Chapter 2. Mathematical Background

1 Fundamental Definitions

2-Norm

$$\|f(t)\|_2 = \sqrt{\int_{-\infty}^{+\infty} f^2(\tau)\, d\tau}$$

∞-norm

$$\|f(t)\|_\infty = \sup_t |f(t)|$$

If $\|f(t)\|_2 \leq \infty$, then we say that $f(t)$ belongs to $\mathcal{L}_2$, i.e., $f(t) \in \mathcal{L}_2$.

If $\|f(t)\|_\infty \leq \infty$, then we say that $f(t)$ belongs to $\mathcal{L}_\infty$, i.e., $f(t) \in \mathcal{L}_\infty$.

Definition 1 *Positive Definite (PD):* $V(x) > 0$ *for* $\forall x \neq 0$.

Definition 2 *Locally Positive Definite (LPD): There exists a ball containing* 0, *for* $\forall x \neq 0$ *in the ball,* $V(x) > 0$.

Definition 3 *Negative Definite (ND): if* $-V(x)$ *is PD.*

Definition 4 *Locally Negative Definite (LND): if* $-V(x)$ *is LPD.*

Definition 5 *fdsa;gfasdj;j dfsa*

Example:

$$V(x_1, x_2) = x_1^2 + x_2^2 \rightarrow \text{PD}$$
$$V(x_1, x_2, x_3) = x_1^2 + x_2^2 \rightarrow \text{NOT PD}$$

$$V(x_1, x_2) = (x_1 + x_2)^2 \rightarrow \text{NOT PD}$$

$$V(x_1, x_2) = (1 - \cos(x_1)) + x_2^2 \rightarrow LPD$$

How about this one?

$$V(x_1, x_2, x_3) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

The state is

$$x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$$

and then

$$V = x^T P x$$

with

$$P = \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

Since the matrix P is positive definite (why?), then V is PD.

$$|10| > 0, \begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} > 0, \begin{vmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{vmatrix} > 0$$

Since all the sucessive principal minors of P are positive, then V is PD.

Definition 6 *Uniformly Continuous (UC): A function $f(t)$ is Uniformly Continuous (UC) if for any positive constant ε, there exists a positive number $\delta(\varepsilon)$ satisfying that for any t_0 and $t \in (t_0 - \delta, t_0 + \delta)$*

$$|f(t) - f(t_0)| \le \varepsilon$$

Note: *In the definition of UC, δ is independent of t_0.*

Example:
The function

$$f(t) = \frac{1}{t}, t > 0$$

is continuous but not UC.
Why? Continuous: First note that

$$\left| \frac{1}{t} - \frac{1}{t_0} \right| = \left| \frac{t - t_0}{t_0 t} \right|$$

If we choose

$$\delta = \min \left\{ \frac{\varepsilon t_0^2}{2}, \frac{t_0}{2} \right\}$$

then $t > \frac{t_0}{2}$, and

$$\left| \frac{1}{t} - \frac{1}{t_0} \right| \le \left| \frac{2(t - t_0)}{t_0^2} \right| \le \varepsilon$$

therefore, $f(t)$ is continuous.

Lemma 7 *If $\dot{f}(t)$ exists and $\in \mathcal{L}_\infty$, then $f(t)$ is UC.*

Why?

$$|f(t) - f(t_0)| = \dot{f}(\xi) |t - t_0|$$

Lemma 8 *If*

$$V(t) > 0$$

and

$$\dot{V}(t) \le -\gamma V(t)$$

where γ is a positive constant, then

$$V(t) \le V(0) e^{-\gamma t}$$

i.e., $V(t)$ goes to zero exponentially fast.

Example:

Tracking problem of the following system

$$\dot{x} = -ax^3 - b\sin(t) + u$$

make $x \to x_d$, where $x_d, \dot{x}_d \in \mathcal{L}_\infty$.

Define the tracking error as

$$e = x_d - x$$

then

$$\begin{aligned} \dot{e} &= \dot{x}_d - \dot{x} \\ &= \left[\dot{x}_d + ax^3 + b\sin(t)\right] - u \end{aligned}$$

Exact Model Knowledge (EMK) Control: Choose u to cancle out the nonlinearities in the dynamics

$$u = \left[\dot{x}_d + ax^3 + b\sin(t)\right] + ke$$

then

$$\dot{e} = -ke$$

and

$$e(t) \leq e(0)e^{-kt}$$

Note: Need to insure that all the signals in the closed-loop system remain bounded.

$$e, x_d \in \mathcal{L}_\infty \Longrightarrow x \in \mathcal{L}_\infty$$

and for the control

$$|u| \leq |\dot{x}_d| + |a| \left|x^3\right| + |b| + k\,|e|$$

therefore, all the signals are bounded.

Question: If a, b unknown, how to design the controller? Adaptive Control.

Lemma 9 *If*

$$V(t) > 0$$

and

$$\dot{V}(t) \leq -\gamma V(t) + \varepsilon$$

where γ is a positive constant, then

$$V(t) \leq V(0)e^{-\gamma t} + \frac{\varepsilon}{\gamma}\left(1 - e^{-\gamma t}\right)$$

that is, $V(t) \in \mathcal{L}_\infty$.

