assginment2

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1. Consider the weak partial order P = $(\{\{1\},\{2\},\{4\},\{1,2\},$ $\{1,4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\},\subseteq)$.

a. Find the maximal elements in P .
b. Find the minimal elements in P .
c. Find the maximum element in P if it exists.
d. Find the minimum element in P if it exists.
e. Find all the upper bounds of $\{\{2\},\{4\}\}$ in P .
f. Find the least upper bound of$\{\{2\},\{4\}\}$ in P if it exists.
g. Find all the lower bounds of $\{\{1,3,4\},\{2,3,4\}\}$ in P .
h. Find the greater lower bound of $\{\{1,3,4\},\{2,3,4\}\}$ in P if it exists.

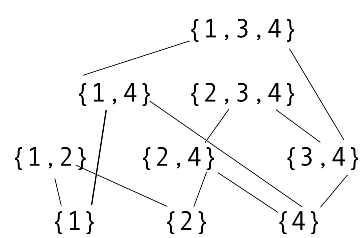

Figure 1: Hasse-diagram

In the figure1, we can know the answer directly.
a.$\{1,2\},\{1,3,4\},\{2,3,4\}$
b.$\{1\},\{2\},\{4\}$
c.NO
d.NO
e.$\{2,4\},\{2,3,4\}$
f.$\{2,4\}$
g.$\{3,4\},\{4\}$
h.NO

5. Show that a total order is a lattice.

Proof:
A total order$(S,\leq)$ must be total ,which means that $\forall x,y,x\leq y\wedge y\leq x$.And then we can find every pair of x,y has a least upper bound and a greatest lower bound, so a totak order is a lattice.

6.Show that $(N, |)$ is a complete lattice.

$$
\begin{aligned}
& a|b \quad is \quad defined \quad as \quad \exists c : ac = b \\
& 1.reflexive : \\
& \forall a \exists c, ac = a, c = 1 \\
& 2.antisymmtric : \\
& \forall a, b, \exists c : a * c = b, \exists a = c\prime * b \\
& => a * c * c\prime = a \\
& => c * c\prime = 1 \\
& => a = b \\
& 3.transitive : \\
& \forall a, b, c \exists d : a * d = b, \exists d\prime : b * d\prime = c \\
& => a * d * d\prime = c \\
& => choose e = d * d\prime
\end{aligned}
$$

Hence, (N, —) is a partially ordered set
The greatest lower bound:0
The least upper bound:1
$=> (N, |)$ is a complete lattice.

7. Show that every finite, nonempty subset of a lattice has a least upper bound and a greatest lower bound.

Proof:
In the definition of 'lattice' we can know that every pair of elements in the set has a least upper bound and a greatest lower bound, and every element in the subsets of the lattice must belong to the lattice, which means that there must be a least upper bound and a greatest lower bound in every finite, nonempty subset of the lattice.

8.Let S be any set. Show that $(\{\emptyset, S\}, \emptyset, S, \cup, \cap, -)$ is a boolean algebra.

$$
\begin{aligned}
& 1.associaty : \\
& (\emptyset \cup \emptyset) \cup S = \emptyset \cup (\emptyset \cup S) \\
& (\emptyset \cap \emptyset) \cap S = \emptyset \cap (\emptyset \cap S) \\
& 2.commtativy : \\
& \emptyset \cap S = S \cap \emptyset \\
& \emptyset \cup S = S \cup \emptyset \\
& 3.distributive : \\
& choose \quad x = y = \emptyset \quad and \quad z = S \quad as \quad as \quad an \quad example \\
& ohters \quad have \quad the \quad same \quad result \\
& \emptyset \cup (\emptyset \cap S) = (\emptyset \cup \emptyset) \cap ((\emptyset \cup \ S)) \\
& \emptyset \cap (\emptyset \cup S) = (\emptyset \cap \emptyset) \cup ((\emptyset \cap \ S)) \\
& 4.identity : \\
& S \cup \emptyset = S \\
& S \cap \{\emptyset, S\} = S \\
& 5.complement : \\
& S \cup \neg S = \{\emptyset, S\} \\
& S \cap \neg S = \emptyset
\end{aligned}
$$

Those five aximos showed above can prove$(\{\emptyset, S\}, \emptyset, S, \cup, \cap, -)$ is a boolean algebra.

10.Show that the idempotent laws follow from the axioms of a boolean algebra.

