Homework Assignment (Problem Set) 2:

Note, Problem Set 2 directly focuses on Modules 3 and 4: Linear Programming and the Economic Interpretation of the Dual and Sensitivity Analysis, and Network Models.

4 Questions

Rubric:
All questions worth 37.5 points
37.5 Points: Answer and solution are fully correct and detailed professionally.
25-37 Points: Answer and solution are deficient in some manner but mostly correct.
15-24 Points: Answer and solution are missing a key element or two.
1-14 Points: Answer and solution are missing multiple elements are significantly deficient/incomprehensible.
0 Points: No answer provided.

Question 1:
1A. Write the general dual problem associated with the given LP.
(Do not transform or rewrite the primal problem before writing the general dual)

Maximize $-4x_1 + 2x_2$
Subject to:
$4x_1 + x_2 + x_3 \geq 20$
$2x_1 - x_2 \leq 6$
$x_1 - x_2 + 5x_3 = -5$
$-3x_1 + 2x_2 + x_3 \geq 4$
$x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted

Minimize W = 20y1 + 6y2 – 5y3 + 4y4
Subject to:
4y1 + 2y2 + y3 -3y4 <= -4 (associated with x1 >= 0)
y1 – y2 – y3 + 2y4 >= 2 (associated with x2 <= 0)
y1 + 5y3 + y4 = 0 (associated with x3 unrestricted)
y1 >= 0, y4 >= 0 (associated with 'greater than or equal to' constraint in primal)
y2 <= 0 (associated with 'less than or equal to' constraint in primal)
y3 unrestricted

1B. Given the following information for a product-mix problem with three products and three resources.
Primal Decision Variables: x_1 = number of units 1 produced; x_2 = # of unit 2 produced; x_3 = # of unit 3 produced

Primal Formulation:

Max Z (Rev.) = $25x_1 + 30x_2 + 20x_3$
Subject To
$8x_1 + 6x_2 + x_3 \leq 50$ (Res. 1 constraint)
$4x_1 + 2x_2 + 3x_3 \leq 20$ (Res. 2 constraint)
$2x_1 + x_2 + 2x_3 \leq 25$ (Res. 3 constraint)
$x_1, x_2, x_3 \geq 0$ (Nonnegativity)

Dual Formulation:

Min W = $50\pi_1 + 20\pi_2 + 25\pi_3$
Subject To
$8\pi_1 + 4\pi_2 + 2\pi_3 \geq 25$
$6\pi_1 + 2\pi_2 + \pi_3 \geq 30$
$\pi_1 + 3\pi_2 + 2\pi_3 \geq 20$
$\pi_1, \pi_2, \pi_3 \geq 0$

Optimal Solution:
Optimal Z = Revenue = $268.75

$x_1 = 0$ (Number of unit 1)	Dual Var. Optimal Value = 22.5 (Surplus variable in 1st dual constraint)
$x_2 = 8.125$ (Number of unit 2)	Dual Var. Optimal Value = 0 (Surplus variable in 2nd dual constraint)
$x_3 = 1.25$ (Number of unit 3)	Dual Var. Optimal Value = 0 (Surplus variable in 3rd dual constraint)
Resource Constraints:	
Resource 1 = 0 leftover units	Dual Var. Optimal Value = 3.125 = π_1

Resource 2 = 0 leftover units Dual Var. Optimal Value = 5.625 = π_2
Resource 3 = 14.375 leftover units Dual Var. Optimal Value = 0 = π_3

***1Bi.* What is the fair-market price for one unit of Resource 3?**

For resource 3, we have 14.375 leftover units. Therefore, if they can be sold for more than $0/unit ($\pi$3), the revenue can increase. So fair market price is $0.

***1Bii.* What is the meaning of the surplus variable value of 22.5 in the 1st dual constraint with respect to the primal problem?**

The dual surplus variable value of 22.5 indicates that the selling price for Product 1 must be raised by at least $22.50 for it to become a part of the optimal solution. Additionally, the primal solution reveals that no units of Product 1 (i.e., x1 = 0) are produced. This suggests that Product 1 is less productive compared to the other products and would require a price increase of $22.50 to be economically viable for production.

Question 2:
Seat and Greet manufactures couches and love seats. Each couch contributes $850 to profit and each love seat, $650. The resource requirements (square feet of fabric, cubic feet of stuffing, and number or workers to complete an item in one day) and availability are shown in the table below. Marketing considerations dictate that at least 50 couches and at least 40 love seats be produced.

