Lyapunov based Nonlinear Control - Assignment4

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Problem 1

For the following linear system

$$r(t) = \dot{e}(t) + \alpha \cdot e(t)$$

where α denotes some positive constant, prove the following properties

- If $r(t) \to 0$ exp. fast, then $e(t), \dot{e}(t) \to 0$ exp. fast;
- If $r(t) \to 0$, then $e(t), \dot{e}(t) \to 0$;
- If $r(t)$ is GUUB, then $e(t), \dot{e}(t)$ is also GUUB.

Solution 1

Solving the differential equation $r(t) = \dot{e}(t) + \alpha \cdot e(t)$ for $e(t)$ yields

$$e(t) = e(0)e^{-\alpha t} + \int_0^t r(\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau \tag{1}$$

(1) If $r(t) \to 0$ exp. fast, then $e(t), \dot{e}(t) \to 0$ exp. fast.

Given $r(t) \to 0$ exp. fast, we know that $|r(t)| \leq r_0 e^{-\gamma t}$, where r_0 and γ are both positive constants. Then we have

$$\begin{aligned}
|e(t)| &\leq |e(0)|e^{-\alpha t} + \int_0^t r_0 e^{-\gamma\tau} e^{-\alpha(t-\tau)}\mathrm{d}\tau \\
&= |e(0)|e^{-\alpha t} + r_0 e^{-\alpha t}\int_0^t e^{(\alpha-\gamma)\tau}\mathrm{d}\tau \\
&= |e(0)|e^{-\alpha t} + \frac{r_0 e^{-\alpha t}}{\alpha-\gamma}\left(e^{(\alpha-\gamma)t} - 1\right) \\
&= \frac{r_0}{\alpha-\gamma}e^{-\gamma t} + \left(|e(0)| - \frac{r_0}{\alpha-\gamma}\right)e^{-\alpha t}.
\end{aligned} \tag{2}$$

From $r(t) = \dot{e}(t) + \alpha \cdot e(t)$, it can be deduced that

$$\begin{aligned}
|\dot{e}(t)| &\leq \alpha \cdot |e(t)| + |r(t)| \\
&= \alpha\left(\frac{r_0}{\alpha-\gamma}e^{-\gamma t} + \left(|e(0)| - \frac{r_0}{\alpha-\gamma}\right)e^{-\alpha t}\right) + r_0 e^{-\gamma t}.
\end{aligned} \tag{3}$$

From (2) and (3) we can see that $e(t), \dot{e}(t) \to 0$ exp. fast.

(2) If $r(t) \to 0$, then $e(t), \dot{e}(t) \to 0$.

Given the condition $r(t) \to 0$, we know that $\lim\limits_{t\to\infty} r(t) = 0$. Calculating the limit of $e(t)$ yields

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{t\to\infty} \left[e(0)e^{-\alpha t} + \int_0^t r(\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau\right] \\ &= \lim_{t\to\infty} e^{-\alpha t} \int_0^t r(\tau)e^{-\alpha\tau}\mathrm{d}\tau.\end{aligned} \tag{4}$$

It's obvious that $\lim\limits_{t\to\infty} e^{-\alpha t} = 0$. So we need to prove $\lim\limits_{t\to\infty} \int_0^t r(\tau)e^{-\alpha\tau}\mathrm{d}\tau$ is bounded.

From the given condition $\lim\limits_{t\to\infty} r(t) = 0$ and the definition of the limitation, we know that for $\forall\varepsilon > 0$, $\exists\delta > 0$ s.t. when $|t| > \delta$, $r(t) < \varepsilon$. So we have

$$\begin{aligned}\lim_{t\to\infty} \int_0^t r(\tau)e^{-\alpha\tau}\mathrm{d}\tau &= \int_0^\infty r(\tau)e^{-\alpha\tau}\mathrm{d}\tau \\ &= \int_0^\delta r(\tau)e^{-\alpha\tau}\mathrm{d}\tau + \int_\delta^\infty r(\tau)e^{-\alpha\tau}\mathrm{d}\tau \\ &\le \left|\int_0^\delta r(\tau)e^{-\alpha\tau}\mathrm{d}\tau\right| + \int_\delta^\infty \varepsilon e^{-\alpha\tau}\mathrm{d}\tau \\ &= \left|\int_0^\delta r(\tau)e^{-\alpha\tau}\mathrm{d}\tau\right| + \frac{\varepsilon}{\alpha}e^{-\alpha\delta}.\end{aligned} \tag{5}$$

