Homework Assignment (Problem Set) 3:

Note, Problem Set 3 directly focuses on Modules 5 and 6: Integer Programs, Nonlinear and Multiobjective Programming.

5 questions

Rubric:

All questions worth 30 points
30 Points: Answer and solution are fully correct and detailed professionally.
26-29 Points: Answer and solution are deficient in some manner but mostly correct.
21-25 Points: Answer and solution are missing a key element or two.
1-20 Points: Answer and solution are missing multiple elements are significantly deficient/incomprehensible.
0 Points: No answer provided.

1. An engineer at Fertilizer Company has synthesized a sensational new fertilizer made of just two interchangeable basic raw materials. The company wants to take advantage of this opportunity and produce as much as possible of the new fertilizer. The company currently has \$180 to buy raw materials at a unit price of \$8 and \$5 per unit, respectively. When amounts x_1 and x_2 of the basic raw materials are combined, a quantity q of fertilizer results given by: $q = 6x_1 + 4x_2 - 0.25x_1^2 - 0.125x_2^2$

Part A: Formulate as a constrained nonlinear program. Clearly indicate the variables, objective function, and constraints.

Variables:
- x1: Amount of raw material 1 used
- x2: Amount of raw material 2 used

Objective function:
Maximize q = 6x1 + 4x2 - 0.25x1^2 - 0.125x2^2

Constraints:
1. Budget constraint: The total cost of raw materials cannot exceed \$180.
   8x1 + 5x2 <= 180
2. Non-negativity constraint: The amount of raw materials used cannot be negative.
   x1, x2 >= 0

Part B: Solve the Program (provide exact values for all variables and the optimal objective function).

Optimal Solution:
Optimal value of x1: 11.999987353128303 (~ 12.00)
Optimal value of x2: 16.000316680853082 (~ 16.00)
Optimal objective function value: 67.99999998742419 (~ 68.00)

Python code is available in attached jupyter notebook which solves this problem and found optimal soluti on.

2. A neighbor is looking to build a rectangular fenced enclosure for his chickens and wants to build using fencing he found in his local farm supply store. He can buy at most 120 feet of fencing, and the price of the fencing is the square root of the length purchased. So he can purchase 100 feet of fencing for $10, 64 feet for $8, etc. He must also purchase fence posts to reinforce the fencing, and due to wind conditions in his yard, he must purchase more posts for east-west fencing than north-south fencing. The cost for fence and posts in the east-west direction is $3 per foot while it is $2 per foot in the north-south direction. You do not need to consider the number of posts, but simply the cost of the fence and reinforcing posts. He has $25 to spend on fencing and $150 to spend on fence posts.

Part A: Formulate the problem as a constrained nonlinear program that will enable us to maximize the area of the fenced area, with constraints. Clearly indicate the variables, objective function, and constraints.

Variables:
Let x be the length of the fence in the north-south direction (feet).
Let y be the length of the fence in the east-west direction (feet).

Objective function:
Maximize $A = x*y$

Constraints:
- Total length of fencing: $2x + 2y \leq 120$
- Cost constraint for fencing: $2\sqrt{(x)} + 2\sqrt{(y)} \leq 25$
- Cost constraint for fence posts in the north-south direction: $4x \leq 150$
- Cost constraint for fence posts in the east-west direction: $6y \leq 150$
  Combined constraints for fence post for all directions: $4x + 6y \leq 150$
- Non Negative constraint: x, y >= 0

Part B: **Solve the Program (provide exact values for all variables and the optimal objective function).**

Optimal Solution:
Optimal value of fence in the north-south direction: 18.750000432908102 (~ 18.75)
Optimal length of fence in the east-west direction: 12.49999971139557 (~12.50)
Optimal area of the fenced enclosure: 234.3750000000181 (~234.38)

Python code is available in attached jupyter notebook which solves this problem and found optimal soluti on.

3. Toy-Vey makes three types of new toys: tanks, trucks, and turtles. It takes two hours of labor to make one tank, two hours for one truck, and one hour for a turtle. The cost of manufacturing one tank is $7, 1 truck is $5 and 1 turtle is $4; a target budget of $164,000 is initially used as a guideline for the company to follow. Material requirements for the toys are shown below.

Toy	Plastic	Rubber	Metal
Tank	2	1	2
Truck	3	1	1

Turtle	4	2	0
Available	16000	5000	9000

Management has rank ordered three goals it wishes to achieve, arranged from highest to lowest priorities.

a) Minimize labor hours over 10,000 hours a week for production (40 hours for each of the 250 employees)
b) Minimize over-utilization of the weekly available supply of materials used in making the toys and place twice as much emphasis on the plastic

c) Minimize the under and over-utilization of the budget. Maximize available labor hour usage

Formulate the above decision problem as a single linear goal program. Do not solve.

