IMU Preintegration on Manifold

Zerolover

June 13, 2016

1 $SO(3)$Group

关于 $SO(3)$ 的介绍略过，这里只列出几个近似的公式：

$$\mathrm{Exp}(\delta\theta) \approx \mathbf{I} + [\theta]_\times \tag{1}$$

$$\mathrm{Exp}(\theta + \delta\theta) \approx \mathrm{Exp}(\theta)\mathrm{Exp}(J_r(\theta)\delta\theta) \tag{2}$$

$$\mathrm{Exp}(\theta)\mathrm{Exp}(\delta\theta) \approx \mathrm{Exp}(\theta + J_r^{-1}(\theta)\delta\theta) \tag{3}$$

$$\mathrm{Exp}(\delta\phi)\mathrm{Exp}(\delta\theta) \approx \mathrm{Exp}(\delta\phi + J_r^{-1}(\delta\phi)\delta\theta) \approx \mathrm{Exp}(\delta\phi + \delta\theta) \tag{4}$$

以及 Adjoint 表示：

$$\mathrm{Exp}(\theta)R = R\mathrm{Exp}(R^T\theta) \tag{5}$$

2 IMU Preintegration measurements

2.1 Integration measurements

给定初值，在 i 和 j 时刻对 imu 的角速度和加速度进行积分，可以计算 j 时刻相对于 i 时刻的姿态：

$$\begin{aligned}
R_j &= R_i \prod_{k=i}^{j-1} \mathrm{Exp}((\tilde{w}_k - b_i^g - \eta_k^{gd})\Delta t) \\
v_j &= v_i + \sum_{k=i}^{j-1} (g + R_k(\tilde{a}_k - b_i^a - \eta_k^{ad}))\Delta t \\
p_j &= p_i + \sum_{k=i}^{j-1} v_k \Delta t + \frac{1}{2} \sum_{k=i}^{j-1} (g + R_k(\tilde{a}_k - b_i^a - \eta_k^{ad}))\Delta t^2
\end{aligned} \tag{6}$$

2.2 Preintegration measurements

在 preintegration 理论中需要将初值 (R_i, v_i, p_i) 和常数项（包含重力 g 的项）分离出来：

$$\begin{aligned}
\Delta R_{ij} &= R_i^T R_j = \prod_{k=i}^{j-1} \mathrm{Exp}((\tilde{w}_k - b_i^g - \eta_k^{gd})\Delta t) \\
\Delta v_{ij} &= R_i^T(v_j - v_i - g\Delta t(j-i)) = \sum_{k=i}^{j-1} \Delta R_{ik}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t \\
\Delta p_{ij} &= R_i^T(p_j - p_i - v_i\Delta t(j-i) - \frac{1}{2} g \Delta t^2 (j-i)^2) = \sum_{k=i}^{j-1} [\Delta v_{ik}\Delta t + \frac{1}{2}\Delta R_{ik}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t^2]
\end{aligned} \tag{7}$$

$\Delta R_{ij}, \Delta v_{ij}, \Delta p_{ij}$ 即为 preintegration measurements，即不考虑初值以及重力加速度项的相对测量。注意到这些项包含有噪声 η，我们也需要将它们分离出来。在分离的过程中发现 preintegration measurements 是近似服从高斯分布的，即：

$$\begin{aligned}
\Delta \tilde{R}_{ij} &\approx \Delta R_{ij} \mathrm{Exp}(\delta\phi_{ij}) \\
\Delta \tilde{v}_{ij} &\approx \Delta v_{ij} + \delta v_{ij} \\
\Delta \tilde{p}_{ij} &\approx \Delta p_{ij} + \delta p_{ij}
\end{aligned} \tag{8}$$

