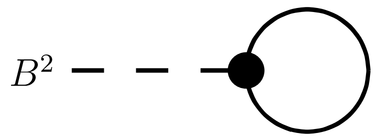
The leading order in g comes from the one-loop diagrams. The simplest (and the most important) of those is

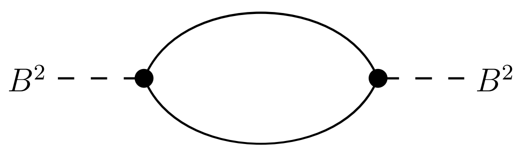


$$= \frac{\delta_{jk}\, B^i_\mu B^i_\mu - B^j_\mu B^k_\mu}{2g} \int \frac{d^2p}{(2\pi)^2}\, \frac{g\,\delta_{jk}}{p^2} =$$

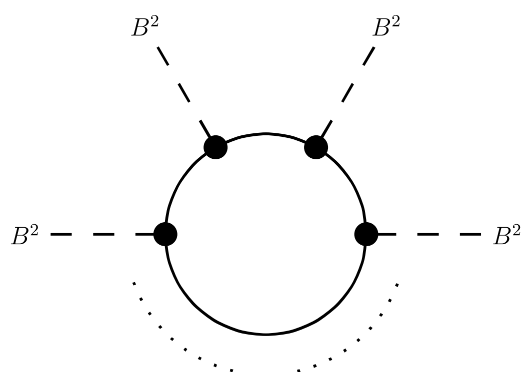
$$= \frac{(N-2)\left(B^i_\mu\right)^2}{2}\, \frac{1}{4\pi} \int_{\Lambda^2/L^2}^{\Lambda^2} \frac{dp^2}{p^2} = \frac{N-2}{4\pi}\left(\partial_\mu \mathbf{n}_0\right)^2 \log L\,.$$

Thus, this diagram brings in a term identical in the form to the original action, i.e. it renormalizes the coupling parameter g.

The diagram

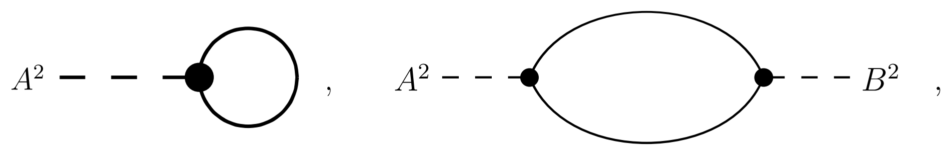


generates term $\sim B^4$ (with the coefficient $\sim 1/\Lambda^2$); the field B^4 has dimension $4 > d$ and hence it is an irrelevant field; it can be neglected in the leading approximation. Likewise, all the diagrams of the form



can be ignored, for the same reason.

Let us take a look at the diagrams with A-vertices. Consider the following ones



Both these diagrams show logarithmic behavior,

$$\int \frac{d^2k}{k^2} \sim \log L\,, \qquad \text{and} \qquad \int \frac{d^2k\ k^\mu k^\nu}{k^2\,k^2} \sim \log L$$

However, it is not difficult to show that these two logarithmic contributions in fact cancel each other. One can do it by a direct calculation, but it is easy to prove this statement on the basis of certain symmetry.

The $\log L$ of each of these two diagrams is of the form

$$\log(L)\ A_\mu^{ij} A_\mu^{ij}\,,$$

where, as before

$$A_\mu^{ij} = \mathbf{e}^j \partial_\mu \mathbf{e}^i$$

Recall that $\mathbf{e}^i(x)$ are arbitrary vectors tangent to $S^{(N-1)}$ at $\mathbf{n}_0(x)$. They satisfy

$$\mathbf{e}^i \mathbf{n}_0 = 0\,, \qquad \mathbf{e}^i \mathbf{e}^j = \delta^{ij}\,,$$

but otherwise arbitrary. One can always choose them differently,

$$\tilde{\mathbf{e}}^i(x) = \Omega_{ij}(x) \mathbf{e}^j(x)\,,$$

where $\Omega_{ij}(x)$ is orthogonal $(N-1) \times (N-1)$ matrix,

$$\Omega(x) \Omega^t(x) = I\,.$$

$\Omega(x)$ is "slow" (for $\mathbf{e}^i(x)$ were assumed to be slow), but otherwise arbitrary function of x. Under this local orthogonal transformations, $A_\mu^{ij}(x)$ as a $O(N-1)$ gauge potential,

$$\tilde{A}_\mu(x) = \Omega(x) A_\mu(x) \Omega^t(x) + \Omega(x) \partial_\mu \Omega^t(x)\,,$$

where $A_\mu(x)$ stands for $(N-1) \times (N-1)$ anti-symmetric matrix $A_\mu^{ij}(x)$.

As this gauge transformation describes just the freedom in the choice of variables, the action $\mathcal{A}_1[\mathbf{n}_0]$ must be gauge invariant. It is easy to check that the combination

$$A_\mu^{ij} A_\mu^{ij} = tr(A_\mu A_\mu^t)$$

is not gauge invariant, and thus it can not appear in $\mathcal{A}_1$. The simplest gauge invariant local combination of course is

$$tr(F_{\mu\nu} F_{\mu\nu}) \,,$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \,.$$

However it involves extra derivative, and thus corresponds to an irrelevant term.

Therefore, in this approximation the only interesting contribution comes from the above loop diagram with single B^2 vertex. We have

$$\mathcal{A}_1 = \int \left(\frac{1}{2g} - \frac{N-2}{4\pi} \log L \right) (\partial_\mu \mathbf{n}_0)^2 \; d^2x \;. \tag{22.8}$$

Thus, the step I of the RG transformation leads to

$$\frac{1}{g} \;\to\; \frac{1}{g_1} = \frac{1}{g} - \frac{N-2}{2\pi} \log L \,.$$

The step 2 is nearly trivial. If one changes

$$\mathbf{n}_0(x) = \mathbf{n}(x/L)$$

the action does not change. Therefore

$$RG_l : \quad \frac{1}{g} \;\to\; \frac{1}{g} - \frac{N-2}{2\pi} \, l \,.$$

In other words, the beta-function is

$$\beta(g) = -\frac{N-2}{2\pi} \, g^2 + O(g^3) \,,$$

and for small g the RG flow equation

$$\frac{dg}{dl} = \frac{N-2}{2\pi} \, g^2$$

shows that $g(l)$ *increases* towards large distances; at $d = 2$ the Goldstone fixed point becomes unstable.

The above flow equation determines (in the leading order in g) the behavior of the action under the RG transformation. One might be interested in the behavior of correlation functions of, say, the field $\mathbf{n}(x)$. To that end one needs then to find the RG function $\gamma_{\mathbf{n}}(g)$ in the CS equation

$$\left(x\frac{\partial}{\partial x} + n\,\gamma_{\mathbf{n}}(g) + \beta(g)\frac{\partial}{\partial g}\right) \langle\, \mathbf{n}(x_1)...\mathbf{n}(x_n)\,\rangle_c = 0\,.$$

It is defined in terms of the action of the RG transformation on the field $\mathbf{n}(x)$.

Consider any correlation function involving $\mathbf{n}(x)$,

$$\langle\, \mathbf{n}(x)...\,\rangle\,,$$

and assume again that the momentum p associated with x is $<< \Lambda/L$. Then, splitting as before

$$\mathbf{n} = \mathbf{n}_0\,\sqrt{1-\phi^2} + \mathbf{e}^i\,\phi_i \simeq \mathbf{n}_0\,\big(1 - \frac{1}{2}\,\phi^2\big) + \mathbf{e}^i\,\phi_i$$

one needs to find expectation value of this combination over the ensemble of the fast modes ϕ. The last term can be ignored (its expectation value is zero), and we have

$$\mathbf{n} \;\to\; \mathbf{n}_0\,\big(1 - \frac{1}{2}\,\langle\,\phi_i^2\,\rangle\big) := Z_{\mathbf{n}}^{-\frac{1}{2}}(L)\,\mathbf{n}_0\,,$$

where

$$Z^{-\frac{1}{2}}(L) = 1 - \frac{1}{2}\,\langle\,\phi_i^2\,\rangle = 1 - \frac{N-1}{2}\,\frac{g}{2\pi}\,l\,.$$

Since at $d = 2$ $\mathbf{n}$ has zero canonical dimension, we then have

$$\gamma_{\mathbf{n}}(g) = D_{\mathbf{n}}(g) = -\frac{d}{dl} Z_{\mathbf{n}}^{-\frac{1}{2}}(l)\bigg|_{l=0} = \frac{N-1}{4\pi}\,g\,.$$

In the above calculation we have assumed that $gl << 1$; that is why the higher-order terms in ϕ were neglected. But, as usual, we can extend it to larger l. Generally, we have

$$RG_l\,\mathbf{n} = Z_{\mathbf{n}}^{-\frac{1}{2}}(l)\,\mathbf{n}\,,$$

where $Z^{-\frac{1}{2}}(l)$ is a solution of the equation

$$-\frac{d}{dl} Z_{\mathbf{n}}^{-\frac{1}{2}}(l) = \gamma_{\mathbf{n}}(g(l))\, Z_{\mathbf{n}}^{-\frac{1}{2}}(l)\,,$$

and $g(l)$ is the integral curve of the equation

$$\frac{dg(l)}{dl} = -\beta(g(l))\,.$$

In our case

$$\frac{1}{g(l)} = \frac{1}{g(0)} - \frac{N-2}{2\pi}\, l\,.$$

Recall that the "flow" of $Z_{\mathbf{n}}^{-\frac{1}{2}}(l)$ determines the scale dependence of the correlation functions

$$\langle\, \mathbf{n}(x_1/L)...\mathbf{n}(x_n/L)\,\rangle_{g(l)} = Z_{\mathbf{n}}^{-\frac{n}{2}}(l)\,\langle\, \mathbf{n}(x_1)...\mathbf{n}(x_n)\,\rangle_{g(0)}\,,$$

where $l = \log L$.

Exercise: For the $O(N)$ nonlinear sigma model,
I. At $d = 2$
a) Assuming $g(l) << 1$, find $Z_{\mathbf{n}}(l)$.
b) Derive the following $|x| \to 0$ asymptotic of the two-point correlation function

$$\langle\, \mathbf{n}(x)\mathbf{n}(0)\,\rangle \;\sim\; \left[\log\left(\frac{1}{x^2}\right)\right]^{\frac{N-1}{N-2}}\,.$$

Explain how this result is in agreement with the exact solution of the model at $N \to \infty$ (PS, §13.3, pages 463-465)

II. Find the flow equation for $g(l)$ and $Z_{\mathbf{n}}(l)$ at $d = 2+\epsilon$, with $\epsilon << 1$. Again, determine the $|x| \to 0$ behavior of the two-point correlation function.

Coming back to the flow equation ($d = 2$)

$$\frac{dg}{dl} = \frac{N-2}{2\pi}\, g^2 + ...$$

one observes that (at $N > 2$) the flow of the coupling $g(l)$ looks like this:

At large distances g increases, and the interaction becomes strong. Our weak coupling analysis is no longer valid (in particular the higher order terms in $\beta(g)$ can not be neglected). On the other hand, if we follow the flow of $g(l)$ backward, to $l \to -\infty$,

$$g(l) = \left(\mathrm{Const} - \frac{N-2}{2\pi}\, l\right)^{-1} \;\to\; \frac{2\pi}{N-2}\,(-l)^{-1}\,,$$

the interaction of the Goldstone modes become weaker, and the short-distance behavior is controlled by a free theory. For this reason, this type of RG flow is called *asymptotic freedom*. Note that the situation is opposite to that in φ^2 theory at $d=4$, where instead $\lambda \to 0$ at long scales, whereas perturbation theory breaks down at short distances.

The asymptotic freedom allows one to establish existence of local field theory; indeed it shows that at short scales the fluctuations are suppressed, and the cutoff can be removed with no violation of locality. On the other hand, at longer scales the fluctuations become stronger. It is not possible to describe these large scale fluctuations by perturbation theory. Nonetheless, one can make intuitive picture based on this pattern of the RG flow. If one concentrate attention on very short scales, the vectors $\mathbf{n}(x)$ are nearly parallel, because at $l \to -\infty$ $g(l)$ is small, thus suppressing the gradients of $\mathbf{n}(x)$. However, if one looks at larger scales, the directions of $\mathbf{n}(x)$ at the points separated by distance $\sim R_c$ are much less correlated, and if the separation id even much larger, there is no correlation.

short scale   large scale

This picture suggests that at $d=2$ the phase with spontaneously broken $O(N)$ symmetry does not exist. In other words, $d=2$ magnet has no ferromagnetic phase with spontaneous magnetization. Let me stress that the perturbative analysis of the RG flow does not prove this statement, which requires much more involved arguments. Note also that this picture is expected to apply only if $N > 2$ [28].

