CAS 701

Logic and Discrete Mathematics

Fall 2017

Exercise Group 5

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You are required to submit solutions to 7 of the following 10 exercises. If you submit more than 7, only the first 7 will be marked. Your solutions should begin with a table of contents that indicates which of the 10 exercises you have chosen to do. Each question is worth 10 points. **The solutions must be written using LaTeX and are due December 8, 2017.**

1. For each of the following first-order formulas X, construct a proof tree of LK whose conclusion is X:

   a. $A[t/x] \supset \exists\, x\, A$ where t is free for x in A.
   b. $\forall\, x\, A \supset \exists\, x\, A$.
   c. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$.

2. For each of the following first-order formulas X, construct a proof tree of LK_e whose conclusion is X:

   a. $\forall\, x\, (x \doteq x)$.
   b. $\forall\, x\, \forall\, y\, ((x \doteq y) \supset (y \doteq x))$.

3. Let $\mathbf{T} = (\mathbf{L}, \Gamma)$ be the first-order theory of monoids presented in the 4 First-Order Logic slides. Prove in LK_e that

   $$\Gamma \vDash \forall\, x\, (\forall\, y\, (x \,\mathsf{mul}\, y \doteq y) \supset (x \doteq e))$$

   is valid.

4. In the STT theory **PA**, define $+$ and $*$ using definite description. Your definitions should be equations of forms

   $$+ = \cdots$$

   and

   $$* = \cdots,$$

   respectively.

5. In the STT theory **PA**, define a predicate of type $\iota \rightarrow *$ that given a natural number n returns true iff n is prime.

6. In the STT theory **COF**, define a function that given a set of reals numbers represented by a predicate of type $\iota \rightarrow *$ returns the minimum of the set.

7. In the STT theory **COF**, define the notion of a derivative of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ at a point $x \in \mathbb{R}$. You many use the lim function defined in section 5 of the Seven Virtues paper.

8. Formalize in STT the theory of well-orders.

9. Formalize in STT the theory of two abstract monoids. You may use two base types of individuals, ι_1 and ι_2, to represent the domains of the two monoids. Define in the theory the notion of a homomorphism from the first monoid to the second and the notion of the kernel of such a homomorphism.

10. Formalize in STT the theory of stacks of abstract elements. You may use two base types of individuals, ι_1 and ι_2, to represent the domain of abstract elements and the domain of stacks of abstract elements, respectively. Model your theory on the STT theory **PA**.