Therefore, $\lim\limits_{t\to\infty} \int_0^t r(\tau)e^{-\alpha\tau}\mathrm{d}\tau$ is bounded. As a result,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} e^{-\alpha t} \int_0^t r(\tau)e^{-\alpha\tau}\mathrm{d}\tau = 0. \tag{6}$$

As for $\dot{e}(t)$, it can be obtained that

$$\lim_{t\to\infty} \dot{e}(t) = \lim_{t\to\infty} \left(-\alpha e(t) + r(t)\right) = 0. \tag{7}$$

From (6) and (7) we know that $e(t), \dot{e}(t) \to 0$.

(3) If $r(t)$ is GUUB, then $e(t), \dot{e}(t)$ is also GUUB.

Given that $r(t)$ is GUUB, $r(t)$ satisfies $\forall t$, $|r(t)| \le c_1$ and $\lim\limits_{t\to\infty} |r(t)| \le c_2$. Then from (1) we know that

$$\begin{aligned}|e(t)| &\le |e(0)|e^{-\alpha t} + \int_0^t |r(t)|e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &\le |e(0)|e^{-\alpha t} + c_1 \int_0^t e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &= e^{-\alpha t}\left(|e(0)| + c_1 \int_0^t e^{\alpha\tau}\mathrm{d}\tau\right) \\ &= e^{-\alpha t}\left(|e(0)| + \frac{c_1}{\alpha}\left(e^{\alpha t} - 1\right)\right) \\ &\le \left||e(0)| - \frac{c_1}{\alpha}\right| e^{-\alpha t} + \frac{c_1}{\alpha} \\ &\le \left||e(0)| - \frac{c_1}{\alpha}\right| + \frac{c_1}{\alpha}.\end{aligned} \tag{8}$$

Then take the limit of $|e(t)|$ and yield

$$\begin{aligned}
\lim_{t\to\infty}|e(t)| &= \lim_{t\to\infty}\left|e(0)e^{-\alpha t}+\int_0^t r(t)e^{-\alpha(t-\tau)}\mathrm{d}\tau\right| \\
&= \lim_{t\to\infty}\int_0^t |r(t)|e^{-\alpha(t-\tau)}\mathrm{d}\tau \\
&\le \lim_{t\to\infty} c_1\int_0^t e^{-\alpha(t-\tau)}\mathrm{d}\tau \\
&= c_1\lim_{t\to\infty} e^{-\alpha t}\int_0^t e^{\alpha\tau}\mathrm{d}\tau \\
&= c_1\lim_{t\to\infty} e^{-\alpha t}\frac{1}{\alpha}\left(e^{\alpha t}-1\right) \\
&= c_1\lim_{t\to\infty}\frac{1}{\alpha}\left(1-e^{-\alpha t}\right) \\
&= \frac{c_1}{\alpha}.
\end{aligned} \tag{9}$$

As for the $\dot{e}(t)$, similarly we can obtain that

$$\begin{aligned}
|\dot{e}(t)| &= |-\alpha e(t)+r(t)| \\
&\le \alpha|e(t)|+|r(t)| \\
&\le \left||e(0)|-\frac{c_1}{\alpha}\right|+\frac{c_1}{\alpha}+c_1.
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\lim_{t\to\infty}|\dot{e}(t)| &= \lim_{t\to\infty}|-\alpha e(t)+r(t)| \\
&\le \lim_{t\to\infty}\alpha|e(t)|+|r(t)| \\
&\le \frac{c_1}{\alpha}+c_2.
\end{aligned} \tag{11}$$

In conclusion, if $r(t)$ is GUUB, then $e(t),\dot{e}(t)$ is also GUUB.