What is different at $d > 2$? This question can be studied in details in formal case of $d = 2 + \epsilon$ with $\epsilon << 1$. In this case we can repeat our calculation of the RG flow at small g. The step 1 (elimination of the fast part) yields essentially the same result,

$$\mathcal{A}_1 = \int \left(\frac{1}{2g} - \frac{N-2}{4\pi} \Lambda^\epsilon \log L \right) (\partial_\mu \mathbf{n}_0)^2 \; d^2x \; . \tag{22.9}$$

However, the step 2 (scale transformation) is no longer exactly trivial. The change of variables

$$\mathbf{n}_0(x) = \mathbf{n}(x/L)$$

leads to

$$\int (\partial_\mu \mathbf{n}_0)^2 \, d^d x = L^{d-2} \int (\partial_\mu \mathbf{n})^2 \, d^d x \, .$$

Therefore

$$\mathcal{A}' = \int L^\epsilon \left(\frac{1}{2g} - \frac{N-2}{4\pi} \Lambda^\epsilon \log L \right) (\partial_\mu \mathbf{n}_0)^2 \; d^2x \; .$$

Since

$$L^\epsilon \approx 1 + \epsilon \, \log L$$

one finds that under the RG transformation

$$\frac{1}{g} \;\to\; \frac{1}{g} + \left(\frac{\epsilon}{g} - \frac{N-2}{2\pi} \right) \log L \, ,$$

[28] At $N = 2$ the g^2 term in the β-function vanishes. In fact, it is easy to see that all higher terms vanish as well. At $N = 2$ the direction $\mathbf{n}(x)$ can be parameterized by a single angle $\theta(x)$, $\mathbf{n}(x) = \big(\cos\theta(x), \sin\theta(x)\big)$, and formally

$$\mathcal{A} = \frac{1}{2g} \int (\partial_\mu \theta)^2 \, d^2x \, ,$$

which is just the action of free massless theory. The action of RG on a free theory is trivial. However, beyond the perturbation theory, the $N = 2$ sigma model has more complex structure. This is because the field $\theta(x)$ is an angular variable, $\theta(x) \sim \theta(x) + 2\pi \, N(x)$, where $N(x)$ is integer-valued. As the result, there are topologically nontrivial field configurations ("vortices", or "instantons"). Their interaction leads to remarkable phase transition at finite $g = g_c$ - the so called Berezinsky - Kosterlitz - Thouless transition (See e.g. Kosterlitz, Thouless, "Ordering, metastability and phase transitions in two-dimensional systems", Journal of Physics C: Solid State Physics, Vol. 6 pages 1181-1203 (1973)).

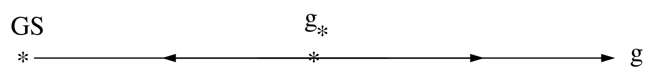
where I have ignored the factor $\Lambda^{\epsilon} \approx 1$ [29]. The associated RG flow equation is

$$\frac{dg}{dl} = -\epsilon\, g + \frac{N-2}{2\pi}\, g^2 = \frac{N-2}{2\pi}\, g(g - g_*)\,,$$

where

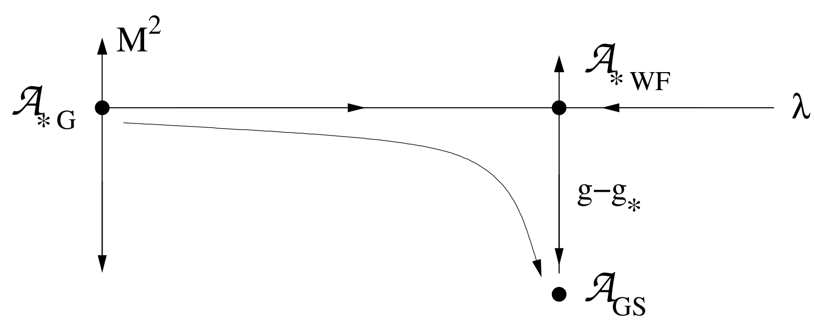
$$g_* = \frac{2\pi\,\epsilon}{N-2}\,.$$

Due to the small linear term, the flow now looks different at small $g \sim \epsilon$,



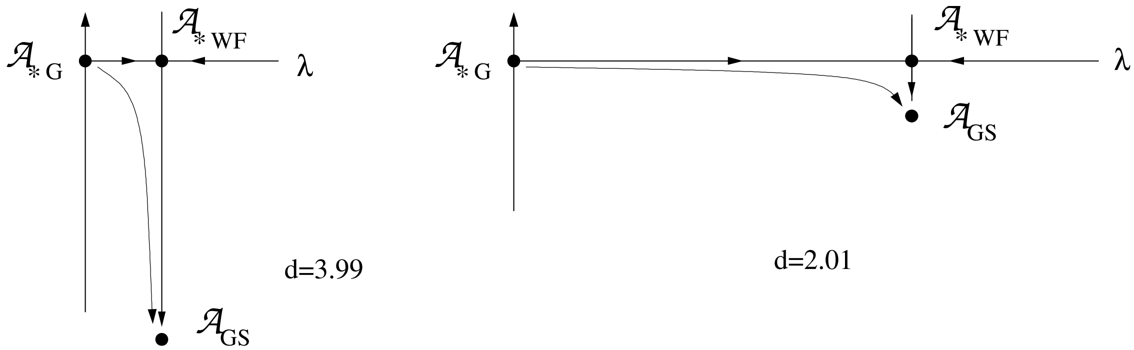
Nontrivial new fixed point at $g = g_*$ appears, and if $g < g_*$ the theory flows towards the Goldstone fixed point at large scale.

It is natural to identify the new fixed point g_* with the Wilson-Fisher fixed point at $d = 2 + \epsilon$. Indeed, general pattern of RG flow in $O(N)$ φ^4 theory suggested by our previous analysis was as follows



Here M^2 and λ are two couplings in φ^4 theory. At closely below 4 the Wilson-Fisher fixed point appears close to $\mathcal{A}_{*,G}$, while the Goldstone fixed point is located far away from it. When d goes down from 4 $\mathcal{A}_{*,G}$ departs from $\mathcal{A}_{*,WF}$, but $\mathcal{A}_{*,GS}$ gets closer.

[29] One can get rid of this factor by changing to a dimensionless coupling $g \to \Lambda^{-\epsilon}\, g$.



We can identify the coordinate along the trajectory going from $\mathcal{A}_{*,WF}$ to $\mathcal{A}_{*,GS}$ with the coupling g of the non-linear sigma model, with the origin $g = 0$ at $\mathcal{A}_{*,GS}$; then $\mathcal{A}_{*,WF}$ appear at some g_*. At $d = 2 + \epsilon$ with small ϵ $\mathcal{A}_{*,GS}$ approaches $\mathcal{A}_{*,WF}$ ($g_* \sim \epsilon$). At $d = 2$ the fixed points $\mathcal{A}_{*,GS}$ and $\mathcal{A}_{*,WF}$ merge, and the flow of g (at positive g) becomes marginally relevant. This pattern is confirmed by exact solution of the theory at $N \to \infty$ (partly described in §13.3 of PS).

23 RG and Energy-Momentum Tensor

Renormalization Group describes the way the scale transformations are realized in QFT. The scale transformations

$$x^\mu \to \tilde{x}^\mu = x^\mu / L$$

constitute particular class of general coordinate transformations

$$x^\mu \to \tilde{x}^\mu = f^\mu(x)\,,$$

where f^μ are arbitrary smooth functions. Variations of the action functional [30]

$$\mathcal{A} = -\int d^d x\, \mathcal{L}(x) \tag{23.1}$$

under these coordinate transformations in a local field theory is generally described in terms of its energy-momentum tensor $T^{\mu\nu}(x)$. Therefore this field, and in particular its trace

$$\Theta(x) := T^\mu_\mu(x)\,,$$

plays important role in RG theory. In fact, all basic equations of RG theory in local QFT can be expressed as certain properties of this field.

In the first lecture of this course we have defined the Energy-Momentum tensor (or EM tensor, for shortness), in a classical field theory, as the Noether's current associated with the translational symmetry $x^\mu \to x^\mu + a^\mu$,

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\, \partial_\nu \varphi - \delta^\mu_\nu\, \mathcal{L}\,,$$

which satisfies the continuity equation

$$\partial_\mu\, T^\mu_\nu = 0 \tag{23.2}$$

[30]The sign minus in this formula corresponds to the relation

$$-\mathcal{A} = iS = i\int dt\, d\mathbf{x}\, \mathcal{L}(x) = \int dx^4\, d\mathbf{x}\, \mathcal{L}(x) = -\int dx^4\, d\mathbf{x}\, \mathcal{L}_E(x)\,.$$

where $\mathcal{L}(x)$ is the usual Lagrangian density with t replaced by $-ix_4$, see Lecture 7.

as a consequence of the classical equations of motion. Generally, there is an important ambiguity in the definition of the local EM tensor. One can redefine T^{μ}_{ν} by adding a total derivative

$$T^{\mu}_{\nu} \to T^{\mu}_{\nu} + \partial_\lambda \psi^{\mu\lambda}_{\nu} \,, \qquad \psi^{\mu\lambda}_{\nu} = -\psi^{\lambda\mu}_{\nu} \,.$$

The new tensor still satisfies the continuity equation (23.2). By appropriate choice of ψ the modyfied EM tensor can always be made symmetric, $T^{\mu\nu} = T^{\nu\mu}$.

More general way to define the EM tensor is to consider the field theory in the space with nontrivial metric

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \,. \tag{23.3}$$

In this case the action

$$\mathcal{A}[\varphi, g] = -\int d^d x \sqrt{g}\, \mathcal{L}(\varphi, \partial_\mu\varphi, g_{\mu\nu})$$

becomes a functional of the metric tensor g also.

For instance, for generic scalar field theory, in the presence of the metric background (23.3), the simplest action is

$$\mathcal{A} = \int \sqrt{g} \left[\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\varphi \partial_\nu\varphi + V(\varphi) \right].$$

This form is called the "minimal coupling to gravity".

In any case, the field $T^{\mu\nu}(x)$ is defined in terms of the variation of the action with respect to the metric (sign!),

$$\delta\mathcal{A} = -\frac{1}{2} \int d^d x \sqrt{g}\; T^{\mu\nu}(x)\, \delta g_{\mu\nu}(x) \,. \tag{23.4}$$

Note that $T^{\mu\nu}$ defined this way is automatically symmetric,

$$T^{\mu\nu}(x) = T^{\nu\mu}(x) \,,$$

as $g_{\mu\nu}$ is.

Note that for flat space this definition is still ambiguous, as one could couple the theory to the external metric in many different ways. For example, in the scalar field theory one could add the "non-minimal" terms like

$$\int d^d x \sqrt{g}\; R\, W(\varphi) \,, \qquad \text{or} \qquad \int d^d x \sqrt{g}\; R_{[\mu\lambda][\nu\rho]}\, W^{[\mu\lambda][\nu\rho]}(\varphi)$$

where R is the curvature, and W is arbitrary. These terms disappears in the flat space, however it affects the variation (23.4) even when the metric is set back to the flat Euclidean one. It is possible to show that all such "non-minimal" modifications of the action in curved space lead to modifications of the EM tensor of the form

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\lambda \partial_\rho Y^{[\mu\lambda][\nu\rho]}\,, \tag{23.5}$$

where Y is antisymmetric in $[\mu\lambda]$ and in $[\nu\rho]$ but symmetric w.r.t. the interchange of these pairs (this is exactly the symmetry of the Riemann's curvature tensor $R_{[\mu\lambda][\nu\rho]}$).

This is intrinsic ambiguity which cannot be resolved unless we specify exactly how the system behaves in curved space. Note that if Y decays sufficiently fast at $x \to \infty$, such modifications do not affect the notions of energy and momentum,

$$E = \int d^{d-1}\mathbf{x}\, T^{00}\,, \qquad P^i = \int d^{d-1}\mathbf{x}\, T^{0i}\,,$$

as they add total derivatives to the integrands. If we are interested just in the theory in flat metric, fixing this ambiguity it is the matter of convenience.