其中 $\Delta\tilde{R}_{ij}, \Delta\tilde{v}_{ij}, \Delta\tilde{p}_{ij}$ 为我们可以计算的测量值，不包含噪声 η。

$$\begin{aligned}\Delta\tilde{R}_{ij} &= \prod_{k=i}^{j-1} \mathrm{Exp}((\tilde{w}_k - b_i^g)\Delta t) \\ \Delta\tilde{v}_{ij} &= \sum_{k=i}^{j-1} \Delta\tilde{R}_{ik}(\tilde{a}_k - b_i^a)\Delta t \\ \Delta\tilde{p}_{ij} &= \sum_{k=i}^{j-1} [\Delta\tilde{v}_{ik}\Delta t + \frac{1}{2}\Delta\tilde{R}_{ik}(\tilde{a}_k - b_i^a)\Delta t^2]\end{aligned} \tag{9}$$

定义 $\eta_{ij}^{\Delta} = [\delta\phi_{ij}^T, \delta p_{ij}^T, \delta v_{ij}^T]_{9\times 1}^T \approx \mathcal{N}(0_{9\times 1}, \sum_{ij})$ 为 noise preintegration vector，它们是和噪声 η 相关的项。这里不会对 η_{ij}^{Δ} 进行求解，因为事实上我们仅需要其递推形式。

2.3 Iterative preintegration measurements

首先给出包含噪声的递推公式:

$$\begin{aligned}\Delta R_{i,k+1} &= \Delta R_{i,k}\mathrm{Exp}((\tilde{w}_k - b_i^g - \eta_k^{gd})\Delta t) \\ \Delta v_{i,k+1} &= \Delta v_{i,k} + \Delta R_{i,k}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t \\ \Delta p_{i,k+1} &= \Delta p_{i,k} + \Delta v_{i,k}\Delta t + \frac{1}{2}\Delta R_{i,k}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t^2\end{aligned} \tag{10}$$

接着给出不含噪声的递推公式:

$$\begin{aligned}\Delta\tilde{R}_{i,k+1} &= \Delta\tilde{R}_{i,k}\mathrm{Exp}((\tilde{w}_k - b_i^g)\Delta t) \\ \Delta\tilde{v}_{i,k+1} &= \Delta\tilde{v}_{i,k} + \Delta\tilde{R}_{i,k}(\tilde{a}_k - b_i^a)\Delta t \\ \Delta\tilde{p}_{i,k+1} &= \Delta\tilde{p}_{i,k} + \Delta\tilde{v}_{i,k}\Delta t + \frac{1}{2}\Delta\tilde{R}_{i,k}(\tilde{a}_k - b_i^a)\Delta t^2\end{aligned} \tag{11}$$

3 IMU Preintegration: Noise Propagation and Bias Updates

3.1 Iterative Noise Propagation

在前面提到 noise preintegration vector $\eta_{ij}^{\Delta} = [\delta\phi_{ij}^T, \delta p_{ij}^T, \delta v_{ij}^T]^T \approx \mathcal{N}(0_{9\times 1}, \sum_{ij})$，这里将证明 preintegration measurements 近似服从高斯分布，并给出 η_{ij}^{Δ} 的递推计算结果。

Rotation

根据 $SO(3)$ 中不确定性的定义，有$\Delta\tilde{R}_{k,k+1} = \Delta R_{k,k+1}\mathrm{Exp}(\delta\phi_{k,k+1})$。
$\Delta R_{k,k+1}$ 表示包含 bias 和 noise 两个相邻时刻的相对旋转，$\Delta\tilde{R}_{k,k+1}$ 表示不包含 noise 两个相邻时刻的相对旋转。

$$\begin{aligned}\Delta R_{k,k+1} &= \mathrm{Exp}((\tilde{w_k} - b_i^g - \eta_k^{gd})\Delta t) \\ &\overset{(2)}{\approx} \mathrm{Exp}((\tilde{w_k} - b_i^g)\Delta t)\mathrm{Exp}(-J_r((\tilde{w_k} - b_i^g)\Delta t)\eta_k^{gd}\Delta t) \\ &= \Delta\tilde{R}_{k,k+1}\mathrm{Exp}(-J_r^k\eta^{gd}\Delta t) \\ &= \Delta\tilde{R}_{k,k+1}\mathrm{Exp}(-\phi_{k,k+1})\end{aligned} \tag{12}$$

