Introduction of Lyapunov-Based Control

1 An Example of Nonlinear Systems

Linear System

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

it has the superposition property. Besides, the stability of the linear system completely depends on its parameters.

Nonlinear System

$$\begin{cases} \dot{x} = f\left(x, u\right) \\ y = g\left(x\right) \end{cases} \tag{2}$$

superposition does not hold for nonlinear systems, and the stability of a nonlinear system depends on both system parameters and initial conditions.

Example: The dynamic model for a 2-DOF overhead crane system (see Figure **??**) can be presented as follows

$$M(q)\ddot{q} + V_m(q, \dot{q})\dot{q} + G(q) = u \tag{3}$$

$$q = [x(t) \quad \theta(t)]^T \tag{4}$$

where $x(t) \in \mathbb{R}^1$ denotes the gantry position, $\theta(t) \in \mathbb{R}^1$ denotes the payload angle with respect to the vertical, and $M(q) \in \mathbb{R}^{2\times 2}$, $V_m(q, \dot{q}) \in \mathbb{R}^{2\times 2}$, $G(q) \in \mathbb{R}^2$, and $u(t) \in \mathbb{R}^2$ are defined as follows

$$\begin{gathered} M(q) = \begin{bmatrix} m_c + m_p & -m_p L\cos\theta \\ -m_p L\cos\theta & m_p L^2 \end{bmatrix}, \\ V_m(q, \dot{q}) = \begin{bmatrix} 0 & m_p L\sin\theta\dot{\theta} \\ 0 & 0 \end{bmatrix}, \\ G(q) = \begin{bmatrix} 0 & m_p gL\sin\theta \end{bmatrix}^T, \ u(t) = \begin{bmatrix} F & 0 \end{bmatrix}^T, \end{gathered} \tag{5}$$

where $m_c, m_p \in \mathbb{R}^1$ represent the gantry mass and the payload mass, respectively, $L \in \mathbb{R}^1$ represents the length of the rod to the payload, $g \in \mathbb{R}^1$ represents the gravity coefficient, and $F(t) \in \mathbb{R}^1$ represents the control force input acting on the gantry (see Figure **??**).

2 Common Nonlinear Systems Behaviors

2.1 Multiple Equilibrium Points

For the system

$$\dot{x} = f(x) \tag{6}$$

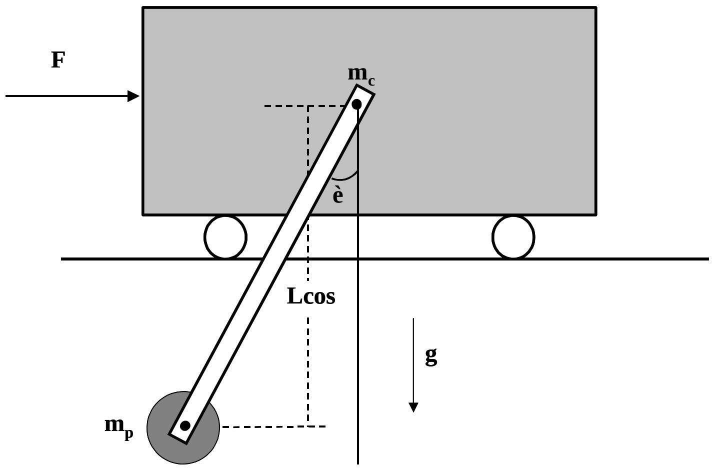

Figure 1:

To obtain equilibrium points, solve the following equation

$$f(x_s) = 0$$

if it has multiple solutions, then the system has multiple equilibrium points.

Example:

$$\dot{x} = -x + x^2 \tag{7}$$

$$-x + x^2 = 0 \Longrightarrow x_s = 1 \text{ or } x_s = 0$$

has too equilibrium points: $x_s = 1$ and $x_s = 0$.

Question: Which one is stable? Why?

Rewrite the equation as follows

$$\dot{x} = x\,(x-1)$$

then if $x > 1$, $\dot{x} > 0, x$ increases with time. For the case of $x < 1$, $(x-1) < 0$, thus

$$\begin{cases} \dot{x} < 0, \text{ for } 1 > x > 0 \\ \dot{x} > 0, \text{ for } x < 0 \end{cases}$$

therefore, $x_s = 1$ is unstable and $x_s = 0$ is stable.

We can solve the system equation to obtain the response as follows

$$x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}} = 1 - \frac{1 - x_0}{1 - x_0 + x_0 e^{-t}}$$

From this formula, it can be seen that if $x_0 < 1$, the denominator $1 - x_0 + x_0 e^{-t} > 0$, thus

$$\lim_{t\to\infty} x(t) = 0.$$

The response of $x_0 = 0.99$ and $x_0 = 1.0001$ is demonstrated in Figure **??**.