Lemma 10 *Let A be a real, symmetric, positive-definite(PD) matrix, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of A, respectively, then*

$$\lambda_{\min}(A)\,\|x\|^2 \leq x^T A x \leq \lambda_{\max}(A)\,\|x\|^2 .$$

Lemma 11 *Nonlinear Damping:*

$$\Omega(x)xy - k_n\Omega^2(x)x^2 \le \frac{y^2}{k_n}$$

why?

$$\Omega(x)xy - k_n\Omega^2(x)x^2 \le -k_n\left(\Omega(x)x - \frac{y}{2k_n}\right)^2 + \frac{y^2}{4k_n} \le \frac{y^2}{k_n}$$

Lemma 12 *Barbalat's Lemma:If $f(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2$, and $\dot{f}(t) \in \mathcal{L}_\infty$, then*

$$\lim_{t\to\infty} f(t) = 0.$$

Lemma 13 *Integral form of Barbalat's Lemma: If $f(t)$ is UC and the intergral*

$$\lim_{t\to\infty} \int_0^t |f(\tau)|\, d\tau$$

exists and is finite, then

$$\lim_{t\to\infty} |f(t)| = 0.$$

Lemma 14 *Extended Barbalat's Lemma: If $f(t)$ is differentiable and it has a finite limit as $t \to \infty$, that is,*

$$\lim_{t\to\infty} f(t) = c$$

where c denotes a constant, and the time derivative $\dot{f}(t)$ can be written as follows

$$\dot{f}(t) = g_1(t) + g_2(t)$$

where $g_1(t)$ is UC and

$$\lim_{t\to\infty} g_2(t) = 0$$

then

$$\lim_{t\to\infty} g_1(t) = 0, \lim_{t\to\infty} \dot{f}(t) = 0$$

Lemma 15 *If*

$$V(t) > 0$$

and

$$\dot{V}(t) \le -f(t), \qquad f(t) > 0$$

and $f(t)$ is UC (or $\dot{f}(t) \in \mathcal{L}_\infty$), then $\lim_{t\to\infty} f(t) = 0$.

1.1 Solution

Choose

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$$

then

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 \\
&= -x_1^2 + x_2\left[-x_1 - x_2 - x_3 - x_1x_3\right] + x_3\left(x_1 + 1\right)x_2 \\
&= -x_1^2 - x_2^2 - x_1x_2 \\
&\le -x_1^2 - x_2^2 + \left[\frac{x_1^2 + x_2^2}{2}\right] \\
&= -\frac{x_1^2 + x_2^2}{2}
\end{aligned}$$

We can then show that

$$x_1 \to 0, x_2 \to 0$$

Utilizing the above fact to the 2nd equation and applying the extended Barbalat's Lemma yields

$$x_3 \to 0$$

therefore, Asymptotic tracking.

Example:

For the same system but a, b unknown, how to construct the controller?

Similarly, we choose

$$u = \left[\dot{x}_d + \hat{a}x^3 + \hat{b}\sin(t)\right] + ke$$

where $\hat{a}, \hat{b}$ denote the on-line estimates for a, b, respectively. Substitute the controller into the dynamics to obtain

$$\begin{aligned}
\dot{e} &= \left[\dot{x}_d + ax^3 + b\sin(t)\right] - \left[\dot{x}_d + \hat{a}x^3 + \hat{b}\sin(t)\right] - ke \\
&= -ke + \tilde{a}x^3 + \tilde{b}\sin(t)
\end{aligned}$$

with $\tilde{a}$, $\tilde{b}$ denote the estimation error.

Choose

$$V(t) = \frac{1}{2}e^2 + \frac{1}{2}\Gamma_1^{-1}\tilde{a}^2 + \frac{1}{2}\Gamma_2^{-1}\tilde{b}^2 \ge 0$$

The Lyapunov function is designed from the control requirement. Objective: make $e \to 0$, and $\tilde{a}$, $\tilde{b} \in \mathcal{L}_\infty$, thus we put it into $V(t)$. Γ_1 and Γ_2 are the subsequently defined update gains for the parameter estimation.

Then

$$\dot{V} = e\dot{e} + \Gamma_1^{-1}\tilde{a}\,\dot{\tilde{a}} + \Gamma_2^{-1}\tilde{b}\,\dot{\tilde{b}}$$

Note

$$\dot{\tilde{a}} = -\dot{\hat{a}}$$

and

$$\dot{\tilde{b}} = -\dot{\hat{b}}$$

(why?, a, b are constants). Therefore, $\dot{V}(t)$ can be rewritten as follows

$$\begin{aligned}\dot{V} &= e\left[-ke + \tilde{a}x^3 + \tilde{b}\sin(t)\right] - \Gamma_1^{-1}\tilde{a}\,\dot{\hat{a}} - \Gamma_2^{-1}\tilde{b}\,\dot{\hat{b}} \\ &= -ke^2 + \tilde{a}\left(ex^3 - \Gamma_1^{-1}\,\dot{\hat{a}}\right) + \tilde{b}\left(e\sin(t) - \Gamma_2^{-1}\,\dot{\hat{b}}\right)\end{aligned}$$

further, make the update laws as

$$\dot{\hat{a}} = \Gamma_1 ex^3$$

and

$$\dot{\hat{b}} = \Gamma_2 e\sin(t)$$

Note: from the update laws, we can see why Γ_1 and Γ_2 are needed in the Lyapunov function.

