Finite Group Lattice Gauge Theories for Quantum Simulation

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Contents

Introduction

chapter one

Introduction to Lattice Gauge Theories

1.1 Review of Yang-Mills theory

A Yang-Mills theory is a gauge field theory on Minkowski space $\mathbb{R}^{1,d}$, where the gauge group $U(1)$ or $SU(N)$, with matter fields, which are defined by a representation of the gauge group. For example, *Quantum Chromodynamics* (QCD) is an $SU(3)$ gauge theory with Dirac spinors in the fundamental representation. Keeping in mind the example of QCD, the Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{1.1}$$

where the fermions ψ are taken in the fundamental representation of $SU(N)$ and the covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. We choose the convention where the Lie algebra generators T^a are Hermitian and $[T^a, T^b] = if^{abc}T^c$, with real structure constants f^{abc}. Furthermore, the generators are such that $\operatorname{tr}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab}$. The strength-field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.2}$$

and transforms in the adjoint representation of $SU(N)$. Both the gauge field A_μ and the curvature tensor $F_{\mu\nu}$ live in the Lie algebra $\mathfrak{su}(N)$.

Under a gauge transformation given by a group-valued function $g(x) \in SU(N)$, such that $\psi \mapsto g(x)\psi(x)$, then

$$A_\mu(x) \mapsto g(x)A_\mu(x)g(x)^{-1} + ig(x)\partial_\mu g(x)^{-1}, \tag{1.3}$$

so that $D_\mu\psi(x) \mapsto g(x)D_\mu\psi(x)$, while

$$F_{\mu\nu} \mapsto g(x)F_{\mu\nu}g(x)^{-1}, \tag{1.4}$$

leaving the action invariant.

The action of the theory in $d+1$ dimensions is the given by

$$S[A, \psi, \overline{\psi}] = \int \mathrm{d}^{d+1}\mathcal{L} \tag{1.5}$$

and the path integral

$$Z = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi \, e^{iS[A,\psi,\overline{\psi}]}. \tag{1.6}$$

1.1.1 Euclidean field theory

In order to work in a Euclidean space-time, we need first to perform a *Wick rotation*, where the time coordinate x_0 is mapped a forth space coordinate x_4, through $x_0 = -ix_4$. The has the effect of changing the path-integral integrand from e^{iS}, which is oscillatory, to e^{-S}, which is positive and can be interpreted as a probability distribution of the configurations of the fields.

The Euclidean path integral is

$$Z_E = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S}, \tag{1.7}$$

so that the Minkowski action and the Euclidean action satisfy $iS_M = -S_E$. The respective actions are given by

$$S_M = \int \mathrm{d}^{d+1}x_M \mathcal{L}_M, \qquad S_E = \int \mathrm{d}^{d+1}x_E \mathcal{L}_E. \tag{1.8}$$

The rotation $x_0 = -ix_4$ leads to $\mathcal{L}_E = -\mathcal{L}_M$.

The Wick rotation does not change the form of the gauge kinetic term, i.e.,

$$-\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{1.9}$$

The sum is a simple Euclidean sum, where there are no minus signs when raising or lowering indices and $\mu = 1, \ldots, d+1$.

Considering now the fermionic part of the Yang-Mills Lagrangian, we need to perform the Wick rotation on the Dirac operator. In Minkowski space-time

$$\overline{\psi}(i\gamma^\mu_M D_\mu - m)\psi = \overline{\psi}(i\gamma^\mu_M \partial_\mu + \gamma^\mu_M A_\mu - m)\psi, \tag{1.10}$$

where γ^μ_M are the gamma matrices of the Clifford algebra, and they satisfy $\{\gamma^\mu_M, \gamma^\nu_M\} = 2\eta^{\mu\nu}$. In the Euclidean Clifford algebra instead the gamma matrices γ^μ_E satisfy $\{\gamma^\mu_E, \gamma^\nu_E\} = 2\delta^{\mu\nu}$. Given the fact that we have $\partial_0 = i\partial_4$ and $A_0 = iA_4$, in order to obtain the correct form we have to put $\gamma^0_M = \gamma^4_E$. This procedure yields

$$\overline{\psi}(i\gamma^\mu_M \partial_\mu + \gamma^\mu_M A_\mu - m)\psi = -\overline{\psi}(\gamma^\mu_E \partial_\mu + i\gamma^\mu_E A_\mu + m)\psi. \tag{1.11}$$

Since $\mathcal{L}_E = -\mathcal{L}_M$, we finally arrive at

$$\mathcal{L}_E = \frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(\gamma^\mu D_\mu)\psi, \tag{1.12}$$

where the indices are all Euclidean and $D_\mu = \partial_\mu + iA_\mu$.

