Chapter 8. Lyapunov-Based Learning Control

1 General Nonlinear System

In the subsequent analysis, we will attack the control problem of the following nonlinear system

$$\dot{e} = f(t) - u_1 \tag{1}$$

where $f(t) \in \mathcal{L}_\infty$. Assume that the system unknown nonlinearity $f(t)$ is periodic and bounded as follows

$$f(t) = f(t-T), \qquad |f(t)| \leq f_o \tag{2}$$

where $T \in \Re^1$ denotes the known period and f_o represents a positive bounding constant.

Question: As the unknown function is periodic, in other words, repeatable, is it possible to somehow learn such a signal so that to compensate it by a feedforward term in the control input? That is the design motivation of the learning control.

Remark 1 *Compared to the adaptive control of learning the system parameters, the subsequently designed learning control intends to learn the uncertain nonlinear function completely. As a result, the LP (linear parameterization) condition for the adaptive control is not needed, instead, some assumption regarding the property of the nonlinear function $f(t)$ is required to make the learning process possible.*

2 Learning Control

For the uncertain nonlinear system of (1), design the following learning controller

$$u_1 = \hat{f}(t) + k_1 e \tag{3}$$

where the feedforward term $\hat{f}(t) \in \Re^1$ represents a learning based estimate for $f(t)$ that is generated on-line via the following expression

$$\hat{f}(t) = \hat{f}(t-T) + k_2 e \tag{4}$$

Remark 2 *Obviously, the present of the first term $\hat{f}(t)$ in the learning controller (3) aims to cancel the uncertain nonlinearity $f(t)$ in the system dynamics (1); hence, the performance of this controller largely depends on whether $f(t)$ can be learned succesfully or not.*

Substituting the controller $u_1(t)$ into the system dynamics (1) yields

$$\begin{aligned} \dot{e} &= f(t) - \hat{f}(t) - k_1 e \\ &= \tilde{f}(t) - k_1 e \end{aligned} \tag{5}$$

where $\tilde{f}(t) \in \Re^1$ denotes the following defined learning error

$$\tilde{f}(t) = f(t) - \hat{f}(t) \tag{6}$$

After substituting (4) into (6) for $\hat{f}(t)$ and performing some mathematical manipulation

$$\begin{aligned} \tilde{f}(t) &= f(t) - \hat{f}(t) \\ &= f(t) - \hat{f}(t-T) - k_2 e \\ &= f(t-T) - \hat{f}(t-T) - k_2 e \\ &= \tilde{f}(t-T) - k_2 e \end{aligned} \tag{7}$$

we obtain

$$\tilde{f}(t-T) = \tilde{f}(t) + k_2 e \tag{8}$$

Define the Lyapunov function as follows

$$V = \frac{1}{2}e^2 + \frac{1}{2k_2}\int_{t-T}^{t}\left[\tilde{f}(\sigma)\right]^2 d\sigma \geqslant 0 \tag{9}$$

Taking the time derivative of (9) yields

$$\begin{aligned} \dot{V} &= e\dot{e} + \frac{1}{2k_2}\left\{\left[\tilde{f}(t)\right]^2 - \left[\tilde{f}(t-T)\right]^2\right\} \\ &= e\dot{e} + \frac{1}{2k_2}\left\{\left[\tilde{f}(t) + \tilde{f}(t-T)\right]\left[\tilde{f}(t) - \tilde{f}(t-T)\right]\right\} \\ &= e\dot{e} + \frac{1}{2k_2}\left[2\tilde{f}(t) + k_2 e\right]\left[-k_2 e\right] \\ &= e\left[\tilde{f}(t) - k_1 e\right] - \frac{e}{2}\left[2\tilde{f}(t) + k_2 e\right] \\ &= e\left[\tilde{f}(t) - k_1 e\right] - e\tilde{f}(t) - \frac{k_2}{2}e^2 \\ &= -\left(k_1 + \frac{k_2}{2}\right)e^2 \leq 0 \end{aligned} \tag{10}$$

where (5) and (8) have been utilized.

