Lyapunov based Nonlinear Control - Assignment5

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1 Problem Statement

For the dynamic system

$$m(x,\theta)\ddot{x} = f(x,\dot{x},t,\theta) + u \tag{1}$$

where $u \in \Re^1$ is the control input into the system, $m(x,\theta)$, $f(x,\dot{x},t,\theta) \in \Re^1$ denote auxiliary functions with $\theta \in \Re^n$ being unknown constant system parameters. Design the following required controllers for the system (1) and simulate for the obtained closed-loop systems to demonstrate the efficacy of the proposed controllers. Make whatever reasonable assumptions you need to fulfill your controllers.

- Design an adaptive controller to drive $x(t)$ to zero;
- Design an sliding mode controller to drive $x(t)$ to zero;
- Design robust controllers including both a high-gain feedback controller and a high-frequency feedback controller to drive $x(t)$ to zero.

Write your control design/analysis and simulation results into a report. For each controller, you need to include controller design, closed-loop system development, stability analysis, and simulation results in the report. Some concluding remarks are required to compare the performance of the aforementioned controllers.

2 Solution

2.1 Adaptive Controller

Assuming that $m(x,\theta) = M(x)^T\theta > 0$ and $f(x,\dot{x},t,\theta) = Y(x,\dot{x},t)^T\theta$, (1) can be rewritten as

$$M(x)^T\theta\ddot{x} = Y(x,\dot{x},t)^T\theta + u. \tag{2}$$

Let $r = \dot{x} + \alpha x$. Taking the derivative of r and substituting it into (2) yields

$$M(x)^T\theta\dot{r} = \left[Y(x,\dot{x},t) + \alpha\dot{x}M(x)\right]^T\theta + u. \tag{3}$$

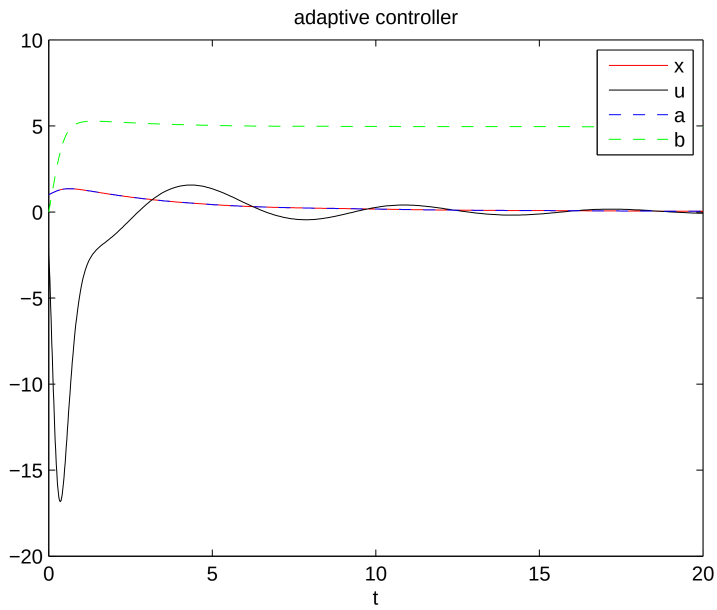

Figure 1: simulation results of the adaptive control.

Design the adaptive controller as

$$u = -\left[\frac{1}{2}\dot{M}(x)r + Y(x,\dot{x},t) + \alpha\dot{x}M(x)\right]^T \hat{\theta} - kr \tag{4}$$

and the parameter update law as

$$\dot{\hat{\theta}} = \Gamma\left[\frac{1}{2}\dot{M}(x)r + Y(x,\dot{x},t) + \alpha\dot{x}M(x)\right] r. \tag{5}$$

Choose the following Lyapunov function

$$V = \frac{1}{2}M(x)^T\theta r^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \geq 0. \tag{6}$$

And taking the derivative of (6) yields

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{M}(x)^T\theta r^2 + M(x)^T\theta r\dot{r} - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &= r\left[\left(\frac{1}{2}\dot{M}(x)r + Y(x,\dot{x},t) + \alpha\dot{x}M(x)\right)^T\tilde{\theta} - kr\right] - \tilde{\theta}^T\left(\frac{1}{2}\dot{M}(x)r + Y(x,\dot{x},t) + \alpha\dot{x}M(x)\right)^T r \\ &= -kr^2 \leq 0.\end{aligned} \tag{7}$$

Since $V \geq 0$ and $\dot{V} \leq 0$, $V \in L_\infty$. Therefore $r \in L_\infty$ and $\tilde{\theta} \in L_\infty$. As a result, $x \in L_\infty$, $\dot{x} \in L_\infty$, $\hat{\theta} \in L_\infty$ and $\dot{r} \in L_\infty$. According to Lemma 15, $kr^2 \to 0$, so $x \to 0$.

