Matrix Expectations

In all the expressions below, **x** is a vector of random variables with whose mean vector and covariance matrix are given by: E(**x**) = **m** and E(**(x-m)(x-m)**^{T}) = **S**.

The expressions for cubic and quartic expectations are restricted to two special cases:

- [x:Independent] means that the components of **x** are independent. In particular, we require that $E(x(i)^p x(j)^q)=E(x(i)^p)E(x(j)^q)$. We define $\mathbf{m}_r=E((\mathbf{x}-\mathbf{m})^r)$ where the r'th power is elementwise. Note that **S**=**DIAG(m**$_2$**)**.
- [x:Gaussian] means that the components of **x** are Real and have a multivariate Gaussian pdf: $(2{*}\mathrm{pi})^{-n/2}\ |\mathbf{S}|^{-1/2}\exp(\ (\mathbf{x}-\mathbf{m})^T\ \mathbf{S}^{-1}\ (\mathbf{x}-\mathbf{m})\ )$ where **x** has dimension n. If **x** is both Gaussian and Independent then $\mathbf{m}_k = \mathbf{diag}(\ (\mathbf{S}/2)\hat{}(k/2) * k!\ /\ (k/2)!)$

Vectors and matrices **a, A, b, B, c, C, d** and **D** are constant (i.e. not dependent on **x**).

General Properties

- The covariance matrix **S** is Hermitian and positive semi-definite.
- **S** is strictly positive definite unless there is a deterministic relation between the elements of **x** of the form **a'x** = 0 for some non-zero **a**.
- If the elements of **x** are uniformly spaced samples from a continuous signal, then **S** is Toeplitz.
- E(tr(**Y**)) = tr(E(**Y**)) where **Y** depends on **x**.

Linear Expectations

- $E(\mathbf{Ax}+\mathbf{b}) = \mathbf{Am}+\mathbf{b}$
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  - $E(\mathbf{Ax}) = \mathbf{Am}$
  - $E(\mathbf{x}+\mathbf{b}) = \mathbf{m}+\mathbf{b}$

Quadratic Expectations

- $E((\mathbf{Ax}+\mathbf{a})(\mathbf{Bx}+\mathbf{b})^T) = \mathbf{ASB}^T + (\mathbf{Am+a})(\mathbf{Bm+b})^T$
-
  - $E(\mathbf{xx}^T) = \mathbf{S} + \mathbf{mm}^T$
  - $E(\mathbf{xa}^T\mathbf{x}) = (\mathbf{S} + \mathbf{mm}^T)\mathbf{a}$
  - $E(\mathbf{x}^T\mathbf{ax}^T) = \mathbf{a}^T(\mathbf{S} + \mathbf{mm}^T)$
  - $E((\mathbf{Ax})(\mathbf{Ax})^T) = \mathbf{A}(\mathbf{S} + \mathbf{mm}^T)\mathbf{A}^T$
  - $E((\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{a})^T) = \mathbf{S} + (\mathbf{m+a})(\mathbf{m+a})^T$
- $E((\mathbf{Ax+a})^T(\mathbf{Bx+b})) = \mathrm{tr}(\mathbf{ASB}^T) + (\mathbf{Am+a})^T(\mathbf{Bm+b})$
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  - $E(\mathbf{x}^T\mathbf{x}) = \mathrm{tr}(\mathbf{S}) + \mathbf{m}^T\mathbf{m}$
  - $E(\mathbf{x}^T\mathbf{Ax}) = \mathrm{tr}(\mathbf{AS}) + \mathbf{m}^T\mathbf{Am}$
  - $E((\mathbf{Ax})^T(\mathbf{Ax})) = \mathrm{tr}(\mathbf{ASA}^T) + (\mathbf{Am})^T(\mathbf{Am})$
  - $E((\mathbf{x+a})^T(\mathbf{x+a})) = \mathrm{tr}(\mathbf{S}) + (\mathbf{m+a})^T(\mathbf{m+a})$

Cubic Expectations

For [x:Independent] :

- $E((\mathbf{Ax}+\mathbf{a})(\mathbf{Bx}+\mathbf{b})^T(\mathbf{Cx}+\mathbf{c})) = \mathbf{A\ DIAG}(\mathbf{B}^T\mathbf{C})\ \mathbf{m}_3 + \mathbf{ASB}^T(\mathbf{Cm+c}) + \mathbf{ASC}^T(\mathbf{Bm+b}) + \mathrm{tr}(\mathbf{BSC}^T){*}(\mathbf{Am+a}) + (\mathbf{Am+a})(\mathbf{Bm+b})^T(\mathbf{Cm+c})$
-
  - $E(\mathbf{xx}^T\mathbf{x}) = \mathbf{m}_3 + 2\mathbf{Sm} + (\mathrm{tr}(\mathbf{S}) + \mathbf{m}^T\mathbf{m}){*}\ \mathbf{m}$
  - $E((\mathbf{Ax}+\mathbf{a})(\mathbf{Ax}+\mathbf{a})^T(\mathbf{Ax}+\mathbf{a})) = \mathbf{A\ DIAG}(\mathbf{A}^T\mathbf{A})\ \mathbf{m}_3 + (2\mathbf{ASA}^T + (\mathbf{Am+a})(\mathbf{Am+a})^T)(\mathbf{Am+a}) + \mathrm{tr}(\mathbf{ASA}^T) * (\mathbf{Am+a})$
- $E((\mathbf{Ax}+\mathbf{a})\mathbf{b}^T(\mathbf{Cx}+\mathbf{c})(\mathbf{Dx}+\mathbf{d})^T) = ?$
  - $E(\mathbf{xa}^T\mathbf{xx}^T) = ?$

