Assginment4

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Contents

1 Question1

1.1 $\neg\neg A \supset A$

$$\dfrac{\dfrac{\dfrac{A \quad \to \quad A}{\neg A, A}\;(\neg : left)}{\neg\neg A \quad \to \quad A}\;(\neg : left)}{\neg\neg A \supset A}\;(\supset : right)$$

1.2 $A \supset (A \vee B)$

$$\dfrac{\dfrac{A \quad \to \quad A, B}{A \quad \to \quad (A \vee B)}\;(\vee : right)}{A \supset (A \vee B)}\;(\supset : right)$$

1.3 $A \vee (A \wedge B) \supset A$

$$\dfrac{\dfrac{A \quad \to \quad A \qquad \dfrac{A, B \quad \to \quad A}{A \quad \to \quad (A \vee B)}\;(\wedge : left)}{A \vee (A \wedge B) \quad \to \quad A}\;(\vee : left)}{A \vee (A \wedge B) \supset A}\;(\supset : right)$$

2 Question2

2.1 $A \supset (B \supset (A \wedge B))$

$$\dfrac{\dfrac{\dfrac{B, A \quad \to \quad A \qquad B, A \quad \to \quad B}{B, A \quad \to \quad (AB)}\;(\wedge : right)}{A \quad \to \quad (B \supset (A \wedge B))}\;(\supset : right)}{A \supset (B \supset (A \wedge B))}\;(\supset : right)$$

2.2 $(A \supset B) \supset (\neg B \supset \neg A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \quad \to \quad A, B \qquad B, A \quad \to \quad B}{A, A \supset B \quad \to \quad B}\;(\supset : left)}{A \supset B \quad \to \quad B, \neg A}\;(\neg : right)}{\neg B, (A \supset B) \quad \to \quad \neg A}\;(\neg : left)}{(A \supset B) \quad \to \quad (\neg B \supset \neg A)}\;(\supset : right)}{(A \supset B) \supset (\neg B \supset \neg A)}\;(\supset : right)$$

2.3 $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

$$\dfrac{\dfrac{A \quad \to \quad A, B, C \qquad C, A \quad \to \quad B, C}{A, A \supset C \quad \to \quad B, C}\;(\supset : left) \qquad \dfrac{C, A \quad \to \quad A, C \qquad C, C, A \quad \to \quad C}{(C, A, (A \supset C) \quad \to \quad C}\;(\supset : left)}{A, (B \supset C), (A \supset C) \quad \to \quad C}\;(\supset : left)$$

$$\dfrac{\dfrac{B \quad \to \quad A, B, C \qquad C, B \quad \to \quad B, C}{B, A \supset C \quad \to \quad B, C}\;(\supset : left) \qquad \dfrac{C, B \quad \to \quad A, C \qquad C, C, B \quad \to \quad C}{(C, B, (A \supset C) \quad \to \quad C}\;(\supset : left)}{B, (B \supset C), (A \supset C) \quad \to \quad C}\;(\supset : left)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A, (B \supset C), (A \supset C) \;\rightarrow\; C \qquad B, (B \supset C), (A \supset C) \;\rightarrow\; C}{(A \vee B), (B \supset C), (A \supset C) \;\rightarrow\; C}\,(\wedge : left)}{(B \supset C), (A \supset C) \;\rightarrow\; (A \vee B) \supset C}\,(\supset\!: right)}{(A \supset C) \;\rightarrow\; ((B \supset C) \supset ((A \vee B) \supset C))}\,(\supset\!: right)}{(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))}\,(\supset\!: right)$$

3 Question 3

3.1 $P \equiv \neg P$

Based on :$P \equiv \neg P = (P \supset \neg P) \wedge (\neg P \supset P)$

$$\dfrac{\dfrac{\dfrac{P, P \;\rightarrow}{P, \;\rightarrow\; \neg P}\,(\neg : right)}{P \supset \neg P}\,(\supset\!: right) \qquad \dfrac{\dfrac{\rightarrow\; P, P}{\neg P \;\rightarrow\; P}\,(\neg : left)}{\neg P \supset P}\,(\supset\!: right)}{(P \supset \neg P) \wedge (\neg P \supset P)}\,(\wedge : right)$$

=> The only axiom is $P \supset P$,which is true, so $P \equiv \neg P$ is valid and satisfiable.