For simulation, we set the system model to $m(x,\theta) = a(x^2+1)+b$ and $f(x,\dot{x},t,\theta) = a\dot{x}+bx\sin(t)$, with a and b being the unknown parameters. Matlab ode45 function is used to simulate the system and test the performance of the adaptive controller. The initial condition of the system is set to $x_0 = 1$, $\dot{x}_0 = 1$, $\hat{a}_0 = 0$ and $\hat{b}_0 = 0$. The control gain and the parameter update gain are set to $k = 1$ and $\Gamma = I$, respectively. The simulation results are shown in Figure 1.

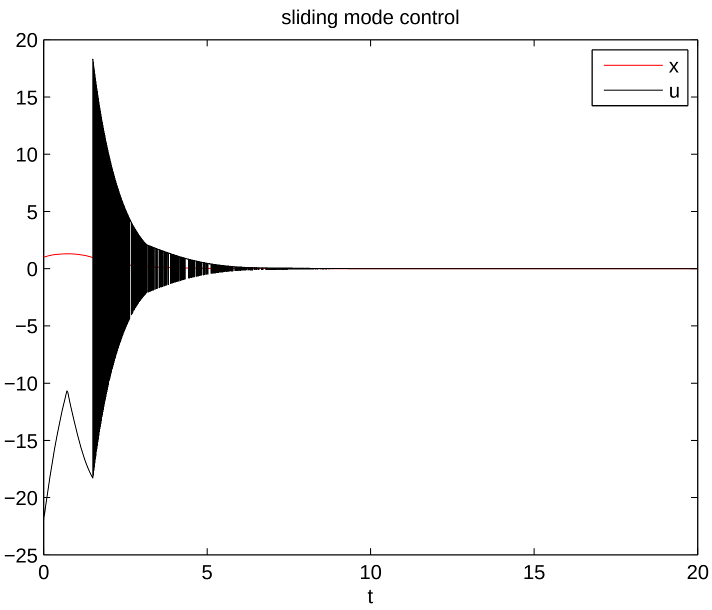

Figure 2: simulation results of the sliding mode control.

2.2 Sliding Mode Controller

Assuming that $\left|\frac{1}{2}\dot{m}(x,\theta)r + \alpha\dot{x}m(x,\theta) + f(x,\dot{x},t,\theta)\right| \leq \rho(x,\dot{x},t)$ and $m(x,\theta) > 0$, design the sliding mode controller as

$$u = -kr - \rho sgn(r). \tag{8}$$

Choose the Lyapunov function as

$$V = \frac{1}{2}m(x,\theta)r^2. \tag{9}$$

Taking the derivative of (9) yields

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{m}(x,\theta)r^2 + m(x,\theta)r\dot{r} \\ &= \frac{1}{2}m(x,\theta)r^2 + r\left[f(x,\dot{x},t,\theta) + u + m(x,\theta)\alpha\dot{x}\right] \\ &= \left[\frac{1}{2}\dot{m}(x,\theta)r + f(x,\dot{x},r,\theta) + m(x,\theta)\alpha\dot{x}\right]r + r\left[-kr - \rho\cdot sgn(r)\right] \\ &\leq \rho|r| + r\left[-kr - \rho\cdot sgn(r)\right] \\ &= -kr^2 \leq 0.\end{aligned} \tag{10}$$

Since $V \geq 0$ and $\dot{V} \leq 0$, $V \in L_\infty$. Therefore $r \in L_\infty$, $x \in L_\infty$, $\dot{x} \in L_\infty$ and $\dot{r} \in L_\infty$. According to Lemma 15, $kr^2 \to 0$, so $x \to 0$.

For the simulation of the sliding mode controller, we use exactly the same system model and control parameters as the ones in the adaptive control. First, the bounding function $\rho(x,\dot{x},t)$ needs to be determined. Assuming that $a \leq \bar{a}$ and $b \leq \bar{b}$,

$$\begin{aligned}&\left|\frac{1}{2}\dot{m}(x,\theta)r + \alpha\dot{x}m(x,\theta) + f(x,\dot{x},t,\theta)\right| \\ &\leq \left|axr + \alpha\dot{x}(ax^2 + a + b) + a\dot{x} + bx\sin(t)\right| \\ &\leq \bar{a}|xr| + \left[\bar{a}x^2 + 2a + b\right]\cdot|\dot{x}| + \bar{b}|x\sin(t)|.\end{aligned} \tag{11}$$

Let $\rho(x, \dot{x}, t) = \bar{a}|xr| + [\bar{a}x^2 + 2a + b] \cdot |\dot{x}| + \bar{b}|x\sin(t)|$, and the sliding mode control can be simulated. The results are shown in Figure 2. Note that though the control error goes to zero, the control input will oscillates around zero.

