CAS 701
Logic and Discrete Mathematics
Fall 2017

Exercise Group 2

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You are required to submit solutions to 10 of the following 15 exercises. If you submit more than 10, only the first 10 will be marked. Your solutions should begin with a table of contents that indications which of the 15 exercises you have chosen to do. Each question is worth 10 points. Some of the questions require the use of mathematical induction. **The solutions must be written using LaTeX and are due October 17, 2017 at the beginning of class.**

1. Consider the weak partial order

$$P = (\{\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\}, \subseteq).$$

   a. Find the maximal elements in P.
   b. Find the minimal elements in P.
   c. Find the maximum element in P if it exists.
   d. Find the minimum element in P if it exists.
   e. Find all the upper bounds of $\{\{2\},\{4\}\}$ in P.
   f. Find the least upper bound of $\{\{2\},\{4\}\}$ in P if it exists.
   g. Find all the lower bounds of $\{\{1,3,4\},\{2,3,4\}\}$ in P.
   h. Find the greater lower bound of $\{\{1,3,4\},\{2,3,4\}\}$ in P if it exists.

2. Suppose $(S_1, \leq_1)$ and $(S_2, \leq_2)$ are partial orders. Prove that $(S_1 \times S_2, \leq)$ is a partial order where $(s_1, s_2) \leq (s'_1, s'_2)$ iff $s_1 \leq_1 s'_1$ and $s_2 \leq_2 s'_2$.

3. The strict total order $(\mathbb{R}, <)$ is both dense and complete. Let $(\mathbb{R} \cup \{-\infty, +\infty\}, <)$ be the strict total order that extends $(\mathbb{R}, <)$ by adding $-\infty$ and $+\infty$ as minimum and maximum elements. Show that $(\mathbb{R} \cup \{-\infty, +\infty\}, <)$ is also dense and complete.

4. Show that $(\mathbb{N} \times \mathbb{N}, \leq)$, where $\leq$ is lexicographical order on $\mathbb{N} \times \mathbb{N}$, is a well-order.

5. Show that a total order is a lattice.

6. Show that $(\mathbb{N}, |)$ is a complete lattice.

7. Show that every finite, nonempty subset of a lattice has a least upper bound and a greatest lower bound.

8. Let S be any set. Show that $(\{\emptyset, S\}, \emptyset, S, \cup, \cap, ^-)$ is a boolean algebra.

9. A *cofinite* subset of a set S is a subset S' of S such that the complement of S' in S is finite. Let S be the finite and cofinite subsets of $\mathbb{N}$. Show that $(S, \emptyset, \mathbb{N}, \cup, \cap, ^-)$ is a boolean algebra.

10. Show that the idempotent laws follow from the axioms of a boolean algebra.

11. Show that the De Morgan's laws follow from the axioms of a boolean algebra.

12. Prove

$$\sum_{i=0}^{n} 2i = n(n+1).$$

13. Prove

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

14. Show that $n! < n^n$ for all $n \in \mathbb{N}$ with $n > 1$.

15. Let S be the set of bit strings defined inductively by:

   a. `"0"` $\in S$.
   b. If $s \in S$, then `"0"` $+ s \in S$ and $s +$ `"0"` $\in S$.
   c. If $s \in S$, then , `"0"` $+ s +$ `"1"` $\in S$ and `"1"` $+ s +$ `"0"` $\in S$.

   $s + t$ denotes the concatenation of s and t. Prove by structural induction that, for all strings $s \in S$, the number of 1s in s is less than or equal to the number of 0s in s.