

Topics on Lattice Gauge Theories

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INTRODUCTION

CHAPTER 1

INTRODUCTION TO LATTICE GAUGE THEORIES

[Ovviamente da scrivere]

1.1 REVIEW OF YANG-MILLS THEORY

[Ovviamente da riscrivere]. A Yang-Mills theory is a gauge field theory on Minkowski space $\mathbb{R}^{1,d}$, where the gauge group $U(1)$ or $SU(N)$, with matter fields, which are defined by a representation of the gauge group. For example, *Quantum Chromodynamics* (QCD) is an $SU(3)$ gauge theory with Dirac spinors in the fundamental representation. Keeping in mind the example of QCD, the Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \quad (1.1)$$

where the fermions ψ are taken in the fundamental representation of $SU(N)$ and the covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. We choose the convention where the Lie algebra generators T^a are Hermitian and $[T^a, T^b] = if^{abc}T^c$, with real structure constants f^{abc} . Furthermore, the generators are such that $\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. The strength-field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad (1.2)$$

and transforms in the adjoint representation of $SU(N)$. Both the gauge field A_μ and the curvature tensor $F_{\mu\nu}$ live in the Lie algebra $\mathfrak{su}(N)$.

Under a gauge transformation given by a group-valued function $g(x) \in SU(N)$, such that $\psi \mapsto g(x)\psi(x)$, then

$$A_\mu(x) \mapsto g(x)A_\mu(x)g(x)^{-1} + ig(x)\partial_\mu g(x)^{-1}, \quad (1.3)$$

so that $D_\mu\psi(x) \mapsto g(x)D_\mu\psi(x)$, while

$$F_{\mu\nu} \mapsto g(x)F_{\mu\nu}g(x)^{-1}, \quad (1.4)$$

leaving the action invariant.

The action of the theory in $d + 1$ dimensions is the given by

$$S[A, \psi, \bar{\psi}] = \int d^{d+1} \mathcal{L} \quad (1.5)$$

and the path integral

$$Z = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS[A, \psi, \bar{\psi}]}. \quad (1.6)$$

1.1.1 EUCLIDEAN FIELD THEORY

In order to work in a Euclidean space-time, we need first to perform a *Wick rotation*, where the time coordinate x_0 is mapped a forth space coordinate x_4 , through $x_0 = -ix_4$. The has the effect of changing the path-integral integrand from e^{iS} , which is oscillatory, to e^{-S} , which is positive and can be interpreted as a probability distribution of the configurations of the fields.

The Euclidean path integral is

$$Z_E = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S}, \quad (1.7)$$

so that the Minkowski action and the Euclidean action satisfy $iS_M = -S_E$. The respective actions are given by

$$S_M = \int d^{d+1} x_M \mathcal{L}_M, \quad S_E = \int d^{d+1} x_E \mathcal{L}_E. \quad (1.8)$$

The rotation $x_0 = -ix_4$ leads to $\mathcal{L}_E = -\mathcal{L}_M$.

The Wick rotation does not change the form of the gauge kinetic term, i.e.,

$$-\frac{1}{2g^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}). \quad (1.9)$$

The sum is a simple Euclidean sum, where there are no minus signs when raising or lowering indices and $\mu = 1, \dots, d + 1$.

Considering now the fermionic part of the Yang-Mills Lagrangian, we need to perform the Wick rotation on the Dirac operator. In Minkowski space-time

$$\bar{\psi}(i\gamma_M^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma_M^\mu \partial_\mu + \gamma_M^\mu A_\mu - m)\psi, \quad (1.10)$$

where γ_M^μ are the gamma matrices of the Clifford algebra, and they satisfy $\{\gamma_M^\mu, \gamma_M^\nu\} = 2\eta^{\mu\nu}$. In the Euclidean Clifford algebra instead the gamma matrices γ_E^μ satisfy $\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}$. Given the fact that we have $\partial_0 = i\partial_4$ and $A_0 = iA_4$, in order to obtain the correct form we have to put $\gamma_M^0 = \gamma_E^4$. This procedure yields

$$\bar{\psi}(i\gamma_M^\mu \partial_\mu + \gamma_M^\mu A_\mu - m)\psi = -\bar{\psi}(\gamma_E^\mu \partial_\mu + i\gamma_E^\mu A_\mu + m)\psi. \quad (1.11)$$