Item	Fabric	Stuffing	Workers
Couch	120	40	3
Love Seat	70	25	2
Available	9010	3500	250

Part A: Formulate the problem as a Linear Program.

Let x1 be the number of couches produced.
Let x2 be the number if love seats produced.

Maximize P = 850x1 + 650x2
Subject to:
120x1 + 70x2 <= 9010
40x1 + 25x2 <= 3500
3x1 + 2x2 <= 250
x1 >= 50
x2 >= 40
x1, x2 >= 0

Part B: Solve the LP (provide exact values for all variables and the optimal objective function).
Hint: The optimal objective function value is $70,450
[Note, I am providing this hint because having the optimal solution is necessary to do Part C.]

Optimal solution is arrived with value of variables as:
x1 = 50
x2 = 43

Max Z = $70,450
Complete code which finds this optimal solution is available in attached python notebook.

Part C: Answer the following questions from your output. ***(Note, do not simply rerun the model – use the Linear Programming output and Sensitivity Analysis to explain your answers.)***

i) If couches contributed $1,000 to profit, what would be the new optimal solution to the problem (decision variables and objective function)?

The new profit can be calculated by taking the new profit per couch and multiplying it with the number of couch and produce, then adding the profit per loveseat multiplied by the number of love seats produced.

The value of decision variables remains same.

Z = (new profit per couch) x1 + (profit per seat) x2
Z = $1000(50) + $650(43)
Z = $50,000 + $27,950
Z = $77,950

ii) What is the most that Seat and Greet should be willing to pay for an extra cubic foot of stuffing?

Based on the sensitivity analysis output we got in python notebook, the shadow price for the stuffing constraint is -0.0. This indicates that the objective function value (profit) does not change with an increase in the right-hand side (RHS) of the stuffing constraint.
Therefore, the most that Seat and Greet should be willing to pay for an extra cubic foot of stuffing is $0.00.

iii) If Seat and Greet were required to produce at least 45 couches, what would their profit be?

Optimal Solution:
Number of Couches (x1): 45.0
Number of Love Seats (x2): 51.571429
Objective Function Value (Profit): $71771.42885

When Seat and Greet are required to produce at least 45 couches, the optimal solution is as follows:

- Number of Couches (x1): 45.0
- Number of Love Seats (x2): 51.571429

The resulting profit for Seat and Greet under this constraint is approximately $71,771.43.

iv) Seat and Greet is considering producing reclining chairs. A reclining chair contributes $500 to profit and requires 30 square feet of fabric, 15 cubic feet of stuffing, and 2 workers to produce a chair in one day. Should Seat and Greet produce any reclining chairs? Again, do not rerun the model.

To determine whether Seat and Greet should produce reclining chairs, we can use the shadow prices (dual values) obtained from the sensitivity analysis. The shadow prices give us the change in profit for each unit increase in the resource availability.
Given the shadow prices from the previous sensitivity analysis:

- Fabric constraint (C1): Shadow Price = -0.0
- Stuffing constraint (C2): Shadow Price = -0.0
- Workers constraint (C3): Shadow Price = -0.0

For the reclining chair:

- Profit per reclining chair = $500
- Fabric requirement per reclining chair = 30 square feet
- Stuffing requirement per reclining chair = 15 cubic feet
- Workers requirement per reclining chair = 2

Let's analyze the impact of producing one more reclining chair on the profit:
Change in Profit = (Profit per reclining chair) + (Shadow Price of Fabric * Fabric requirement per reclining chair) + (Shadow Price of Stuffing * Stuffing requirement per reclining chair) + (Shadow Price of Workers * Workers requirement per reclining chair)
Change in Profit = $500 + (-0.0 * 30) + (-0.0 * 15) + (-0.0 * 2) Change in Profit = $500

Since producing one more reclining chair would increase the profit by $500, which is a positive change, Seat and Greet should produce reclining chairs to maximize their profit.

Question 3:
Suppose you are in the market to buy a new car for $20,000. The total maintenance costs for this car are dependent on its age in years (see table below). So, the total maintenance of a car after 3 years is $7,000, not ($3,000 + $5,000 + $7,000 = $15,000). However, you can avoid growing maintenance costs by trading in the car at any point, the value of which is also dependent on its age in years (see same table below). Suppose that if you trade a car in at any point in the next 5 years, the cost of the new car you purchase is still $20,000. You hope to minimize the net cost of having a car over the next five years (purchase costs + maintenance costs – trade-in value).