*Bonus (5 points): Solve the problem and give the number of each toy to produce as well as any violations of the goals (weights don't have to add up to 1; use simple weights – 1, 2, 3, etc.).

Decision Variables:

- Let T be the number of tanks produced.
- Let R be the number of trucks produced.
- Let U be the number of turtles produced.

First writing LP form:

- Labor hours constraint: $2T + 2R + U \leq 10,000$
- Plastic constraint: $2T + 3R + 4U \leq 16,000$
- Rubber constraint: $T + R + 2U \leq 5,000$
- Metal constraint: $2T + U \leq 9,000$
- Budget constraint: $7T + 5R + 4U \leq 164,000$
- Non-negativity constraint: $T,R,U \geq 0$

GP Form:

Decision Variables:

- ρ_i - amount by which we numerically exceed the ith goal (positive deviation)
- η_i - amount by which we are numerically under the ith goal (negative deviation)

Goals:

- Labor Constraint: $2T + 2R + U + \eta 1 - \rho 1 = 10000$
- Plastic Constraint: $2T + 3R + 4U + \eta_2 - \rho_2 = 16000$
- Rubber Constraint: $T + R + 2U + \eta_3 - \rho_3 = 5000$
- Metal Constraint: $2T + U + \eta_4 - \rho_4 = 9000$
- Budget Constraint: $7T + 5R + 4U + \eta_5 - \rho_5 = 164000$
- Non-Negativity Constraints: $T,R,U,\eta_i,\rho_i \geq 0$

Objective function:

- $\text{Min}(Z) = \rho 1 + 2\rho 2 + \rho 3 + \rho 4 + \eta 5 + \rho 5 + \eta 1$

Bonus problem output:

Optimal - objective value 126333.33
After Postsolve, objective 126333.33, infeasibilities - dual 0 (0), primal 0 (0)
Optimal objective 126333.3333 - 6 iterations time 0.002, Presolve 0.00
Option for printingOptions changed from normal to all
Total time (CPU seconds):   0.00   (Wallclock seconds):   0.02

Optimal Production Plan:
Tanks: 4500.00
Trucks: 2333.33
Turtles: 0.00

Goal Deviations:
Labor Excess: 3666.67
Labor Shortfall: 0.00
Plastic Excess: 0.00
Plastic Shortfall: 0.00
Rubber Excess: 1833.33
Rubber Shortfall: 0.00
Metal Excess: 0.00
Metal Shortfall: 0.00
Budget Excess: 0.00
Budget Shortfall: 120833.33

Python code is available in attached jupyter notebook which solves this problem and found optimal solution.

4. Breaking Ad is planning its advertising campaign for a customer's new product and is going to leverage podcasts and YouTube for its advertisements. The total number of exposures per $1,000 is estimated to be 10,000 for podcasts and 7,500 for YouTube. The customer sees the campaign as successful if 750,000 people are reached and consider the campaign to be superbly successful if the exposures exceed 1 million people. The customer also wants to target its two largest age groups: 18 – 21 and 25 – 30. The total number of exposures per $1,000 for these age groups are shown below.

Age Group	Podcasts	YouTube
18 – 21	2,500	3,000
25 – 30	3,000	1,500
Total Exposures	10,000	7,500

Management has rank ordered five goals it wishes to achieve, arranged from highest to lowest priorities.

a) Successful campaign – at least 750,000 exposures
b) Limit advertising costs to $100,000.
c) Limit podcast advertising costs to $70,000
d) Superbly successful campaign – at least 1 million exposures
e) Achieve at least 250,000 exposures for each of the two age groups. However, as the 25 – 30 age group has more buying power, double the emphasis on this age group over the 18 – 21 age group

Formulate the above decision problem as a single linear goal program. Do not solve.

*Bonus (5 points): Solve the problem and give the expenditures for each media advertising campaign as well as any violations of the goals (weights don't have to add up to 1; use simple weights – 1, 2, 3, etc.).