3.2 $(P \wedge Q) \supset (P \vee Q)$

$$\dfrac{\dfrac{\dfrac{P, Q \;\rightarrow\; P, Q}{P, Q \;\rightarrow\; (P \vee Q)}\,(\vee : right)}{(P \wedge Q) \;\rightarrow\; (P \vee Q)}\,(\wedge : left)}{(P \wedge Q) \supset (P \vee Q)}\,(\supset\!: right)$$

=> The only axiom is $P, Q \supset P, Q$,which is true, so $(P \wedge Q) \supset (P \vee Q)$ is valid and satisfiable.

3.3 $(Q \supset \neg P) \equiv (P \equiv Q)$

Based on $(Q \supset \neg P) \equiv (P \equiv Q) = ((Q \supset \neg P) \supset ((P \supset Q) \wedge (Q \supset P))) \wedge (((P \supset Q) \wedge (Q \supset P)) \supset (Q \supset \neg P))$

$$\dfrac{\dfrac{\dfrac{\dfrac{P \;\rightarrow\; Q, Q \qquad \dfrac{P \;\rightarrow\; P, Q}{\neg P, P \;\rightarrow\; Q}\,(\neg : left)}{P, (Q \supset \neg P) \supset Q}\,(\supset\!: left)}{(Q \supset \neg P) \supset (P \supset Q)}\,(\supset\!: right) \qquad \dfrac{\dfrac{Q \;\rightarrow\; Q, P \qquad \dfrac{Q \;\rightarrow\; P, P}{\neg P, Q \;\rightarrow\; P}\,(\neg : left)}{Q, (Q \supset \neg P) \supset P}\,(\supset\!: left)}{(Q \supset \neg P) \supset (Q \supset P)}\,(\supset\!: right)}{(Q \supset \neg P) \;\rightarrow\; ((P \supset Q) \wedge (Q \supset P)))}\,(\wedge : right)}{(Q \supset \neg P) \supset ((P \supset Q) \wedge (Q \supset P)))}\,(\supset\!: right)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, Q \;\rightarrow\; Q, P \qquad P, P, Q \;\rightarrow\; P}{P, Q, (Q \supset P) \supset P}\,(\supset\!: left)}{Q, (Q \supset P) \;\rightarrow\; P, \neg P}\,(\neg : right) \qquad \dfrac{\dfrac{P, Q, Q \;\rightarrow\; Q \qquad P, P, Q, Q \;\rightarrow}{P, Q, Q, (Q \supset P) \;\rightarrow}\,(\supset\!: left)}{Q, Q, (Q \supset P) \supset \neg P}\,(\neg : right)}{Q, (P \supset Q), (Q \supset P) \;\rightarrow\; \neg P}\,(\supset\!: left)}{(P \supset Q), (Q \supset P) \;\rightarrow\; (Q \supset \neg P)}\,(\supset\!: right)}{((P \supset Q) \wedge (Q \supset P)) \;\rightarrow\; (Q \supset \neg P)}\,(\wedge : left)}{((P \supset Q) \wedge (Q \supset P)) \supset (Q \supset \neg P)}\,(\supset\!: right)$$

$$\dfrac{(Q \supset \neg P) \supset ((P \supset Q) \wedge (Q \supset P))) \qquad ((P \supset Q) \wedge (Q \supset P)) \supset (Q \supset \neg P)}{(Q \supset \neg P) \supset ((P \supset Q) \wedge (Q \supset P))) \wedge ((P \supset Q) \wedge (Q \supset P)) \supset (Q \supset \neg P)}\,(\wedge : right)$$

This formula is satisfiable and invalid.