Consider now quantum field theory. The correlation functions are defined through the functional integrals

$$\langle \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\rangle = Z^{-1} \int [\mathcal{D}\varphi]\ \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\ e^{-\mathcal{A}[\varphi,g]}\ , \tag{23.6}$$

where for the moment I consider the system in external gravitational field $g_{\mu\nu}(x)$. Here $\mathcal{O}(x)$ are some local fields, which can be the field $\varphi(x)$ itself, or some composite fields built from φ and its derivatives at the point x. In the presence of the external metric these composite fields may depend on this metric in a local way, i.e. thay may involve $g_{\mu\nu}(x)$ and its derivatives at the point x.

Consider a small variation around the flat metric,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \delta g_{\mu\nu}(x)\,.$$

For generic variation the correlation functions change in a nontrivial way because the theory indeed depends on the background metric. However, the change is simple if we take $\delta g_{\mu\nu}(x)$ corresponding to pure coordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu(x) \tag{23.7}$$

with infinitesimal $\varepsilon^\mu(x)$; in this case

$$-\delta g_{\mu\nu}(x) = \partial_\mu \varepsilon_\nu(x) + \partial_\nu \varepsilon_\mu(x)\,.$$

The coordinate transformation is just a relabeling of the degrees of freedom $\varphi(x)$, so it amounts to a change of variables in the functional integral (23.6). Therefore the numerical value of this integral does not change. This leads to the identity

$$0 = \sum_{i=1}^{n} \langle \mathcal{O}_1(x_1)\cdots \delta_\varepsilon \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n)\rangle + \int d^d x\, \partial_\mu \varepsilon_\nu(x)\, \langle T^{\mu\nu}(x)\, \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\rangle\ , \tag{23.8}$$

where $\delta_\varepsilon \mathcal{O}_i(x)$ stand for local variations of the fields $\mathcal{O}_i(x_i)$ under the coordinate transformation (23.7). The identities of this type are generally known as the *Ward identities*.

What can be said about the variations $\delta_\varepsilon \mathcal{O}_i(x)$ on general grounds? These again are local fields which depend on $\varepsilon^\mu(x)$

(i) linearly, i.e. $\delta_{\varepsilon_1+\varepsilon_2}\mathcal{O} = \delta_{\varepsilon_1}\mathcal{O} + \delta_{\varepsilon_2}\mathcal{O}$, and

(ii) localy, i.e. $\delta_\varepsilon \mathcal{O}(x)$ depends on $\varepsilon^\mu(x)$ and its derivatives taken at the point x.

In fact, the part of $\delta_\varepsilon \mathcal{O}(x)$ which contains just $\varepsilon^\mu(x)$, with no derivatives, is completely fixed,

$$\delta_\varepsilon \mathcal{O}(x) = \varepsilon^\mu(x)\, \partial_\mu \mathcal{O}(x) + \text{terms with } \partial_\nu \varepsilon^\mu(x) \text{ and higher derivatives}\ ,$$

because for constant $\varepsilon^\mu(x) = \varepsilon^\mu$ we have

$$\mathcal{O}(x) \to \mathcal{O}(x+\varepsilon) = \mathcal{O}(x) + \varepsilon^\mu\, \partial_\mu \mathcal{O}(x)\ .$$

Moreover, the part which contains the first derivatives $\partial_\nu \varepsilon^\mu(x)$ is also severely restricted. For instance, if $\varphi(x)$ is a scalar field, and we assume that it transforms undependently on other fields, we can write

$$\delta_\varepsilon \varphi(x)(x) = \varepsilon^\mu(x)\, \partial_\mu \varphi(x) + \frac{D}{d}\, (\partial_\mu \varepsilon^\mu)\, \varphi(x)\,,$$

where d is the dimensionality of the space, and the parameter D is related to the dimension of φ. This generalizes to more complicated scalar composite fields $\mathcal{O}_\alpha$ as follows

$$\delta_\varepsilon \mathcal{O}_\alpha(x) = \varepsilon^\mu(x)\, \partial_\mu \mathcal{O}_\alpha(x) + \frac{D_\alpha^\beta}{d}\, (\partial_\mu \varepsilon^\mu)\, \mathcal{O}_\beta(x) + \text{terms with } \partial_\nu \partial_\lambda \varepsilon^\mu\ , \text{ etc}\ .$$

For tensor fields the terms involving local rotations

$$\omega_{\mu\nu}(x) = \partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x)$$

will also appear.

Going back to the Ward Identity (23.8), we observe that integral there can be transformed by parts, hence we have

$$0 = \sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \cdots \delta_\varepsilon \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle - \int d^d x \, \varepsilon_\nu(x) \, \langle \partial_\mu T^{\mu\nu}(x) \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \; . \tag{23.9}$$

As the first term here depends only on $\varepsilon(x)$ (and its derivatives) at the points $x_1, \cdots, x_n$ only, we conclude that

$$\langle \partial_\mu T^{\mu\nu}(x) \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = 0 \quad \text{for} \quad x \neq x_1, \cdots, x_n \, ,$$

i.e. in quantum field theory the continuity equation (23.2) is satisfied in the weak sense,

$$\partial_\mu T^{\mu\nu}(x) \simeq 0 \, ,$$

exactly as the equations of motion do.

Now let us consider particular case of the coordinate transformations, the scale transformations (also called the dilations)

$$x^\mu \to x^\mu + \varepsilon \, x^\mu \, .$$

In this case

$$\partial_\mu \varepsilon_\nu(x) = \delta_{\mu\nu} \, \varepsilon \, ,$$

while all higher derivatives of $\varepsilon(x)$ vanish, and so for scalar fields $\mathcal{O}_\alpha$ we have

$$\delta_\varepsilon \mathcal{O}_\alpha(x) = \varepsilon \, (\mathcal{D}\mathcal{O})_\alpha(x) \, ,$$

where by definition

$$(\mathcal{D}\mathcal{O})_\alpha(x) = x^\mu \frac{\partial}{\partial x^\mu} \, \mathcal{O}_\alpha(x) + D_\alpha^\beta \, \mathcal{O}_\beta(x) \, . \tag{23.10}$$

Then the Ward identity (23.8) reduces to

$$\sum_{i=1}^{n} \langle \mathcal{O}_{\alpha_1}(x_1) \cdots (\mathcal{D}\mathcal{O})_{\alpha_i}(x_i) \cdots \mathcal{O}_{\alpha_n}(x_n)\rangle = \int d^d x \langle \Theta(x) \mathcal{O}_{\alpha_1}(x_1) \cdots \mathcal{O}_{\alpha_n}(x_n)\rangle \quad (23.11)$$

where

$$\Theta(x) = T_\mu^\mu(x)$$

is the trace component of the EM tensor.

Now let us assume that our QFT depends on some number N of parameters (the "coupling constants") $\lambda^k, k = 1, \cdots, N$. These parameters enter the action $\mathcal{A}$. The derivatives

$$\Phi_k(x) = -\frac{\partial}{\partial \lambda^k} \mathcal{L}(x)$$

are in general some scalar composite fields. Then [31]

$$\frac{\partial}{\partial \lambda^k} \langle \mathcal{O}_{\alpha_1}(x_1) \cdots \mathcal{O}_{\alpha_n}(x_n)\rangle = -\int d^d x \, \langle \Phi_k(x) \mathcal{O}_{\alpha_1}(x_1) \cdots \mathcal{O}_{\alpha_n}(x_n)\rangle$$

If we assume that $\Theta(x)$ can be expanded in terms of $\Phi_k(x)$,

$$\Theta(x) = -\sum_{k=1}^{N} \beta^k(\lambda) \, \Phi_k(x) \, , \quad (23.12)$$

one arrives again at the Callan-Symanzik equation, i.e.

$$\sum_{i=1}^{n} \langle \mathcal{O}_{\alpha_1}(x_1) \cdots (\mathcal{D}\mathcal{O})_{\alpha_i}(x_i) \cdots \mathcal{O}_{\alpha_n}(x_n)\rangle + \sum_{k=1}^{N} \beta^k(\lambda) \frac{\partial}{\partial \lambda^k} \langle \mathcal{O}_{\alpha_1}(x_1) \cdots \mathcal{O}_{\alpha_n}(x_n)\rangle = 0 \, , \quad (23.13)$$

where the coefficients $D_\alpha^\beta(\lambda)$ in the operator $\mathcal{D}$ in (23.10) can of course depend on the parameters λ. The possibility to expand the field $\Theta(x)$ in terms of finitely many "basic" fields $\Phi_k(x)$ is the statement of renormalizability of the field theory.

[31] Of course, in general the fields $O_\alpha(x)$ may depend on the couplings λ. If such dependence is present, the derivatives of O_α would add to the matrix D_α^β, but otherwise would not change our argument. Here I ignore this contribution for simplicity.

The energy-momentum tensor $T^{\mu\nu}(x)$ is special composite field. As we know, in interacting field theory composite fields in general may require infinite renormalizations. However, the above derivation of the Ward identity (23.8) suggests that in the case of the EM tensor no infinite renormalization is needed. Indeed, the correlation functions in (23.8) are understood as the correlation functions of fully renormalized fields $\mathcal{O}_\alpha$, they remain finite when the regularization is removed. Therefore the EM tensor defined through the variation (23.4) must have finite correlation functions without additional renormalization. It is possible to check this statement in renormalized perturbation theory in $d = 4$ φ^4 theory.

Conformal invariance at the fixed-point

If $\lambda^k = \lambda^k_*$ such that

$$\beta^k(\lambda_*) = 0\,,$$

the trace of the EM tensor vanishes in view of (23.12)

$$\Theta(x) = 0\,. \tag{23.14}$$

Remark: The above statement ignores important subtlety. The derivation of the Callan-Symanzik equation (23.13) relies in fact on the integrated version of (23.12). Locally, Eq.(23.12) may hold only up to total derivatives, i.e. the terms of the form $\partial_\mu K^\mu(x)$, with $K^\mu(x)$ being some local field. Thus, vanishing of the beta-functions only implies that $\Theta(x) = \partial_\mu K^\mu(x)$. Unless K^μ is itself a total derivative (i.e. $\Theta = \partial_\mu\partial_\nu L$, in which case the ambiguity (23.5) can be used to redefine $T_{\mu\nu}$ to eliminate its trace), presence of such a term would spoil the argument for conformal invariance. In some cases (in particular, in $d = 2$ unitary quantum field theories) this possibility can be excluded. Generally it is not excluded, and therefore vanishing of the beta-functions does not guarantee conformal invariance. Moreover, in general, counterexamples can be found. For more on this subtlety, see *J.Polchinski, Renormalization and effective Lagrangians, Nucl.Phys. B231, 269 (1984).* In the following discussion I assume that the fixed point in question admits the EM tensor which satisfies (23.14).

The correlation functions of this fixed-point theory enjoy scale invariance with the dimensions of the fields $\mathcal{O}_\alpha$ determined by the eigenvalues of the matrix

$$D^\beta_\alpha = D^\beta_\alpha(\lambda_*)\,.$$

In fact, vanishing of the trace component $\Theta(x)$ in the fixed-point theory guarantees much bigger symmetry. Namely, let $v^\mu(x)$ be any vector field which satisfies the equation

$$\partial_\mu v_\nu(x) + \partial_\nu v_\mu(x) = \rho(x)\,\delta_{\mu\nu} = \frac{2}{d}\,\delta_{\mu\nu}\,(\partial_\lambda v^\lambda(x))\,. \tag{23.15}$$

Consider the coordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu(x) \quad \text{with} \quad \varepsilon^\mu(x) = \varepsilon\, v^\mu(x)\,, \tag{23.16}$$

where ε is infinitesimal parameter. For such ε^μ

$$\partial_\mu \varepsilon_\nu\, T^{\mu\nu} = \rho(x)\,\Theta(x) = 0\,,$$

and the second term in the Ward identity (23.8) disappears, i.e.

$$\sum_{i=1}^{n} \langle \mathcal{O}_1(x_1)\cdots \delta_\varepsilon \mathcal{O}_i(x_i)\cdots \mathcal{O}_n(x_n)\rangle = 0\,. \tag{23.17}$$

We obtain the closed equation

$$\sum_{i=1}^{n} \langle \Phi_{\alpha_1}(x_1)\cdots \mathcal{D}_v \Phi_{\alpha_i}(x_i)\cdots \Phi_{\alpha_n}(x_n)\rangle = 0\,, \tag{23.18}$$

where

$$\mathcal{D}_v \Phi_\alpha(x) = \left(v^\mu(x)\,\frac{\partial}{\partial x^\mu} + \frac{D_\alpha}{d}\,\left(\partial_\mu v^\mu(x)\right)\right)\Phi_\alpha(x)\,,$$

and $\Phi_\alpha(x)$ are the fields which diagonalize the matrix D_α^β, and D_α are corresponding eigenvalues.

