Topics on Lattice Gauge Theories

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Contents

Introduction

Chapter 1

Introduction to lattice gauge theories

[Ovviamente da scrivere]

1.1 Review of Yang-Mills theory

[Ovviamente da riscrivere]. A Yang-Mills theory is a gauge field theory on Minkowski space $\mathbb{R}^{1,d}$, where the gauge group $U(1)$ or $SU(N)$, with matter fields, which are defined by a representation of the gauge group. For example, *Quantum Chromodynamics* (QCD) is an $SU(3)$ gauge theory with Dirac spinors in the fundamental representation. Keeping in mind the example of QCD, the Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{1.1}$$

where the fermions ψ are taken in the fundamental representation of $SU(N)$ and the covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. We choose the convention where the Lie algebra generators T^a are Hermitian and $[T^a, T^b] = if^{abc}T^c$, with real structure constants f^{abc}. Furthermore, the generators are such that $\operatorname{tr}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab}$. The strength-field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.2}$$

and transforms in the adjoint representation of $SU(N)$. Both the gauge field A_μ and the curvature tensor $F_{\mu\nu}$ live in the Lie algebra $\mathfrak{su}(N)$.

Under a gauge transformation given by a group-valued function $g(x) \in SU(N)$, such that $\psi \mapsto g(x)\psi(x)$, then

$$A_\mu(x) \mapsto g(x)A_\mu(x)g(x)^{-1} + ig(x)\partial_\mu g(x)^{-1}, \tag{1.3}$$

so that $D_\mu\psi(x) \mapsto g(x)D_\mu\psi(x)$, while

$$F_{\mu\nu} \mapsto g(x)F_{\mu\nu}g(x)^{-1}, \tag{1.4}$$

leaving the action invariant.

The action of the theory in $d+1$ dimensions is the given by

$$S[A,\psi,\overline{\psi}] = \int \mathrm{d}^{d+1}\mathcal{L} \tag{1.5}$$

and the path integral

$$Z = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi\, e^{iS[A,\psi,\overline{\psi}]}. \tag{1.6}$$

1.1.1 EUCLIDEAN FIELD THEORY

In order to work in a Euclidean space-time, we need first to perform a *Wick rotation*, where the time coordinate x_0 is mapped a forth space coordinate x_4, through $x_0 = -ix_4$. The has the effect of changing the path-integral integrand from e^{iS}, which is oscillatory, to e^{-S}, which is positive and can be interpreted as a probability distribution of the configurations of the fields.

The Euclidean path integral is

$$Z_E = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S}, \tag{1.7}$$

so that the Minkowski action and the Euclidean action satisfy $iS_M = -S_E$. The respective actions are given by

$$S_M = \int \mathrm{d}^{d+1}x_M\mathcal{L}_M, \qquad S_E = \int \mathrm{d}^{d+1}x_E\mathcal{L}_E. \tag{1.8}$$

The rotation $x_0 = -ix_4$ leads to $\mathcal{L}_E = -\mathcal{L}_M$.

The Wick rotation does not change the form of the gauge kinetic term, i.e.,

$$-\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{1.9}$$

The sum is a simple Euclidean sum, where there are no minus signs when raising or lowering indices and $\mu = 1,\dots,d+1$.

Considering now the fermionic part of the Yang-Mills Lagrangian, we need to perform the Wick rotation on the Dirac operator. In Minkowski space-time

$$\overline{\psi}\big(i\gamma^\mu_M D_\mu - m\big)\psi = \overline{\psi}\big(i\gamma^\mu_M\partial_\mu + \gamma^\mu_M A_\mu - m\big)\psi, \tag{1.10}$$

where γ^μ_M are the gamma matrices of the Clifford algebra, and they satisfy $\{\gamma^\mu_M,\gamma^\nu_M\} = 2\eta^{\mu\nu}$. In the Euclidean Clifford algebra instead the gamma matrices γ^μ_E satisfy $\{\gamma^\mu_E,\gamma^\nu_E\} = 2\delta^{\mu\nu}$. Given the fact that we have $\partial_0 = i\partial_4$ and $A_0 = iA_4$, in order to obtain the correct form we have to put $\gamma^0_M = \gamma^4_E$. This procedure yields

$$\overline{\psi}\big(i\gamma^\mu_M\partial_\mu + \gamma^\mu_M A_\mu - m\big)\psi = -\overline{\psi}\big(\gamma^\mu_E\partial_\mu + i\gamma^\mu_E A_\mu + m\big)\psi. \tag{1.11}$$

