Chapter 4. Exact Model Knowledge Controller Design

1 Exact Model Knowledge Controller Design

As stated earlier, the Lyapunov techniques often aim to construct model-based controllers. In this chapter, we will introduce the process of designing a suitable controller for a nonlinear system with exact model knowledge.

For the following system

$$\dot{x} = f(x) + u \tag{1}$$

where $x \in \Re^1$ represents the system state, $f(x) \in \Re^1$ denotes the exactly known system dynamics, $u \in \Re^1$ is the control input. The objective is to design the control input u to make the system state track some desired trajector $x_d \in \Re^1$.

Define the tracking error $e \in \Re^1$ as follows

$$e = x_d - x \tag{2}$$

then by taking its time derivative and substituting (1) into the resulting expression for $\dot{x}$, the open loop error system can be obtained as

$$\dot{e} = \dot{x}_d - f(x) - u \tag{3}$$

Based on the control objective that we need to push the tracking error e to zero, we choose the following Lyapunov function

$$V = \frac{1}{2}e^2 \tag{4}$$

Taking the time derivative of (4) yields

$$\begin{aligned} \dot{V} &= e\dot{e} \\ &= e\left(\dot{x}_d - f(x) - u\right) \end{aligned} \tag{5}$$

Obviously, if we can design the controller u to make

$$\dot{x}_d - f(x) - u = -ke \tag{6}$$

with $k \in \Re^1$ being a positive control gain, then

$$\dot{V} = -ke^2 \tag{7}$$

and the tracking error e goes to zero exponentially fast. From (6), it can be seen that the controller we need to construct is

$$u = (\dot{x}_d - f(x)) + ke \tag{8}$$

The controller (8) consists of two parts, the first part is the terms in the bracket $(\dot{x}_d - f(x))$, they are feedforward terms targeting to cancel the system dynamics, while the second part is a standard feedback term ke.

Example 1 *For the following nonlinear system*

$$\dot{x} = ax^2 + \frac{u}{b + c\sin^2 t}$$

where a, b, $c \in \Re^1 > 0$ denote known positive constants, design a nonlinear control $u \in \Re^1$ to drive $x \in \Re^1$ to zero exponentially fast.

Based on the dynamics of the system, design the following controller

$$u = \left(b + c\sin^2 t\right)\left[-ax^2 - kx\right]$$

then the closed-loop dynamics can be developed as

$$\begin{aligned} \dot{x} &= ax^2 + \frac{u}{b + c\sin^2 t} \\ &= -kx \end{aligned}$$

the solution to the above differential equation is

$$x(t) = x_0 e^{-kt}$$

therefore, $x \to 0$ exponentially fast.

2 Backstepping Method Introduction

Backstepping is a very useful method for designing controls, especially for cascaded systems, it constructs a controller for a given system via a step-by-step process. We will use examples to demonstrate the design process.

Example 2 *For the following system*

$$\dot{x} = \sin(x) - y$$

where $x \in \Re^1$ denotes the system state, while $y \in \Re^1$ is the control input. To push x to zero, we design a EMK controller as

$$y = \sin(x) + kx$$

then the closed-loop dynamics can be obtaind

$$\dot{x} = -kx$$

and the state x goes to zero exponentially fast.

Example 3 *Now we modify the system as follows*

$$\begin{cases} \dot{x} = \sin(x) - y \\ \dot{y} = u \end{cases}$$

where $x \in \Re^1$ and $y \in \Re^1$ represent the system state, $u \in \Re^1$ is the control input. Now we need to design u to drive x to zero asymptotically fast. We will use the backstepping method to design the *controller by the following two steps: Step 1, assume that y_d is the virtual input, and based on the above example, we can design y_d as*

$$y_d = \sin(x) + kx \tag{9}$$

to make x go to zero. However, since y is only "virtual", in other words, it cannot be chosen freely, how to deal with that? To aid our analysis, we rewrite the first equation of the system dynamics as