The coordinate transformations (23.16) which satisfy the equation (23.15) are called the *conformal transformations*. It is possible to find all solutions of this equation. The situation turns out to be rather different in the cases $d > 2$ and $d = 2$.

If $d > 2$ the general solution has the form[32]

$$v^\mu(x) = a^\mu + \omega^{\mu\nu}\,x^\nu + l\,x^\mu + (b^\mu\,x^2 - 2\,x^\mu\,(bx))\,,$$

[32]The Lie algebra of these vector fields is isomorphic to $so(d+1,1)$, the algebra of infinitesimal Lorentz transformations of $d+2$-dimensional space-time.

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. The parameters a, ω, l, b are interpreted as follows:

a^μ	$-$	translations
$\omega^{\mu\nu}$	$-$	rotations
l	$-$	dilations
b^μ	$-$	special conformal transformations

As the first three are very familiar, I will present only the finite version of the special conformal transformations,

$$x^\mu \to \tilde{x}^\mu = \frac{x^\mu + B^\mu\, x^2}{1 + 2\,(Bx) + B^2\, x^2}\,,$$

where B^μ is an arbitrary vector.

Invariance of the fixed-point theory w.r.t. the special conformal transformations provides further limitations on the correlation functions. Consider the Ward identity (23.18) with

$$v^\mu(x) = b^\mu\, x^2 - 2\, x^\mu\, (bx)\,.$$

In this case

$$\mathcal{D}_v \Phi_\alpha(x) = \left[(b^\mu\, x^2 - 2\, x^\mu\, (bx))\, \frac{\partial}{\partial x^\mu} - 2\, D_\alpha\, (bx) \right] \Phi_\alpha(x)\,,$$

and (23.18) becomes nontrivial closed differential equation for the correlation functions. In fact, this equation completely determines one- , two- and three-point correlation functions, up to overall constant factors (Homework Problem 10)

If $d = 2$ the equation

$$\partial_\mu v_\nu + \partial_\mu v_\nu - \delta_{\mu\nu}\,(\partial_\lambda v^\lambda) = 0 \tag{23.19}$$

has infinitely many solutions. This is very easy to see by going to the complex coordinates in $d = 2$ space,

$$z = x^1 + i\, x^2\,, \qquad \bar{z} = x^1 - i\, x^2\,.$$

Then (23.19) leads to two equations

$$\partial_{\bar{z}} v^z = 0\,, \qquad \partial_z v^{\bar{z}} = 0\,,$$

which shows that $v^z(z)$ ($v^{\bar{z}}(\bar{z})$) can be arbitrary holomorphic (anti-holomorphic) function. This rich symmetry has important consequences for the structure of $d = 2$ conformal field theories.

Scale invariance of the fixed-point theory allows one to establish some important properties of QFT associated with its unstable manifold.

Scale invariance and density of states

Important conclusion can be made about its density of states. The density of states is directly related to the equilibrium state entropy $S(E)$ of the system. The form of $S(E)$ in a scale invariant quantum field theory can be established directly by general arguments. However, I will use this opportunity to display some general features of QFT at finite temperature.

The functional integral of d-dimensional Euclidean quantum field theory

$$\langle \, ... \, \rangle = Z^{-1} \int D[\varphi] \, (...) \, e^{-\mathcal{A}[\varphi]}$$

has formal interpretation as expectation value over a thermal equilibrium ensemble in *classical* statistical mechanics in d spatial dimensions. We have used this interpretation many times already. Here I would like to discuss Euclidean functional integral representation of *quantum* statistical mechanics of fields in d dimensional space-time (i.e. $d-1$ dimensional space).

Recall that the density matrix of thermal equilibrium state in quantum statistics has the general form[33]

$$\hat{P} = Z^{-1}(\beta) \, e^{-\beta \hat{H}} \, , \tag{23.20}$$

where $\beta = (kT)^{-1}$. The exponential here coincides with the time evolution operator at pure imaginary value of time $t = -i\hbar\beta$ (I set $\hbar = 1$ henceforth). This form is still valid if one is dealing with the equilibrium state of quantum fields. The imaginary-time evolution operator has representation as the functional integral,

$$\langle \, \varphi_f(\mathbf{x}) | e^{-\beta \hat{H}} | \varphi_i(\mathbf{x}) \, \rangle = \int D[\varphi(\mathbf{x}, \tau)]_{\substack{\varphi(\mathbf{x},0)=\varphi_i(\mathbf{x}) \\ \varphi(\mathbf{x},\beta)=\varphi_f(\mathbf{x})}} \, e^{-\mathcal{A}[\varphi]} \tag{23.21}$$

[33] The normalization factor $Z(\beta)$ here is different from the factor Z in e.g. Eq.(7.1). $Z(\beta)$ goes to Z in the low temperature limit $\beta \to \infty$ (see below).

Here $\mathbf{x}$ are $d-1$ dimensional "spatial" coordinates, and τ is the imaginary time (so that $x = (\mathbf{x}, \tau)$). The integration is over the functions $\varphi(x) = \varphi(\mathbf{x}, \tau)$ in the slab $0 \leq \tau \leq \beta$, with the boundary conditions

$$\varphi(\mathbf{x}, 0) = \varphi_i(\mathbf{x}) \,, \qquad \varphi(\mathbf{x}, \beta) = \varphi_f(\mathbf{x}) \,.$$

The partition sum of the quantum system at the finite temperature is the trace

$$Z(\beta) = \mathrm{tr}\big(e^{-\beta \hat{H}}\big) \,, \tag{23.22}$$

i.e. the sum of its diagonal matrix elements

$$\int D[\varphi(\mathbf{x})] \, \langle \, \varphi(\mathbf{x}) | \, e^{-\beta \hat{H}} | \varphi(\mathbf{x}) \, \rangle \,. \tag{23.23}$$

If the representation (23.21) is used, the integration over $\varphi(\mathbf{x})$ in (23.23) combines with the integral $D[\varphi(x)]$ in (23.21) to produce integral over all fields $\varphi(\mathbf{x}, \tau)$, with $0 \leq \tau \leq \beta$, satisfying the periodic boundary condition

$$\varphi(\mathbf{x}, 0) = \varphi(\mathbf{x}, \beta) \,. \tag{23.24}$$

Thus, we have

$$Z(\beta) = \int D[\varphi(\mathbf{x}, \tau)]_{\varphi(\mathbf{x},0)=\varphi(\mathbf{x},\beta)} \, e^{-\mathcal{A}[\varphi]} \,. \tag{23.25}$$

Geometrically, this is the functional integral over the functions $\varphi(x)$ on a d-dimensional cylinder $\mathbb{R}^{d-1} \times S^1$, with τ being the coordinate on the circle of circumference β, $\tau \sim \tau + \beta$. Of course this is nothing but the Matsubara representation specified to the quantum field theory at hand.

Now, let us change interpretation of the functional integral (23.25). We can take one of the coordinates on $\mathbb{R}^{d-1}$, say x_1, and claim it to be the imaginary time. The coordinate τ will be treated as one of the spatial coordinates. In this interpretation the imaginary time spans infinite range, $-\infty < x_1 < \infty$, while the "space" is $\mathbb{R}^{d-2} \times S^1$ is the $d-1$ dimensional cylinder, with the coordinates $(\mathbf{x}_{\|}, x_{\perp})$. Here $\mathbf{x}_{\|} = (x_2, ..., x_{d-1})$, and $x_{\perp} = \tau$. The later is compactified,

$$x_{\perp} \sim x_{\perp} + \beta \,.$$

Since now the imaginary time spans large (in the limit, infinite) range, (23.25) is dominated by the contribution from the ground state,

$$Z(\beta) \sim e^{-L \, E_0(\beta)} \,, \tag{23.26}$$

where $L \to \infty$ is the range of x_1, and $E_0(\beta)$ is the ground state energy of the quantum field in the spatial geometry $\mathbb{R}^{d-2} \times S^1$.

This geometric setting is similar to that we discussed in the context of the Casimir effect, with β replacing the separation between the plates a. The difference is that in the Casimir case we had a fixed boundary condition $\varphi = 0$ at the plates, which now is replaced by the periodic boundary condition (23.24). For the free massless field it is easy to repeat the calculation of the Casimir energy (I recommend it as an Exercise), with the result

$$E_0(\beta) = -\frac{C\, V_\parallel}{\beta^{d-1}}\,, \tag{23.27}$$

where C is some dimensionless (d-dependent) constant, and

$$V_\parallel = \int d^{d-2}\mathbf{x}_\parallel$$

is the volume of the $\mathbb{R}^{d-2}$ component of the space. In fact, apart from the value of the constant, the form (23.27) is dictated by scale invariance of the theory. Indeed, the ground state energy is an extensive quantity, which at large $V_\parallel$ must be proportional to this volume. Since the theory has no intrinsic scale, the β-dependence then follows from the dimensional analysis. This simple argument shows that the form (23.27) applies to any scale-invariant field theory, id est fixed point field theory. The constant C of course would be different for different fixed points.

Returning to original interpretation - the quantum field at finite temperature $kT = \beta^{-1}$, and using (23.26) we have for the free energy of a scale-invariant field theory

$$\beta\, F(\beta) = -\log(Z(\beta)) = -\frac{C\, V_{d-1}}{\beta^{d-1}}\,, \tag{23.28}$$

where $V_{d-1} = V_\parallel \times L$ is the volume of the space. Of course, one could have established the form (23.28) directly, by dimensional analysis (the free energy F, being extensive characteristic, must be $\sim V_{d-1}$, whereas the quantity (23.28) is dimensionless), but I believe our detour through the Casimir energy was instructive.

The finite temperature partition function is closely related to the energy density of states. We have

$$Z(\beta) = \int_0^\infty e^{-\beta\, E}\, \rho(E)\, dE\,,$$

where $\rho(E)\,\Delta E$ is the number of states in the energy interval between E and $E+\Delta E$. Mathematically, $Z(\beta)$ is the Laplace transform of $\rho(E)$. The inverse transform is

$$\rho(E) = \int_{-i\infty}^{+i\infty} e^{\beta\,E}\, Z(\beta)\,\frac{d\beta}{2\pi i} = \int_{-i\infty}^{+i\infty} e^{\beta\,E}\, e^{\frac{C\,V_{d-1}}{\beta^{d-1}}}\,\frac{d\beta}{2\pi i}\,.$$

At large V_{d-1} the integral is dominated by the saddle point at

$$\beta = \beta_0(E) := \left(\frac{C\,(d-1)\,V_{d-1}}{E}\right)^{1/d}\,,$$

and we find

$$\rho(E) \sim e^{S(E)}\,, \tag{23.29}$$

where

$$S(E) = \tilde{C}\;V_{d-1}^{1/d}\;E^{1-1/d} \tag{23.30}$$

where $\tilde{C}$ is another constant, related to C in (23.27)[34].

By the definition, the quantity $S(E)$ in (23.29) coincides with the equilibrium state entropy of the system at the temperature $kT = 1/\beta_0(E)$. The equation (23.30) expresses the universal scale invariant form of the entropy. Indeed, the entropy is an extensive property which must have the form $V_{d-1}\,\sigma$, where the entropy density $\sigma = \sigma(\varepsilon)$ can depend on the energy density $\varepsilon := E/V_{d-1}$. On the other hand $S(E)$, being a dimensionless quantity, can only depend on the dimensionless combination $V_{d-1}E^{d-1}$. Eq. (23.30) is the only form that satisfies these conditions. Thus, the asymptotic density of states (23.29)(23.30) can be deduced directly, from the scale invariance of the fixed point theory. Nonetheless, our previous discussion is instructive as it displays some basics of the finite-temperature formalism of QFT.

Note that in any dimensionality $S(E)$ grows slower then the exponential $\sim E$. This observation is useful in comparing to the string theories, whose typical energy density is the exponential[35].