2.0.1 Signal Chasing

Question: Can we show all the signals bounded so far?

Based on (9) and (10), we know that $e(t) \in \mathcal{L}_\infty \Longrightarrow x(t) \in \mathcal{L}_\infty \overset{?????}{\Longrightarrow} u_1(t) \in \mathcal{L}_\infty$ or $\tilde{f}(t)$ or $\hat{f}(t) \in \mathcal{L}_\infty$?

2.0.2 Error Tracking

From (9) and (10), and **if** we can further show that $\dot{e} \in \mathcal{L}_\infty$, then it is straightforward to conclude that

$$\lim_{t\to\infty} e(t) = 0. \tag{11}$$

Now, it still comes to the question of the boundedness of the signal $\dot{e}$. Unfortunately, with the controller proposed in (3), there is no way to prove that $\dot{e} \in \mathcal{L}_\infty$.

Question: Can we modify (3) to make it bounded? If yes, how?

2.1 Saturated Learning Controller

Apparently, a choice to ensure the boundedness of u_1 is to saturate the controller itself or the learning based signal $\hat{f}(t)$. Based on this thought, the learning based estimate $\hat{f}(t)$ is modified as follows

$$\hat{f}(t) = sat_{f_o}\left(\hat{f}(t-T)\right) + k_2 e \tag{12}$$

where the saturation function $sat_{f_o}(\cdot)$ is defined in the following manner

$$sat_{f_o}(\varepsilon) = \begin{cases} \varepsilon & \text{for } |\varepsilon| \leq f_o \\ sgn(\varepsilon)\,\delta_o & \text{for } |\varepsilon| > f_o \end{cases} \tag{13}$$

Remark 3 *The structure of (12) and the definition of (13) provide a bounded estimation signal $\hat{f}(t)$ in the sense that*

$$\left|\hat{f}(t)\right| \leq f_o + k_2 \tag{14}$$

As the controller has been changed, we need to repeat the stability analysis for the new controller (12). To do that, we first perform the following calculation

$$\begin{aligned} f(t-T) - sat_{f_o}\left(\hat{f}(t-T)\right) &= f(t) - \left[\hat{f}(t) - k_2 e\right] \\ &= f(t) - \hat{f}(t) + k_2 e \end{aligned} \tag{15}$$

To show the stability of the closed-loop system, we choose the following Lyapunov function

$$V_2 = \frac{1}{2}e^2 + \frac{1}{2k_2}\int_{t-T}^{t}\left[f(\sigma) - sat_{f_o}\hat{f}(\sigma)\right]^2 d\sigma \geqslant 0 \tag{16}$$

and calculate its time derivative as follows

$$\begin{aligned} \dot{V}_2 &= e\dot{e} + \frac{1}{2k_2}\left\{\left[f(t) - sat_{f_o}\hat{f}(t)\right]^2 - \left[f(t-T) - sat_{f_o}\hat{f}(t-T)\right]^2\right\} \\ &= e\left[\tilde{f}(t) - k_1 e\right] + \frac{1}{2k_2}\left\{\left[f(t) - sat_{f_o}\hat{f}(t)\right]^2 - \left[f(t) - \hat{f}(t) + k_2 e\right]^2\right\} \\ &= e\tilde{f}(t) - k_1 e^2 + \frac{1}{2k_2}\left\{\left[f(t) - sat_{f_o}\hat{f}(t)\right]^2 - \left[f(t) - \hat{f}(t)\right]^2\right\} \\ &\quad -\frac{1}{2k_2}\left[k_2^2 e^2 + 2k_2 e\left(f(t) - \hat{f}(t)\right)\right] \\ &\leq -\left(k_1 + \frac{k_2}{2}\right)e^2 \leq 0 \end{aligned} \tag{17}$$

where the following fact has been utilized (see Appendix for the proof)

$$\left\{\left[f(t) - sat_{f_o}\hat{f}(t)\right]^2 - \left[f(t) - \hat{f}(t)\right]^2\right\} \leq 0 \tag{18}$$

2.1.1 Signal Chasing

Based on (16) and (17), we know that $e(t) \in \mathcal{L}_\infty \Longrightarrow x(t) \in \mathcal{L}_\infty \Longrightarrow u_1(t) \in \mathcal{L}_\infty$ since $\hat{f}(t)$, $\tilde{f}(t) \in \mathcal{L}_\infty \Longrightarrow$all the signals during closed-loop operation remain bounded.