Problem 2

For the following linear system

$$\ddot{x}=ax^3+bxe^{-t}+\frac{c\ln\left(|x|+1\right)}{\dot{x}^2+2}+\left(x^2+\cos^2 x\right)u$$

where $a,b,c>0$ denotes known positive constants, design a nonlinear control to drive x to the desired trajectory

$$x_d=10\sin(t)$$

exponentially fast.

- Show that your controller achieves the desired control performance;
- Demonstrate that all the signals during closed-loop operation remain bounded, and there is no singularity presented with your controller.

Solution 2

Define the tracking error as follows

$$e = x_d - x. \tag{12}$$

Then define the following filtered error signal

$$r = \dot{e} + \alpha e. \tag{13}$$

Taking the derivative of r yields

$$\begin{aligned}
\dot{r} &= \ddot{e} + \alpha \dot{e} \\
&= \ddot{x}_d - \ddot{x} + \alpha(\dot{x}_d - \dot{x}) \\
&= \ddot{x}_d - \left[ax^3 + bxe^{-t} + \frac{c\ln(|x|+1)}{\dot{x}^2+2} + \left(x^2 + \cos^2 x\right) u\right] + \alpha(\dot{x}_d - \dot{x}).
\end{aligned} \tag{14}$$

Design the following EMK controller ($x^2 + \cos^2 x \neq 0$)

$$u = \frac{\ddot{x}_d - \left(ax^3 + bxe^{-t} + \frac{c\ln(|x|+1)}{\dot{x}^2+2}\right) + \alpha(\dot{x}_d - \dot{x}) + kr}{x^2 + \cos^2 x}. \tag{15}$$

Then the closed-loop dynamics is

$$\dot{r} = -kr. \tag{16}$$

Therefore, $r \to 0$ exponentially fast. And from the conclusion drawn from Problem 1, we know that both $e(t)$ and $\dot{e}(t)$ go to zero exponentially fast.

From (13) and (16), we know that $r(t) \in \mathcal{L}_\infty$. And due to the third conclusion from Problem 1, it can be deduced that $e(t), \dot{e}(t) \in \mathcal{L}_\infty$. As can be seen from (12) as well as the derivative of (12) that $x(t), \dot{x}(t) \in \mathcal{L}_\infty$. As a result, we know that $u(t) \in \mathcal{L}_\infty$. Hence, all the signals during closed-loop operation remain bounded.

Furthermore, $x^2 + \cos^2 x \neq 0$ and $\dot{x}^2 + 2 \geq 2$ hold true. So there is no singularity presented with the controller.

A simulation result is illustrated in Figure 1, in which the system parameters are set to $a = 1.7$, $b = -2.4$ and $c = 2.1$, the controller parameters are $\alpha = 1$ and $k = 1$, and the initial state is set to $x(0) = 0$ and $\dot{x} = 0$.

Problem 3

For the following linear system

$$\begin{cases} \dot{x} = x\cos(x) + x^2 - y \\ \dot{y} = u \end{cases}$$

where x and y represent the system state, u is the control input. Design the control u to drive x to zero asymptotically fast.

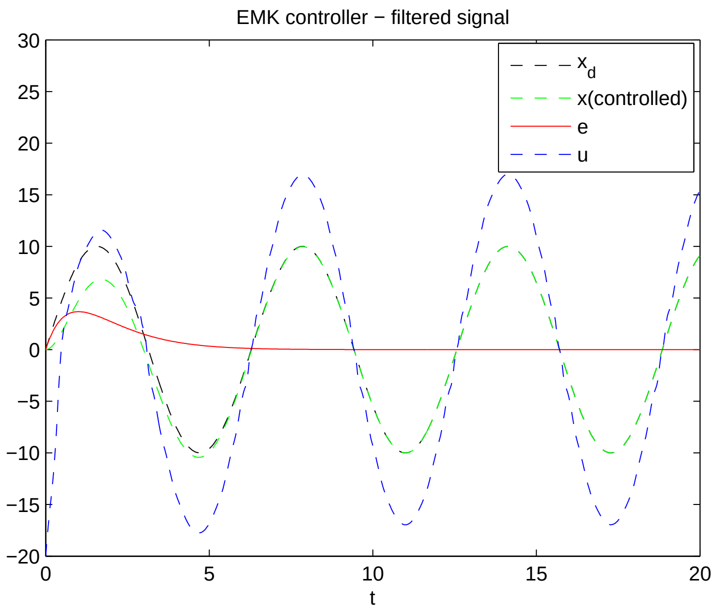

Figure 1: Simulation result - EMK controller with filtered signal.