1.1.2 Hamiltonian formulation

The Hamiltonian formulation of a Yang-Mills theory can be tricky, especially the part about the gauge field. Usually, one has to procede by computing the conjugate momenta and performing a Legendre transform. The main issue here is that the gauge field component A_0, does not have a conjugate momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0. \tag{1.13}$$

Hence, the transformation is not invertible. The easiest way to remedy to the situation is to impose the gauge condition $A_0 = 0$, which is called *canonical gauge* or *temporal gauge*. With this condition, the kinetic term for the gauge fields can be written as

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{g^2}\left(\mathrm{E}^2 - \mathrm{B}^2\right) = \frac{1}{g^2}(E_i^a E_i^a - B_i^a B_i^a), \tag{1.14}$$

{eq:YM_lagrang_temporal_ga

where E and B are, respectively, the "chromoelectric" and the "chromomagnetic" fields. In the temporal gauge $\mathrm{E} = \dot{\mathrm{A}}$, the time derivative of the spatial components A of the gauge field A_μ, while B corresponds to the spatial components of the strength-field tensor $F^{\mu\nu}$ and does not involve any time derivative. From the Legendre transformation of (1.14) we obtain the Hamiltonian density:

$$\mathcal{H} = \frac{1}{g^2}E_i^a \dot{A}_i^a - \frac{1}{2g^2}(E_i^a E_i^a - B_i^a B_i^a) = \frac{1}{2g^2}\operatorname{tr}\left(\mathrm{E}^2 + \mathrm{B}^2\right), \tag{1.15}$$

hence the Hamiltonian in d spatial dimensions is

$$H = \int \mathrm{d}^d x \frac{1}{2g^2}\operatorname{tr}\left(\mathrm{E}^2 + \mathrm{B}^2\right). \tag{1.16}$$

In the Hamiltonian formulation, the fields A and E are now operators, satisfying the commutation relations

$$\begin{aligned}\left[A_i^a(x), E_j^b(y)\right] &= ig^2\delta_{ij}\delta_{ab}\delta(x-y)\\ \left[E_i^a(x), E_j^b(y)\right] &= \left[A_i^a(x), A_j^b(y)\right] = 0.\end{aligned} \tag{1.17}$$

{eq:comm_rel_E_A_continuu

The B operator is defined from A.

In order to impose the canonical gauge $A_0 = 0$, the equation of motion for A_0 has to be satisfied. This leads [citation?]to the fact that one must have

$$D_i E_i = 0, \tag{1.18}$$

where D_i are the spatial components of the covariant derivative and E_i the spatial components of the chromoelectric field. Unfortunately, the equation

above is inconsistent with the commutation relation (1.17) and so it cannot be implemented as an operator equation. The easiest solution is to impose it on states that are considered *physical*:

$$D_i E_i \left|\psi_{\text{phys}}\right\rangle = 0, \tag{1.19}$$

for each component of $D_i E_i$. The constraints select a subspace of the overall Hilbert space, which will be labeled as the *physical Hilbert space*. The condition above for a $U(1)$ theory reduces to the well known *Gauss law* $\nabla \cdot \mathrm{E} = 0$.

1.1.3 The Sign Problem

Even though interesting phases have been predicted for QCD in the $\mu - T$ plane[citation?], such as quark-gluon plasma[citation?]o superconductivity[citation?], detailed quantitative analysis has been limited to the $\mu = 0$ region only. This is due mainly to the difficulty of studying QCD in the low energy regime, where the perturbative approach fails[citation?]. Moreover, even *lattice gauge theories*, a non-perturbative approach to Yang-Mills theory (and gauge theories in general), is not applicable for $\mu \neq 0$ due to the *sign problem*.

In the Hamiltonian formulation, the chemical potential in the same manner as statistical mechanics. If $\hat{H}$ is the Hamiltonian operator and $\hat{N}$ a number operator, then one can simply replace $\hat{H}$ with $\hat{H} - \mu\hat{N}$. In the case of a Yang-Mills theory, the number operator corresponds to fermion number, i.e., $\hat{N} = \psi^\dagger\psi$.

In the path integral formalism, fermions are Grassmann variables, which we can integrate over:

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \exp\left(-\int d^{d+1}\overline{\psi} K \psi\right) = \det K, \tag{1.20}$$

where K is the kinetic operator for the fermions. If the fermions are coupled to gauge field, as one would expect from a Yang-Mills theory, then K has some complicated dependence on the fields A_μ. If one includes the chemical potential term $\mu\overline{\psi}^\dagger\psi$ in the Lagrangian, then the fermion determinant $\det K$ turns out to be complex[citation?], with a non-trivial phase factor.

As a result, the integrand of the path-integral is no longer positive, and it canot be interpreted as a probability distribution. This is the infamous *sign problem* and poses severe limitation to Monte Carlo simulations in the finite μ region.