	Age of Car	Maintenance	Trade-in
1	0	$3000	N/A
2	1	$5000	$12000
3	2	$7000	$10000
4	3	$13000	$6000
5	4	$20000	$2000
6	5	N/A	$1000

Part A: Formulate this as a shortest-path network problem and draw the network. As a hint, think about the cost associated with having the car from year 1 to year 2 - $20,000 + $3,000 - $12,000 = $11,000. What is it from year 1 to year 3, or year 2 to year 3?

Nodes:
In this problem, the nodes represent the different years of owning the car.
Our network will have six nodes (1, 2, 3, 4, 5, and 6).

Arcs and Costs:
- An arc (i, j) represents purchasing a new car at the beginning of year i and trading it in at the beginning of year j.
- The cost cij of arc (i, j) is calculated as: cij=cost of purchasing car at the beginning of year i + maintenance cost during year i −trade-in value received at the beginning of year j

Using this formula, we can calculate the costs **cij** for all valid pairs of nodes

Calculation of Costs:
- c12 = 11,000
- c13 = 15,000
- c14 = 21,000
- c15 = 31,000
- c16 = 39,000
- c23 = 11,000
- c24 = 15,000
- c25 = 21,000
- c26 = 31,000
- c34 = 11,000
- c35 = 15,000
- c36 = 21,000
- c45 = 11,000
- c46 = 15,000
- c56 = 11,000

Draw the network:

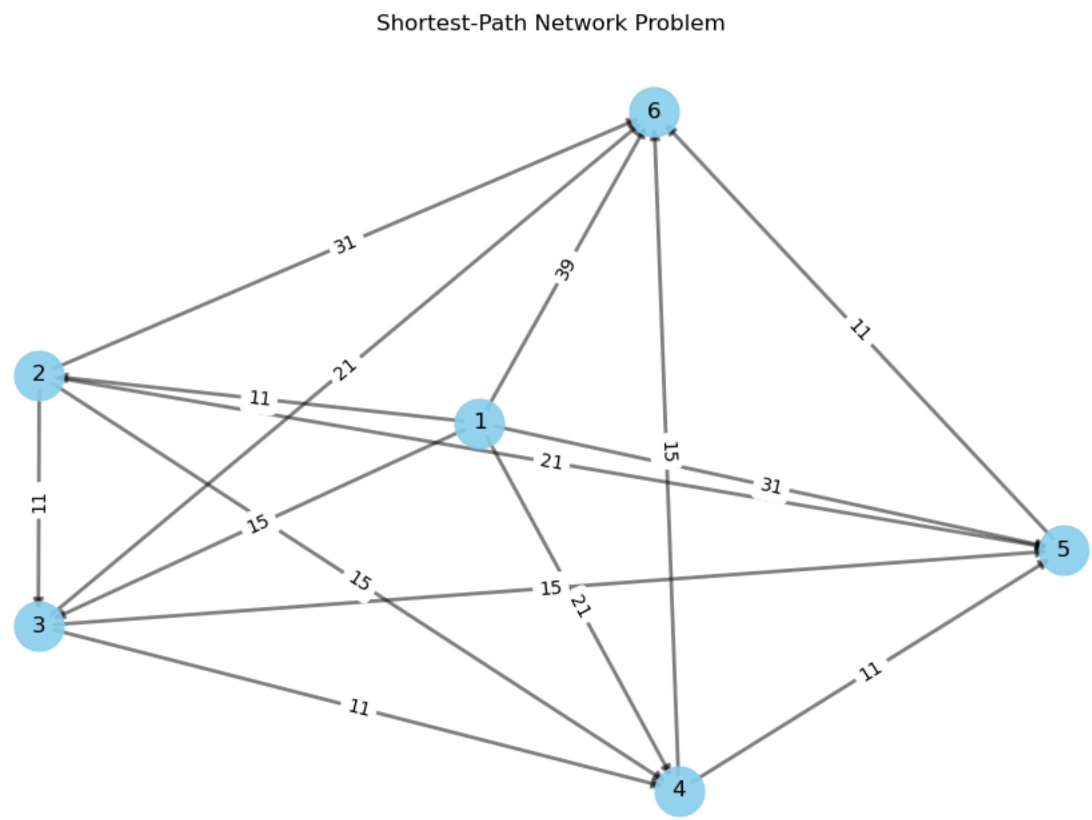


Fig 1: Network flow has been drawn using python networkx package. All values are in thousands(1000's)