Then

$$\dot{V} = -ke^2 \le 0$$

Therefore, $V(t)$ is nonincreasing and thus remain bounded.

$$V(t) \in \mathcal{L}_\infty \Longrightarrow e, \tilde{a}, \tilde{b} \in \mathcal{L}_\infty \Longrightarrow x, \hat{a}, \hat{b}, \dot{\hat{a}}, \dot{\hat{b}} \in \mathcal{L}_\infty \Longrightarrow u \in \mathcal{L}_\infty \Longrightarrow \dot{x}, \dot{e} \in \mathcal{L}_\infty$$

and all the signals remain bounded during closed-loop operation.

Define

$$f(t) = ke^2 > 0$$

then

$$\dot{f}(t) = 2ke\dot{e} \in \mathcal{L}_\infty$$

we already have:

$$V(t) > 0$$

$$\dot{V}(t) = -ke^2 \le -f(t), \qquad f(t) = ke^2 > 0$$

$$\dot{f}(t) \in \mathcal{L}_\infty$$

then the lemma above can be utilized to conclude that

$$\lim_{t\to\infty} f(t) = 0 \Longrightarrow \lim_{t\to\infty} e(t) = 0$$

or we can directly use Barbalat's Lemma:$f(t) = e(t)$

$$\int_0^{+\infty} f^2(\tau)\,d\tau = \int_0^{+\infty} e^2(\tau)\,d\tau = \int_0^{+\infty} \frac{-\dot{V}(\tau)}{k} d\tau = \frac{1}{k}\left[V(0) - V(\infty)\right] \in \mathcal{L}_\infty$$

we have $f(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2,\ \dot{f}(t) \in \mathcal{L}_\infty \Longrightarrow f(t) = e(t) \to 0$

Question: How about $\tilde{a}, \tilde{b}$?

$$\hat{a} \to a?? \quad \hat{b} \to b??$$

NO, Not Necessary! Some additional conditions are needed to ensure the exact estimate of the system parameters

Let's give a brief analysis:

Apply the extended Barbalat's Lemma: We have already shown that

$$\lim_{t\to\infty} e(t) = 0$$

and

$$\begin{aligned}\dot{e} &= -ke + \tilde{a}x^3 + \tilde{b}\sin(t) \\ &= g_1(t) + g_2(t)\end{aligned}$$

where $g_1(t) = \tilde{a}x^3 + \tilde{b}\sin(t)$ is UC (since $\dot{x}(t), \dot{\hat{a}}\ (t), \dot{\hat{b}}\ (t) \in \mathcal{L}_\infty$) and

$$g_2(t) = -ke,\ \lim_{t\to\infty} g_2(t) = 0$$

then

$$\lim_{t\to\infty} \tilde{a}x^3 + \tilde{b}\sin(t) = 0$$

Quesion: Notice that since $e \to 0$, then

$$\dot{\hat{a}} = \Gamma_1 e x^3 e \to 0$$

and

$$\dot{\hat{b}} = \Gamma_2 e \sin(t) e \to 0$$

does it mean that

$$\hat{a},\ \hat{b} \to \text{constant??}$$

NO, Not Necessary!

Actually

$$\dot{x}(t) \to 0 \ \ ; \ \ x(t) \to \text{constant}$$

Counterexample:

$$x(t) = \ln(t+1)$$

then

$$\dot{x}(t) = \frac{1}{t+1} \to 0$$

but $x(t)$ has no finite limit.

Question: How about the reverse part

$$x(t) \to \text{constant} \Longrightarrow \dot{x}(t) \to 0?$$

NO, Under General Situation, Not True!

Counterexample: Find it yourself.

Why are we interested in this? Needed for its velocity analysis of some system!

Will there be some additional condition to make the fact true? How about this one:

$$\left.\begin{array}{c} x(t) \to \text{constant} \\ \dot{x}(t) \ \text{ is UC or } \ddot{x}(t) \in \mathcal{L}_\infty \end{array}\right\} \Longrightarrow \dot{x}(t) \to 0?$$

Can we apply the Extended Barbalat's Lemma to show this?

$$x(t) \Longleftrightarrow f(t)$$

To show

$$\lim_{t\to\infty} \dot{x}(t) = 0$$

What about $g_1(t)$ and $g_2(t)$? Definitions for them?

$$g_1(t) = \dot{x}(t)$$

$$g_2(t) = 0$$

then satisfies all the conditions, and thus $\dot{x}(t) \to 0$.