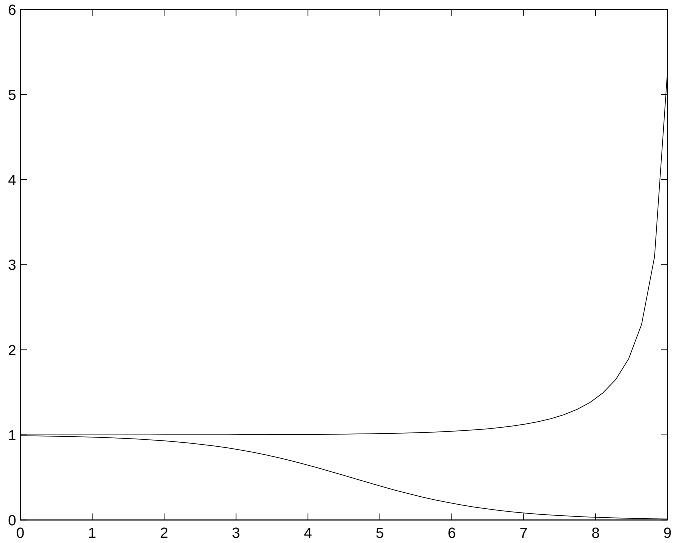

Figure 2:

2.2 Limit Cycle

Limit Cycle: Oscillation of fixed amplitude and fixed frequency without external excitation.

Example:

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

if $|x| > 1, x^2 - 1 > 0$, the damper consumes energy. If $|x| < 1, x^2 - 1 < 0$, the damper produces energy. Therefore, the system has a tendency to make x oscillate around some region.

Response of

$$x_0 = 5,\ \dot{x}_0 = 5$$

and

$$x_0 = -1,\ \dot{x}_0 = -3$$

is shown in Figure 3

2.3 Chaos

The system output is extremely sensitive to initial conditions.

Example:

$$\ddot{x} + 0.1\dot{x} + x^5 = 6\sin(t)$$

Figure 4 shows the response of the system to two almost identical initial conditions

$$x_0 = 2, \dot{x}_0 = 3$$

and

$$x_0 = 2.01, \dot{x}_0 = 3.01.$$

Due to the high nonlinearity in x^5, the response become apparently different after a certain time.

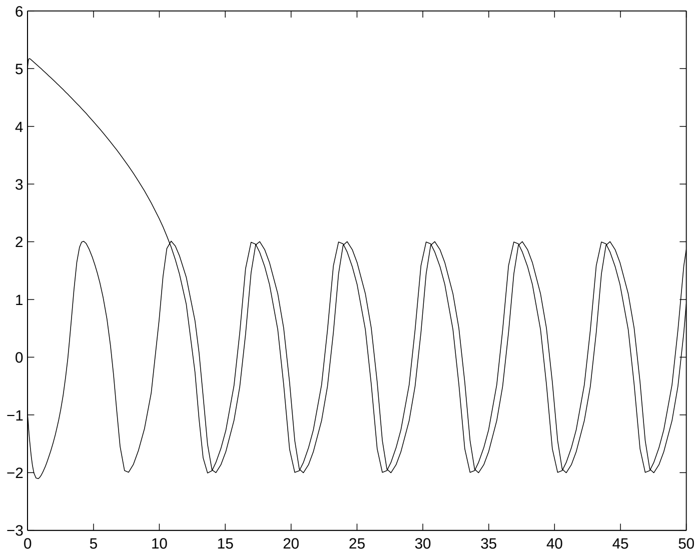

Figure 3:

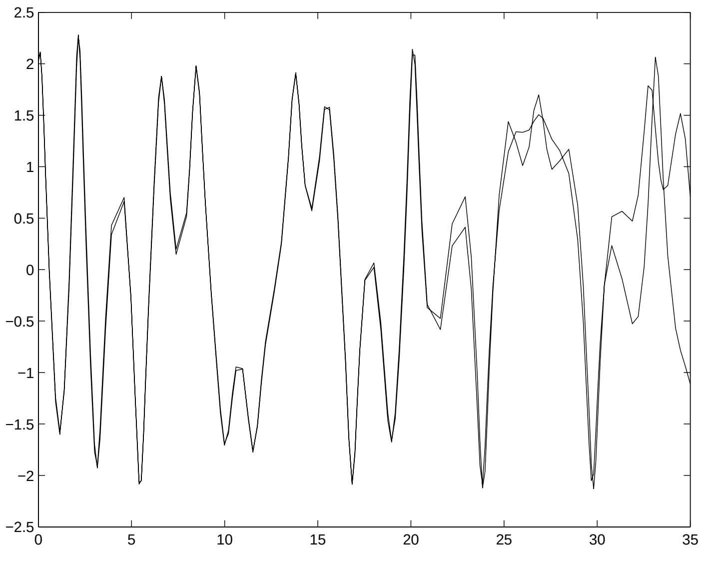

Figure 4: Chaotic Behavior of a Nonlinear System

2.4 Other Behaviors

Saturation, dead zone, hysteresis, and so on.