Decision Variables:

x_1 – Ad expenditure on Podcast (in Thousand Dollars)

x_2 – Ad expenditure on YouTube (in Thousand Dollars)

Constraints:

- Total Exposures 1: $10000x_1 + 7500x_2 \geq 750000$
- Total Expenditure: $1000(x_1 + x_2) \leq 100{,}000$
- Podcast Expenditure: $1000x_1 \leq 70{,}000$
- Total Exposures 2: $10{,}000x_1 + 7{,}500x_2 \geq 1000000$
- 18-21 Age Group: $2500x_1 + 3000x_2 \geq 250000$
- 25-30 Age Group: $3000x_1 + 1500x_2 \geq 250000$
- Purchasing power between Age Groups:
  - $3000x_1 + 1500x_2 = 2(2500x_1 + 3000x_2)$
  - Rewriting above equation: $2x_1 + 4.5x_2 = 0$
- Non-Negativity Constraints: $x1, x2 \geq 0$

GP Form:

Decision Variables:

- ρ_i - amount by which we numerically exceed the ith goal (positive deviation)
- η_i - amount by which we are numerically under the ith goal (negative deviation)

Goals:

- $10000x_1 + 7500x_2 + \eta_1 - \rho_1 = 750000$
- $x_1 + x_2 + \eta_2 - \rho_2 = 100$
- $x_1 + \eta_3 - \rho_3 = 70$
- $10000x_1 + 7500x_2 + \eta_4 - \rho_4 = 1000000$
- $2500x_1 + 3000x_2 + \eta_5 - \rho_5 = 250000$
- $3000x_1 + 1500x_2 + \eta_6 - \rho_6 = 250000$
- $2x_1 + 4.5x_2 + \eta_7 - \rho_7 = 0$
- $x_1, x_2 \geq 0$ and $\eta_i, \rho_i \geq 0$*, for all i*

Objective function:

- $\text{Min } (Z) = \eta 1 + \rho 2 + \rho 3 + \eta 4 + \eta 5 + 2\eta 6$

Bonus Problem Solution:

Optimal - objective value 10
After Postsolve, objective 10, infeasibilities - dual 0 (0), primal 0 (0)
Optimal objective 10 - 3 iterations time 0.002, Presolve 0.00
Option for printingOptions changed from normal to all
Total time (CPU seconds): 0.00 (Wallclock seconds): 0.01

Optimal Advertising Campaign Plan:
Podcast Expenditure: 70.00 Thousand Dollars
YouTube Expenditure: 40.00 Thousand Dollars

Goal Deviations:
Successful Campaign Excess: 0.00
Cost Limit Shortfall: 10.00
Podcast Cost Limit Excess: 0.00
Podcast Cost Shortfall: 0.00
Superbly Successful Campaign Excess: 0.00
Superbly Successful Shortfall: 0.00
Age Group Exposure Excess (18-21): 0.00
Age Group Exposure Shortfall (18-21): 45000.00
Age Group Exposure Excess (25-30): 0.00
Age Group Exposure Shortfall (25-30): 20000.00
Purchasing Power Violation: 0.00
Purchasing Power Shortfall: 320.00

Python code is available in attached jupyter notebook which solves this problem and found optimal solution.

5. A local farmer's market sells, among other things, fresh apples during the harvest season. The market has \$750 to purchase bushels of apples from orchard 1 at \$5 per bushel, orchard 2 at \$6 per bushel, or orchard 3 at \$8 per bushel. However, the quality of the apples varies by orchard and the market can earn (in profit) \$10 per bushel from orchard 1, \$11 per bushel from orchard 2, and \$20 per bushel from orchard 3. Orchard 3 is selective with its sales and will only sell between 20 and 40 bushels to the market. That is, it will not sell to the farmer's market if they order fewer than 20 bushels and will not sell more than 40 bushels to the market. Further, orchard 1 only has 50 bushels available to sell.

Part A: Formulate this as a nonlinear program by using an indicator variable to maximize the market's profit

Let's define some variables to represent the decision-making process:

x1 = number of bushels purchased from orchard 1
x2 = number of bushels purchased from orchard 2
x3 = number of bushels purchased from orchard 3

Objective function:

- Maximize: 10x1 + 11x2 + 20x3

Constraints:

- The total cost of purchasing apples cannot exceed \$750:
  5x1 + 6x2 + 8x3 <= 750
- Orchard 1 has a maximum of 50 bushels available:
  x1 <= 50
- Orchard 3 will only sell between 20 and 40 bushels:
  20y <= x3 <= 40y

- Non-negativity constraint x_i:
  x1, x2, x3 >= 0
- Non-negativity constraint y:
  $y \in \{0,1\}$

Part B: Solve the problem and give the solution (decision variables and objective function).

Optimal Solution:
Number of bushels purchased from orchard 1: 49.99999999983959 (~ 50.00)
Number of bushels purchased from orchard 2: 29.999999999839382 (~ 30.00)
Number of bushels purchased from orchard 3: 39.99999999989095 (~ 40.00)
Indicator value for orchard 3 selection?: **Selected**
Maximum Profit: $ 1629.999999994448 (~ $1630.00)

Python code is available in attached jupyter notebook which solves this problem and found optimal solution.