其中 $\Delta\tilde{R}_{k,k+1} = \mathrm{Exp}((\tilde{w_k} - b_i^g)\Delta t)$，$J_r^k = J_r((\tilde{w_k} - b_i^g)\Delta t)$，$\phi_{k,k+1} = J_r^k\eta^{gd}\Delta t$。

两个相邻时刻的相对旋转是服从高斯分布的，可以证明在积分后 i 时刻和 j 时刻的相对旋转也是近似服从高斯的，即 $\Delta\tilde{R}_{ij} \approx \Delta R_{ij}\mathrm{Exp}(\delta\phi_{ij})$，下面进行推导并求出 $\delta\phi_{ij}$ 的递推公式：
设初始时刻 $\Delta R_{ii} = \mathbf{I}_{3\times3}, \delta\phi_{ii} = \mathbf{0}_{3\times3}$

$$
\begin{aligned}
&\Delta R_{i,i+1} \approx \Delta R_{ii}\Delta\tilde{R}_{i,i+1}\mathrm{Exp}(-J_r^i\eta_i^{gd}\Delta t)\\
&\implies \delta\phi_{i,i+1} = J_r^i\eta_i^{gd}\Delta t\\
&\Delta R_{i,i+2} = \Delta R_{i,i+1}\Delta R_{i+1,i+2}\\
&\qquad \approx \Delta\tilde{R}_{i,i+1}\mathrm{Exp}(-\delta\phi_{i,i+1})\Delta\tilde{R}_{i+1,i+2}\mathrm{Exp}(-J_r^{i+1}\eta_{i+1}^{gd}\Delta t)\\
&\qquad \overset{(5)}{=} \Delta\tilde{R}_{i,i+1}\Delta\tilde{R}_{i+1,i+2}\mathrm{Exp}(-\Delta\tilde{R}_{i+1,i+2}^T\delta\phi_{i,i+1})\mathrm{Exp}(-J_r^{i+1}\eta_{i+1}^{gd}\Delta t)\\
&\qquad \overset{(4)}{\approx} \Delta\tilde{R}_{i,i+2}\mathrm{Exp}(-\Delta\tilde{R}_{i+1,i+2}^T\delta\phi_{i,i+1} - J_r^{i+1}\eta_{i+1}^{gd}\Delta t)\\
&\implies \phi_{i,i+2} = \Delta\tilde{R}_{i+1,i+2}^T\delta\phi_{i,i+1} + J_r^{i+1}\eta_{i+1}^{gd}\Delta t\\
&\Delta R_{i,i+3} = \Delta R_{i,i+2}\Delta R_{i+2,i+3}\\
&\qquad \approx \Delta\tilde{R}_{i,i+2}\mathrm{Exp}(-\phi_{i,i+2})\Delta\tilde{R}_{i+2,i+3}\mathrm{Exp}(-J_r^{i+2}\eta_{i+2}^{gd}\Delta t)\\
&\qquad \overset{(5)}{=} \Delta\tilde{R}_{i,i+2}\Delta\tilde{R}_{i+2,i+3}\mathrm{Exp}(-\Delta\tilde{R}_{i+2,i+3}^T\phi_{i,i+2})\mathrm{Exp}(-J_r^{i+2}\eta_{i+2}^{gd}\Delta t)\\
&\qquad \overset{(4)}{\approx} \Delta\tilde{R}_{i,i+3}\mathrm{Exp}(-\Delta\tilde{R}_{i+2,i+3}^T\phi_{i,i+2} - J_r^{i+2}\eta_{i+2}^{gd}\Delta t)\\
&\implies \phi_{i,i+3} = \Delta\tilde{R}_{i+2,i+3}^T\phi_{i,i+2} + J_r^{i+2}\eta_{i+2}^{gd}\Delta t
\end{aligned}
$$