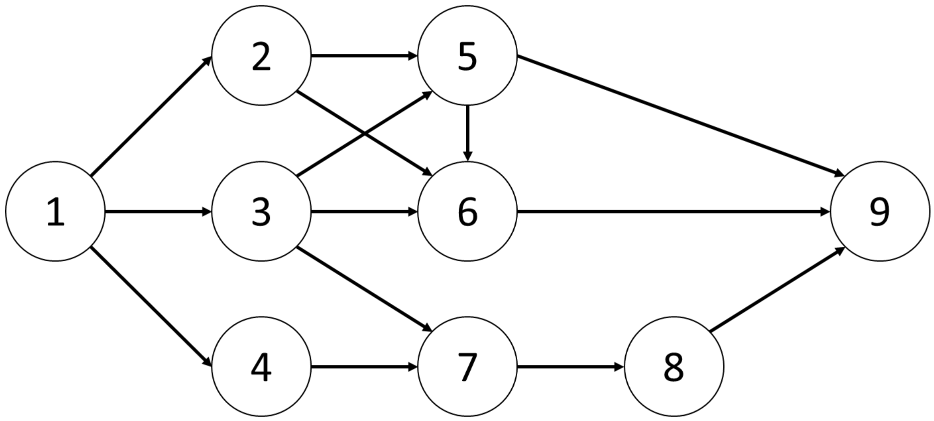
Part B: Solve the problem and give the best plan (shortest path) to have a car for the next 5 years at the lowest cost.

The shortest path from node 1 to node 6 is: [1, 3, 6]
The total cost for the shortest path is: $36000

Complete code which finds this optimal solution is available in attached python notebook.

Question 4:
An average of 900 cars enter a traffic network (shown below) each hour, trying to make it from node 1 to node 9. The maximum number of cars that can pass over each arc and the time it takes a car to drive each arc is shown below. Use a minimum cost network flow model to minimize the total time for the 900 cars to drive from node 1 to node 9 (assuming all enter at the same time and traffic jams do not exist).

i	j	Max cars	Time (min)
1	2	500	16
1	3	500	30
1	4	400	7
2	5	300	17
2	6	350	12
3	5	400	1
3	6	300	4



3	7	250	18
4	7	400	20
5	6	200	13
5	9	600	5
6	9	900	5
7	8	700	2
8	9	700	7

Part A: Formulate the problem as a minimum cost network flow problem

Nodes and Arcs:
- We have nodes from **1 to 9**. These are the points in the network, where car can enter or leave, or be transferred from one arc to another.
- The arcs (directed edges) represent the roads connecting the nodes, with each arc having a certain capacity (maximum number of cars that can pass per hour) and travel time (cost).

Cost:
- This is the time it takes for a car to drive each arc. In a minimum cost network flow problem, we aim to minimise the total cost.

Supply and Demand:
- **Supply**: node 1 has a supply of 900 cars, which represents the cars entering the network.
- **Demand:** node 9 has a demand of 900 cars, which represents the cars that need to arrive at the destination.

  The formulations involves setting up a flow variable for each arc, representing the number of cars, travelling that arc, the objective is to minimise the total travel time to the constraint of the capacities and the conservation of flow at each note (except for the source and sink nodes).

Objective Function:
- We want to minimize the total time for the 900 cars to drive from node 1 to node 9.
- The total time is the sum of the time taken on each arc, which can be expressed as:

Minimize Z = 16f_12 + 30f_13 + 7f_14 + 17f_25 + 12f_26 + 1f_35 + 4f_36 + 18f_37 + 20f_47 + 13f_56 + 5f_59 + 5f_69 + 2f_78 + 7f_89

Constraints:
- Flow conservation at intermediate nodes (for each except 1 and 9):
  Node 2: (f_12 = f_25 + f_26)
- Node 3: (f_13 = f_35 + f_36 + f_37)
- Node 4: (f_14 = f_47)
- Node 5: (f_25 + f_35 = f_56 + f_59)
- Node 6: (f_26 + f_36 + f_56 = f_69)
- Node 7: (f_37 + f_47 = f_78)
- Node 8: (f_78 = f_89)