It is straightforward to argue that the energy density (23.29), (23.30) is valid asymptotically, at large E, in any quantum field theory originating at certain critical

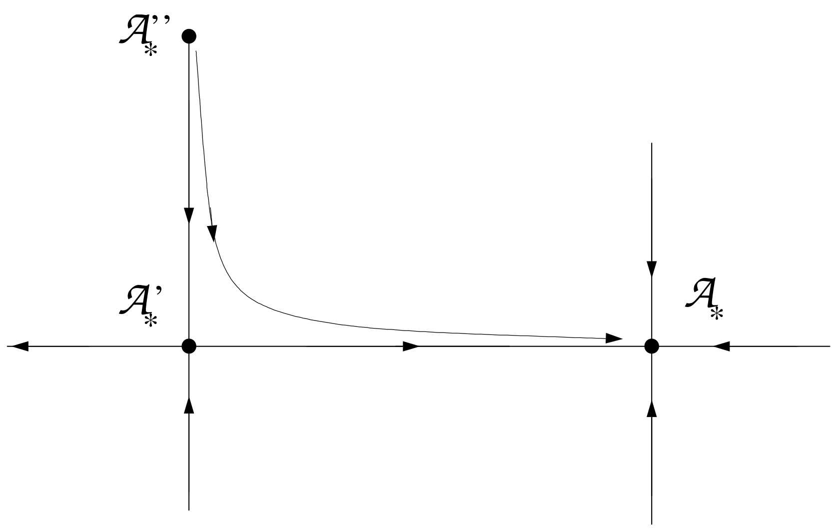
[34] $\tilde{C} = \frac{d}{d-1}\,(C\,(d-1))^{1/d}$

[35] Stringy density of states $S(E) \simeq E/kT_H$ has manifestation in the so called *Hagedorn transition* at $T = T_H$.

fixed point (i.e. associated with $U(\mathcal{A}_*)$. And if we adopt the Wilsonian idea of quantum field theories as RG flows on the unstable manifolds of fixed points, the above equation (23.30) become general limitation on the asymptotic density of states in quantum field theory.

Let me make here a remark on the status of this idea. In our discussion of the Renormalization Group we (following Wilson) found it natural to identify local field theories with the special class of RG flows in Σ, the flows that can be "integrated backward" without limit. A critical fixed point $\mathcal{A}_*$ gives simple mechanism for generating such flows. There can be several fixed points, or even continuous manifold of fixed points[36]. In any case, the above conclusion about the high energy density of states, which follows from the scale invariance of the fixed point theory, remains valid.

However, another, somewhat more intricate behavior is conceivable. Let me briefly discuss it, just in the way of entertainment. To start with, let us imagine a slice of the critical surface passing through a fixed point $\mathcal{A}_*$. All RG trajectories in some domain around $\mathcal{A}_*$ flow to $\mathcal{A}_*$ at $l \to +\infty$, see Figure below. Let's also assume that there is another fixed point $\mathcal{A}'_*$ laying in the same critical surface, such that $\mathcal{A}_* \in U(\mathcal{A}'_*)$. Furthermore, assume there is yet another fixed point $\mathcal{A}''_* \in U(\mathcal{A}'_*)$.



This Figure shows an RG trajectory which originates at $\mathcal{A}''_*$, then passes very close to $\mathcal{A}'_*$, finally converging to $\mathcal{A}_*$. Starting from any point on this trajectory, one can "integrate backward" without limit, i.e. defines local QFT. While the large scale

[36] About limit cycles, C-theorems, etc.

properties of this QFT are controlled by the "infrared" fixed point $\mathcal{A}_*$, its behavior at shorter scales is more intricate. When traced backwards, the RG trajectory stays long "time" near the fixed point $\mathcal{A}'_*$, before departing towards $\mathcal{A}''_*$ at $l \to -\infty$. The associated QFT has its asymptotic short distance behavior determined by the "ultraviolet" fixed point $\mathcal{A}''_*$, but in some wide range of intermediate scales it has scaling properties associated with $\mathcal{A}'_*$.

One can imagine yet more complicated sequence of nested fixed points, giving rise to QFT with many different intermediate regimes corresponding to the trajectory passing close to consecutive fixed points. Ultimately, then, a flow is conceivable which passes successively near infinitely many fixed points. While such trajectory can be integrated backward with no limit, its short distance behavior is not controlled by any given fixed point.

24 Operator Formalism vs Functional Integral in QFT

So far we have discussed QFT mostly from the perspective of (Euclidean space-time) functional integral. There are aspects which are better understood within the formalism based on the notions of the Hilbert space of states which supports the algebra of local field operators. In the previous discussion I made occasional references to statements that follow from the operator formalism, mostly from the positivity of norms in the Hilbert space, as well as from the so called spectral condition. Here I would like to recollect the basics of that formalism, and clarify its relation to the functional integral.

Operator formalism

As was stated before (Lecture 7), the correlation functions defined by the functional integrals

$$\langle\, O_1(x_1)...O_N(x_N)\,\rangle := Z^{-1}\int [D\Phi]\, O_1(x_1)...O_N(x_N)\, e^{-\mathcal{A}[\Phi]} \tag{24.1}$$

can be interpreted as the vacuum-vacuum matrix element of the "T-ordered" product

$$\langle\, O_1(x_1)...O_N(x_N)\,\rangle = \langle 0 \mid T_\tau \left(\hat{O}_1(x_1)...\hat{O}_N(x_N)\right) \mid 0\rangle \tag{24.2}$$

of the Heisenberg field operators $\hat{O}(x)$. The latter notions involve nominating one of the coordinates to the role of the "Euclidean time" τ, so that $x = (\mathbf{x}, \tau)$. Usually, the Heisenberg picture is developed in the real time t which relates to τ by the analytic continuation to $\tau = it$, but in most of the development below this specification is not necessary; we generally will regard τ as a complex number. Then

$$\hat{O}(\mathbf{x}, \tau) = e^{\hat{H}\tau}\, \hat{O}(\mathbf{x})\, e^{-\hat{H}\tau}\,. \tag{24.3}$$

By construction, the field operators satisfy Heisenberg "equation of motion"

$$\frac{\partial}{\partial\tau}\hat{O}(\mathbf{x}, \tau) = [\hat{H}, \hat{O}(\mathbf{x}, \tau)] \tag{24.4}$$

In fact, most of the calculations can be done at real τ ("imaginary time"). In that case the "time ordering" T_τ means that the operators in (24.2) are ordered according decreasing values of τ - the operators with greater τ are placed to the left.

We have derived this relation in quantum mechanics (Lecture 7), and then assumed that it can be extended to QFT. In fact, such extension is not straightforward, and involves many subtleties mostly related to regularization and renormalization (Our brief inspection of the Wilson's RG gives the idea of the size of the problem). Here I would like to revisit the relation (24.2) from more general perspective. Starting from the functional integral definition of the correlation functions, we will try to recover basic notions of the operator formalism. Eq.(24.3) involves the Hamiltonian, and we would like to see how the Hamiltonian, or more generally, the Energy-Momentum P^μ, can be introduced in terms of the correlation functions.

Correlation Functions and Energy-Momentum

Consider the correlation function

$$\langle\, T^{\mu\nu}(x)\, O_1(x_1) ... O_N(x_N)\,\rangle$$

involving the energy-momentum tensor field $T^{\mu\nu}(x)$. As we have seen in the previous Lecture, it satisfies the Ward Identity (23.9), in particular

$$\langle \partial_\mu T^{\mu\nu}(x)\, O_1(x_1)\cdots O_N(x_N)\rangle = 0 \quad \text{for} \quad x \neq x_1, \cdots, x_N\,,$$

As a consequence, the integral

$$\int_\Sigma d\Sigma_\mu(x)\,\langle\, T^{\mu\nu}(x)\, O_1(x_1) ... O_N(x_N)\,\rangle$$

taken over a $d-1$-hypersurface Σ does not change under topologically trivial deformations of Σ. The hypersurface Σ divides the space $\mathbb{R}^d$ into two parts, and correspondingly splits the set of insertion points $x_1, ..., x_N$ into two subsets, according to their locations within the parts. Topologically trivial are those deformations which do not change this splitting, or, in other words the deformations which do not make the hypersurface to "cross over" any of the insertion points. For the sake of our present discussion it is convenient to choose take Σ to be an "equal-time slice". As before, we nominate one of the coordinates in $\mathbb{R}^d$ to be the (imaginary) "time" τ i.e. $x = (\mathbf{x}, \tau)$, and define $\Sigma(\tau)$ as the union of points with given τ. Define[37]

$$iP^\mu(\tau) := \int_{\Sigma(\tau)} d\Sigma_\nu(x)\, T^{\mu\nu}(x) = \int_{\Sigma(\tau)} d^{d-1}\mathbf{x}\; T^{\mu\tau}(\mathbf{x}, \tau)\,, \tag{24.5}$$

[37] The factor i is added to this notation to agree with the conventional notation P^μ for the energy-momentum in real Minkowski space time. Upon continuation $\tau = it$ the components of the energy-momentum tensor transform so that $T^{\tau k} = iT^{0k}$, $k = 1, 2, ..., d-1$, and $T^{\tau\tau} = -T^{00}$.

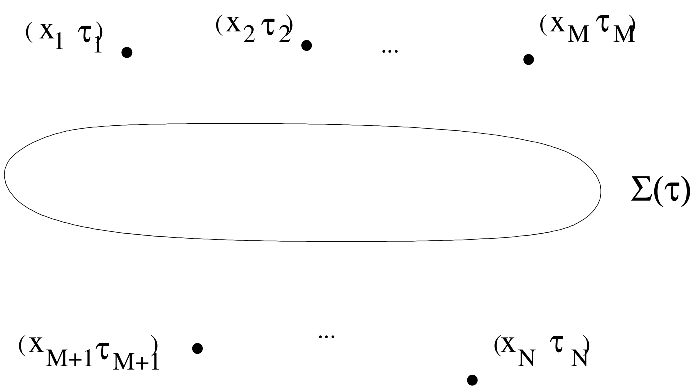


Fig.25.1

and assume that

$$\tau_1 > \tau_2 > ... > \tau_M > \tau > \tau_{M+1} > ... > \tau_N \,, \tag{24.6}$$

as illustrated here

Then the correlation function

$$\langle\, P^\mu(\tau)\, O_1(x_1) \cdots O_N(x_N)\rangle \tag{24.7}$$

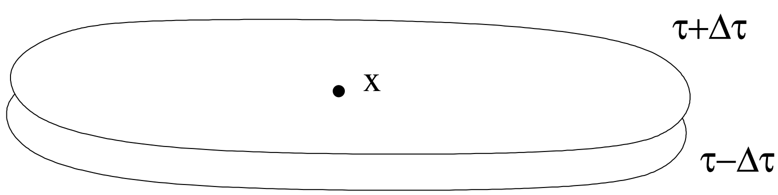
does not change if one changes the value of τ, as long as the order (24.6) is preserved. We therefore drop the argument τ writing in the notation for $P^\mu := P^\mu(\tau)$, but adopt the convention that the position of the insertion P^μ in the correlation function corresponds to the order (24.6), i.e. (24.7) is written as

$$\langle\, O_1(x_1) \cdots O_M(x_M)\, P^\mu\, O_{M+1}(x_{M+1}) ... O_N(x_N)\rangle \tag{24.8}$$

With this convention the position of the insertion P^μ represents the topological class of the surface Σ in (24.5) relative the points x_i.

Now that the position of P^μ in the correlation function (24.7) matters, one may ask about the "commutators" of P^μ with the local field insertions. We have formally (see the Figure below)

$$iP^\mu O(\mathbf{x},\tau) - O(\mathbf{x},\tau) iP^\mu = \int d^3\mathbf{y}\, T^{\tau\mu}(\mathbf{y},\tau+\Delta\tau) O(\mathbf{x},\tau) - \int d^3\mathbf{y}\, T^{\tau\mu}(\mathbf{y},\tau-\Delta\tau) O(\mathbf{x},\tau)\,,$$



or, by closing the hypersurface

$$i[P^\mu, O(x)] = \int_{\Sigma_x} d\Sigma_\nu(y)\, T^{\mu\nu}(y)\, O(x) \tag{24.9}$$

where Σ_x is an arbitrary small closed hypersurface wrapped around the point x. Again, this relation is understood as being valid for the insertions in the correlation functions, with the above ordering convention.

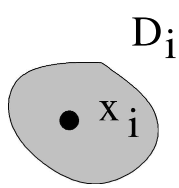
The integral in the r.h.s. can be calculated as follows. Consider again the Ward identity (23.8),

$$\sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \cdots \delta_\varepsilon \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle = \\ - \int d^d x\, \partial_\mu \varepsilon_\nu(x)\, \langle T^{\mu\nu}(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle\ . \tag{24.10}$$

where $\delta_\varepsilon O(x)$ stand for the variations of the fields $O(x)$ under the coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu(x)$. The integration in the r.h.s. is over the whole space $D := \mathbb{R}^d$. Let us split this domain into several pieces, as follows

$$D := \mathbb{R}^d = \sum_{i=1}^{N} D_i + D_{out}\,,$$

where D_i is a small domain containing the point x_i, and D_{out} is the remaining part of D left after removing all D_i [38].