Since $\mathcal{L}_E = -\mathcal{L}_M$, we finally arrive at

$$\mathcal{L}_E = \frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(\gamma^\mu D_\mu)\psi, \tag{1.12}$$

where the indices are all Euclidean and $D_\mu = \partial_\mu + iA_\mu$.

1.1.2 HAMILTONIAN FORMULATION

The Hamiltonian formulation of a Yang-Mills theory can be tricky, especially the part about the gauge field. Usually, one has to procede by computing the conjugate momenta and performing a Legendre transform. The main issue here is that the gauge field component A_0, does not have a conjugate momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0. \tag{1.13}$$

Hence, the transformation is not invertible. The easiest way to remedy to the situation is to impose the gauge condition $A_0 = 0$, which is called *canonical gauge* or *temporal gauge*. With this condition, the kinetic term for the gauge fields can be written as

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{g^2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \frac{1}{g^2}(E_i^a E_i^a - B_i^a B_i^a), \tag{1.14}$$

where $\mathbf{E}$ and $\mathbf{B}$ are, respectively, the "chromoelectric" and the "chromomagnetic" fields. In the temporal gauge $\mathbf{E} = \dot{\mathbf{A}}$, the time derivative of the spatial components $\mathbf{A}$ of the gauge field A_μ, while $\mathbf{B}$ corresponds to the spatial components of the strength-field tensor $F^{\mu\nu}$ and does not involve any time derivative. From the Legendre transformation of (1.14) we obtain the Hamiltonian density:

$$\mathcal{H} = \frac{1}{g^2}E_i^a \dot{A}_i^a - \frac{1}{2g^2}(E_i^a E_i^a - B_i^a B_i^a) = \frac{1}{2g^2}\operatorname{tr}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \tag{1.15}$$

hence the Hamiltonian in d spatial dimensions is

$$H = \int \mathrm{d}^d x \frac{1}{2g^2}\operatorname{tr}\left(\mathbf{E}^2 + \mathbf{B}^2\right). \tag{1.16}$$

In the Hamiltonian formulation, the fields $\mathbf{A}$ and $\mathbf{E}$ are now operators, satisfying the commutation relations

$$\begin{aligned}\left[A_i^a(x), E_j^b(y)\right] &= ig^2\delta_{ij}\delta_{ab}\delta(x-y)\\ \left[E_i^a(x), E_j^b(y)\right] &= \left[A_i^a(x), A_j^b(y)\right] = 0.\end{aligned} \tag{1.17}$$

The $\mathbf{B}$ operator is defined from $\mathbf{A}$.

In order to impose the canonical gauge $A_0 = 0$, the equation of motion for A_0 has to be satisfied. This leads [citation?]to the fact that one must have

$$D_i E_i = 0, \tag{1.18}$$

where D_i are the spatial components of the covariant derivative and E_i the spatial components of the chromoelectric field. Unfortunately, the equation above is inconsistent with the commutation relation (1.17) and so it cannot be implemented as an operator equation. The easiest solution is to impose it on states that are considered *physical*:

$$D_i E_i \left|\psi_{\text{phys}}\right\rangle = 0, \tag{1.19}$$

for each component of $D_i E_i$. The constraints select a subspace of the overall Hilbert space, which will be labeled as the *physical Hilbert space*. The condition above for a $U(1)$ theory reduces to the well known *Gauss law* $\nabla \cdot \mathbf{E} = 0$.