For [x:Gaussian] :

- $E((\mathbf{Ax}+\mathbf{a})(\mathbf{Bx}+\mathbf{b})^T(\mathbf{Cx}+\mathbf{c})) = \mathbf{ASB}^T(\mathbf{Cm+c}) + \mathbf{ASC}^T(\mathbf{Bm+b}) + \mathrm{tr}(\mathbf{BSC}^T){*}(\mathbf{Am+a}) + (\mathbf{Am+a})(\mathbf{Bm+b})^T(\mathbf{Cm+c})$
-
  - $E(\mathbf{xx}^T\mathbf{x}) = 2\mathbf{Sm} + (\mathrm{tr}(\mathbf{S}) + \mathbf{m}^T\mathbf{m}){*}\ \mathbf{m}$
  - $E((\mathbf{Ax}+\mathbf{a})(\mathbf{Ax}+\mathbf{a})^T(\mathbf{Ax}+\mathbf{a})) = (2\mathbf{ASA}^T + (\mathbf{Am+a})(\mathbf{Am+a})^T)(\mathbf{Am+a}) + \mathrm{tr}(\mathbf{ASA}^T) * (\mathbf{Am+a})$
- $E((\mathbf{Ax}+\mathbf{a})\mathbf{b}^T(\mathbf{Cx}+\mathbf{c})(\mathbf{Dx}+\mathbf{d})^T) = ?$
  - $E(\mathbf{xb}^T\mathbf{xx}^T) = ?$

Quartic Expectations

For [x:Independent] :

For [x:Gaussian] :

- $E((Ax + a)(Bx + b)^T(Cx + c)(Dx + d)^T) = (ASB^T+(Am+a)(Bm+b)^T)(CSD^T+(Cm+c)(Dm+d)^T) + (ASC^T+(Am+a)(Cm+c)^T)(BSD^T+(Bm+b)(Dm+d)^T + (Bm+b)^T(Cm+c)* (ASD^T - (Am+a)(Dm+d)^T) + tr(BSC^T)*(ASD^T + (Am+a)(Dm+d)^T)$
-
    - $E(xx^Txx^T) = 2(S+mm^T)$^2 $+ m^Tm* (S - mm^T) + tr(S)* (S + mm^T)$
    - $E(xx^TAxx^T) = (S+mm^T)(AS+Amm^T) + (SA^T+mAm^T)(S+mm^T) + m^TAm * (S - mm^T) + tr(AS)*(S + mm^T)$
    -
        - [m=0]: $E(xx^TAxx^T) = SAS + SA^TS + tr(AS)*S$
    - $E((x^TAx) * xx^T) =(S+mm^T)(AS+Amm^T) + (SA^T+mAm^T)(S+mm^T) + m^TAm * (S - mm^T) + tr(AS)*(S + mm^T)$
    -
        - [m=0]: $E((x^TAx) * xx^T) =SAS + SA^TS + tr(AS)*S$
    - $E((Ax + a)(Ax + a)^T(Ax + a)(Ax + a)^T) = 2(ASA^T+(Am+a)(Am+a)^T)^2 + (Am+a)^T(Am+a)* (ASA^T - (Am+a)(Am+a)^T) + tr(ASA^T)*(ASA^T + (Am+a)(Am+a)^T)$
- $E((Ax + a)^T(Bx + b)(Cx + c)^T(Dx + d)) = tr(AS(C^TD+D^TC)SB^T) + ((Am+a)^TB + (Bm+b)^TA)S(C^T(Dm+d) + D^T(Cm+c)) + (tr(ASB^T)+(Am+a)^T(Bm+b))(tr(CSD^T)+(Cm+c)^T(Dm+d))$
-
    - $E(x^Txx^Tx) = 2tr(S^2) + 4m^TSm + (tr(S) + m^Tm)^2$
    - $E(x^TAxx^TBx) = tr(AS(B+B^T)S) + m^T(A + A^T)S(B + B^T)m + (tr(AS)+m^TAm)(tr(BS)+m^TBm)$
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    - [m=0]: $E(x^TAxx^TBx) = tr(AS(B+B^T)S) + tr(AS)*tr(BS)$
- $E(a^Txb^Txc^Txd^Tx) = (a^T(S+mm^T)b)(c^T(S+mm^T)d)+(a^T(S+mm^T)c)(b^T(S+mm^T)d)+(a^T(S+mm^T)d)(b^T(S+mm^T)c)-2a^Tmb^Tmc^Tmd^Tm$
- $E((Ax + a)^T(Ax + a)(Ax + a)^T(Ax + a)) = 2tr(ASA^TASA^T) + 4(Am+a)^TASA^T(Am+a) + (tr(ASA^T) + (Am+a)^T(Am+a))^2$