The integral splits accordingly, as

$$\int d^d x = \sum_{i=1}^{N} \int_{D_i} d^d x + \int_{D_{out}} d^d x$$

For all $x \in D_{out}$ we have

$$\partial_\mu T^{\mu\nu}(x) = 0 \qquad \text{for} \quad x \in D_{out}\,,$$

[38] In conventional set theory symbolics the decomposition reads $D_{out} := D \backslash (\cup_i D_i)$.

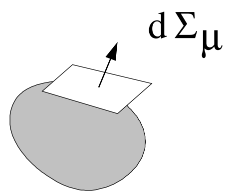
and hence the integral over D_{out} can be transformed via the Stokes theorem, as

$$\int_{D_{out}} d^d x\, \partial_\mu \langle\, (\varepsilon_\nu(x) T^{\mu\nu}(x))\; O_1(x_1)...O_N(x_N)\,\rangle = -\sum_{i=1}^{N} I_i\,,$$

where I_i stand for the surface integrals over the boundaries of D_i,

$$I_i = \int_{\partial D_i} d\Sigma_\mu(x)\, \varepsilon_\nu(x)\, \langle\, T^{\mu\nu}(x)...O_i(x_i)...\rangle$$

and $d\Sigma_\mu(x)$ is the normal element of the boundary ∂D_i.



Hence, the r.h.s. of the Eq.(24.10) splits into the sum

$$\sum_{i=1}^{N} \left(I_i - \int_{D_i} d^d x\, \partial_\mu \varepsilon_\nu(x)\, \langle\, T^{\mu\nu}(x)...\rangle \right)\,. \tag{24.11}$$

By construction, each of the terms here is independent on the shape of the domain D_i, which can be made arbitrary small. This implies that each of these terms depend on $\varepsilon(x)$ in a local way, i.e. the term associated with D_i depends on $\varepsilon(x_i)$, $\partial\varepsilon(x_i)$, ... Comparing (24.11) with the l.h.s. of (24.10) we conclude that the equality is satisfied term by term in the sum over i. This leads to the following representation of the variations $\delta_\varepsilon O(x)$

$$\delta_\varepsilon O(x) = \int_{\partial D_x} d\Sigma_\mu(y)\, \varepsilon_\nu(y)\, T^{\mu\nu}(y) O(x) - \int_{D_x} d^d y\, \partial_\mu \varepsilon_\nu(y)\, T^{\mu\nu}(y)\, O(x)\,. \tag{24.12}$$

The relation is understood as valid for the correlation functions. Let me note again that the r.h.s. is in fact independent on the choice of the (small) domain D_x.

As was already mentioned, the variation $\delta_\varepsilon O(x)$ admits expansion in the derivatives of ε taken at the point x, with the leading term

$$\delta_\varepsilon O(x) = \varepsilon^\mu(x)\, \partial_\mu O(x) + \text{terms involving } \partial\varepsilon(x), ...$$

In particular, for a constant $\varepsilon^\mu(x) = \varepsilon^\mu$ the terms with the derivatives do not contribute. On the other hand, in this case the second term in (24.12) vanishes, and we obtain

$$\partial^\mu O(x) = \int_{\partial D_x} d\Sigma_\nu(y) T^{\mu\nu}(y)\, O(x)$$

The r.h.s. here coincides with the expression in (24.9). Finally, we find

$$[P^\mu, O(x)] = i\, \partial^\mu O(x)\,.$$

Taking μ to be the "time" index τ, and recalling that $P^\tau = iP^0 = iH$, one recovers (24.4)[39]. The "spatial" components of this equation

$$i\nabla O(\mathbf{x},\tau) = [\mathbf{P}, O(\mathbf{x},\tau)]$$

reproduce the well known (see e.g. Chapter 2 of PS) representation of the Heisenberg field operators

$$\hat{O}(\mathbf{x},t) = e^{-i\mathbf{P}\mathbf{x}+iHt}\hat{O}(\mathbf{0},0)\, e^{i\mathbf{P}\mathbf{x}-iHt}$$

which I chose here to write directly in Minkowski space-time.

Space of States[40]

The functional integral is particularly suitable for dealing with the correlation functions (24.1). Calculations in the previous paragraphs suggest a natural way to construct the space of states $\mathcal{H}$ and an algebra of operators, in terms of the Euclidean space correlation functions, as follows.

Let us again split the whole space $\mathbb{R}^d$, with the coordinates $x^\mu = (\mathbf{x},\tau)$, into two half-spaces

$$\mathbb{R} = \mathbb{R}^d_+ \cup \mathbb{R}^d_-$$

separated by the "equal time" slice, the hypersurface $\tau = 0$. Namely

$$\begin{aligned} \mathbb{R}^d_+ \ &: \quad (\mathbf{x},\tau),\ \tau \geq 0\,, \\ \mathbb{R}^d_- \ &: \quad (\mathbf{x},\tau),\ \tau \leq 0\,, \end{aligned}$$

We associate states with the product of field insertions $\varphi(x_1)...\varphi(x_M)$ at the points

$$x_1, x_2, ..., x_N \in \mathbb{R}^d_-\,,$$

[39] sign

[40] The ideas displayed in this section are borrowed from the construction in *K. Osterwalder, R. Schrader, "Axioms for Euclidean Greens functions," Comm. Math. Phys. 31 (1973), 83-112; 42 (1975), 281-305.*

$$\varphi(x_1)...\varphi(x_M) \; \to | \, \varphi(x_1)...\varphi(x_M) \, \rangle \, . \tag{24.13}$$

At this point, the above equation just provides notation for the states. We define $\mathcal{H}$ as the linear envelope of the above states

$$\mathcal{H} = \mathrm{Span} \left\{ | \, \varphi(x_1)...\varphi(x_M) \, \rangle \; : \; x_i \in \mathbb{R}^d_- \right\}$$

The intuition behind this construction comes from the simple minded idea of $\mathcal{H}$ as the space of functionals

$$\mathcal{H} = \text{Space of} \;\; \Psi[\varphi(\mathbf{x})]$$

of the field configurations $\varphi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{d-1}$. Let $x_i = (\mathbf{x}_i, \tau_i) \in \mathbb{R}_-$, and consider the functional integral

$$\int_{\varphi(\mathbf{x},0)=\varphi(\mathbf{x})} [D\varphi(x)] \, \varphi(x_1)...\varphi(x_M) \, e^{-\mathcal{A}[\varphi(x)]} \tag{24.14}$$

over all field configurations $\varphi(x)$ in the half-space $x \in \mathbb{R}_-$. This integral defines (up to overall normalization) the functional

$$\Psi_{\varphi(x_1)...\varphi(x_M)}[\varphi(\mathbf{x})]$$

which may be regarded as the Schroedinger wave function representing the state (24.13). In particular, the integral (24.14) with no insertions represents the vacuum state,

$$\Psi_0[\varphi(\mathbf{x})] = \int_{\varphi(\mathbf{x},0)=\varphi(\mathbf{x})} [D\varphi(x)] \, e^{-\mathcal{A}[\varphi(x)]} \quad \to \quad | \, 0 \rangle \in \mathcal{H} \, .$$

The problem with this representation is in the definition of the scalar products. The simple minded definition

$$\langle \Psi_2 \mid \Psi_1 \rangle := \int [D\varphi(\mathbf{x})] \, \Psi_2^*[\varphi(\mathbf{x})] \Psi_1[\varphi(\mathbf{x})] \tag{24.15}$$

has all required formal properties, but involves the functional integration which, as we already know, requires regularization and renormalization. In this context the problem of renormalization looks much harder then that in the functional integral (24.1) because the functionals $\Psi[\varphi(\mathbf{x})]$ do not have (quasi) local structure, and generally are far more complicated then $e^{-\mathcal{A}}$.

Instead, one defines the scalar product of the states (24.13) as

$$\langle \varphi(x_{M+1})...\varphi(x_N) \mid \varphi(x_1)...\varphi(x_M) \rangle := \langle \varphi(\bar{x}_{M+1})...\varphi(\bar{x}_N) \varphi(x_1)...\varphi(x_M) \rangle \tag{24.16}$$

Here the "mirror image" points $\bar{x}$ are defined as follows:

$$\text{"Reflection"}: \qquad x = (\mathbf{x}, \tau) \quad \leftrightarrow \quad \bar{x} = (\mathbf{x}, -\tau) \tag{24.17}$$

Then, if all the points $x_1, ..., x_M$, $y_1, ..., y_N$ in (24.16) are in $\mathbb{R}^d_-$, the "mirror" points $\bar{y}_1, ..., \bar{y}_N$ are all in $\mathbb{R}^d_+$. The configuration of the points in the correlation function in the r.h.s. of (24.16) is exactly as in (24.9), see Fig.25.1 above.

The most important property of the scalar product is its *positivity*,

$$\langle \Psi \mid \Psi \rangle > 0$$

for any $\mid \Psi\rangle \in \mathcal{H}$, $\mid \Psi\rangle \neq 0$. For the formal representation (24.16) the positivity is obvious: Eq.(24.15) with $\Psi_1 = \Psi_2$ is explicitly positive. For the scalar product defined in terms of the correlation functions this property is postulated as the **reflection positivity**

$$\sum_{N,M} \int_{x_i, y_k \in \mathbb{R}^d_-} \langle \varphi(\bar{y}_1)...\varphi(\bar{y}_N)\varphi(x_1)...\varphi(x_M)\rangle \, f^*_N(y_1, ..., y_N) f_M(x_1, ..., x_M) \geq 0$$

for any sequence of functions $f_N(x_1, ..., x_N)$, which expresses positivity of the norm of the state

$$\mid \Psi\rangle = \sum_N \int_{x_i \in \mathbb{R}^d_-} f_N(x_1, ..., x_N) \mid \varphi(x_1)...\varphi(x_N)\rangle \, .$$

More generally, we can associate states with general products of local fields $O_a(x)$,

$$O_{a_1}(x_1)...O_{a_N}(x_N) \quad \to \quad \mid O_{a_1}(x_1)...O_{a_N}(x_N)\rangle \in \mathcal{H} \, , \qquad x_1, ..., x_N \in \mathbb{R}^d_- \tag{24.18}$$

according to the same idea

$$\Psi_{O_1(x_1)...O_N(x_N)}[\varphi(\mathbf{x})] = \int_{\varphi(\mathbf{x},0)=\varphi(\mathbf{x})} [D\varphi(x)] \, O_1(x_1)...O_N(x_N) \, e^{-\mathcal{A}[\varphi(x)]}$$

with the scalar product product related to their correlation functions similarly,

$$\langle O_{b_1}(y_1)...O_{b_M}(y_M) \mid O_{a_1}(x_1)...\varphi(x_M)\rangle := \langle \bar{O}_{b_1}(\bar{y}_1)...\bar{O}_{b_M}(\bar{y}_M) O_{a_1}(x_1)...O_{a_N}(x_N)\rangle$$

where again $x_1, ...x_N; y_1, ..., y_M \in \mathbb{R}^d_-$, and the points $\bar{y}_i \in \mathbb{R}^d_+$ are the "mirror images" of y_i, as in Eq.(24.17). However, when the fields $O_i(x_i)$ are not all scalars, the fields $\bar{O}_b$ are related to O_b by certain "reflection". To get an idea assume that the field $O(x)$ is real-valued vector with the components $O^\mu(x)$. Then, if $\mu = 1, 2, ..., d-1$ represent "spatial" components,

$$\bar{O}^\mu(\bar{x}) = O^\mu(\bar{x})\,, \quad \bar{O}^\tau(\bar{x}) = -O^\tau(\bar{x})\,,$$

i.e. the sign of the "temporal" component O^τ gets flipped upon reflection. For example,

$$\overline{\partial_\tau\varphi}(x) = -\partial_\tau\varphi(x)$$

so that

$$\langle\partial_\nu\varphi(y)|\partial_\mu\varphi(x)\rangle = \pm\,\langle\partial_\nu\varphi(\bar{y})\partial_\mu\varphi(x)\rangle$$

where the sign $+$ applies when both μ and ν are "spatial" or both are "temporal", otherwise the sign is $-$. Similar rule applies to higher rank tensors. For complex values fields the "reflection" involves complex conjugation. The correlation functions of composite fields O_a obey then the suitably generalized reflection positivity conditions, which follow from the positivity of the norms of the states (24.18).