1.1.3 THE SIGN PROBLEM

Even though interesting phases have been predicted for QCD in the $\mu - T$ plane[citation?], such as quark-gluon plasma[citation?]o superconductivity[citation?], detailed quantitative analysis has been limited to the $\mu = 0$ region only. This is due mainly to the difficulty of studying QCD in the low energy regime, where the perturbative approach fails[citation?]. Moreover, even *lattice gauge theories*, a non-perturbative approach to Yang-Mills theory (and gauge theories in general), is not applicable for $\mu \neq 0$ due to the *sign problem*.

In the Hamiltonian formulation, the chemical potential in the same manner as statistical mechanics. If $\hat{H}$ is the Hamiltonian operator and $\hat{N}$ a number operator, then one can simply replace $\hat{H}$ with $\hat{H} - \mu\hat{N}$. In the case of a Yang-Mills theory, the number operator corresponds to fermion number, i.e., $\hat{N} = \psi^\dagger\psi$.

In the path integral formalism, fermions are Grassmann variables, which we can integrate over:

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \exp\left(-\int d^{d+1}\overline{\psi}K\psi\right) = \det K, \tag{1.20}$$

where K is the kinetic operator for the fermions. If the fermions are coupled to gauge field, as one would expect from a Yang-Mills theory, then K has some complicated dependence on the fields A_μ. If one includes the chemical potential term $\mu\overline{\psi}^\dagger\psi$ in the Lagrangian, then the fermion determinant $\det K$ turns out to be complex[citation?], with a non-trivial phase factor.

As a result, the integrand of the path-integral is no longer positive, and it canot be interpreted as a probability distribution. This is the infamous *sign problem* and poses severe limitation to Monte Carlo simulations in the finite μ region.

1.2 LATTICE FIELD THEORY

Starting from the path integral formulation, the first step in the formulation of a *lattice field theory* (LFT) is the discretization of space-time, where a discrete $d+1$-dimensional lattice substitutes the continuum space-time. The simplest choice in this regard is a hypercubic lattice with lattice spacing a, but in theory an LFT can be defined on any type of lattice. An immediate advantage of using a lattice instead of a continuum is the natural ultraviolet cutoff given by the inverse of the lattice spacing.

Formally a lattice Λ is defined as

$$\Gamma = \left\{ x \in \mathbb{R}^4 : x = \sum_{\mu=1}^{d+1} a n_\mu \hat{\mu} \quad n_\mu \in \mathbb{Z} \right\}, \tag{1.21}$$

where $\mu = 1, \ldots, d+1$ and $\hat{\mu}$ is the unit vector in the direction μ. The edges will be labeled by a pair $(x, \hat{\mu})$, meaning that we are referring to the edge in the $\hat{\mu}$ direction from the vertex x. It is important to fix an orientation for each direction in the lattice. The most natural choice is to choose $+\hat{\mu}$ for each $\hat{\mu}$. So, even though $(x, \hat{\mu})$ and $(x + \hat{\mu}, -\hat{\mu})$ refers to the same link, the former is traversed in the positive direction while the latter in the negative direction.

In an LFT, both the vertices and edges (also called links) host degrees of freedom (d.o.f). In particular, the matter fields lives on the vertices while the gauge fields live on the links

between vertices. However, the definition of these d.o.f. will need some care, because we have two main requirements, especially if we are interested in Yang-Mills theory:

- The lattice action should reduce to the continuum action in the continuum limit, i.e., $a \to 0$;
- The lattice action should respect the gauge symmetry.

Lorentz invariance is naturally broken on a lattice but we expect to recover it in the continuum limit.

1.2.1 GAUGE FIELDS ON A LATTICE

[spiegazione perché usiamo il gruppo di Lie e non l'algebra di Lie] Considering a general group G, we associate an element $U_\mu(x) \in G$ to each link (x, μ). If one traverse the link in the opposite direction, one should obtain the inverse element U^{-1}. In the case of $SU(N)$, we take $U_\mu(x)$ to be the matrices in the fundamental representation. We can obtain a vector potential in the continuum limit by writing

$$U_\mu(x) = e^{iaA_\mu(x)}, \tag{1.22}$$

where a is the lattice spacing.

