Chapter 3. Lyapunov Stability

1 Definitions of Various Stability

Definition 1 *Stable: The equilibrium state*

$$x = 0$$

is defined as stable if, for any $R > 0$, there esists $r > 0$, such that if

$$\|x_0\| \leq r$$

then for $\forall t > 0$

$$\|x(t)\| \leq R.$$

Otherwise, the equilibrium point is unstable.

Definition 2 *Asymptotically stable: An equilibrium point $x = 0$ is asymptotically stable if it is stable, and if in addition, there exists some $r > 0$, such that*

$$\|x_0\| \leq r$$

implies that $x(t) \to 0$ as $t \to \infty$.

Definition 3 *Exponentially Stable: An equilibrium point $x = 0$ is exponentially Stable if there exists two strictly positive numbers α and λ, such that for $\forall t > 0$*

$$\|x(t)\| \leq \alpha \|x_0\| e^{-\lambda t}$$

in some ball B_r around the origin.

Examples of various stability

2 Lyapunov's Direct Method

Theorem 4 *Local Stability: If in a ball B_{R_o}, there exists a scalar function $V(x)$ satisfies that:*

a) $V(x)$ is positive definite (PD) in B_{R_o};
b) $\dot{V}(x)$ is negative semi-definite in B_{R_o};
then the equilibrium point o is locally stable. If, actually, $\dot{V}(x)$ is locally negative definite in B_{R_o}, then the stability is asymptotic.

Example:

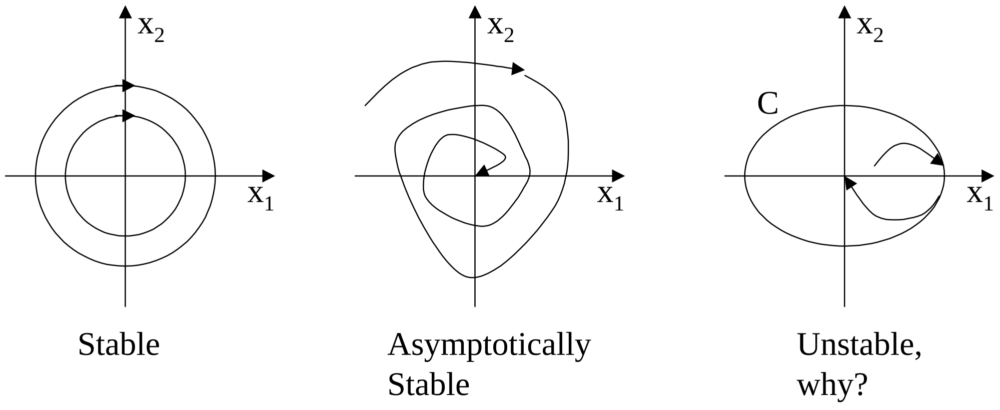

Figure 1: Stability Demonstration Examples

System Dynamics:

$$\ddot{\theta} + \dot{\theta} + \sin(\theta) = 0$$

choose the state as

$$x_1 = \theta, x_2 = \dot{\theta}$$

and the region is defined as

$$-\pi < x_1 < \pi$$

then the dynamics is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) - x_2 \end{cases}$$

choose

$$V = (1 - \cos(x_1)) + \frac{x_2^2}{2}, \text{LPD around } (0,\, 0)$$

then

$$\begin{aligned} \dot{V} &= x_2 \sin(x_1) + x_2 \dot{x}_2 \\ &= x_2 \sin(x_1) - \sin(x_1)\, x_2 - x_2^2 = -x_2^2 \le 0 \end{aligned}$$

$$\text{Locally, } \dot{V} = 0 \Longleftrightarrow x_2 = 0, \text{ but } x_1 \text{ can be of any value}$$

thus, $\dot{V}$ is negative semi-definite and the equilibrium point is locally stable. Is this the strongest conclusion we can obtain for this system? Let's try another Lyanpunov function

$$\begin{aligned} V(x_1, x_2) &= \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.9 & 0.9 \\ 0.9 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1 - \cos(x_1)) \\ &= \frac{1}{2} \left[0.9x_1^2 + 1.8x_1x_2 + x_2^2 \right] + (1 - \cos(x_1)) \Longrightarrow \text{PD} \end{aligned}$$

Taking its time derivative yields

$$\begin{aligned} \dot{V} &= 0.9x_1 (\dot{x}_1 + \dot{x}_2) + x_2 (0.9\dot{x}_1 + \dot{x}_2) + \dot{x}_1 \sin(x_1) \\ &= -0.9x_1 \sin(x_1) - x_2 (\sin(x_1) + 0.1x_2) + x_2 \sin(x_1) \\ &= -0.9x_1 \sin(x_1) - 0.1x_2^2 \end{aligned}$$

$$\text{Locally, } \dot{V} = 0 \Longleftrightarrow x_1 = 0,\ x_2 = 0$$

therefore, $\dot{V}$ is negative definite and the equilibrium point is locally asymptotically stable.

What have we learned from the example: a). Lyapunov's method is conservative; b). A suitable Lyapunov function is of the most importance.

Example:

System Dynamics:

$$\begin{cases} \dot{x}_1 = x_1\left(x_1^2 + x_2^2 - 1\right) - x_2 \\ \dot{x}_2 = x_1 + x_2\left(x_1^2 + x_2^2 - 1\right) \end{cases}$$

choose

$$V = x_1^2 + x_2^2 \Longrightarrow \text{PD}$$

then

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1^2\left(x_1^2 + x_2^2 - 1\right) - 2x_1x_2 + 2x_1x_2 + 2x_2^2\left(x_1^2 + x_2^2 - 1\right) \\ &= 2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 1\right) \Longrightarrow \text{LPD} \end{aligned}$$

thus in the ball $x_1^2 + x_2^2 \leq 1$, $\dot{V}$ is locally negative definite and the equilibrium point $(0,0)$ is locally asymptotically stable.

Theorem 5 *Global Stability: If a) $V(x)$ is positive definite (PD); b) $\dot{V}(x)$ is negative definite (ND); c) $V(x) \to \infty$ as $\|x\| \to \infty$(Radially Unbounded), then the equilibrium point is globally asymptotically stable.*

Example:

System Dynamics:

$$\begin{cases} \dot{x}_1 = -x_1\left(x_1^2 + x_2^2\right) - x_2 \\ \dot{x}_2 = x_1 - x_2\left(x_1^2 + x_2^2\right) \end{cases}$$

choose

$$V = x_1^2 + x_2^2 \Longrightarrow \text{PD}$$

then

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= -2x_1^2\left(x_1^2 + x_2^2\right) - 2x_1x_2 + 2x_1x_2 - 2x_2^2\left(x_1^2 + x_2^2\right) \\ &= -2\left(x_1^2 + x_2^2\right)^2 \Longrightarrow \text{ND} \end{aligned}$$

thus the equilibrium point $(0,0)$ is globally asymptotically stable.

Assignment: Find the equilibrium point(s) of the following friction system

$$\dot{v} + 2av\,|v| + bv = c, \quad a > 0, b > 0, c > 0$$

and analyze the corresponding stability.