3 Why Nonlinear Control

All physical systems are nonlinear in nature. Linear model is only an approximation. Conventionally, the nonlinear system is linearized around its operating point and a linear controller is then designed for the obtained simplified model. In some cases, this cannot achieve satisfactory performance.

1. To improve system performance;
2. Linerization cannot provide correct solution

Example: for the same system

$$\dot{x} = -x + x^2$$

Linearizing it yields

$$\dot{x} = -x$$

it has only 1 stable equilibrium point at $x_s = 0$??

3. To deal with model uncertainties:

Example:

$$\dot{x} = f(x, u) + g(x, u)$$

where $f(x, u)$ denotes the modeled dynamics, while $g(x, u)$ represents unmodeled uncertainties including noise, disturbance, etc.

4 General Nonlinear Control

4.1 Phase Plane Analysis:

A graphical method mainly for 2nd-order systems.

4.2 Describing System Analysis

An extended version of the frequency response method, it can be used to approximately analyze and predict nonlinear behavior.

Main Use: Prediction of limit cycles in nonlinear systems, unfortunately only an approximate way.

4.3 Lyanpunov Method

It is first introduced to judge the stability of a nonlinear system.

Stable: state not blow up, how to judge? Linear System: Route Criteria, Nyquist Criteria

Example for Stability Analysis:

$$\dot{x} = \frac{1}{2}\sin(x) - x$$

Is this system stable?

Choose

$$V = \frac{1}{2}x^2$$

then $V \in \mathcal{L}_\infty$ (V remains bounded) implies $x \in \mathcal{L}_\infty$. Taking the time derivative of V yields

$$\begin{aligned}
\dot{V} &= x\dot{x} = x\left(\frac{1}{2}\sin(x) - x\right) \\
&\leq -x^2 + \frac{1}{2}\left|x\sin(x)\right| \\
&\leq -x^2 + \frac{1}{2}x^2 \\
&\leq -\frac{1}{2}x^2
\end{aligned}$$

therefore,

$$\dot{V} \leq -V$$

and

$$V(t) \leq V_0 e^{-t}$$

$$x(t) \leq x_0 e^{-\frac{1}{2}t}$$

Question: If V cannot be solved out, how to judge the stability from the differential equation or inequalities? Lyapunov Theorem, Babalat's Lemma, and so on.

Besides stability analysis, the Lyapunov method is also a powerful tool to design nonlinear controllers.

Example: Suppose we have the following system

$$\dot{x} = f(x) + g(x)u$$

where $f(x)$, $g(x)$ are known functions and u is the control input. Besides, $g(x) \geqslant g_0$ with g_0 being positive constant. How to design a controller to regulate x?

Choose

$$V = \frac{1}{2}x^2$$

then we take its time derivative and substitute into the system dynamics to obtain

$$\dot{V} = x\dot{x} = x\left(f(x) + g(x)u\right)$$

Can we choose a suitable controller u to make $\dot{V} \leq 0$ or further, $\dot{V} \leq -x^2$?

Make

$$u = -\frac{f(x)}{g(x)} - \frac{x}{g(x)}$$

with k representing a positive control gain. Then

$$\dot{V} = -x^2$$

Similarly as the example above, x goes to zero exponentially fast.

Question 1. Why do we need the assumption of $g(x) \geqslant g_0$? To make the controller free of singularity, or make the system controllable.

Question 2. There are two terms within the controller u, which one is feedback, which one is feedforwad? What are they for?

Advantage: Implement the controller design and stability analysis simultaneously; Backbone of the existing nonlinear controllers; Heart and soul of model control.

Disadvantage: hard to construct a suitable Lyapunov function for a give complex dynamic system, conservative method: the conclusion you made for a system based on the Lyapunov analysis can be weaker than its actual situation. We will show this in the future class.

5 Differences between Linear Control and Lyapunov Control

a) Model-Free Control vs Model-Based Control

b) Feedback Control vs Feedforward Control

6 Homework

Use Matlab/Simulink to simulate the following system

$$\ddot{x} + 0.1\dot{x} + x^5 = 6\sin(t)$$

for two set of initial conditions

$$x_0 = 2, \dot{x}_0 = 3$$

and

$$x_0 = 2.01, \dot{x}_0 = 3.01$$