As an elementary application, let us show that in a scale invariant (i.e. RG fixed point) theory the mass dimensions D_α of all nontrivial fields $\Phi_\alpha(x)$ are positive (we denote $\Phi_\alpha(x)$ the scalar fields that diagonalize the matrix of anomalous dimensions, see Lecture 20). From the scale invariance

$$\langle\Phi_\alpha(x)\Phi_\alpha(y)\rangle = \frac{C_\alpha}{|x-y|^{2D_\alpha}}$$

where $|x-y|$ is the Euclidean distance between the points x and y. Let us take $\tau > 0$, so that $x = (\mathbf{x}, -\tau/2) \in \mathbb{R}^d_-$. Set $y = \bar{x} = (\mathbf{x}, \tau/2)$. We have

$$\langle\Phi_\alpha(x)\mid\Phi_\alpha(x)\rangle = \frac{C_\alpha}{(\tau^2)^{D_\alpha}}\,.$$

The positivity requires $C_\alpha > 0$; One can set $C_\alpha = 1$ by adjusting normalization of Φ_α. Consider the norm

$$\left\langle\frac{\partial}{\partial x^1}\Phi_\alpha(\mathbf{x}, -\tau/2)\middle|\frac{\partial}{\partial x^1}\Phi_\alpha(\mathbf{x}, \tau/2)\right\rangle \tag{24.19}$$

where x^1 is one of the spatial coordinates in $\mathbf{x} = (x^1, x^2, ..., x^{d-1})$. This norm is expressed as the second derivative of the correlation function

$$\frac{\partial^2}{\partial x^1\partial y^1}\langle\Phi_\alpha(\mathbf{x}, \tau/2)\Phi_\alpha(\mathbf{y}, -\tau/2)\rangle$$

evaluated at $\mathbf{y} = \mathbf{x}$. Elementary calculation yields for (24.19)

$$\frac{2D_\alpha}{(\tau^2)^{D_\alpha+1}}$$

which must be positive, hence $D_\alpha \geq 0$.

The case $D = 0$ is instructive. If the field Φ has mass dimension $D = 0$ it follows that

$$|\frac{\partial}{\partial x^1}\Phi(x)\rangle = 0\,.$$

It is then easy to prove that any correlation function involving the derivative of $\Phi(x)$ vanishes

$$\langle \partial_\mu \Phi(x)\, O_1(x_1)...O_N(x_N)\rangle = 0\,, \quad \Rightarrow \quad \partial_\mu \Phi(x) \simeq 0\,.$$

Then one can show (using cluster factorization, see below) that

$$\langle \Phi(x)\, O_1(x_1)...O_N(x_N)\rangle = \langle O_1(x_1)...O_N(x_N)\rangle$$

i.e. the only field with zero mass dimension is the identity operator.

25 Energy-Momentum Operators

The expression (24.9) can be now promoted to the definition of the operators $\hat{P}^\mu$ acting in the space $\mathcal{H}$. Specifically, for any $x_1, ..., x_N, y_1, ...y_M$ in $\mathbb{R}^d_-$ define

$$\langle \varphi(y_1)...\varphi(y_M) \mid \hat{P}^\mu \mid \varphi(x_1)...\varphi(x_N)\rangle = \langle \varphi(\bar{y}_1)...\varphi(\bar{y}_M)\, P^\mu\, \varphi(x_1)...\varphi(x_N)\rangle \quad (25.1)$$

where the correlation function in the r.h.s. involves the integral of the energy-momentum tensor over the equal-time slice $\tau = 0$,

$$P^\mu(\tau) := -i \int_\Sigma d\Sigma_\mu(x)\, T^{\mu\nu}(x) = -i \int_{\Sigma(\tau=0)} d^{d-1}\mathbf{x}\; T^{\mu\tau}(\mathbf{x}, \tau = 0)\,, \qquad (25.2)$$

It is straightforward to check, using the definition of the inner product, that the "spatial" components $\mathbf{P} = (P^1, ..., P^{d-1})$ are Hermitian, while P^τ is anti-Hermitian. We therefore define d Hermitian operators $(\hat{\mathbf{P}}, \hat{H})$ as

$$\hat{P}^k := -i \int_{\Sigma(\tau=0)} d^{d-1}\mathbf{x}\; T^{k\tau}(\mathbf{x}, \tau)\,, \qquad (25.3)$$

$$\hat{H} = -\int_{\Sigma(\tau=0)} d^{d-1}\mathbf{x}\; T^{\tau\tau}(\mathbf{x}, \tau)\,. \qquad (25.4)$$

Obviously, $\hat{H}$ coincides with $\hat{P}^0 := -i\hat{P}^\tau$, the real-time component of $\hat{P}^\mu$.

A remark is in order. If we are dealing with the field theory in the infinite space-time $\mathbb{R}^d$, the matrix elements of these operators diverge because of the contribution of generally nonzero expectation value

$$\langle T^{\mu\nu}(x)\rangle = g^{\mu\nu}\,\varepsilon$$

where $g^{\mu\nu} - \delta^{\mu\nu}$ is the Euclidean metric tensor, and ε is the vacuum energy density. It is convenient to get rid of this divergence by defining reduced energy-momentum tensor

$$T^{\mu\nu}(x) \;\to\; T^{\mu\nu}(x) - g^{\mu\nu}\varepsilon$$

which has zero expectation value. This can be interpreted as adding the "cosmological term" to the action

$$\mathcal{A} \;\to\; \mathcal{A} + \varepsilon \int d^dx\, \sqrt{g}(x)$$

For the fixed metric background such term has no effect on the content of the theory, but brings in the above subtraction when $T^{\mu\nu}(x)$ is defined as the response to the variation of the background metric,

$$\delta\mathcal{A} = -\frac{1}{2}\int d^d x\,\sqrt{g}\;T^{\mu\nu}(x)\,\delta g_{\mu\nu}(x)\,.$$

In what follows we adopt this convention, assuming that $T^{\mu\nu}(x)$ is chosen in such a way that

$$\langle T^{\mu\nu}(x)\rangle = 0\,. \tag{25.5}$$

With this convention, it is easy to prove that the vacuum $\mid 0\rangle$ is the eigenstate of $\hat{P}^\mu$ with the eigenvalue 0, i.e.

$$\hat{H}\mid 0\rangle = 0\,, \qquad \hat{\mathbf{P}}\mid 0\rangle = 0\,. \tag{25.6}$$

Indeed, consider the element

$$\langle O_1(x_1)...O_N(x_N)\mid \hat{P}^\mu \mid 0\rangle = \langle O_1(\bar{x}_1)...O_N(\bar{x}_N)\,P^\mu\rangle \tag{25.7}$$

where P^μ is the surface integral (25.5). As all insertions $\bar{x}_1, ..., \bar{x}_N$ are in $\mathbb{R}^d_+$, the hypersurface Σ involved in the definition (25.1) can be deformed away to the domain $\tau \to \infty$, where the integral vanishes. As the result, all matrix elements (25.7) are equal to zero. Eq.(25.6) follows.

Similar calculation shows that the operators P_μ act on the states $O_1(x_1)...O_N(x_N)\rangle$ as follows

$$\hat{P}_\mu \mid O_1(x_1)...O_N(x_N)\rangle = i\left[\sum_{k=1}^{N}\frac{\partial}{\partial x_k^\mu}\right]\mid O_1(x_1)...O_N(x_N)\rangle$$

Indeed, this time deforming Σ to the $\tau \to -\infty$ domain leaves behind integrals around the insertion points x_k, which, according to

$$\partial^\mu O(x) = \int_{\partial D_x} d\Sigma_\nu(y) T^{\mu\nu}(y)\,O(x)$$

leaves behind the derivatives in the r.h.s. As usual, the infinitesimal translations generated by $\hat{P}_\mu$ can be exponentiated to finite translations. In terms of the operators (25.3),(25.4) we have

$$\begin{aligned} e^{-i\hat{\mathbf{P}}\mathbf{X}}\mid O_1(\mathbf{x}_1,\tau_1)...O_N(\mathbf{x}_N,\tau_N)\rangle &= \mid O_1(\mathbf{x}_1+\mathbf{X},\tau_1)...O_N(\mathbf{x}_N+\mathbf{X},\tau_N)\rangle \\ e^{-\hat{H}L}\mid O_1(\mathbf{x}_1,\tau_1)...O_N(\mathbf{x}_N,\tau_N)\rangle &= \mid O_1(\mathbf{x}_1,\tau_1-L)...O_N(\mathbf{x}_N,\tau_N-L)\rangle \end{aligned} \tag{25.8}$$

Stationary States. Energy Positivity

The operators $\hat{P}^\mu$ commute, and thus can be simultaneously diagonalized in $\mathcal{H}$. It is convenient to discuss the spectrum of the Hermitian operators $\hat{P}^\mu$, $\mu = 0, ..., d-1$ with $\hat{P}^0 := -i\hat{P}^\tau = \hat{H}$, which, of course, coincide with the conventional Energy-Momentum operators in the Minkowski space-time. Let $\mid \alpha\rangle$ be an eigenstate

$$\hat{P}^\mu \mid \alpha\rangle = p^\mu_\alpha \mid \alpha\rangle \,.$$

We will make few assumptions about the spectrum of these operators. One is about the spectrum of $\hat{H}$. We assume that the eigenvalues E_α of $\hat{H}$ are satisfy $E_\alpha \geq 0$, with the only zero eigenvalue $E_0 = 0$ being associated with the vacuum state, Eq.(25.6). This assumption implies

$$\lim_{L\to\infty} \; e^{-L\hat{H}} = \mid 0\rangle\langle 0 \mid \tag{25.9}$$

There are several simple consequences from this condition alone. One is the so-called cluster factorization property of the correlation functions, which states that if a collection of N points consists of two groups with very large separation between the groups, correlations between the fields in the different groups tend to zero, that is

$$\begin{aligned}\lim_{L\to\infty} \langle O_1(x_1)...O_M(x_M)O_{M+1}(x_M+L)...O_N(x_N+L)\rangle = \\ \langle O_1(x_1)...O_M(x_M)\rangle\langle O_{M+1}(x_M)...O_N(x_N)\rangle\end{aligned} \tag{25.10}$$

To prove, take the direction of L to be the "time", then the l.h.s. can be represented as

$$\langle O_1(x_1)...O_M(x_M) \mid e^{-L\hat{H}} \mid O_{M+1}(x_M)...O_N(x_N)\rangle$$

and (25.10) follows from (25.9).

Less obvious consequence concerns analytic properties of the correlation functions

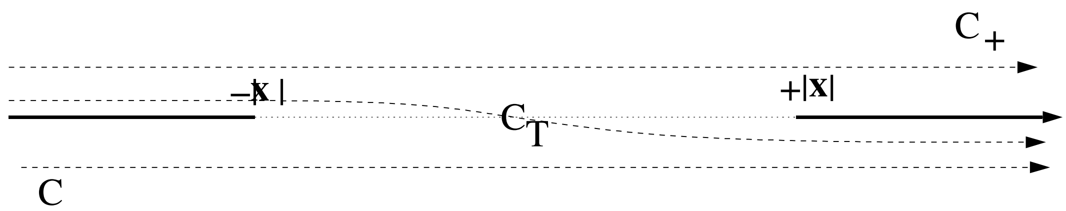
$$\langle O_1(\mathbf{x}_1, \tau_1)...O_N(\mathbf{x}_N, \tau_N)\rangle$$

at complex values of $\tau_1, ..., \tau_N$. We have briefly studied this question at the beginning of the course, for the case of the Klein-Gordon propagator (Lecture 4). In fact, our arguments there depend only on the condition of the positivity of the energy spectrum, which was explicit in the KG theory. Therefore, the analysis can be repeated word for word in general case. Let me restate the result for the two point function

$$\langle O_1(\mathbf{x}, \tau)O_2(\mathbf{0}, 0)\rangle$$

In the complex plane of $t = -i\tau$ we have

$$\langle 0 \mid \hat{O}_2(\mathbf{0}, 0)\, \hat{O}_1(\mathbf{x}, t) \mid 0 \rangle$$



$$\langle 0 \mid \hat{O}_1(\mathbf{x}, t)\, \hat{O}_2(\mathbf{0}, 0) \mid 0 \rangle$$

Here

$$\hat{O}(\mathbf{x}, t) = e^{-i\mathbf{P}\mathbf{x}+iHt}\hat{O}(\mathbf{0}, 0)\, e^{i\mathbf{P}\mathbf{x}-iHt}$$

are conventional Heisenberg field operators. The discontinuity across the branch cuts represents the commutators which lay within the future and past light cones (”local commutativity”).