1.2 Lattice Field Theory

Starting from the path integral formulation, the first step in the formulation of a *lattice field theory* (LFT) is the discretization of space-time, where a discrete $d+1$-dimensional lattice substitutes the continuum space-time. The simplest choice in this regard is a hypercubic lattice with lattice spacing a, but in theory an LFT can be defined on any type of lattice. An immediate advantage of using a lattice instead of a continuum is the natural ultraviolet cutoff given by the inverse of the lattice spacing.

Formally a lattice Λ is defined as

$$\Gamma = \left\{ x \in \mathbb{R}^4 : x = \sum_{\mu=1}^{d+1} a n_\mu \hat{\mu} \quad n_\mu \in \mathbb{Z} \right\}, \tag{1.21}$$

where $\mu = 1, \dots, d+1$ and $\hat{\mu}$ is the unit vector in the direction μ. The edges will be labeled by a pair $(x, \hat{\mu})$, meaning that we are referring to the edge in the $\hat{\mu}$ direction from the vertex x. It is important to fix an orientation for each direction in the lattice. The most natural choice is to choose $+\hat{\mu}$ for each $\hat{\mu}$. So, even though $(x, \hat{\mu})$ and $(x + \hat{\mu}, -\hat{\mu})$ refers to the same link, the former is traversed in the positive direction while the latter in the negative direction.

In an LFT, both the vertices and edges (also called links) host degrees of freedom (d.o.f). In particular, the matter fields lives on the vertices while the gauge fields live on the links between vertices. However, the definition of these d.o.f. will need some care, because we have two main requirements, especially if we are interested in Yang-Mills theory:

- The lattice action should reduce to the continuum action in the continuum limit, i.e., $a \to 0$;
- The lattice action should respect the gauge symmetry.

Lorentz invariance is naturally broken on a lattice but we expect to recover it in the continuum limit.

1.2.1 Gauge fields on a lattice

[spiegazione perché usiamo il gruppo di Lie e non l'algebra di Lie] Considering a general group G, we associate an element $U_\mu(x) \in G$ to each link (x, μ). If one traverse the link in the opposite direction, one should obtain the inverse element U^{-1}. In the case of $SU(N)$, we take $U_\mu(x)$ to be the matrices in the fundamental representation. We can obtain a vector potential in the continuum limit by writing

$$U_\mu(x) = e^{iaA_\mu(x)}, \tag{1.22}$$

where a is the lattice spacing.

It is necessary to discuss about gauge invariance before moving to the dynamics of these gauge fields. A gauge transformation is described by a group-valued function $g(x)$ (in the appropriate representation), which acts on the vertices x. The variable $U_\mu(x)$ sits in the middle of the site x and $x+\hat{\mu}$, then it is reasonable to think its transformation is

$$U_\mu(x) \mapsto g(x) U_\mu(x) g(x+a\hat{\mu})^\dagger. \tag{1.23}$$ {eq:gauge_transf_field_lattice}

In order to introduce a dynamics for the gauge fields $U_\mu(x)$, we need to define their action which need two satisfy two requirements: it has to be gauge-invariant and reduce to the pure gauge Yang-Mills action in the continuum limit. From (1.23), we can immediately deduce that taking the product of $U_\mu(x)$ along a closed curve will yield a gauge-invariant quantity. The simplest close curve we can consider is a *plaquette*, i.e., the smallest square face. Hence, on a plaquette $\square$ we introduce $W_\square$ to put in the action, defined as

$$W_\square = U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu(x+a\hat{\nu})^\dagger U_\nu(x)^\dagger. \tag{1.24}$$ {eq:single_plaquette_Wilson_

Notice that we do not have any sum in the indices μ and ν because they are not Lorentz indices. The quantity in (1.24) is called a single plaquette *Wilson loop*.

We can only have scalar quantities in the action, so we need to take the trace of $W_\square$. Then, our lattice action will be defined as the sum over the plaquettes of $\operatorname{tr} W_\square$ (and its Hermitian conjugate):

$$\mathcal{S} = -\frac{1}{g^2} \sum_\square \left(\operatorname{tr} W_\square + \operatorname{tr} W_\square^\dagger \right). \tag{1.25}$$ {eq:wilson_action}

This is known as the *Wilson action* [citation?]. The quantity $\operatorname{tr} W_\square$ behaves as expected in the continuum limit, where we have to work with the strength field $F^{\mu\nu}$:

$$\operatorname{tr} W_\square \approx N - \frac{a^4}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^6), \tag{1.26}$$

The lattice action is not unique. The Wilson action in (1.25) is the simplest choice that one can make that satisfy our requirement. Some other modification, for example, can include other types of closed loops and these modification can have their place. However, they will not be considered here.

