Homework 2

problem 1

a) $q(\phi) = p(\phi \mid r_{ij}, u_i, v_j)$

$$= \prod_{(i,j)\in\Omega} p(\phi_{ij} \mid r_{ij}, u_i, v_j)$$

While

$$p(\phi_{ij} \mid r_{ij}, u_i, v_j) = \frac{p(r_{ij} \mid \phi_{ij})\, p(\phi_{ij} \mid u_i, v_j)}{\int p(r_{ij} \mid \phi_{ij})\, p(\phi_{ij} \mid u_i, v_j)\, d\phi_{ij}}$$

$$= \frac{\mathbb{1}\{\mathrm{sign}(\phi_{ij}) = r_{ij}\}\, e^{-\frac{1}{2}(\phi_{ij} - u_i^T v_j)^2}}{\int_{-\infty}^{\infty} \mathbb{1}\{\mathrm{sign}(\phi_{ij}) = r_{ij}\}\, e^{-\frac{1}{2}(\phi_{ij} - u_i^T v_j)^2}\, d\phi_{ij}}$$

Then $q(\phi) = \prod_{(i,j)\in\Omega} q(\phi_{ij})$

and $q(\phi_i) = p(\phi_{ij} \mid r_{ij}, u_i, v_j) = \dfrac{\mathbb{1}\{\mathrm{sign}(\phi_{ij}) = r_{ij}\}\, e^{-\frac{1}{2}(\phi_{ij} - u_i^T v_j)^2}}{\int_{-\infty}^{\infty} \mathbb{1}\{\mathrm{sign}(\phi_{ij}) = r_{ij}\}\, e^{-\frac{1}{2}(\phi_{ij} - u_i^T v_j)^2}\, d\phi_{ij}}$

b) Derive $L(U,V)$

$$L(u,v) = \ln p(V) + \ln p(U) + \sum_{(i,j)\in\Omega} E_q[\ln p(r_{ij}, \phi_{ij} \mid v_j, u_i)] + \text{constant}$$

$$p(r_{ij}, \phi, u_i, v_j) = p(u_i)p(v_j) \prod p(r_{ij} \mid \phi_{ij})\, p(\phi_{ij} \mid u_i, v_j)$$

Then we have

$$L(u,v) = -\frac{1}{2c} u_i^T u_i - \frac{1}{2c} v_j^T v_j - \sum_{(i,j)\in\Omega} \frac{1}{2\sigma^2}\left(v_j^T u_i u_i^T v_j - 2 v_j^T u_i E_q[\phi_{ij}]\right) + \text{constant}$$

$$E_q[\phi_{ij}] = \begin{cases} u_i^T v_j + \sigma \times \dfrac{\Phi'(-u_i^T v_j/\sigma)}{1 - \Phi(-u_i^T v_j/\sigma)} & \text{if } r_{ij} = 1 \\[2ex] u_i^T v_j + \sigma \times \dfrac{-\Phi'(-u_i^T v_j/\sigma)}{\Phi(-u_i^T v_j/\sigma)} & \text{if } r_{ij} = -1 \end{cases}$$

(c).

Solving for $\nabla L(U,V)$,

we find that

$$u_i = \left(\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T / \sigma^2\right)^{-1} \left(\sum_{(i,j)\in\Omega} v_j \, E_q[\phi_{ij}] / \sigma^2\right)$$

$$v_j = \left(\frac{1}{c}I + \sum_{j=1}^{N} u_i u_i^T / \sigma^2\right)^{-1} \left(\sum_{(i,j)\in\Omega} u_i \, E_q[\phi_{ij}] / \sigma^2\right)$$

(d)