Capacity constraints:

- $f\_12 \le 500$
- $f\_13 \le 500$
- $f\_14 \le 400$
- $f\_25 \le 300$
- $f\_26 \le 350$
- $f\_35 \le 400$
- $f\_36 \le 300$
- $f\_37 \le 250$
- $f\_47 \le 400$
- $f\_56 \le 200$
- $f\_59 \le 600$
- $f\_69 \le 900$
- $f\_78 \le 700$
- $f\_89 \le 700$

Supply and demand constraints:
- Supply at node 1: (f_12 + f_13 + f_14 = 900)
- Demand at node 9: (f_59 + f_69 + f_89 = 900)

Non-negativity constraints:
- $f\_ij \ge 0$ for all arcs (i,j)

Part B: Solve the problem and provide the solution (decision variables and objective function).

Status: Optimal
Flow_('1',_'2') = 350.0
Flow_('1',_'3') = 400.0
Flow_('1',_'4') = 150.0
Flow_('2',_'5') = 0.0
Flow_('2',_'6') = 350.0
Flow_('3',_'5') = 400.0
Flow_('3',_'6') = 0.0
Flow_('3',_'7') = 0.0
Flow_('4',_'7') = 150.0
Flow_('5',_'6') = 0.0
Flow_('5',_'9') = 400.0
Flow_('6',_'9') = 350.0
Flow_('7',_'8') = 150.0
Flow_('8',_'9') = 150.0

Total travel time: 31350.0 minutes

Complete code which finds this optimal solution is available in attached python notebook.

Question 5:
A university has three professors who each teach four courses per year. Each year, four sections of marketing, finance, and production must be offered. At least one section of each class must be offered during each semester (fall and spring). Each professor's time preferences and preference for teaching various courses are given below.
The total satisfaction a professor earns teaching a class is the sum of the semester satisfaction and the course satisfaction. Thus, professor 1 derives a satisfaction of 3 + 3 = 6 from teaching marketing during the fall semester.

	Professor 1	Professor 2	Professor 3
Fall Term	3	5	4
Spring Term	4	3	5
Marketing	3	5	5
Finance	7	4	7
Production	5	7	6

Part A: Formulate the problem as a minimum cost network flow problem that can be used to assign professors to courses to maximize the total satisfaction of the three professors. Draw the network and identify the nodes and arcs.

- P1, P2, P3 are the nodes for the three professors.
- MF stands for the combination (Marketing, Fall),
- MS for (Marketing, Spring),
- FF for (Finance, Fall),
- FS for (Finance, Spring),
- PF for (Production, Fall),
- PS for (Production, Spring) .
- The numbers on the edges are cost(capacities). The numbers on the nodes are the net flow. The cost of an edge is computed as follows.
- Cost of edge P1 → MF is (Marketing, Professor 1) + (Fall, Professor 1), where (Marketing, Professor 1), (Fall, Professor 1) are entries in the table.

Objective Function:
Max Z = 6p1mf + 7p1ms + 10p1ff + 11p1fs + 8p1pf + 9p1ps + 10p2mf + 8p2ms + 9p2ff + 7p2fs + 12p2pf + 10p2ps + 9p3mf + 10p3ms + 11p3ff + 12p3fs + 10p3ps + 11p3ps

Subject to:
- P1 = 4
- P2 = 4
- P3 = 4
- Mf >=1
- Ms >=1
- FF>=1
- FS >=1
- PF>=1
- PS>=1