Since $\mathcal{L}_E = -\mathcal{L}_M$, we finally arrive at

$$\mathcal{L}_E = \frac{1}{2g^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) + \bar{\psi}(\gamma^\mu D_\mu)\psi, \quad (1.12)$$

where the indices are all Euclidean and $D_\mu = \partial_\mu + iA_\mu$.

1.1.2 HAMILTONIAN FORMULATION

The Hamiltonian formulation of a Yang-Mills theory can be tricky, especially the part about the gauge field. Usually, one has to proceed by computing the conjugate momenta and performing a Legendre transform. The main issue here is that the gauge field component A_0 , does not have a conjugate momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0. \quad (1.13)$$

Hence, the transformation is not invertible. The easiest way to remedy to the situation is to impose the gauge condition $A_0 = 0$, which is called *canonical gauge* or *temporal gauge*. With this condition, the kinetic term for the gauge fields can be written as

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) = \frac{1}{g^2} (\mathbf{E}^2 - \mathbf{B}^2) = \frac{1}{g^2} (E_i^a E_i^a - B_i^a B_i^a), \quad (1.14)$$

where \mathbf{E} and \mathbf{B} are, respectively, the “chromoelectric” and the “chromomagnetic” fields. In the temporal gauge $\mathbf{E} = \dot{\mathbf{A}}$, the time derivative of the spatial components \mathbf{A} of the gauge field A_μ , while \mathbf{B} corresponds to the spatial components of the strength-field tensor $F^{\mu\nu}$ and does not involve any time derivative. From the Legendre transformation of (1.14) we obtain the Hamiltonian density:

$$\mathcal{H} = \frac{1}{g^2} E_i^a \dot{A}_i^a - \frac{1}{2g^2} (E_i^a E_i^a - B_i^a B_i^a) = \frac{1}{2g^2} \text{tr}(\mathbf{E}^2 + \mathbf{B}^2), \quad (1.15)$$

hence the Hamiltonian in d spatial dimensions is

$$H = \int d^d x \frac{1}{2g^2} \text{tr}(\mathbf{E}^2 + \mathbf{B}^2). \quad (1.16)$$

In the Hamiltonian formulation, the fields \mathbf{A} and \mathbf{E} are now operators, satisfying the commutation relations

$$\begin{aligned} [A_i^a(x), E_j^b(y)] &= ig^2 \delta_{ij} \delta_{ab} \delta(x - y) \\ [E_i^a(x), E_j^b(y)] &= [A_i^a(x), A_j^b(y)] = 0. \end{aligned} \quad (1.17)$$

The \mathbf{B} operator is defined from \mathbf{A} .

In order to impose the canonical gauge $A_0 = 0$, the equation of motion for A_0 has to be satisfied. This leads [\[citation?\]](#) to the fact that one must have

$$D_i E_i = 0, \quad (1.18)$$

where D_i are the spatial components of the covariant derivative and E_i the spatial components of the chromoelectric field. Unfortunately, the equation above is inconsistent with the commutation relation (1.17) and so it cannot be implemented as an operator equation. The easiest solution is to impose it on states that are considered *physical*:

$$D_i E_i |\psi_{\text{phys}}\rangle = 0, \quad (1.19)$$

for each component of $D_i E_i$. The constraints select a subspace of the overall Hilbert space, which will be labeled as the *physical Hilbert space*. The condition above for a $U(1)$ theory reduces to the well known *Gauss law* $\nabla \cdot \mathbf{E} = 0$.