1.

$$x + (x * \neg x) = (x + x) * (x + \neg x) \qquad (distributive \quad and \quad complement)$$
$$x + 0 = (x + x) * 1 \qquad (identity)$$
$$x = x + x$$

2.

$$x * (x + \neg x) = (x * x) + (x * \neg x) \qquad (distributive)$$
$$x * 1 = x * x + 0 \qquad (complement)$$
$$x = x * x \qquad (identity)$$

11.Show that the De Morgan's laws follow from the axioms of a boolean algebra.

0.1 Based on:If $a * \neg b = 0$ and $a + \neg b = 1$,then $a = b$

1.

$$\begin{aligned} &(\neg x * \neg y) * (x + y) \\ &= (\neg x * \neg y) * x + (\neg x * \neg y) * \neg y \\ &= (\neg x * x) * \neg y + \neg x * (\neg y * y) \\ &= 0 * \neg y + \neg x * 0 \\ &= 0 \end{aligned}$$

2.

$$\begin{aligned} &(\neg x * \neg y) + (x + y) \\ &= (\neg x * \neg x + x) + y \\ &= ((\neg x + x) * (x + \neg y)) + y \\ &= (1 * (x + \neg y)) + y \\ &= x + \neg y + y \\ &= x + 1 \\ &= 1 \end{aligned}$$

3.

$$(\neg x * \neg y) = \neg(x + y)$$

0.2 Based on:If $a * \neg b = 0$ and $a + \neg b = 1$,then $a = b$

1.

$$\begin{aligned} &(\neg x + \neg y) * (x * y) \\ &= (\neg x * x * y) + (\neg y * x * y) \\ &= (0 * y) + (0 * x) \\ &= 0 \end{aligned}$$

2.

$$\begin{aligned} &(x * y) + (\neg x + \neg y) \\ &= ((x + \neg x) * (y + \neg x)) + \neg y \\ &= (1 * (y + \neg x)) + \neg y \\ &= y + \neg x + \neg y \\ &= 1 \end{aligned}$$

3.

$$\neg x + \neg y = \neg(x * y)$$

12.Prove $\sum_{i=0}^{n} 2i = n(n+1)$

Assume $p(n) = \sum_{i=0}^{n} 2i = n(n+1)$ **holds, then show** $p(n+1)$**holds**

$$\begin{aligned}
p(0) = \textstyle\sum_{i=0}^{0} 2i &= 2*0 \\
&= 0*(0+1) \\
&=> p(0) holds \\
p(n+1) &= p(n) + 2*(n+1) \\
&= n(n+1) + 2*(n+1) \ \&= n^2+3n+2 \\
&= (n+1)(n+2) \\
&=> p(n+1) holds \\
&=> p(n) = \sum_{i=0}^{n} 2i = n(n+1) \quad is \quad proved
\end{aligned}$$

13.Prove $\sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6$

Assume $p(n) = \sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6$ **holds, then show** $p(n+1)$**holds**

$$\begin{aligned}
p(0) &= \sum_{i=0}^{0} i^2 \\
&= 0*(0+1)*(2*0+1)/6 \\
&=> p(0) holds \\
p(n+1) &= p(n) + (n+1)(n+2)(2(n+1)+1)/6 \\
&= n(n+1)(2n+1)/6 + (n+1)(n+2)(2(n+1)+1)/6 \\
&= (n+1)(n+2)(2(n+1)+1)/6 \\
&=> p(n+1) holds \\
&=> p(n) = \sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6 \quad is \quad proved
\end{aligned}$$

14. Show that $n! < n^n$ for all $n \in N$with $n > 1$

Assume $p(n) : n! < n^n$ **for all** $n \in N$**with** $n > 1$ **holds, then show** $p(n+1)$**holds**

$$\begin{aligned}
1.when \quad n = 2, n! = 2, n^n &= 4 \\
&=> n! < n^n \\
&=> p(2) holds \\
2.assume \quad p(n) : n! < n^n \quad holds \\
=> (n+1)n! < (n+1)n^n \\
=> (n+1)! < (n+1)n^n \\
=> since (n+1)n^n < (n+1)(n+1)^n \\
=> (n+1)! < (n+1)n^n < (n+1)(n+1)^n \\
=> (n+1)! < (n+1)^(n+1)
\end{aligned}$$

$$=> p(n) = \sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6 \quad is \quad proved$$