因此 $\delta\phi_{i,k+1} = \Delta\tilde{R}_{k,k+1}^T\delta\phi_{i,k} + J_r^k\eta_k^{gd}\Delta t$。

velocity

对于速度而言，其高斯分布为 $\Delta\tilde{v}_{k,k+1} = \Delta v_{k,k+1} + \delta v_{k,k+1}$。
根据公式 (10) 中的 $\Delta v_{i,k+1} = \Delta v_{i,k} + \Delta R_{i,k}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t$ 进行推导。
假设初始时刻 $\Delta\tilde{v}_{i,k} = \Delta v_{i,k} + \delta v_{i,k}$。

$$
\begin{aligned}
\Delta v_{i,k+1} &= \Delta v_{i,k} + \Delta R_{i,k}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t\\
&= \Delta\tilde{v}_{i,k} - \delta v_{i,k} + \Delta\tilde{R}_{i,k}\mathrm{Exp}(-\delta\phi_{i,k})(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t\\
&\overset{(1)}{\approx} \Delta\tilde{v}_{i,k} - \delta v_{i,k} + \Delta\tilde{R}_{i,k}(\mathbf{I}_{3\times3} - [\delta\phi_{i,k}]_\times)(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t\\
&\approx \Delta\tilde{v}_{i,k} + \Delta\tilde{R}_{i,k}(\tilde{a}_k - b_i^a)\Delta t - (\delta v_{i,k} - \Delta\tilde{R}_{i,k}[\tilde{a}_k - b_i^a]_\times\delta\phi_{i,k}\Delta t + \Delta\tilde{R}_{i,k}\eta_k^{ad}\Delta t)\\
&= \Delta\tilde{v}_{i,k+1} - (\delta v_{i,k} - \Delta\tilde{R}_{i,k}[\tilde{a}_k - b_i^a]_\times\delta\phi_{i,k}\Delta t + \Delta\tilde{R}_{i,k}\eta_k^{ad}\Delta t)
\end{aligned}
$$

因此 $\delta v_{i,k+1} = \delta v_{i,k} - \Delta\tilde{R}_{i,k}^T[\tilde{a}_k - b_i^a]_\times\delta\phi_{i,k}\Delta t + \Delta\tilde{R}_{i,k}^T\eta_k^{ad}\Delta t$。

Position

对于位移而言，其高斯分布为 $\Delta\tilde{p}_{k,k+1} = \Delta p_{k,k+1} + \delta p_{k,k+1}$。
根据公式 (10) 中的 $\Delta p_{i,k+1} = \Delta p_{i,k} + \Delta v_{i,k}\Delta t + \frac{1}{2}\Delta R_{i,k}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t^2$ 进行推导。
假设 $\Delta\tilde{p}_{i,k} = \Delta p_{i,k} + \delta p_{i,k}$

$$
\begin{aligned}
\Delta p_{i,k+1} &= \Delta p_{i,k} + \Delta v_{i,k}\Delta t + \frac{1}{2}\Delta R_{i,k}(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t^2\\
&= \Delta\tilde{p}_{i,k} - \delta p_{i,k} + (\Delta\tilde{v}_{i,k} - \delta v_{i,k})\Delta t + \frac{1}{2}\Delta\tilde{R}_{i,k}\mathrm{Exp}(-\delta\phi_{i,k})(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t^2\\
&\overset{(1)}{\approx} \Delta\tilde{p}_{i,k} + \Delta\tilde{v}_{i,k}\Delta t - \delta p_{i,k} - \delta v_{i,k}\Delta t + \frac{1}{2}\Delta\tilde{R}_{i,k}(\mathbf{I}_{3\times3} - [\delta\phi_{i,k}]_\times)(\tilde{a}_k - b_i^a - \eta_k^{ad})\Delta t^2\\
&\approx \Delta\tilde{p}_{i,k} + \Delta\tilde{v}_{i,k}\Delta t + \frac{1}{2}\Delta\tilde{R}_{i,k}(\tilde{a}_k - b_i^a)\Delta t^2 - \delta p_{i,k} - \delta v_{i,k}\Delta t + \frac{1}{2}\Delta\tilde{R}_{i,k}[\tilde{a}_k - b_i^a]_\times\delta\phi_{i,k}\Delta t^2 - \frac{1}{2}\Delta\tilde{R}_{i,k}\eta_k^{ad}\Delta t^2\\
&\approx \Delta\tilde{p}_{i,k+1} - (\delta p_{i,k} + \delta v_{i,k}\Delta t - \frac{1}{2}\Delta\tilde{R}_{i,k}[\tilde{a}_k - b_i^a]_\times\delta\phi_{i,k}\Delta t^2 + \frac{1}{2}\Delta\tilde{R}_{i,k}\eta_k^{ad}\Delta t^2)
\end{aligned}
$$