Solution 3

The back-stepping method can be used to design the controller for this system. Assume that y_d is a virtual input and can be designed as

$$y_d = x\cos(x) + x^2 + kx \tag{17}$$

to make x go to zero. Rewrite the first equation of the system dynamics and substitue y_d into it as

$$\begin{aligned}\dot{x} &= x\cos(x) + x^2 - y_d + (y_d - y)\\ &= x\cos(x) + x^2 - y_d + e_y\\ &= -kx + e_y.\end{aligned} \tag{18}$$

Then design the control input u to drive e_y to zero. The dynamics of e_y is

$$\begin{aligned}\dot{e}_y &= \dot{y}_d - \dot{y}\\ &= \dot{x}\cos(x) - x\sin(x)\dot{x} + 2x\dot{x} + k\dot{x} - \dot{y}\\ &= \dot{x}\left(\cos(x) - x\sin(x) + 2x + k\right) - u\\ &= (-kx + e_y)\left(\cos(x) - x\sin(x) + 2x + k\right) - u.\end{aligned} \tag{19}$$

Design

$$u = (-kx + e_y)\left(\cos(x) - x\sin(x) + 2x + k\right) + k_u e_y + x. \tag{20}$$

Then

$$\dot{e}_y = -k_u e_y - x. \tag{21}$$

Choose the Lyapunov function as follows

$$V = \frac{1}{2}e_y^2 + \frac{1}{2}x^2 \geq 0. \tag{22}$$

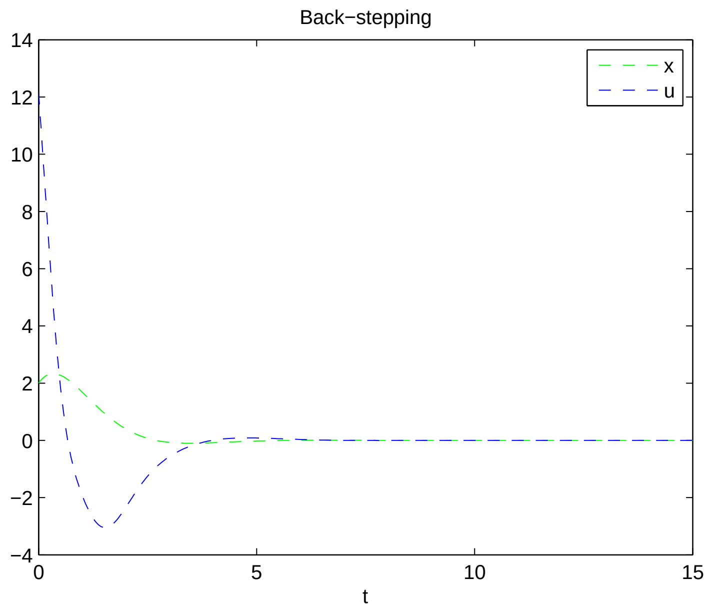

Figure 2: Simulation result - Back-stepping.

And take its time derivative as

$$\begin{aligned} \dot{V} &= e_y \dot{e}_y + x\dot{x} \\ &= -k_u e_y^2 - x e_y + x(-kx + e_y) \\ &= -k_u e_y^2 - kx^2 \leq 0. \end{aligned} \tag{23}$$

As a result, both x and e_y go to zero asymptotically fast.

A simulation result is illustrated in Figure 2, in which the controller parameters are set to $k = 1$ and $ku = 1$, and the initial state is set to $x(0) = 2$ and $y(0) = 1$.