m \\ &= e_1(e_2m'') = \underbrace{(e_1e_2)m''}_{=0} = 0 \cdot m'' = 0 \end{aligned}$$

$$\begin{aligned} &\implies m = 0. \text{ hence } e_1M \cap e_2M = (0) \\ &\implies M = e_1M \oplus e_2M \end{aligned}$$

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References

[1] Auslander, M., Reiten, I., Smalø, S. O., *Representation theory of Artin algebras.* Corrected reprint of the 1995 original. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997. xiv+425 pp. ISBN: 0-521-41134-3; 0-521-59923-7.

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