2.3 Robust Controller

2.3.1 High Gain Feedback Robust Control

Assuming that $\left|\frac{1}{2}\dot{m}(x,\theta)r + \alpha\dot{x}m(x,\theta) + f(x,\dot{x},t,\theta)\right| \leq \rho(x,\dot{x},t)$ and $m(x,\theta) > 0$, design the high gain feedback robust controller as

$$u = -kr - k_n\rho^2 r. \tag{12}$$

Choose the Lyapunov function as

$$V = \frac{1}{2}m(x,\theta)r^2. \tag{13}$$

Taking the derivative of (9) yields

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\dot{m}(x,\theta)r^2 + m(x,\theta)r\dot{r} \\
&= \frac{1}{2}m(x,\theta)r^2 + r\left[f(x,\dot{x},t,\theta) + u + m(x,\theta)\alpha\dot{x}\right] \\
&= \left[\frac{1}{2}\dot{m}(x,\theta)r + f(x,\dot{x},r,\theta) + m(x,\theta)\alpha\dot{x}\right]r + r\left[-kr - k_n\rho^2 r\right] \\
&\leq \rho|r| + r\left[-kr - k_n\rho^2 r\right] \\
&= -kr^2 + (\rho|r| - k_n\rho^2 r^2) \\
&\leq -kr^2 + \frac{1}{k_n} \\
&= -2kV + \frac{1}{k_n}.
\end{aligned} \tag{14}$$

Lemma 9 can be applied to (14) to obtain the upper bound of V as follows

$$V \leq V_0 e^{-2kt} + \frac{1}{2kk_n}\left(1 - e^{-2kt}\right). \tag{15}$$

As a result, $V \in L_\infty$ satisfies

$$\lim_{t\to\infty} V \leq \frac{1}{2kk_n}. \tag{16}$$

From (13) we know that $r, m \in L_\infty$. Therefore $x \in L_\infty$, $\dot{x} \in L_\infty$, $u \in L_\infty$ and $\dot{r} \in L_\infty$. All the signals in the closed-loop operation are bounded.

From (13) and (17), the r satisfies

$$|r| \leq \sqrt{r_0^2 e^{-2kt} + \frac{1}{kk_n}\left(1 - e^{-2kt}\right)}. \tag{17}$$

thus

$$\lim_{t\to\infty} |r| \leq \sqrt{\frac{1}{kk_n}} \tag{18}$$

Hence, r is GUUB. Therefore, $x, \dot{x}$ are both GUUB.

The experimental setup of the high gain feedback robust control is the same as the sliding mode control. The simulation results are shown in Figure 3. From the results we can see that there is no oscillation phenomenon in the high gain feedback control. However, the control gain need to be fairly large to drive the control error to zero.

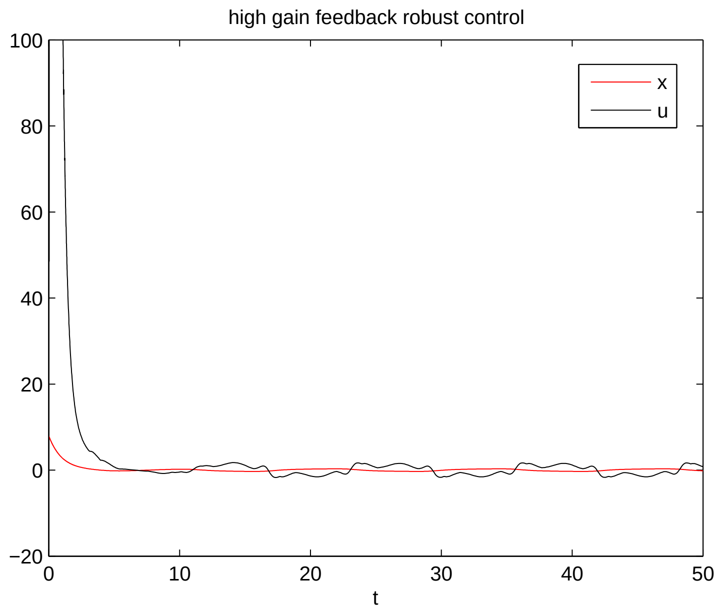

Figure 3: simulation results of the high gain feedback robust control.