因此 $\delta p_{i,k+1} = \delta p_{i,k} + \delta v_{i,k}\Delta t - \frac{1}{2}\Delta\tilde{R}_{i,k}[\tilde{a}_k - b_i^a]_\times\delta\phi_{i,k}\Delta t^2 + \frac{1}{2}\Delta\tilde{R}_{i,k}\eta_k^{ad}\Delta t^2$。

综上，可以得到 η_{ij}^{Δ} 的递推计算公式：

$$
\begin{aligned}
\delta\phi_{i,k+1} &= \Delta\tilde{R}_{k,k+1}^T \delta\phi_{i,k} + J_r^k \eta_k^{gd} \Delta t \\
\delta v_{i,k+1} &= \delta v_{i,k} - \Delta\tilde{R}_{i,k}^T [\tilde{a}_k - b_i^a]_\times \delta\phi_{i,k} \Delta t + \Delta\tilde{R}_{i,k}^T \eta_k^{ad} \Delta t \\
\delta p_{i,k+1} &= \delta p_{i,k} + \delta v_{i,k} \Delta t - \frac{1}{2} \Delta\tilde{R}_{i,k} [\tilde{a}_k - b_i^a]_\times \delta\phi_{i,k} \Delta t^2 + \frac{1}{2} \Delta\tilde{R}_{i,k} \eta_k^{ad} \Delta t^2
\end{aligned}
\tag{13}
$$

写成矩阵的形式:

$$
\begin{bmatrix} \delta\phi_{i,k+1} \\ \delta p_{i,k+1} \\ \delta v_{i,k+1} \end{bmatrix} = \begin{bmatrix} \Delta\tilde{R}_{k,k+1}^T & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -\frac{1}{2}\Delta\tilde{R}_{i,k}[\tilde{a}_k - b_i^a]_\times \Delta t^2 & \mathbf{I}_{3\times3} & \mathbf{I}_{3\times3}\Delta t \\ -\Delta\tilde{R}_{i,k}^T[\tilde{a}_k - b_i^a]_\times \Delta t & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix}_{9\times9} \begin{bmatrix} \delta\phi_{i,k} \\ \delta p_{i,k} \\ \delta v_{i,k} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3\times3} \\ \frac{1}{2}\Delta\tilde{R}_{i,k}\Delta t^2 \\ \Delta\tilde{R}_{i,k}^T\Delta t \end{bmatrix}_{9\times3} \eta_k^{ad} + \begin{bmatrix} J_r^k \Delta t \\ \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} \end{bmatrix}_{9\times3} \eta_k^{gd}
$$

$\eta_{i,k+1}^{\Delta} = A_k \eta_{i,k+1}^{\Delta} + B_k \eta_k^{ad} + C_k \eta_k^{gd}$

3.2 Bias Correction via First-Order Updates

在前面的推导中，我们假定在时刻 i 和 j 之间的偏置是相同的，然而在优化过程中偏置会得到修正。一种简单的方式是将更新后的偏置代入上面的方程中，再对时刻 i 和 j 之间的测量进行积分，这样显然不是高效的。设 $\hat{b} \leftarrow \bar{b} + \delta b$，其中 $\bar{b}$ 为上一次的估计，δb 为微小的增量更新。对公式 (9) 在 $\bar{b}$ 处进行一阶泰勒展开:

$$
\begin{aligned}
\Delta\tilde{R}_{ij}(\hat{b}_i^g) &= \prod_{k=i}^{j-1} \mathrm{Exp}((\tilde{w}_k - \bar{b}_i^g - \delta b_i^g)\Delta t) = \Delta\tilde{R}_{ij}(\bar{b}_i^g)\mathrm{Exp}(\frac{\partial\Delta\bar{R}_{ij}}{\partial b^g}\delta b_i^g) \\
\Delta\tilde{v}_{ij}(\hat{b}_i^g, \hat{b}_i^a) &= \sum_{k=i}^{j-1} \Delta\tilde{R}_{ik}(\hat{b}_i^g)(\tilde{a}_k - \bar{b}_i^a - \delta b_i^a)\Delta t = \Delta\tilde{v}_{ij}(\bar{b}_i^g, \bar{b}_i^a) + \frac{\partial\Delta\bar{v}_{ij}}{\partial b^g}\delta b_i^g + \frac{\partial\Delta\bar{v}_{ij}}{\partial b^a}\delta b_i^a \\
\Delta\tilde{p}_{ij}(\hat{b}_i^g, \hat{b}_i^a) &= \sum_{k=i}^{j-1} [\Delta\tilde{v}_{ik}(\hat{b}_i^g, \hat{b}_i^a)\Delta t + \frac{1}{2}\Delta\tilde{R}_{ik}(\hat{b}_i^g)(\tilde{a}_k - \bar{b}_i^a - \delta b_i^a)\Delta t] = \Delta\tilde{p}_{ij}(\bar{b}_i^g, \bar{b}_i^a) + \frac{\partial\Delta\bar{p}_{ij}}{\partial b^g}\delta b_i^g + \frac{\partial\Delta\bar{p}_{ij}}{\partial b^a}\delta b_i^a
\end{aligned}
\tag{14}
$$

因此当偏置更新后，我们只需要更新 preintegration measurements 中由于微小增量 δb 带来的更新。上式与式 (7) 的形式相同，仿照式 (10) 可以写出其迭代形式:

$$
\begin{aligned}
\Delta\tilde{R}_{i,k+1}(\hat{b}_i^g) &= \Delta\tilde{R}_{i,k}(\hat{b}_i^g)\mathrm{Exp}((\tilde{w}_k - \bar{b}_i^g - \delta b_i^g)\Delta t) \\
\Delta\tilde{v}_{i,k+1}(\hat{b}_i^g, \hat{b}_i^a) &= \Delta\tilde{v}_{i,k}(\hat{b}_i^g, \hat{b}_i^a) + \Delta\tilde{R}_{i,k}(\hat{b}_i^g)(\tilde{a}_k - \bar{b}_i^a - \delta b_i^a)\Delta t \\
\Delta\tilde{p}_{i,k+1}(\hat{b}_i^g, \hat{b}_i^a) &= \Delta\tilde{p}_{i,k}(\hat{b}_i^g, \hat{b}_i^a) + \Delta\tilde{v}_{i,k}(\hat{b}_i^g, \hat{b}_i^a)\Delta t + \frac{1}{2}\Delta\tilde{R}_{i,k}(\hat{b}_i^g)(\tilde{a}_k - \bar{b}_i^a - \delta b_i^a)\Delta t^2
\end{aligned}
\tag{15}
$$

按照 3.1 节的推导方法可以写出 $\frac{\partial\Delta\bar{R}_{ij}}{\partial b^g}, \frac{\partial\Delta\bar{v}_{ij}}{\partial b^g}, \frac{\partial\Delta\bar{v}_{ij}}{\partial b^a}, \frac{\partial\Delta\bar{p}_{ij}}{\partial b^g}, \frac{\partial\Delta\bar{p}_{ij}}{\partial b^a}$ 的递推形式，由于式 (14) 与式 (8) 的定义不同，因此递推公式在符号上和式 (13) 不太一样，但形式是相同的:

$$
\begin{aligned}
\frac{\partial\Delta\bar{R}_{i,k+1}}{\partial b^g} &= \Delta\tilde{R}_{k,k+1}^T(\bar{b}_i^g)\frac{\partial\Delta\bar{R}_{i,k}}{\partial b^g} - J_r^k\Delta t \\
\frac{\partial\Delta\bar{v}_{i,k+1}}{\partial b^g} &= \frac{\partial\Delta\bar{v}_{i,k}}{\partial b^g} - \Delta\tilde{R}_{i,k}^T(\bar{b}_i^g)[\tilde{a}_k - \bar{b}_i^a]_\times\frac{\partial\Delta\bar{R}_{i,k}}{\partial b^g}\Delta t \\
\frac{\partial\Delta\bar{v}_{i,k+1}}{\partial b^a} &= \frac{\partial\Delta\bar{v}_{i,k}}{\partial b^a} - \Delta\tilde{R}_{i,k}^T(\bar{b}_i^g)\Delta t \\
\frac{\partial\Delta\bar{p}_{i,k+1}}{\partial b^g} &= \frac{\partial\Delta\bar{p}_{i,k}}{\partial b^g} + \frac{\partial\Delta\bar{v}_{i,k}}{\partial b^g}\Delta t - \frac{1}{2}\Delta\tilde{R}_{i,k}(\bar{b}_i^g)[\tilde{a}_k - \bar{b}_i^a]_\times\frac{\partial\Delta\bar{R}_{i,k}}{\partial b^g}\Delta t^2 \\
\frac{\partial\Delta\bar{p}_{i,k+1}}{\partial b^a} &= \frac{\partial\Delta\bar{p}_{i,k}}{\partial b^a} + \frac{\partial\Delta\bar{v}_{i,k}}{\partial b^a}\Delta t - \frac{1}{2}\Delta\tilde{R}_{i,k}(\bar{b}_i^g)\Delta t^2
\end{aligned}
\tag{16}
$$

令 $H_k^a=[\frac{\partial\Delta\bar{R}_{i,k}}{\partial b^a}^T,\frac{\partial\Delta\bar{p}_{i,k}}{\partial b^a}^T,\frac{\partial\Delta\bar{v}_{i,k}}{\partial b^a}^T]_{9\times3}^T, H_k^g=[\frac{\partial\Delta\bar{R}_{i,k}}{\partial b^g}^T,\frac{\partial\Delta\bar{p}_{i,k}}{\partial b^g}^T,\frac{\partial\Delta\bar{v}_{i,k}}{\partial b^g}^T]_{9\times3}^T, H_k=\begin{bmatrix} H_k^a & H_k^g \end{bmatrix}_{9\times6}$, 结果整理可以得到其矩阵形式:

$$
\begin{aligned}
H_{k+1}^a &= A_k H_k^a - B_k \\
H_{k+1}^g &= A_k H_k^g - C_k
\end{aligned} \tag{17}
$$

4 IMU Factors: Residual Errors and Jacobians

由式子 (8) 给定的 preintegration measurements 模型中，测量噪声是零均值高斯分布的（在一阶近似时）。这里定义由 preintegration measurements 定义的误差向量 $r\mathcal{I}_{ij}=[r_{\Delta R_{ij}}^T,r_{\Delta v_{ij}}^T,r_{\Delta p_{ij}}^T]^T\in\mathbb{R}^9$:

$$
\begin{aligned}
r_{\Delta R_{ij}} &= \mathrm{Log}\bigg(\big(\Delta\tilde{R}_{ij}(\bar{b}_i^g)\mathrm{Exp}(\frac{\partial\Delta\bar{R}_{ij}}{\partial b^g}\delta b_i^g)\big)^T R_i^T R_j\bigg) \\
r_{\Delta v_{ij}} &= R_i^T(v_j-v_i-g\Delta t(j-i))-[\Delta\tilde{v}_{ij}(\bar{b}_i^g,\bar{b}_i^a)+\frac{\partial\Delta\bar{v}_{ij}}{\partial b^g}\delta b_i^g+\frac{\partial\Delta\bar{v}_{ij}}{\partial b^a}\delta b_i^a] \\
r_{\Delta p_{ij}} &= R_i^T(p_j-p_i-v_i\Delta t(j-i)-\frac{1}{2}g\Delta t^2(j-i)^2)-[\Delta\tilde{p}_{ij}(\bar{b}_i^g,\bar{b}_i^a)+\frac{\partial\Delta\bar{p}_{ij}}{\partial b^g}\delta b_i^g+\frac{\partial\Delta\bar{p}_{ij}}{\partial b^a}\delta b_i^a]
\end{aligned} \tag{18}
$$