CAS 701
Logic and Discrete Mathematics
Fall 2017

Exercise Group 4

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You are required to submit solutions to 10 of the following 15 exercises. If you submit more than 10, only the first 10 will be marked. Your solutions should begin with a table of contents that indicates which of the 15 exercises you have chosen to do. Each question is worth 10 points. **The solutions must be written using LaTeX and are due November 24, 2017 at the beginning of class.**

1. For each of the following propositional formulas X, construct a proof tree of G' whose conclusion is X:

   a. $\neg\neg A \supset A$.
   b. $A \supset (A \vee B)$.
   c. $A \vee (A \wedge B) \supset A$.

2. For each of the following propositional formulas X, construct a proof tree of G' whose conclusion is X:

   a. $(A \supset B) \supset (\neg B \supset \neg A)$.
   b. $A \supset (B \supset (A \wedge B))$.
   c. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$.

3. For each of the following propositional formulas A, construct a proof tree or counterexample of G' that shows A is valid, satisfiable but not valid, or unsatisfiable:

   a. $P \equiv \neg P$.
   b. $(P \wedge Q) \supset (P \vee Q)$.
   c. $(Q \supset \neg P) \equiv (P \equiv Q)$.

4. For each of the following propositional formulas A, construct a proof tree or counterexample of G' that shows A is valid, satisfiable but not valid, or unsatisfiable:

a. $(P \supset Q) \wedge (\neg P \supset Q)$.
b. $((P \supset Q) \supset R) \supset S$.
c. $(P \wedge Q) \supset (P \supset Q)$.

5. For each of the following propositional formulas A, construct a propositional formula B in *conjunctive normal form* and a counterexample tree of G' such that the counterexample tree shows A and B are logically equivalent (see Example 3.4.5 in Gallier 2015):

   a. $(P \supset R) \supset ((Q \supset S) \supset ((P \vee Q) \supset R))$.
   b. $(P \supset Q) \supset ((Q \supset \neg R) \supset \neg P)$.

6. For each of the following propositional formulas A, construct a propositional formula B in *disjunctive normal form* and a counterexample tree of G' such that the counterexample tree shows A and B are logically equivalent (see Example 3.4.6 in Gallier 2015):

   a. $(P \supset R) \supset ((Q \supset S) \supset ((P \vee Q) \supset R))$.
   b. $(P \supset Q) \supset ((Q \supset \neg R) \supset \neg P)$.

7. Let $\mathbf{L}$ be a first-order language, t be an $\mathbf{L}$-term, and A be an $\mathbf{L}$-formula such that t is free for x in A. Show that the following formulas are valid:

   a. $\forall x\, A \supset A[t/x]$.
   b. $A[t/x] \supset \exists x\, A$.

8. Let $\mathbf{L}$ be a first-order language. Define what it means for an $\mathbf{L}$-formula to be in *prenex normal form*. Implement a program, as succinct as possible, in your programming language of choice that, given an $\mathbf{L}$-formula A as input, returns an $\mathbf{L}$-formula A' as output such that A' is in prenex normal form and $A \equiv A'$ is valid. Haskell would be an excellent choice the programming language.

9. Express the theory of lattices as a first-order axiomatic theory $(\mathbf{L}, \Gamma)$ where $\mathbf{L} = (\emptyset, \emptyset, \{\leq\})$ and $\leq$ is a weak partial order.

10. Express the theory of lattices as a first-order axiomatic theory $(\mathbf{L}, \Gamma)$ where $\mathbf{L} = (\emptyset, \{\wedge, \vee\}, \emptyset)$ and $\wedge$ and $\vee$ denote the meet and join of a lattice, respectively.

11. Express the theory of simple directed graphs as a first-order axiomatic theory.

12. Express the theory of vector spaces as a first-order axiomatic theory.

13. Let $\mathbf{PA}' = (\mathbf{L}, \Gamma)$ be first-order Peano arithmetic. Show that $\mathbf{PA}' \vDash S^2\, 0 * S^3\, 0 = S^6\, 0$ where $S^n\, 0$ is an abbreviation for $S \cdots S\, 0$ (n times).

14. Let **L** be a first-order language. Assume that there is a proof system for **L** that is sound and complete. From this assumption prove the compactness theorem for **L**, i.e., if Γ is a finitely satisfiable set of **L**-formulas, then Γ is satisfiable.

15. Use the compactness theorem for first-order logic to show that every first-order axiomatic theory $\mathbf{T} = (\mathbf{L}, \Gamma)$ that has arbitrarily large finite models has an infinite model. Is there a first-order axiomatic theory of all finite **L**-structures? Hint: Add a infinite number of new constants to **L** and add axioms to Γ that say that these new constants have distinct values.