Obviously, in a path-integral formulation of lattice gauge theories we need to define the path integral, which is immediate. The partition function is given by

$$Z = \int \prod_{(x,\hat{\mu})} \mathrm{d}U_\mu(x) e^{-\mathcal{S}}, \tag{1.27}$$

where the integration measure $\mathrm{d}U_\mu(x)$ is understood to be the Haar measure. In case of a compact group, like $SU(N)$, it is well defined and yields a finite value. Now that the path integral measure has been defined, the average of an observable $\mathcal{O}$ can be computed as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{(x,\hat{\mu})} \mathrm{d}U_\mu(x) \mathcal{O} e^{-\mathcal{S}} \tag{1.28}$$

1.2.2 Order parameters and gauge invariance

The Wilson formulation of lattice gauge theories can resemble spin models studied in statistical mechanics. The link variables $U_\mu(x)$ can be thought as some sort of generalization of the spin degrees of freedom. They are distributed in a crystal-like structure and interact with their nearest neighbours, in this case through a four-body interaction (the plaquette action), instead of two-body interaction (like the Ising model). If one wants to pursue this analogy, then it is reasonable to look at order parameters that behaves like the spontaneous magnetization, where a non-vanishing expectation value signals a phase transition. The analogue of such an order parameter in lattice gauge theory would be something like

$$\langle U_\mu(x) \rangle \neq 0, \tag{1.29}$$ {eq:non_zero_link_var_expt_

rut it has been shown[citation?][teorema di Elitzur] that this is impossible in Wilson theory.

In standard spin models, a non-zero magnetization represents a spontaneous breaking of the global symmetry of the system. Consider the simplest case of the classical Ising model, where the degrees of freedom are binary variables $\sigma = \pm 1$. Without an external field, the energy is given by the interaction of nearest neighbouring spins, i.e., $\sigma_i \sigma_j$. This system has an obvious global $\mathbb{Z}_2$ symmetry, that corresponds to the inversion $\sigma_i \mapsto -\sigma_i$ off all the spins. A ferromagnetic phase is, by definition, signaled by $\langle \sigma \rangle \neq 0$, which necessarily breaks the global $\mathbb{Z}_2$ symmetry of the model. Once a direction is selected by $\langle \sigma \rangle \neq 0$, it remains stable under thermal fluctuations because they cannot coherently shift of the magnetization of an infinite (or rather large) number of spins.

In a lattice gauge theory, an expectation value like (1.29) would *breaks* gauge invariance, which is a *local symmetry*, not a global one. As explained previously, gauge invariance means that the action is unchanged under local arbitrary "rotations" of the link variables $U_\mu(x)$, see (1.23). Hence, thermal fluctuations will induce such rotations and in the long rung it will average on

all the possible gauges. This leads to

$$\langle U_\mu \rangle = \int \mathrm{d}U_\mu U_\mu = 0 \tag{1.30}$$

if U_μ contains only non-trivial irreducible representations of the group. This means that "magnetization" is always vanishing in a lattice gauge theory and gauge invariance cannot be spontaneously broken, which is the contents of the Elitzur theorem[citation?].

The conclusion of this brief discussion may seem rather grim, as magnetization in spin models is the most convenient and used order parameter. But this does not mean that there are no other order parameters in a lattice gauge theory. We have just showed that the problem when considering something like $\langle U_\mu \rangle$ is gauge invariance. So, the most reasonable step forward is to consider *gauge-invariant quantities*. We have already seen that tracing over a product of U_μ variables along a closed curve is a gauge-invariant quantity, called *Wilson loop*.

In so far, we have considered only single plaquette loops but nothing restraints us from considering arbitrary large loops, indeed it serves as a *confinement test* for pure gauge theories. It has been shown [citation?]that confinement is equivalent to the *area law* behaviour of Wilson loops, i.e.,

$$\langle W(\mathcal{C}) \rangle \sim \exp(-\sigma A(\mathcal{C})), \tag{1.31}$$ {eq:wilson_area_law}

where $A(\mathcal{C})$ is the minimal area inside the closed path $\mathcal{C}$ and σ the *string tension* (the coefficient of the linear potential between two quarks). On the other hand, in the absence of confinement one finds instead the *perimeter law*

$$\langle W(\mathcal{C}) \rangle \sim \exp(-kP(\mathcal{C})), \tag{1.32}$$ {eq:wilson_perimeter_law}

where $P(\mathcal{C})$ is the perimeter of the curve $\mathcal{C}$ and k just some constant.

