Lyapunov based Nonlinear Control - Assignment3

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Problem 1

Find the equilibrium point(s) of the following friction system

$$\dot{v} + 2av|v| + bv = c, a > 0, b > 0, c > 0$$

and analyze the corresponding stability.

Solution 1

Rewrite the system equation as follows

$$\dot{v} = -2av|v| - bv + c. \tag{1}$$

Solve (2) to get the equilibrium points.

$$-2av|v| - bv + c = 0. \tag{2}$$

Considering the absolute value of v, (2) turns out to be in the form

$$\begin{cases} -2av^2 - bv + c = 0 & v \geq 0 \\ 2av^2 - bv + c = 0 & v < 0 \end{cases}. \tag{3}$$

Solve (3) and we know that the system has only one equilibrium point

$$v_s = \frac{b - \sqrt{b^2 + 8ac}}{-4a}. \tag{4}$$

To analyze the stability on this equilibrium point, we first define the translation

$$x \triangleq v - v_s, \tag{5}$$

then for the system $\dot{x} = -a(x + v_s)|x + v_s| - b(x + v_s) + c$, it has one equilibrium point $x_s = 0$. Likewise, the system equation can be rewritten as

$$\dot{x} = \begin{cases} -2a(x + v_s)^2 - b(x + v_s) + c & x \geq -v_s \\ 2a(x + v_s)^2 - b(x + v_s) + c & x < -v_s \end{cases}. \tag{6}$$

Choose the Lyapunov function

$$V(x) = \frac{1}{2}x^2. \tag{7}$$

Obviously, $V(x)$ is positive definite and goes to infinity when $x \to \infty$. Then

$$\dot{V}(x) = x\dot{x} = \begin{cases} -2ax(x+v_s)^2 - bx(x+v_s) + cx & x \geq -v_s \\ 2ax(x+v_s)^2 - bx(x+v_s) + cx & x < -v_s \end{cases}. \tag{8}$$

We can tell from (8) that when $x < -v_s$, each item of $\dot{V}(x)$ is non-positive. And they don't take the value of zero at the same x. So $\dot{V}(x)$ is negative definite when $x < -v_s$. As for the case of $x \geq -v_s$, known that $-2av_s^2 - bv_s + c = 0$, $\dot{V}(x)$ can be written as follows

$$\dot{V}(x) = -x^2[2a(x+2v_s)+b], x \geq -v_s. \tag{9}$$

It can be seen from (9) that when $x \geq -v_s$, $\dot{V}(x) \leq 0$ holds true. And $\dot{V}(x) = 0$ if and only if $x = 0$. So $\dot{V}(x)$ is also negative definite. Therefore, the equilibrium point $v = v_s$ is globally asymptotically stable.

Problem 2

For the following nonlinear system

$$\dot{x} = -kx + \sin^3(x) + x\cos^2(x)$$

where $k > 2$ denotes a positive constant. Find its equilibrium point and use Lyapunov method to analyze its stability (get a conclusion as strong as possible).

- Show that there is **only** one equilibrium point at

$$x = 0.$$

- Utilize the Lyapunov method to analyze its stability around origin.

Solution 2

(1) Show that there is only one equilibrium point at $x = 0$.

It's obvious that when $x = 0$, $\dot{x} = 0$. So $x = 0$ is one of the system's equilibrium points. In what follows, we'll prove that $x = 0$ is the only equilibrium point.
When $x > 0$, the system equation can be rewritten as

$$\dot{x} = (1-k)x + \sin^2(x)(\sin(x) - x) \tag{10}$$

Let

$$f(x) = \sin(x) - x. \tag{11}$$

Then take the derivative of $f(x)$ w.r.t. x.

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \cos(x) - 1 < 0. \tag{12}$$

From (12) we can see that $f(x)$ is decreasing for $x \in \mathbb{R}$. Since $f(0) = 0$, $f(x)$ satisfied

$$f(x) \begin{cases} > 0 & x < 0 \\ = 0 & x = 0 \\ < 0 & x > 0 \end{cases}. \tag{13}$$

And also $(1-k)x < 0, \sin^2(x) > 0$. As a result, $\dot{x} < 0$ holds true when $x > 0$. Likewise, when $x < 0$ we can deduce that $\dot{x} > 0$.

From all above, we can conclude that

$$\dot{x} \begin{cases} > 0 & x < 0 \\ = 0 & x = 0 \\ < 0 & x > 0 \end{cases}. \tag{14}$$

So the system has only one equilibrium point $x = 0$.

(2) Utilize the Lyapunov method to analyze its stability around origin.

Choose the Lyapunov function

$$V(x) = \frac{1}{2}x^2. \tag{15}$$

Obviously, $V(x)$ is positive definite and goes to infinity when $x \to \infty$. Then

$$\dot{V}(x) = x\dot{x}. \tag{16}$$

As shown from (14), $\dot{V}(x)$ satisfied $\dot{V}(x) \leq 0$. And $\dot{V}(x) = 0$ holds true if and only if $x = 0$. So $\dot{V}(x)$ is negative definite. Therefore, the equilibrium point $x = 0$ is globally asymptotically stable.

Furthermore,

$$\dot{V}(x) = x\dot{x} = (-k+1)x^2 + x\sin^2(x)(sin(x) - x). \tag{17}$$

From (13), we know that for $x \in \mathbb{R}$, $x\sin^2(x)(sin(x) - x) \leq 0$. Therefore, $\dot{V}(x)$ satisfies

$$\dot{V}(x) \leq -(k-1)x^2 = -2(k-1)V(x). \tag{18}$$

Lemma 8

if $V(t) > 0$ and $\dot{V}(t) \leq -\gamma V(t)$, where γ is a positive constant, then

$$V(t) \leq V(0)d^{-\gamma t}.$$

As known from Lemma 8,

$$V(t) \leq V(t_0)e^{-2(k-1)(t-t_0)}. \tag{19}$$

As a result,

$$x(t) \leq x(t_0)e^{-(k-1)(t-t_0)}. \tag{20}$$

According to the definition of **Exponentially Stable**, the equilibrium point $x = 0$ is also globally exponentially stable.