Spectrum of Energy-Momentum

Some properties of the spectrum of the operator $\hat{P}^\mu = (\hat{H}, \hat{\mathbf{P}})$ can be established on general grounds. Let $\mathcal{P}$ be the spectrum of this operator, i.e. the set of its eigenvalues

$$p^\mu \in \mathcal{P} \;\Rightarrow\; \exists \mid \alpha\rangle \in \mathcal{H} \text{ such that } \; p^\mu = p^\mu_\alpha : \quad \hat{P}^\mu|\alpha\rangle = p^\mu_\alpha|\alpha\rangle\,.$$

Lorentz invariance suggests that

$$p^\mu \in \mathcal{P} \;\Rightarrow\; \Lambda^\mu_\nu\, p^\nu \in \mathcal{P}\,,$$

where Λ is a Lorenz transformation. This means that any $p^\mu \in \mathcal{P}$ such that $p^\mu p_\mu = m^2$ enters $\mathcal{P}$ together with the entire hyperboloid

$$\mathcal{P}_{m^2} = \{p^\mu : p^\mu p_\mu = (p^0)^2 - \mathbf{p}^2 = m^2\}\,.$$

It follows then from the positivity of the energy that $\mathcal{P}$ lays entirely within the future light cone $p^\mu p_\mu > 0,\; p^0 > 0$.

Another important property of $\mathcal{P}$ is its additivity, i.e.

$$p^\mu_\alpha, p^\mu_\beta \in \mathcal{P} \Rightarrow p^\mu_\alpha + p^\mu_\beta \in \mathcal{P}\,.$$

Intuitively, if we think of the state $\mid \alpha\rangle$ carrying the momentum p^μ_α as a local excitations over the ground state, it is possible to add another excitation, with the momentum p^μ_β arbitrary far away from the first one. At large separations the excitations do not interact, and their energies and momenta add up. Formal derivation using the cluster property will be considered later.

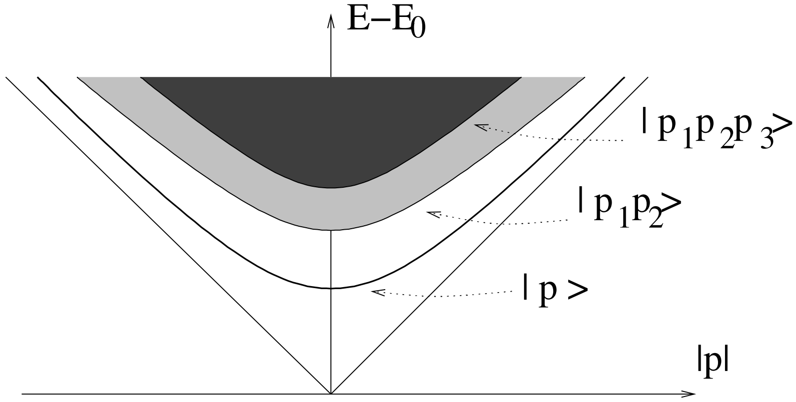
The additivity implies that if $\mathcal{P}$ contains $\mathcal{P}_{m^2}$, it also contains all p^μ of the form

$$p^\mu = p^\mu_1 + \cdots + p^\mu_N\,; \qquad p^\mu_i \in \mathcal{P}_{m^2}\,.$$

The totality of such vectors p^μ will be denoted $\bar{\mathcal{P}}_{m^2}$,

$$\bar{\mathcal{P}}_{m^2} = \{p^\mu = \cup_N \cup_{p_1,\cdots,p_N}\ \{p^\mu = p^\mu_1 + \cdots + p^\mu_N\,; p^\mu_i \in \mathcal{P}_{m^2}\}\,.$$

The set $\bar{\mathcal{P}}_{m^2}$ looks like this



Note that $\bar{\mathcal{P}}_{m^2}$ is exactly the spectrum of $\hat{P}^\mu$ in the Klein-Gordon fields theory, where we have found that

$$\hat{P}^\mu = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, p^\mu\, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\,,$$

where $p^\mu = (\omega_{\mathbf{p}}, \mathbf{p})$, $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, and $a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}$ are particle creation and annihilation operators. There these states are interpreted as N-particle states of free bosons.

The spectrum $\bar{\mathcal{P}}_{m^2}$ in some sense is the minimal Lorentz-invariant set compatible with the above additivity condition. If this spectrum appears in an interacting QFT we say that the theory contains just one massive particle of the mass m. The states associated with the hyperboloid $\mathcal{P}_{m^2}$ are the one-particle states, and the states with

$(p^\mu)^2 \geq 4m^2$ are interpreted as the scattering states of two or more interaction particles.

In general case the spectrum can be more complicated (even in a scalar QFT like we consider here). It can involve more then one kind of stable particles, and hence would contain several hyperboloids, say $\mathcal{P}_{m_1^2}$ and $\mathcal{P}_{m_2^2}$, and correspondingly its continuous part would look more complicated (as the drawing shows).

In any case, if we form the operator

$$\hat{P}^2 = \hat{H}^2 - \hat{\mathbf{P}}^2\,,$$

the discrete components of its spectrum are associated with the one-particle states, and the continuous spectrum incorporates all the multi-particle scattering states.

"Asymptotic completeness": $\mathcal{P} = \cup_a \mathcal{P}_{m_a^2}$

Lorentz invariance of the spectrum has certain implications for the general structure of the correlation functions in QFT. As the simplest example, consider the two-point correlation function

$$\langle \varphi(\mathbf{x},\tau)\,\varphi(\mathbf{0},0)\rangle = \langle 0 \mid T_\tau\big(\hat{\varphi}(\mathbf{x},\tau)\,\hat{\varphi}(\mathbf{0},0)\big) \mid 0\rangle\,,$$

where $(\mathbf{x},\tau)$ are the Euclidean space-time coordinates. Assuming $\tau > 0$ we can write it as

$$\sum_\alpha \langle 0 \mid \hat{\varphi}(\mathbf{x},\tau) \mid \alpha\rangle\langle\alpha \mid \hat{\varphi}(\mathbf{0},0) \mid 0\rangle\,.$$

Using the fact that

$$\hat{\varphi}(\mathbf{x},\tau) = e^{\hat{H}\,\tau - i\hat{\mathbf{P}}\,\mathbf{x}}\,\hat{\varphi}(\mathbf{0},0)\,e^{-\hat{H}\,\tau + i\hat{\mathbf{P}}\mathbf{x}}\,.$$

we can write the above sum as

$$\sum_\alpha \langle 0 \mid \hat{\varphi}(\mathbf{0},0) \mid \alpha\rangle\langle\alpha \mid \hat{\varphi}(\mathbf{0},0) \mid 0\rangle\,e^{-E_\alpha\,\tau + i\mathbf{P}_\alpha\,\mathbf{x}}\,,$$

where E_α, $\mathbf{P}_\alpha$ are the energy and momentum of the state $\mid \alpha\rangle$. In fact, we have continuously many states, so that

$$\sum_\alpha \to \int d\mu(\alpha)\,.$$

where α denotes complete set of quantum numbers characterizing the state. Define

$$\rho(q) = \int d\mu(\alpha)\, |\langle 0 \mid \hat{\varphi}(0) \mid \alpha\rangle|^2\, (2\pi)^4\, \delta^{(4)}(q^\mu - p^\mu_\alpha) \tag{25.11}$$

Then, assuming $\tau > 0$, we have

$$\begin{aligned}\langle \varphi(\mathbf{x},\tau)\, \varphi(\mathbf{0},0)\rangle &= \int d\mu(\alpha)\, \langle 0 \mid \hat{\varphi}(\mathbf{x},\tau)\, \hat{\varphi}(\mathbf{0},0) \mid 0\rangle = \\ &= \int \frac{d^4q}{(2\pi)^4}\, \rho(q)\, e^{-q^0\tau + i\mathbf{qx}} \end{aligned} \tag{25.12}$$

Evidently,

$$\begin{aligned}\rho(q) = 0 \quad &\text{if } q^\mu \notin \mathcal{P} \\ \rho(q) \geq 0 \quad &\text{if } q^\mu \in \mathcal{P}\end{aligned}$$

so that $\rho(q) \geq 0$ everywhere.

$\rho(q)$ vanishes if $q^0 < 0$ (Hamiltonian is assumed to be positive defined), and at $q^0 > 0$ it depends on $q^2 = (q^0)^2 - (\mathbf{q})^2$ (Lorentz invariance); I will write

$$\rho = \theta(q^0)\, \rho(q^2)\,.$$

Then, for $\tau > 0$ we have

$$\begin{aligned}\langle\, \varphi(\mathbf{x},\tau)\varphi(\mathbf{0},0)\,\rangle &= \int_0^\infty d\mu^2\, \rho(\mu^2)\, \int \frac{d^4q}{(2\pi)^4}\, \delta^{(4)}((q^0)^2 - \mathbf{q}^2 - \mu^2)\theta(q^0)\, e^{-q^0\tau + i\mathbf{qx}} = \\ &= \int_0^\infty \frac{d\mu^2}{2\pi}\, \rho(\mu^2)\, \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, \frac{1}{2\omega_\mathbf{q}}\, e^{-\omega_\mathbf{q}\tau + i\mathbf{qx}}\,,\end{aligned}$$

where

$$\omega_\mathbf{q} = \sqrt{\mathbf{q}^2 + \mu^2}\,.$$

Proceeding similarly, the case $\tau < 0$ we obtain the same expression with

$$e^{-\omega_q\tau} \;\to\; e^{\omega_q\tau}$$

Both cases $\tau > 0$ and $\tau < 0$ can be put together by using

$$\frac{1}{2\omega_q}\, e^{-\omega_q|\tau|} = \int \frac{d^4q}{(2\pi)^4}\, \frac{e^{iq_4\tau}}{q_4^2 + \mathbf{q}^2 + \mu^2}$$

Finally

$$\langle\, \varphi(\mathbf{x},\tau)\varphi(\mathbf{0},0)\,\rangle = \int_0^\infty \frac{d\mu^2}{2\pi}\, \rho(\mu^2)\, D(|x|),\mu^2)\,, \tag{25.13}$$

where $|x| = \sqrt{\tau^2 + \mathbf{x}^2}$, and $D(|x),\mu^2)$ is the Klein-Gordon propagator with the mass μ,

$$D(|x),\mu^2) = \int \frac{d^4k}{(2\pi)^2}\, \frac{e^{i\mathbf{k}\mathbf{x}+ik_4\tau}}{k_4^2 + \mathbf{k}^2 + \mu^2}\,.$$

Eq.(25.13) is known as the *Källen-Lehmann representation.*

For $\mathcal{P}_{m^2}$

$$\rho(\mu^2) = (2\pi)Z\,\delta(\mu^2 - m^2) + \tilde\rho(\mu^2)$$

where $\tilde\rho(\mu^2) = 0$ at $\mu^2 < 4m^2$,

$$Z^{1/2} = \langle 0 \mid \hat\varphi(0) \mid p\rangle \tag{25.14}$$

$$\tilde W^{(2)}(p) = \frac{Z}{p^2 + m^2} + \int_{4m^2}^\infty \frac{\tilde\rho(\mu^2)}{p^2+\mu^2}\,\frac{d\mu^2}{2\pi} \tag{25.15}$$

which is another form of the Källen-Lehmann representation.

Note the pole at $p^2 = -m^2$ explicit in this formula, appearing entirely from the contributions of the one-particle states in the intermediate-state decomposition. The residue Z is positive (as it is the square of the absolute value of the matrix element (25.14)), but its value depends of the normalization of the field φ. That normalization was subject to one of the normalization conditions in our calculations in the perturbation theory. Thus, under the normalization condition

$$\frac{d}{dp^2}\Gamma^{(2)}(p^2) = p^2 + m^2 + O((p^2+m^2)^2) \tag{25.16}$$

of the scheme SI in the φ^4 theory (see Eqs.(14.8),(14.9)) we just choose $Z = 1$. Let us also note that $\tilde\rho(\mu^2)$ is non-negative at all μ^2, as the consequence of the positivity of the norms in the Hilbert space, see Eq.(25.11).

At $|x| \to \infty$ we have

$$D(|x|, \mu^2) \simeq \frac{\mu^{1/2}}{|x|^{3/2}} \, e^{-\mu|x|} \,,$$

and the form (25.15) suggests that the large $|x|$ the asymptotic of $\langle \varphi(x)\varphi(0) \rangle$ is always dominated by one-particle contribution, provided the matrix element (25.14) is not zero.