The reason behind this behaviour can be seen with a simple qualitative picture. A closed timelike Wilson loop basically represents a process in which a quark-antiquark pair is produced, moved along the sides of the loop and annihilated. If we are confining phase we can then expect a liner potential between the quark and antiquark. We can imagine a flux tube *binding* the two charges, which swoops the whole inside the loop. Then, it is easy to image that the energy of this whole process will necessarily depend on the area of the loop. On the other hand, if we are in a deconfined phase then there is no potential binding the two quarks. In this case the energy of the whole process depends only on the self-energy of quarks, which move along the sides of the loop. Therefore, the leading energy contribution of this process depends on the perimeter, instead of the area. Obviously, this picture is no longer valid

when dynamical matter is involved. In a confining phase, pair production is always preferred when separating two quarks at large distances. [inserire immagine]

From (1.31) and (1.32), we can deduce that the string tension σ can be used as an order parameter. It is non-zero for a confining phase, while it vanishes for a deconfined phase. But it is *non-local* in nature, as it involves the asymptotic behaviour of potential, and therefore of the correlation functions of the theory.

1.2.3 Fermions on a lattice

Defining fermions is not an easy task due to the known *doubling problem*. In simple terms, when introducing fermions on a lattice, instead of a continuous space, it leads to a extra spurious fermions, which are just lattice artifacts.

In order to briefly see this, consider the correlation function for a single fermionic species. If K is the kinetic matrix for the fermions, then $G = K^{-1}$ gives their correlation matrix. One finds [citation?]that the correlation function between two sites x and y has the form

$$(G)_{x,y} = \frac{1}{a^d L^d} \sum_k \tilde{G}_k e^{2\pi i k \cdot (x-y)/L}, \tag{1.33}$$ {eq:fermionic_corr_func_real}

where a is the lattice spacing, L^d the total volume and $\tilde{G}_k$ the correlation function in momentum space:

$$\tilde{G}_k^{-1} = m + \frac{i}{a} \sum_\mu \gamma_\mu \sin(2\pi k_\mu / L). \tag{1.34}$$ {eq:fermionic_corr_func_latt}

It involves a trigonometric function because the derivative term involves nearest neighbouring sites. One can then take the model to a large lattice, which justifies in substituting the discrete sums with integrals:

$$\frac{2\pi k_\mu}{La} \to q_\mu \qquad \text{and} \qquad \frac{1}{a^d L^d} \sum_k \to \int \frac{\mathrm{d}q^d}{(2\pi)^d}, \tag{1.35}$$ {eq:limit_large_lattice}

where the q_μ's are continuous momentum variables. This substitution maps (1.34) into

$$\tilde{G}_k^{-1} = m + \frac{i}{a} \sum_\mu \gamma_\mu \sin(a q_\mu). \tag{1.36}$$ {eq:fermionic_corr_func_larg}

One can naively think of taking the limit $a \to 0$ and expand $\sin(aq_\mu)$ around the zero and obtain something that look like the correct continuum limit:

$$\tilde{G}_\mu^{-1} = m + i\not{q} + \mathcal{O}(a^2). \tag{1.37}$$

But one should not be fooled by this sloppy procedure just because it appears to give the wanted result. Each component q_μ takes values in the region $[-\pi/a, +\pi/a]$, hence we have to integrate on the whole volume $[-\pi/a, +\pi/a]^d$. Looking at (1.36), it is clear that the major contributions to G in (1.33) comes from the zeros of $\tilde{G}_k^{-1}$. This, not only vanishes in the region $q_\mu \sim 0$ but also for large momentum $q_\mu \sim \pi/a$. The propagator has no suppression of momentum values near π/a. We can isolate the large momenta region by considering

$$\tilde{q}_\mu = q_\mu - \pi/a \tag{1.38}$$

for each direction in space. In this way, we de facto half the integration region,

$$\int_{-\pi/a}^{\pi/a} \mathrm{d}q_\mu \;\to\; \int_{-\pi/2a}^{\pi/2a} (\mathrm{d}q_\mu + \mathrm{d}\tilde{q}_\mu) \tag{1.39}$$

and now the limit $a \to 0$ can be taken safely, but it comes with a price to pay. For each direction in space, we have two independent regions that gives a free fermion contribution to the propagator in the continuum limit. We have effectively *doubled* the number of fermions for each direction. In a d-dimensional lattice we end up with 2^d independent fermions, even though we initially started with just one. [qualcosa a che fare con la chiralità]

There are many solutions to this fermion doubling problem[citation?], but we will focus only on one in this manuscript: *the staggered fermions*[citation?]. We have seen that these fictitious fermions come from the large momenta regions, where $q_\mu \sim \pi/a$. Brutally cutting out this large momenta regions spoils the completeness of the Fourier transform, so it is not a solution, but a smarter solution can give out the same effect. The idea is to spread the fermionic degrees of freedom over multiple lattice sites, reducing effectively the momenta space. For example, in two dimensions it would correspond to placing the *particles* on *even* sites and the *antiparticles* on *odd* sites. A site is considered even or odd when $(-1)^x = (-1)^{x_1+\cdots+x_d} = +1$ or -1.

