Topics on Lattice Gauge Theories

Sunny Pradhan

Contents

Introduction

Chapter 1

Review of continuum gauge theory

Ovviamente da scrivere

1.1 Yang-Mills theory

Ovviamente da riscrivere. A Yang-Mills theory is a gauge field theory on Minkowski space $\mathbb{R}^{1,d}$, where the gauge group $U(1)$ or $SU(N)$, with matter fields, which are defined by a representation of the gauge group. For example, *Quantum Chromodynamics* (QCD) is an $SU(3)$ gauge theory with Dirac spinors in the fundamental representation. Keeping in mind the example of QCD, the Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{1.1}$$

where the fermions ψ are taken in the fundamental representation of $SU(N)$ and the covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. We choose the convention where the Lie algebra generators T^a are Hermitian and $[T^a, T^b] = if^{abc}T^c$, with real structure constants f^{abc}. Furthermore, the generators are such that $\operatorname{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The strength-field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.2}$$

and transforms in the adjoint representation of $SU(N)$. Both the gauge field A_μ and the curvature tensor $F_{\mu\nu}$ live in the Lie algebra $\mathfrak{su}(N)$.

Under a gauge transformation given by a group-valued function $g(x) \in SU(N)$, such that $\psi \mapsto g(x)\psi(x)$, then

$$A_\mu(x) \mapsto g(x)A_\mu(x)g(x)^{-1} + ig(x)\partial_\mu g(x)^{-1}, \tag{1.3}$$

so that $D_\mu\psi(x) \mapsto g(x)D_\mu\psi(x)$, while

$$F_{\mu\nu} \mapsto g(x)F_{\mu\nu}g(x)^{-1}, \tag{1.4}$$

leaving the action invariant.

The action of the theory in $d+1$ dimensions is the given by

$$S[A,\psi,\overline{\psi}] = \int \mathrm{d}^{d+1}\mathcal{L} \tag{1.5}$$

and the path integral

$$Z = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi\, e^{iS[A,\psi,\overline{\psi}]}. \tag{1.6}$$

1.2 Euclidean field theory

In order to work in a Euclidean space-time, we need first to perform a *Wick rotation*, where the time coordinate x_0 is mapped a forth space coordinate x_4, through $x_0 = -ix_4$. The has the effect of changing the path-integral integrand from e^{iS}, which is oscillatory, to e^{-S}, which is positive and can be interpreted as a probability distribution of the configurations of the fields.

The Euclidean path integral is

$$Z_E = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S}, \tag{1.7}$$

so that the Minkowski action and the Euclidean action satisfy $iS_M = -S_E$. The respective actions are given by

$$S_M = \int \mathrm{d}^{d+1}x_M\mathcal{L}_M, \qquad S_E = \int \mathrm{d}^{d+1}x_E\mathcal{L}_E. \tag{1.8}$$

The rotation $x_0 = -ix_4$ leads to $\mathcal{L}_E = -\mathcal{L}_M$.

The Wick rotation does not change the form of the gauge kinetic term, i.e.,

$$-\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{1.9}$$

The sum is a simple Euclidean sum, where there are no minus signs when raising or lowering indices and $\mu = 1, \ldots, d+1$.

Considering now the fermionic part of the Yang-Mills Lagrangian, we need to perform the Wick rotation on the Dirac operator. In Minkowski space-time

$$\overline{\psi}\big(i\gamma_M^\mu D_\mu - m\big)\psi = \overline{\psi}\big(i\gamma_M^\mu\partial_\mu + \gamma_M^\mu A_\mu - m\big)\psi, \tag{1.10}$$

where γ_M^μ are the gamma matrices of the Clifford algebra, and they satisfy $\{\gamma_M^\mu, \gamma_M^\nu\} = 2\eta^{\mu\nu}$. In the Euclidean Clifford algebra instead the gamma matrices γ_E^μ satisfy $\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}$. Given the fact that we have $\partial_0 = i\partial_4$ and $A_0 = iA_4$, in order to obtain the correct form we have to put $\gamma^0_M = \gamma^4_E$. This procedure yields

$$\overline{\psi}(i\gamma^\mu_M \partial_\mu + \gamma^\mu_M A_\mu - m)\psi = -\overline{\psi}(\gamma^\mu_E \partial_\mu + i\gamma^\mu_E A_\mu + m)\psi. \tag{1.11}$$

Since $\mathcal{L}_E = -\mathcal{L}_M$, we finally arrive at

$$\mathcal{L}_E = \frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(\gamma^\mu D_\mu)\psi, \tag{1.12}$$

where the indices are all Euclidean and $D_\mu = \partial_\mu + iA_\mu$.

1.3 Hamiltonian formulation

The Hamiltonian formulation of a Yang-Mills theory can be tricky, especially the part about the gauge field. Usually, one has to procede by computing the conjugate momenta and performing a Legendre transform. The main issue here is that the gauge field component A_0, does not have a conjugate momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0. \tag{1.13}$$

Hence, the transformation is not invertible. The easiest way to remedy to the situation is to impose the gauge condition $A_0 = 0$, which is called *canonical gauge* or *temporal gauge*. With this condition, the kinetic term for the gauge fields can be written as

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{g^2}\big(\mathbf{E}^2 - \mathbf{B}^2\big) = \frac{1}{g^2}(E^a_i E^a_i - B^a_i B^a_i), \tag{1.14}$$

where $\mathbf{E}$ and $\mathbf{B}$ are, respectively, the "chromoelectric" and the "chromomagnetic" fields. In the temporal gauge $\mathbf{E} = \dot{\mathbf{A}}$, the time derivative of the spatial components $\mathbf{A}$ of the gauge field A_μ, while $\mathbf{B}$ corresponds to the spatial components of the strength-field tensor $F^{\mu\nu}$ and does not involve any time derivative. From the Legendre transformation of (1.14) we obtain the Hamiltonian density:

$$\mathcal{H} = \frac{1}{g^2}E^a_i \dot{A}^a_i - \frac{1}{2g^2}(E^a_i E^a_i - B^a_i B^a_i) = \frac{1}{2g^2}\operatorname{tr}\big(\mathbf{E}^2 + \mathbf{B}^2\big), \tag{1.15}$$

hence the Hamiltonian in d spatial dimensions is

$$H = \int \mathrm{d}^d x \frac{1}{2g^2}\operatorname{tr}\big(\mathbf{E}^2 + \mathbf{B}^2\big) \tag{1.16}$$

Chapter 2

Introduction to lattice gauge theories

Chapter 3

Dualities in lattice gauge theories

Chapter 4

Spin networks and finite group lattice gauge theories