2.3.2 High Frequency Feedback Robust Control

Assuming that $\left|\frac{1}{2}\dot{m}(x,\theta)r + \alpha\dot{x}m(x,\theta) + f(x,\dot{x},t,\theta)\right| \leq \rho(x,\dot{x},t)$ and $m(x,\theta) > 0$, design the high frequency feedback robust controller as

$$u = -kr - \frac{\rho^2 r}{\rho|r| + \varepsilon}. \tag{19}$$

Choose the Lyapunov function as

$$V = \frac{1}{2}m(x,\theta)r^2. \tag{20}$$

Taking the derivative of (9) yields

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\dot{m}(x,\theta)r^2 + m(x,\theta)r\dot{r} \\
&= \frac{1}{2}m(x,\theta)r^2 + r\left[f(x,\dot{x},t,\theta) + u + m(x,\theta)\alpha\dot{x}\right] \\
&= \left[\frac{1}{2}\dot{m}(x,\theta)r + f(x,\dot{x},r,\theta) + m(x,\theta)\alpha\dot{x}\right]r + r\left[-kr - \frac{\rho^2 r}{\rho|r| + \varepsilon}\right] \\
&\leq \rho|r| + r\left[-kr - \frac{\rho^2 r}{\rho|r| + \varepsilon}\right] \\
&= -kr^2 + \frac{\rho|r|\varepsilon}{\rho|r| + \varepsilon} \\
&\leq -kr^2 + \varepsilon \\
&= -2kV + \varepsilon.
\end{aligned} \tag{21}$$

Lemma 9 can be applied to (21) to obtain the upper bound of V as follows

$$V \leq V_0 e^{-2kt} + \frac{\varepsilon}{2k}\left(1 - e^{-2kt}\right). \tag{22}$$

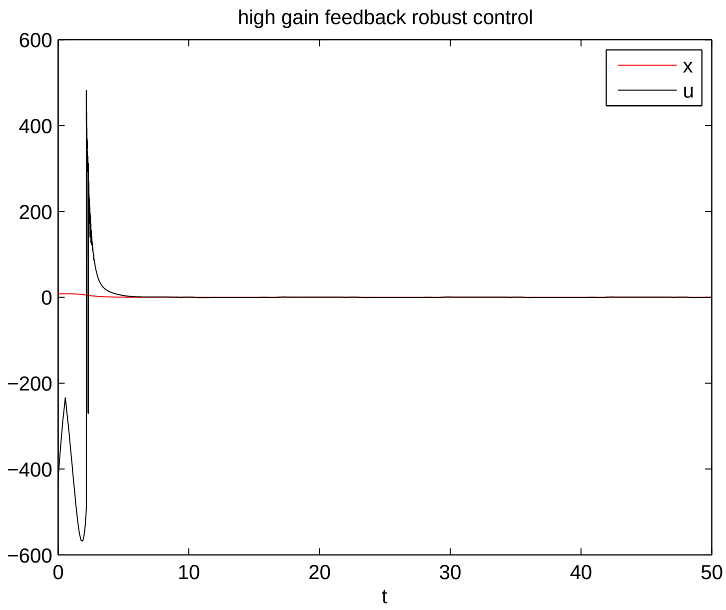

Figure 4: simulation results of the high frequency feedback robust control.

As a result, $V \in L_\infty$ satisfies

$$\lim_{t\to\infty} V \leq \frac{\varepsilon}{2k}. \tag{23}$$

From (13) we know that $r, m \in L_\infty$. Therefore $x \in L_\infty$, $\dot{x} \in L_\infty$, $u \in L_\infty$ and $\dot{r} \in L_\infty$. All the signals in the closed-loop operation are bounded.

From (20) and (24), the r satisfies

$$|r| \leq \sqrt{r_0^2 e^{-2kt} + \frac{\varepsilon}{k}(1 - e^{-2kt})}. \tag{24}$$

thus

$$\lim_{t\to\infty} |r| \leq \sqrt{\frac{\varepsilon}{k}} \tag{25}$$

Hence, r is GUUB. Therefore, $x, \dot{x}$ are both GUUB.

The experimental setup is the same as the high gain feedback robust control. The parameter ε is set to 0.1. The simulation results are shown in Figure 4. The control gain is not as large as the one in the high gain feedback robust control. In Figure 3, we manually bound the y axis for the sake of visualization.