To obtain a staggered fermion, we define a new fermionic species $\chi(x)$ such that

$$\psi(x) = \prod_\mu (\gamma^\mu)^{n_\mu} \chi(x), \tag{1.40}$$

where $x_\mu = a n_\mu$. Now, if we want to express the discredited covariant derivative, the term $\gamma^\mu \psi(x + a n_\mu)$ have two extra powers of γ^μ compared to $\overline{\psi}$. Since $(\gamma^\mu)^2 = \pm 1$, we have therefore

$$\overline{\psi}(x)\gamma^\mu \psi(x) = (-1)^{\eta_\mu(x)} \chi(x)^\dagger \chi(x + a n_\mu), \tag{1.41}$$

where $\eta_\mu(x)$ is some sign function depending on the site x. This function can be obtained from the commutation relations of the gamma matrices. In

particular, in two dimensions we have

$$\eta_1(x) = 1 \quad \text{and} \quad \eta_2(x) = (-1)^{n_1}, \tag{1.42}$$

while in four (Euclidean) dimension we have instead [Citare Tong, gauge theories]

$$\eta_1(x) = 1, \quad \eta_2(x) = (-1)^{n_1}, \quad \eta_3(x) = (-1)^{n_1+n_2}, \quad \eta_4(x) = (-1)^{n_1+n_2+n_3}. \tag{1.43}$$

A similar reasoning can be applied to the mass term $m\overline{\psi}(x)\psi(c)$, where it becomes

$$m\overline{\psi}(x)\psi(x) = (-1)^{\eta(x)}\chi(x)^\dagger\chi(x), \tag{1.44}$$

for some sign function $\eta(x)$ that can be obtained from the commutation relations of the gamma functions.

chapter two

Lattice Gauge Theories for Quantum Simulation

2.1 Quantum simulation

Simulating quantum mechanics is a very challenging task, in particular if one is interested in many-body systems. The description of the state requires a large number of parameters, for keeping track of all the quantum amplitudes, that grows exponentially with the system size. Hence, one would have an *exponential explosion* in terms of *classical* resources (i.e., computer memory). If simulating a quantum system is not a task for classical computers, then it should be a task for *quantum computers*. This kind of devices, first envisioned from Feynman[citation?], promises much more than simulating quantum mechanics. Indeed, quantum computation and quantum information theory are, still today, very active research fields.

A quantum computer can encodes the large amount of information of a quantum system in its large number of amplitudes. So, the size of a quantum computer would only be proportional to the size of the quantum system it intends to simulate, *without* an exponential explosion in *quantum* resources. In fact, a quantum computer can indeed act as a *universal quantum simulator* (Lloyd 1996[citation?]). This is basically the idea behind *digital quantum simulations*.

However, another approach is possible. One can mimic the evolution of a given quantum system by means of another *analogous* and *controllable* quantum system. Hence, we will only need a specific quantum machine for a specific class of problems. This is, instead, the idea behind *analog quantum simulations*. In this case, for a specific set of problems the full implementation of a quantum computer may not be necessary.

In general, *quantum simulation* can be (loosely) defined as simulating a quantum system by quantum mechanical means, There are three paths that can be taken in this regard:

- digital quantum simulation

- analog quantum simulation
- quantum-information inspired algorithms for classical simulation

We will discuss briefly each one of them. By *quantum simulator* we mean a *controllable* quantum system used to simulate or emulate other quantum systems. We see that only digital and analog quantum simulations employ a quantum simulator. The last option employs techniques, inspired by quantum information theory, that make it possible to truncate and approximates quantum states in order to have efficient classical simulations.

[Inserire figura schematica di un quantum simulator]

2.1.1 Digital quantum simulations

This approach employs the circuit model for quantum computation. Generally, the quantum simulator is a collection of *qubits* (i.e., two-level quantum systems). This constitutes the *quantum register*. A wave function of the simulated system using the computation basis, in other words a superposition of binary bit string. Each bit of a string refers to the state of a qubit in the registry, which can be either $|0\rangle$ or $|1\rangle$ in the computational basis. On this register, any many-qubit unitary transformation U is implemented through the application of a sequence of single- and two-qubit unitary operations, called *quantum gates*. An immediate example of relevant unitary transformation is the time-evolution operator, that solves the time-independent Schrödinger equation.