1.1.3 THE SIGN PROBLEM

Even though interesting phases have been predicted for QCD in the $\mu - T$ plane^[citation?], such as quark-gluon plasma^[citation?] or superconductivity^[citation?], detailed quantitative analysis has been limited to the $\mu = 0$ region only. This is due mainly to the difficulty of studying QCD in the low energy regime, where the perturbative approach fails^[citation?]. Moreover, even *lattice gauge theories*, a non-perturbative approach to Yang-Mills theory (and gauge theories in general), is not applicable for $\mu \neq 0$ due to the *sign problem*.

In the Hamiltonian formulation, the chemical potential in the same manner as statistical mechanics. If \hat{H} is the Hamiltonian operator and \hat{N} a number operator, then one can simply replace \hat{H} with $\hat{H} - \mu\hat{N}$. In the case of a Yang-Mills theory, the number operator corresponds to fermion number, i.e., $\hat{N} = \psi^\dagger\psi$.

In the path integral formalism, fermions are Grassmann variables, which we can integrate over:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int d^{d+1}\bar{\psi} K \psi\right) = \det K, \quad (1.20)$$

where K is the kinetic operator for the fermions. If the fermions are coupled to gauge field, as one would expect from a Yang-Mills theory, then K has some complicated dependence on the fields A_μ . If one includes the chemical potential term $\mu\bar{\psi}^\dagger\psi$ in the Lagrangian, then the fermion determinant $\det K$ turns out to be complex^[citation?], with a non-trivial phase factor.

As a result, the integrand of the path-integral is no longer positive, and it cannot be interpreted as a probability distribution. This is the infamous *sign problem* and poses severe limitation to Monte Carlo simulations in the finite μ region.

1.2 LATTICE FIELD THEORY

Starting from the path integral formulation, the first step in the formulation of a *lattice field theory* (LFT) is the discretization of space-time, where a discrete $d + 1$ -dimensional lattice substitutes the continuum space-time. The simplest choice in this regard is a hypercubic lattice with lattice spacing a , but in theory an LFT can be defined on any type of lattice. An immediate advantage of using a lattice instead of a continuum is the natural ultraviolet cutoff given by the inverse of the lattice spacing.

Formally a lattice Λ is defined as

$$\Gamma = \left\{ x \in \mathbb{R}^4 : x = \sum_{\mu=1}^{d+1} a n_\mu \hat{\mu} \quad n_\mu \in \mathbb{Z} \right\}, \quad (1.21)$$

where $\mu = 1, \dots, d + 1$ and $\hat{\mu}$ is the unit vector in the direction μ . The edges will be labeled by a pair $(x, \hat{\mu})$, meaning that we are referring to the edge in the $\hat{\mu}$ direction from the vertex x . It is important to fix an orientation for each direction in the lattice. The most natural choice is to choose, obviously, $+\hat{\mu}$ for each $\hat{\mu}$. So, even though $(x, \hat{\mu})$ and $(x + \hat{\mu}, -\hat{\mu})$ refers to the same link, the former is traversed in the positive direction while the latter in the negative direction.

In an LFT, both the vertices and edges (also called links) host degrees of freedom (d.o.f). In particular, the matter fields live on the vertices while the gauge fields live on the links

between vertices. However, the definition of these d.o.f. will need some care, because we have two main requirements:

- The lattice action should reduce to the continuum action in the continuum limit, i.e., $a \rightarrow 0$;
- The lattice action should respect the gauge symmetry.

Lorentz invariance is naturally broken on a lattice but we expect to recover it in the continuum limit.

1.2.1 GAUGE FIELDS ON A LATTICE

1.2.2 FERMIONS ON A LATTICE

1.3 QUANTUM SIMULATION

1.4 FINITE-GROUP APPROACH

CHAPTER 2

DUALITIES IN LATTICE GAUGE THEORIES

2.1 BOND-ALGEBRA APPROACH TO DUALITIES

2.2 GAUGE-REDUCING DUALITIES

2.3 DUALITIES OF TWO-DIMENSIONAL PURE LGTS

2.4 DUALITIES OF LADDER LGTS