4 Question4

4.1 $(P \supset Q) \wedge (\neg P \supset Q)$

$$\dfrac{\dfrac{P \;\rightarrow\; Q}{P \supset Q}\,(\supset: right) \qquad \dfrac{\dfrac{\rightarrow\; P, Q}{\neg P \;\rightarrow\; Q}\,(\neg: left)}{\neg P \supset Q}\,(\supset: right)}{(P \supset Q) \wedge (\neg P \supset Q)}\,(\wedge: right)$$

This formula is unsatisfiable and invalid.

4.2 $((P \supset Q) \supset R) \supset S$

$$\dfrac{\dfrac{\dfrac{P \;\rightarrow\; Q, S}{(P \supset Q), S}\,(\supset: right) \qquad R \;\rightarrow\; S}{((P \supset Q) \supset R) \;\rightarrow\; S}\,(\supset: left)}{((P \supset Q) \supset R) \supset S}\,(\supset: right)$$

This formula is unsatisfiable and invalid.

4.3 $(p \wedge Q) \supset (P \supset Q)$

$$\dfrac{\dfrac{\dfrac{P, P, Q \;\rightarrow\; Q}{P, (PQ) \;\rightarrow\; Q}\,(\wedge: left)}{(p \wedge Q) \supset (P \;\rightarrow\; Q) S}\,(\supset: right)}{(p \wedge Q) \supset (P \supset Q)}\,(\supset: right)$$

This formula is satisfiable and valid.

5 Question5

5.1 $(P \supset R) \supset ((Q \supset S) \supset ((P \wedge Q) \supset R))$

$$\dfrac{\dfrac{P \;\rightarrow\; P, Q, R \qquad R, P \;\rightarrow\; Q, R}{P, (P \supset R) \;\rightarrow\; Q, R}\,(\supset: left) \qquad \dfrac{P, S \;\rightarrow\; P, R \qquad R, P, S \;\rightarrow\; R}{P, S, (P \supset R) \;\rightarrow\; R}\,(\supset: left)}{P, (Q \supset S), (P \supset R) \;\rightarrow\; R}\,(\supset: left)$$

$$\dfrac{\dfrac{Q \;\rightarrow\; P, Q, R \qquad R, Q \;\rightarrow\; Q, R}{Q, (P \supset R) \;\rightarrow\; Q, R}\,(\supset: left) \qquad \dfrac{S, Q \;\rightarrow\; P, R \qquad R, S, Q \;\rightarrow\; R}{S, Q, (P \supset R) \;\rightarrow\; R}\,(\supset: left)}{Q, (Q \supset S), (P \supset R) \;\rightarrow\; R}\,(\supset: left)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P, (Q \supset S), (P \supset R) \;\rightarrow\; R \qquad Q, (Q \supset S), (P \supset R) \;\rightarrow\; R}{(P \vee Q), (Q \supset S), (P \supset R) \;\rightarrow\; R}\,(\wedge: left)}{(Q \supset S), (P \supset R) \;\rightarrow\; (P \vee Q) \supset R}\,(\supset: right)}{(P \supset R) \;\rightarrow\; ((Q \supset S) \supset ((P \vee Q) \supset R))}\,(\supset: right)}{(P \supset R) \supset ((Q \supset S) \supset ((P \vee Q) \supset R))}\,(\supset: right)$$

The false axiom is $S, Q \;\rightarrow\; P, R$, so the conjunctive normal form is : $\neg S \vee \neg Q \vee P \vee R$

5.2 $(P \supset Q) \supset ((Q \supset \neg R) \supset \neg P)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{P \;\to\; P, Q \qquad Q, P \;\to\; Q}{P, (P \supset Q) \;\to\; Q}\,(\supset\!: left)
}{(P \supset Q) \;\to\; Q, \neg P}\,(\neg : right)
\qquad
\dfrac{
\dfrac{
\dfrac{P \;\to\; P, R \qquad Q, P \;\to\; R}{P, (P \supset Q) \;\to\; R}\,(\supset\!: left)
}{P, \neg R, (P \supset Q) \;\to\;}\,(\neg : right)
}{\neg R, (P \supset Q) \;\to\; \neg P}\,(\neg : right)
}{(Q \supset \neg R), (P \supset Q) \;\to\; \neg P}\,(\supset\!: left)
}{
\dfrac{(P \supset Q) \;\to\; ((Q \supset \neg R) \supset \neg P)}{(P \supset Q) \supset ((Q \supset \neg R) \supset \neg P)}\,(\supset\!: right)
}\,(\supset\!: right)
$$