Even though it has been proven (Lloyd 1996[citation?]) that "anything" can be simulated on a quantum computer, not all unitary operations can be simulated *efficiently*. Therefore, there are *mathematically possible* Hamiltonians that cannot be efficiently simulated with a circuit-based model. Luckily, it is believed that *physically relevant* Hamiltonians can indeed be simulated efficiently. Furthermore, it should be stressed that the implemented unitary operations are often just approximations of the desired unitary operation. With greater precision comes a greater number of gates.

The typical setup for a digital simulation is made of three steps:

- *Initial-state preparation* — where the quantum register has to prepared in the state $|\psi(0)\rangle$. This step can be by itself difficult, and it is not always guaranteed that an efficient algorithm may exist.
- *Unitary evolution* — where the circuit has to reproduce or simulate the action of a unitary operator U. In case of a unitary time evolution

and local Hamiltonian this can be achieved approximately with some "trotterization" scheme.

- *Final measurement* — after obtained the wanted state $|\psi(t)\rangle = U|\psi(0)\rangle$, a *measurement* is needed in order to extract the relevant physical information. Instead of capturing the whole wave function $|\psi(t)\rangle$, with for example quantum tomography, one may proceed with the direct estimation of certain physical quantities, such as correlation functions or spectra of operators.

2.1.2 Analog quantum simulations

Analog quantum simulation (AQS) in another possible approach to quantum simulation, where a one quantum system mimics or emulate another. The Hamiltonian of the system to be simulated H_{sys} is directly mapped onto the Hamiltonian of the simulator H_{sim}. Obviously, this can be done if there is a mapping between the system and the simulator. Note that the simulator may only partly reproduce the dynamics of the system, or simulate some effective description of the system.

An important advantage of AQS is that it does not require a full quantum computer, even more the simulator does not even need to be a computer at all. Finding the mapping in an AQS might look, at first, simpler than finding the most efficient gate decomposition of a Hamiltonian, but it is not always guaranteed and there are no recipes ready for such mappings. The obvious drawback of AQS is that the quantum simulators are problem specific.

[Aggiungere altro]

2.1.3 Quantum-inspired algorithms

Classical numerical algorithms for the simulation of quantum many-body systems came out of research on quantum information theory in these later years. The most important examples of quantum-inspired algorithms are *tensor networks* methods. Tensor networks make it possible to compress the information about a many-body wave function by expressing it as a contraction of a network of tensors (as suggested by the name). For a large class of physically relevant models, the ground state is gapped and has, in a certain sense, a finite amount of entanglement. This fact is expressed by the so-called *area law*, where the entanglement between two partitions of the system grows with size of the boundary, the area between the two partitions, and not with the size of the partition itself. The main advantage of tensor networks is their ability to capture this area law.

[Aggiungere altro]

2.2 Quantum simulation of lattice gauge theories

The problem of lattice gauge theories is arguably one of the most computationally intensive quantum many-body problem of all, due to the large numbers of degrees of freedom per site and the necessity of simulation in three spatial dimensions. We have already show a formulation of LGT in the path integral formalism. It can be used for simulations with Monte Carlo methods and indeed it already quite some results [inserire quali][citation?]. However, Monte Carlo methods suffers some problem. Simulation in Euclidean space time cannot approach several problems. For example, we already shown that the presence of fermionic matter leads to the so-called sign problem, which makes it very difficult to simulate situations with finite chemical potential. Another desirable feature it the real time evolution in Minkowski space time, which is absent when time is imaginary.

Recently, different approaches have been proposed for the quantum simulation of LGTs, from different communities, one for each possible path in quantum simulation (showed previously). For quantum-inspired classical simulation, different methods have been proposed for the simulation of LGTs using *tensor networks states*, to study the ground state, time evolution, and phase structure with both numerical and analytical models. The second type of approach relies on *analog simulation* with different kind of controllable experimental devices. The options ranges from ultracold atoms in optical lattices[citation?], trapped ions[citation?], or superconducting qubits[citation?]. The proposals have addressed LGTs of different levels of complexity, Abelian or non-Abelian, with or without dynamical matter, etc.... The last but not the least type of approach is digital quantum simulation, where the task of simulating the theory is done by a quantum computer. We will mainly focus on this last approach.

In order to be able to simulate a LGT on a quantum computer, some kind of *digitization* of the fields is necessary. By digitization, we mean the task of formulation, representing, and encoding QFT (choosing the basis) in ways useful for computational calculations. The lattice field theory, presented in Sec. 1.2, is the most conventional digitization scheme of non-perturbative field theory but it is only feasible for classical computers. It relies on resources far beyond near-term quantum computers. In conventional LGT, fermionic fields are integrated out, leaving a non-local action. A direct application of this procedure to quantum computers would require a high connectivity between qubits. Furthermore, for bosons, LGT works with bosons (i.e., the

gauge fields) which have a infinite-dimensional local Hilbert space. This is prohibited on a real quantum computer, where we have only finite quantum registers. We will show the different tactics for solving this issue later in the chapter