Before moving on the dynamics of these gauge fields we have talk on the meaning of gauge invariance on the lattice. A gauge transformation is described by a group-valued function $g(x)$ (in the appropriate representation), which acts on the vertices x. The variable $U_\mu(x)$ sits in the middle of the site x and $x + \hat{\mu}$, then it is reasonable to think its transformation is

$$U_\mu(x) \mapsto g(x) U_\mu(x) g(x + a\hat{\mu}). \tag{1.23}$$

In order to introduce a dynamics for the gauge fields $U_\mu(x)$, we need to define their action which need two satisfy two requirements: it has to be gauge-invariant and reduce to the pure gauge Yang-Mills action in the continuum limit. From (1.23), we can immediately deduce that taking the product of $U_\mu(x)$ along a closed curve will yield a gauge-invariant quantity. The simplest close curve we can consider is a *plaquette*, i.e., the smallest square face. Hence, on a plaquette $\square$ we introduce $W_\square$ to put in the action, defined as

$$W_\square = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu(x + a\hat{\nu})^\dagger U_\nu(x)^\dagger. \tag{1.24}$$

Notice that we do not have any sum in the indices μ and ν because they are not Lorentz indices. The quantity in (1.24) is called a single plaquette *Wilson loop*.

We can only have scalar quantities in the action, so we need to take the trace of $W_\square$. Then, our lattice action will be defined as the sum over the plaquettes of $\operatorname{tr} W_\square$ (and its Hermitian conjugate):

$$\mathcal{S} = -\frac{1}{g^2} \sum_\square \left(\operatorname{tr} W_\square + \operatorname{tr} W_\square^\dagger \right). \tag{1.25}$$

This is known as the *Wilson action* [citation?]. The quantity $\operatorname{tr} W_\square$ behaves as expected in the continuum limit, where we have to work with the strength field $F^{\mu\nu}$:

$$\operatorname{tr} W_\square \approx N - \frac{a^4}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^6), \tag{1.26}$$

The lattice action is not unique. The Wilson action in (1.25) is the simplest choice that one can make that satisfy our requirement. Some other modification, for example, can include other types of closed loops and these modification can have their place. However, they will not be considered here.

Obviously, in a path-integral formulation of lattice gauge theories we need to define the path integral, which is immediate. The partition function is given by

$$Z = \int \prod_{(x,\hat{\mu})} \mathrm{d}U_\mu(x) e^{-\mathcal{S}}, \tag{1.27}$$

where the integration measure $\mathrm{d}U_\mu(x)$ is understood to be the Haar measure. In case of a compact group, like $SU(N)$, it is well defined and yields a finite value. Now that the path integral measure has been defined, the average of an observable $\mathcal{O}$ can be computed as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{(x,\hat{\mu})} \mathrm{d}U_\mu(x) \mathcal{O} e^{-\mathcal{S}} \tag{1.28}$$

1.2.2 WILSON CONFINEMENT TEST

One of the most important quantity that can be computed for a pure lattice gauge theory is the Wilson loop for generic closed paths $\mathcal{C}$. It serves as *confinement* test. It has been shown [citation?] that confinement is equivalent to the *area law* behaviour of Wilson loops, i.e.,

$$\langle W(\mathcal{C}) \rangle \sim \exp(-\sigma A(\mathcal{C})), \tag{1.29}$$

where $A(\mathcal{C})$ is the area inside the closed path $\mathcal{C}$ and σ the *string tension* (the coefficient of the linear potential between two quarks).

[motivazione fisica del legame tra confinamento e area law del Wilson loop]

On the other hand, in the absence of confinement one finds instead the *perimeter law*

$$\langle W(\mathcal{C}) \rangle \sim \exp(-kP(\mathcal{C})), \tag{1.30}$$

where $P(\mathcal{C})$ is the perimeter of the curve $\mathcal{C}$ and k just some constant.

In this picture, the string tension σ can be interpreted as *order parameter* for confinement: it is non-zero and finite in a confined phase, while it is zero in a deconfined phase.

1.2.3 FERMIONS ON A LATTICE

1.3 QUANTUM SIMULATION

1.4 FINITE-GROUP APPROACH

Chapter 2

Dualities in lattice gauge theories