$$\begin{aligned} \dot{x} &= \sin(x) - y \\ &= \sin(x) - y_d + (y_d - y) \\ &= \sin(x) - y_d + e_y \end{aligned} \tag{10}$$

where y_d is an auxiliary signal chosen to make x go to zero providing the mismatch term e_y doesn't exist. In other words, we choose y_d, instead of y, as our virtual control, since it can be chosen freely. In (10), e_y represents the mismatch between y_d and y

$$e_y = y_d - y. \tag{11}$$

According to the above analysis, y_d can be design as

$$y_d = \sin(x) + kx$$

then

$$\dot{x} = -kx + e_y \tag{12}$$

and, if e_y disappears, then the above design of y_d pushes x to zero. Therefore, step 2 aims to design the real control input u to drive e_y to zero (that is, to make the actual y track the desired y_d). To do that, we first need to deduct the dynamics of $\dot{e}_y$. Taking the time derivative of (11) and substituting (12) into the resulting expression yields

$$\begin{aligned} \dot{e}_y &= \dot{y}_d - \dot{y} \\ &= \dot{x}\cos(x) + k\dot{x} - \dot{y} \\ &= \dot{x}\left(\cos(x) + k\right) - u \\ &= \left(-kx + e_y\right)\left(\cos(x) + k\right) - u \end{aligned}$$

if we design

$$u = \left(-kx + e_y\right)\left(\cos(x) + k\right) + k_u e_y$$

then

$$\dot{e}_y = -k_u e_y$$

and e_y goes to zero exponentially fast. Will this controller work?

Choose

$$V = \frac{1}{2}e_y^2 + \frac{1}{2}x^2$$

then taking its time derivative yields

$$\begin{aligned} \dot{V} &= e_y\dot{e}_y + x\dot{x} \\ &= -k_u e_y^2 + x\left(-kx + e_y\right) \\ &= \left(-k_u e_y^2 - kx^2\right) + xe_y \end{aligned}$$

now how do we deal with the additional interconnecting term xe_y? Do we definitely need to cancel our this term? How? Suppose we modify our controller slightly

$$u = (-kx + e_y)(\cos(x) + k) + k_u e_y + x$$

then

$$\dot{e}_y = -k_u e_y - x$$

after substituting it into the expression of $\dot{V}(t)$, it is easy to see that the latter term in $\dot{e}_y$ will eventually cancel out the troublesome interconnecting term xe_y, thus $\dot{V}(t)$ becomes

$$\dot{V} = -k_u e_y^2 - kx^2$$

Therefore, both e_y and x will go to zero exponentially fast.

3 Filtered Signal Techniques

In this section, we will demonstrate how a filtered signal can be introduced to convert a second-order system into a first-order system. For the following linear system

$$r(t) = \dot{e}(t) + \alpha \cdot e(t)$$

where α denotes some positive constant, the following properties can be proven

- If $r(t) \to 0$ exp. fast, then $e(t)$, $\dot{e}(t) \to 0$ exp. fast;
- If $r(t) \to 0$, then $e(t)$, $\dot{e}(t) \to 0$;
- If $r(t)$ is GUUB, then $e(t)$, $\dot{e}(t)$ is also GUUB.

Example 4 *For the following nonlinear system*

$$\ddot{x} = -\dot{x}^2 \sin(x) + u$$

Assume both $x \in \Re^1$ and $\dot{x} \in \Re^1$ are measurable, design a EMK controller to push the state x to zero exp. fast.

Define the following filtered error signal

$$r = \dot{x} + \alpha x$$

then

$$\begin{aligned} \dot{r} &= \ddot{x} + \alpha\dot{x} \\ &= -\dot{x}^2 \sin(x) + (\alpha\dot{x} + u) \end{aligned}$$

Based on the open-loop dynamics, design the following EMK controller

$$u = \dot{x}^2 \sin(x) - \alpha\dot{x} - kr$$

The closed-loop dynamics is

$$\dot{r} = -kr$$

therefore, $r \to 0$ exp. fast$\Longrightarrow \dot{x}$ and $x \to 0$ exp. fast.