2.1.2 Error Tracking

From (16) and (17), and the fact that $\dot{e} \in \mathcal{L}_\infty$, it is straightforward to conclude that

$$\lim_{t \to \infty} e\left(t\right) = 0. \tag{19}$$

2.1.3 Tracking Result

Asymptotically tracking

Question: Is the controller of (3) continuous?

Question: Can the estimation signal $\hat{f}\left(t\right)$ learn the unknown nonlinearity $f(t)$? Extended Barbalat's Lemma to equation (5).

3 Comparision of Several Nonlinear Controllers

Nonlinear Control	**Assumption Needed**	**Result**	**Something Related**
Adaptive	LP: $f(x,t,\theta) = Y\left(x,t\right)\theta$	Asymptotic	Identification (PE Needed)
Sliding Mode	$\left\|f(x,t,\theta)\right\| \leq \rho\left(\left\|x\right\|,t\right)$	Exponential	Chattering
High Frequency	$\left\|f(x,t,\theta)\right\| \leq \rho\left(\left\|x\right\|,t\right)$	GUUB	Chattering
High Gain	$\left\|f(x,t,\theta)\right\| \leq \rho\left(\left\|x\right\|,t\right)$	GUUB	Large Control Input
Learning	$f\left(t\right) = f\left(t-T\right), \left\|f\left(t\right)\right\| \leq f_o$	Asymptotic	Nonlinearity Learned

A Inequality Proof

After expanding (18) and cancelling common terms, the inequality of (18) can be rewritten as follows

$$\left(sat_{f_0}\hat{f}(t)\right)^2 - 2f(t)\cdot sat_{f_0}\hat{f}(t) \leq \left(\hat{f}(t)\right)^2 - 2f(t)\cdot\hat{f}(t) \tag{20}$$

We can then substract $\left[\left(\hat{f}(t)\right)^2 - 2f(t)\cdot\hat{f}(t)\right]$ from both sides of (20) and factor the resulting expression to obtain the following equivalent inequality

$$\left(sat_{f_0}\hat{f}(t) - \hat{f}(t)\right)\left(sat_{\delta_0}\hat{f}(t) + \hat{f}(t) - 2f(t)\right) \leq 0. \tag{21}$$

To show the inequality of (21), we then need to divide the proof into three possible cases as follows.

A.1 Case 1: $\left|\hat{f}(t)\right| \leq f_o$

Based on the definition of the function (13), it is easy to see that for this case

$$sat_{f_0}\hat{f}(t) = \hat{f}(t); \tag{22}$$

hence, the inequality (21) holds for Case 1.

A.2 Case 2: $\hat{f}(t) > f_o$

From the definition of the function (13), it is clear that for this case

$$sat_{f_0}\hat{f}(t) = f_0 \tag{23}$$

which leads to the following fact

$$sat_{f_0}\hat{f}(t) - \hat{f}(t) \leq 0, \qquad sat_{f_0}\hat{f}(t) + \hat{f}(t) \geq 2f_0. \tag{24}$$

The inequality of (21) then directly follows from (24) and the fact that

$$f(t) \leq f_0. \tag{25}$$

A.3 Case 3: $\hat{f}(t) < -f_o$

From the definition of the function (13), it is clear that for this case

$$sat_{f_0}\hat{f}(t) = -f_0 \tag{26}$$

which leads to the following fact

$$sat_{f_0}\hat{f}(t) - \hat{f}(t) \geq 0, \qquad sat_{f_0}\hat{f}(t) + \hat{f}(t) \leq -2f_0. \tag{27}$$

The inequality of (21) then directly follows from (27) and the fact that

$$f(t) \geq -f_0. \tag{28}$$

Therefore, (18) is true for all the three possible cases.