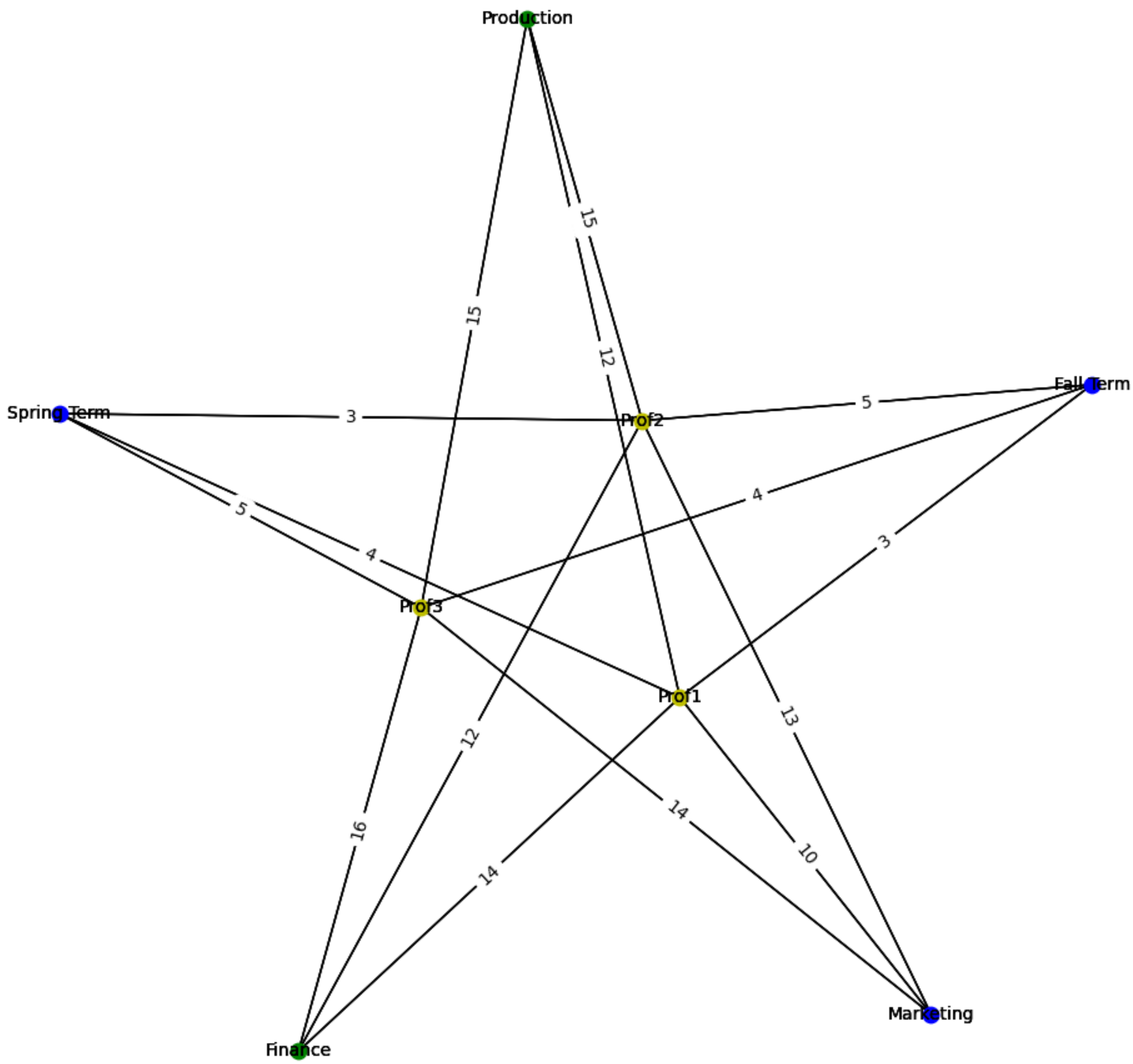


Fig 2: Network Diagram with total satisfaction score labels on edges

Part B: Solve the problem (provide exact values for all variables and the optimal objective function).

Optimal - objective value 130
Optimal objective 130 - 8 iterations time 0.002
Option for printingOptions changed from normal to all
Total time (CPU seconds): 0.00 (Wallclock seconds): 0.00

Finance_Fall_Prof1 = 1.0
Finance_Fall_Prof2 = 0.0
Finance_Fall_Prof3 = 0.0
Finance_Spring_Prof1 = 3.0
Finance_Spring_Prof2 = 0.0

Finance_Spring_Prof3 = 0.0
Marketing_Fall_Prof1 = 0.0
Marketing_Fall_Prof2 = 1.0
Marketing_Fall_Prof3 = 0.0
Marketing_Spring_Prof1 = 0.0
Marketing_Spring_Prof2 = 0.0
Marketing_Spring_Prof3 = 3.0
Production_Fall_Prof1 = 0.0
Production_Fall_Prof2 = 3.0
Production_Fall_Prof3 = 0.0
Production_Spring_Prof1 = 0.0
Production_Spring_Prof2 = 0.0
Production_Spring_Prof3 = 1.0

Objective function value = 130.0

Complete code which finds this optimal solution is available in attached python notebook.