The conjunctive form is $\neg Q \vee \neg P \vee R$

6 Question6

6.1

Based on Question5.1, the disjunctive normal form is$(\neg S \wedge \neg P) \vee Q \vee R$

6.2

Based on Question5.2, the disjunctive normal form is$(\neg P \vee Q) \vee (P \wedge \neg R) \vee ((Q \wedge P) \wedge \neg R)$

7 Question7

7.1 $\forall x A \supset A[t/x]$

$$
\dfrac{\dfrac{A[t/x] \;\to\; A[t/x]}{\forall x A \;\to\; A[t/x]}\,(\forall : left)}{\forall x A \supset A[t/x]}\,(\supset\!: right)
$$

7.2 $A[t/x] \supset \exists x A$

$$
\dfrac{\dfrac{A[t/x] \;\to\; A[t/x]}{A[t/x] \;\to\; \exists x A}\,(\exists : left)}{A[t/x] \supset \exists x A}\,(\supset\!: right)
$$

8 Question9

Let L =$\{CS, FS, PS\}$
1. CS = $\emptyset$
2. FS = $\emptyset$
3. PS = $\{\leq\}$ with r($\leq$) = 2

Let Γ be the following set of axioms:
1. $\forall x \neg(x \leq x)$,
2. $\forall x \forall y((x < y) \supset \neg(y < x))$
3. $\forall x \forall y \forall z(((x < y) \wedge (y < z)) \supset (x < z))$

Then T = (L, Γ) is a first-order theory of the weak partial orders.

9 Question11

The first-order language of (directed) graphs is L = r, where r is a binary relation symbol.
The only terms are the variables x.
Atomic formulas look like $(x \approx y)$ or (rxy).
Example:
The subformulas of $\forall x((rxy) \;\to\; \exists y(ryx))$:
$\forall x((rxy) \;\to\; \exists y(ryx))$
$rxy(rxy) \;\to\; \exists y(ryx)$

$\exists y(ryx)$
$ryx.$

10 Question12

define a first-order language LK for K-vector spaces (or left K-modules) as follows. LK has all the usual logical symbols from first-order logic as well as a constant symbol 0 for the zero vector, a binary function symbol + for vector addition, a unary function symbol for additive inverse of vectors, and for each λ ∈K a unary function symbol λ for scalar mulitiplication by λ. In this language we axiomatise the notion of K-vector space of positive dimension. The axioms are,
Axioms for an abelian group:
$\forall x\forall y\forall z(x+(y+z)=(x+y)+z)\forall x\forall y\forall z(x+(y+z)=(x+y)+z)$
$\forall x\forall y(x+y=y+x)\forall x\forall y(x+y=y+x)$
$\forall x(x+0=x)\forall x(x+0=x)$
$\forall x(x+(x)=0)\forall x(x+(-x)=0)$
$Axioms for scalar multiplication:$
$\forall x(s\lambda(s\mu(x))=s\lambda(x))\forall x(s\lambda(s\mu(x))=s\lambda\mu(x)), for all \lambda,\mu\in K\lambda,\mu\in K$
$\forall x(s\lambda(x)+s\mu(x)=s(\lambda+\mu)(x))\forall x(s\lambda(x)+s\mu(x)=s(\lambda+\mu)(x)), for all \lambda,\mu\in K\lambda,\mu\in K$
$\forall x(s\lambda(x+y)=s\lambda(x)+s\lambda(x))\forall x(s\lambda(x+y)=s\lambda(x)+s\lambda(x)), for all \lambda\in K\lambda\in K$
$\forall x(s1(x)=x)$