1. Initialize u_0, v_0 to a vector of all zeros.
   set σ.

2. For iteration $t=1,\dots,T$

E step:

$$E_{q_t}[\phi_{ij}] = \begin{cases} u_i^T V_{t-1} + \sigma \times \dfrac{\Phi'(-u_i^T V_{t-1}/\sigma)}{1-\Phi(-u_i^T V_{t-1}/\sigma)} & \text{if } r_{ij}=1 \\ u_i^T V_{t-1} + \sigma \times \dfrac{-\Phi'(-u_i^T V_{t-1}/\sigma)}{\Phi(-u_i^T V_{t-1}/\sigma)} & \text{if } r_{ij}=-1 \end{cases}$$

$$E_{q_t}[\phi_{ij}] = \begin{cases} u_{t-1}^T v_j + \sigma \times \dfrac{\Phi'(-u_{t-1}^T v_j/\sigma)}{1-\Phi(-u_{t-1}^T v_j/\sigma)} & \text{if } r_{ij}=1 \\ u_{t-1}^T v_j + \sigma \times \dfrac{-\Phi'(-u_{t-1}^T v_j/\sigma)}{\Phi(-u_{t-1}^T v_j/\sigma)} & \text{if } r_{ij}=-1 \end{cases}$$

M step:

$$w \quad V_t = \left(\frac{I}{c} + \sum_{i=1}^{N} u_i u_i^T/\sigma^2\right)^{-1} \left(\sum_{i=1}^{N} u_i E_{q_t}[\phi_{ij}]/\sigma^2\right)$$

$$x \quad U_t = \left(\frac{I}{c} + \sum_{j=1}^{M} v_j v_j^T/\sigma^2\right)^{-1} \left(\sum_{j=1}^{M} v_j E_{q_t}[\phi_{ij}]/\sigma^2\right)$$