The starting point for a digital simulation of a LGT is its Hamiltonian formulation. This has been worked by Kogut and Susskind in their seminal paper[citation?], the starting point for any endeavour in quantum simulation of LGTs, which we will review in the following section

2.2.1 Kogut-Susskind Hamiltonian formulation

The classical formulation of Hamiltonian LTG is due to Kogut and Susskind, in [citation?]. It can be regarded as the Hamiltonian corresponding to the Wilson action [inserire eqref]. The former can be obtained from the latter by the transfer matrix technique[citation?], where two different lattice spacing are assigned to time and spatial dimensions and then the continuum limit for the time direction is taken. Another derivation can be done by means of Legendre transform, but in this text we will adopt a more modern approach, based on [citare Milsted, Osborne 2018].

As one can expect, the Kogut-Susskind Hamiltonian H_{KS} is made of two terms, the electric part and the magnetic part:

$$H_{\mathrm{KS}} = H_E + H_B. \tag{2.1}$$

These will be constructed separately and for a simple reason. The magnetic term involves only the spatial component of the field strength tensor, i.e., $\boldsymbol{B}^2 \sim F^{ij}F_{ij}$, while the electric term involves also the temporal components, i.e., $\boldsymbol{E}^2 \sim F^{0i}F_{0i}$. Given that in the Hamiltonian formalism time is continuous while space is discrete, the two terms cannot be treated on the same footing.

Starting from the magnetic term, this can be taken to be the same as the single-plaquette term in the Wilson action (1.25), but we have to limit ourselves to purely spatial plaquettes. In fact, the magnetic energy is $\boldsymbol{B}^2 = \frac{1}{2}F^{ij}F_{ij}$, which is just the spatial part of $F^{\mu\nu}F_{\mu\nu}$ of the continuum action. Given that the spatial directions are kept discrete, we do not need to make any modification. The same cannot be said for the electric energy, which is $\boldsymbol{E}^2 = F_{0i}F^{0i}$. With continuous time we cannot construct plaquettes that extends in the time direction.

Another point of divergence with the path-integral approach is that now both the electric field and magnetic term are operators, not simple variables. In order to have well-defined, or defined at all, operators we have to define the appropriate Hilbert space on which these operators act.

We start by considering a single link ℓ and define the gauge degrees of freedom. In the construction in Sec. 1.2.1, given a gauge group G we have associated an element $g \in G$ to the link ℓ. Hence, the configuration space for each link ℓ is exactly G. When quantizing, the configuration space G has to be elevated to a Hilbert space $\mathcal{H}^G$ which is spanned by the elements $g \in G$:

$$\mathcal{H}^G \equiv \text{span}\{|g\rangle : g \in G\}, \tag{2.2}$$ {eq:single_link_Hilbert_space}

where the set $\{|g\rangle\}$ is an orthonormal basis. Therefore, an element $|\psi\rangle$ of $\mathcal{H}^G$ can be written as

$$|\psi\rangle = \int \mathrm{d}g \; \psi(g) \, |g\rangle \, , \tag{2.3}$$

where $\int \mathrm{d}g$ is a proper measure (usually the Haar measure) on the group in the case of continuous groups, like $\mathrm{SU}(N)$, or a simple sum in case of finite groups. In the former case $\mathcal{H}^G$ is equivalent to the space $L^2(G)$ of square-integrable functions of G, i.e., each element $|\psi\rangle$ can be identified with the functions $\psi(g)$. While in the latter case, the space $\mathcal{H}^G$ is equivalent to the so-called group algebra $\mathbb{C}[G]$. The total Hilbert space of the model is simply given by the tensor product

$$\mathcal{H}_{\text{tot}} = \bigotimes_{\ell} \mathcal{H}^G_\ell \tag{2.4}$$

Focusing now on the case of continuous groups, like $\mathrm{SU}(N)$ Given a single link Hilbert space $\mathcal{H}^G$, the first set of operators we can define on it are the *position observables* $\hat{u}_{mn}$, via

$$\hat{u}_{mn} \, |g\rangle = U(g)_{mn} \, |g\rangle \, , \tag{2.5}$$

where $U(g)_{mn}$ is the matrix element (m, n) of the matrix $U(g)$, which is the image of g in the fundamental representation U of G. One can then define a matrix of operators $\hat{u}$, whose elements are precisely the operators $\hat{u}_{mn}$. Note that $\hat{u}$ is unitary as long as the chosen representation is unitary, but this not guarantee that each operator $\hat{u}_{mn}$ is unitary. Indeed, it can be shown that $(\hat{u}_{mn})^\dagger = (\hat{u}^\dagger)_{mn}$.

chapter three

Dualities in Lattice Gauge Theories