Quantum Fields I, Spring 2025

Final Exam (Due by May 19)

Consider φ^3 theory described by the bare action

$$\mathcal{A}[\varphi_0] = \int d^6x \left[\frac{1}{2}\,(\partial\varphi_0)^2 \; + \; \frac{m_0^2}{2}\,\varphi_0^2 \; + \; \frac{\lambda_0}{3!}\varphi_0^3 + \kappa_0\,\varphi_0\right] \tag{1}$$

in the (Euclidean) space-time of the dimension $d = 6$. Ignore the fact that the action is not bounded from below[1]. An auxiliary "coupling" κ_0 is introduced to simplify bookkeeping: it is assumed that $\kappa_0 = \kappa_0(m_0^2, \lambda_0)$ is adjusted in such a way that

$$\langle\varphi_0(x)\rangle = 0 \tag{2}$$

order by order in the perturbation theory.

I. **Perturbation theory:** Define, as usual, the momentum-space correlation functions $W^{(n)}(p_1, ..., p_n)$, and the proper vertices $\Gamma^{(n)}(p_1, ..., p_n)$ as

$$\Gamma^{(1)} = 0\,, \tag{2'}$$

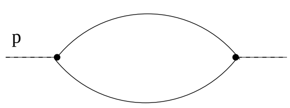
$$\Gamma^{(2)}(p) = \left[W^{(2)}(p)\right]^{-1} := p^2 + m_0^2 + \Sigma_0(p)\,,$$

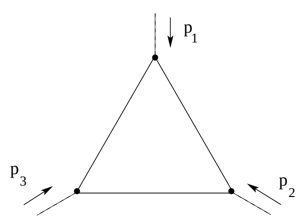
and

$$-\Gamma^{(n)}(p_1, ..., p_n) =$$
$$\{\text{sum of 1-p irreducible diagrams contributing to } \; W^{(n)}(p_1, ..., p_n)\}\,.$$

(Eq.(2') expresses the condition (2).)

(a)[20pt] Consider the diagrams representing the one-loop contributions to $\Sigma_0(p)$ and $\Gamma^{(3)}(p_1, p_2, p_3)$ (Figures below). Write down the contributions, including the combinatorial factors, in the form of 6-momentum integrals (you are not required to evaluate the momentum integrals at this stage).





[1]Although the full functional integral $\int D[\varphi_0](...)e^{-\mathcal{A}[\varphi_0]}$ does not converge, the expansion around the saddle point makes sense order by order in the coupling constant.

(b)[10pt] Discuss the question of UV (non)convergence of the diagrams in (a), and when divergent, estimate dominating cutoff dependence of the regularized integrals.

II. **Renormalized perturbation theory:**

(a)[15pt] Determine superficial degree of divergence for all proper vertices $\Gamma^{(n)}(p_1, ..., p_n)$. Identify primitive divergences, and write down all counterterms needed to absorb the UV divergences.

(b)[20pt] Write the action (1) as

$$\mathcal{A} = \int d^6x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{3!}\varphi^3 + \text{counterterms} \right] . \tag{3}$$

where the normalization of the field

$$\varphi = Z^{-1/2}\varphi_0 \tag{4}$$

and the renormalized parameters m^2 and λ are defined according to the normalization conditions

$$\Gamma^{(2)}(p) = p^2 + m^2 + O\left((p^2)^2\right) , \tag{5}$$

$$\Gamma^{(3)}(p_1, p_2, p_3)\big|_{p_1=0, p_2=0} = \lambda , \tag{6}$$

and Eq.(2'). The proper vertices are defined in terms of the correlation functions of the re-normalized field (4). Using dimensional regularization, find explicitly the leading-order contributions to the counterterm coefficients.

III. **Perturbative renormalization group:**

(a) [15pt] For *massless* theory (3) (defined by the condition $\Gamma^{(2)}(p)\big|_{p^2=0} = 0$) consider scale-dependent renormalization scheme, defined by the conditions (instead of (5),(6))

$$\frac{d}{dp^2}\Gamma^{(2)}(p)\big|_{p^2=\mu^2} = 1 , \tag{5'}$$

$$\Gamma^{(3)}(p_1, p_2, p_3)\big|_{p_1^2=p_2^2=p_3^2=\mu^2} = \lambda_{(\mu)} , \tag{6'}$$

Find the leading order contributions to all conterterm coefficients (again, use dimensional regularization). At $d = 6$ calculate $\Sigma(p)$ and

$$\Gamma^{(3)}(p^2) := \Gamma^{(3)}(p_1, p_2, p_3)\big|_{p_1^2=p_2^2=p_3^2=p^2}$$

in the renormalized perturbation theory, to the order λ^2 (for Σ) and λ^3 (for $\Gamma^{(3)}(p^2)$). (You don't have to evaluate the full momentum integral for $\Gamma^{(3)}$, just isolating its dependence on the normalization scale would suffice.)

(b) [20pt] In the massless theory, consider renormalized proper vertices $\Gamma^{(n)}(p_1, ..., p_n)$ as the functions of the renormalized coupling constant $\lambda := \lambda_{(\mu)}$. Write down the Callan-Symanzick equation. From the results of IIIa, derive the Renormalization Group beta function $\beta(\lambda)$ to the order λ^3, and gamma function $\gamma(\lambda)$ to the order λ^2. Discuss the properties of the RG flow generated by this β-function at small λ. Compare to the flow in $d = 4$ φ^4 theory (See Lecture 18).

IV [extra credit]. Consider the theory (1) in d-dimensional space-time with $d = 6 - \epsilon$ with $\epsilon << 1$. Find the RG β-function and γ-function in the leading approximation at $\epsilon << 1$. Observe that at $d < 6$ there is a non-trivial fixed point at pure imaginary $\lambda = ig$ (It is known as the Lee-Yang fixed point in the theory of phase transitions.) Find the anomalous dimensions of all relevant operators at this non-trivial point, in the leading order in ϵ.

Useful equations

1. Volume of $S^{(d-1)}$, a unit sphere in d dimensions

$$\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}\,, \qquad \Omega_6 = \pi^3\,.$$

2. Feynman parameterizations

$$\frac{1}{AB} = \int_0^1 \frac{du}{(uA + (1-u)B)^2}\,.$$

3. Some dimensionally regularized integrals

$$\int \frac{d^d k}{(2\pi)^d}\,\frac{1}{(k^2+\Delta)^a} = \frac{\Delta^{\frac{d}{2}-a}}{(4\pi)^{\frac{d}{2}}}\,\frac{\Gamma(a-d/2)}{\Gamma(a)}\,.$$