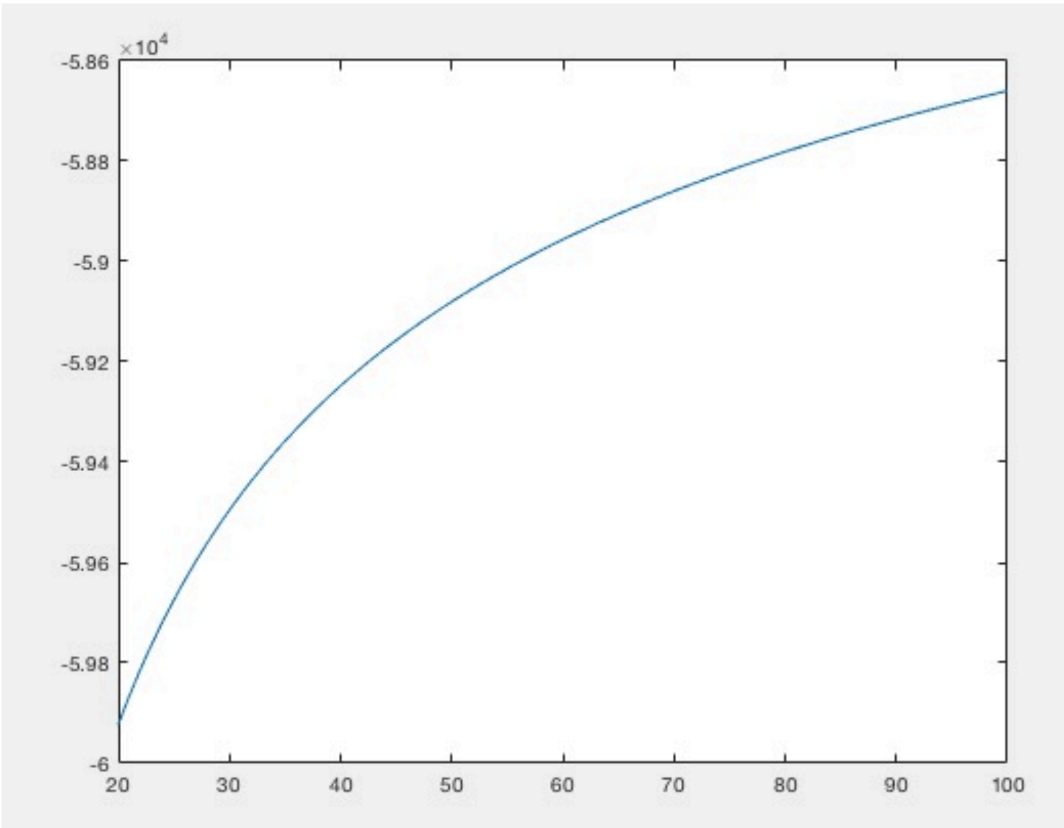
Calculate

$$\ln P(R,U,V) = (M+N)\frac{d}{2}\ln\left(\frac{1}{2\pi c}\right) - \frac{1}{2c}\left(\sum_{i=1}^{N} u_i^T u_i + \sum_{j=1}^{M} v_j^T v_j\right) + \sum_{i=1}^{N}\sum_{j=1}^{M}\left(\frac{1+r_{ij}}{2}\right)\ln\Phi(u_i^T v_j/\sigma) + \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{1-r_{ij}}{2}\ln(1-\Phi(u_i^T v_j/\sigma))$$

Problem 2

(a)

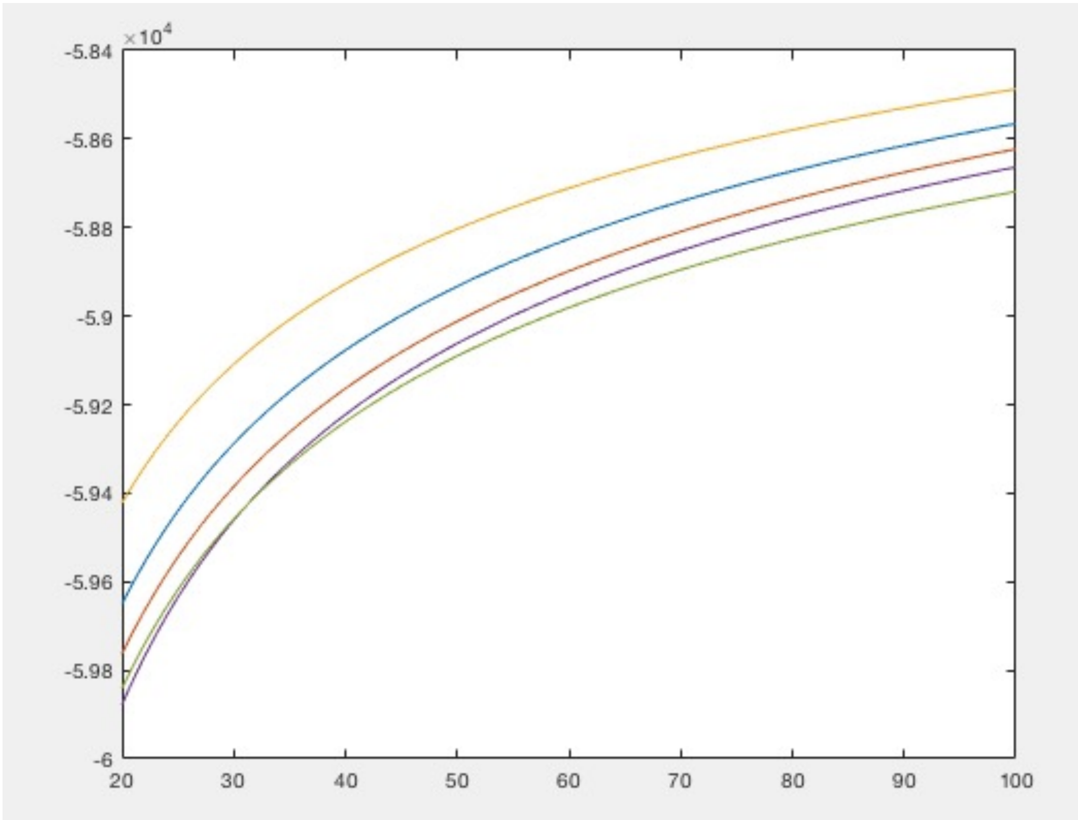
100 iterations:



(b)

Using 5 different random input points, each for 100 iterations:

With 5 different random starting points:



×10^4
-5.84
-5.86
-5.88
-5.9
-5.92
-5.94
-5.96
-5.98
-6
20
30
40
50
60
70
80
90
100