Chapter 7. Lyapunov-Based Robust Control

1 General Nonlinear System

In the subsequent analysis, we will attack the control problem of the following nonlinear system

$$\dot{e} = f(x, t, \theta) - u_1 \tag{1}$$

where $f(x, t, \theta)$ represents an unknown nonlinear function, and

$$x(t) \in \mathcal{L}_\infty \Longrightarrow f(x, t, \theta) \in \mathcal{L}_\infty. \tag{2}$$

2 Robust Control

Assume that the system nonlinearity $f(x, t, \theta)$ can be upper bounded as follows

$$|f(x, t, \theta)| \leq \rho(|x|, t) \tag{3}$$

where $\rho(|x|, t)$ denotes some known positive function which satisfies the property of

$$x(t) \in \mathcal{L}_\infty \Longrightarrow \rho(|x|, t) \in \mathcal{L}_\infty \tag{4}$$

2.1 High Frequency Feedback Robust Control

Design the controller as follows

$$u_1 = \frac{\rho^2(|x|, t)\, e}{\rho(|x|, t)\,|e| + \varepsilon} + ke \tag{5}$$

where ε denotes a small positive constant.

Question: Why is the controller of (5) called high-frequency feedback robust control?

Question: In the controller of (5), if $\varepsilon \to 0$, what happens to the controller? In this case, which kind of controller the high frequency controller of (5) converts to?

Substituting the controller $u_1(t)$ into the system dynamics (1) yields

$$\dot{e} = f(x, t, \theta) - \frac{\rho^2(|x|, t)\, e}{\rho(|x|, t)\,|e| + \varepsilon} - ke \tag{6}$$

Define the Lyapunov function as follows

$$V_2 = \frac{1}{2}e^2 \geqslant 0 \tag{7}$$

Taking the time derivative of (7) yields

$$\begin{aligned}
\dot{V}_2 &= e\dot{e} \quad (8)\\
&= -ke^2 + ef(x,t,\theta) - \frac{\rho^2\left(|x|,t\right)e^2}{\rho\left(|x|,t\right)|e|+\varepsilon}\\
&\leq -ke^2 + \left[|e|\,\rho\left(|x|,t\right) - \frac{\rho^2\left(|x|,t\right)e^2}{\rho\left(|x|,t\right)|e|+\varepsilon}\right]\\
&= -ke^2 + \left[\frac{\left(e^2\rho^2\left(|x|,t\right)+\varepsilon\,|e|\,\rho\left(|x|,t\right)\right)-\rho^2\left(|x|,t\right)e^2}{\rho\left(|x|,t\right)|e|+\varepsilon}\right]\\
&= -ke^2 + \left[\frac{|e|\,\rho\left(|x|,t\right)}{\rho\left(|x|,t\right)|e|+\varepsilon}\varepsilon\right]\\
&\leq -ke^2+\varepsilon\\
&= -2kV_2+\varepsilon
\end{aligned}$$

Lemma 8 can then be applied to (8) to obtain the upper bound of $V_2\left(t\right)$ as follows

$$V_2(t) \leq V_{20}e^{-2kt} + \frac{\varepsilon}{2k}\left(1-e^{-2kt}\right) \quad (9)$$

therefore, $V_2\left(t\right) \in \mathcal{L}_\infty$. In fact,

$$\lim_{t\to\infty} V_2(t) \leq \frac{\varepsilon}{2k} \quad (10)$$

2.1.1 Signal Chasing

Based on (7) and (9), we know that $e\left(t\right) \in \mathcal{L}_\infty \Longrightarrow x\left(t\right) \in \mathcal{L}_\infty \Longrightarrow \rho\left(|x|,t\right) \in \mathcal{L}_\infty \Longrightarrow u_1\left(t\right), u\left(t\right) \in \mathcal{L}_\infty$; hence $\dot{e}\left(t\right), \dot{x}\left(t\right) \in \mathcal{L}_\infty \Longrightarrow$all the signals in the closed-loop operation remain bounded.

2.1.2 Error Tracking

From the definition of $V_2\left(t\right)$ and the fact of (9), after some mathematical manipulation, we know

$$|e(t)| \leq \sqrt{e_0^2e^{-2kt} + \frac{\varepsilon}{k}\left(1-e^{-2kt}\right)} \quad (11)$$

with e_0 being the initial error. It can then be obtaind from (11) that

$$\lim_{t\to\infty}|e(t)| \leq \sqrt{\frac{\varepsilon}{k}}$$

Hence, the tracking error $e(t)$ is global uniformly ultimately bounded (GUUB).

Remark 1 *The upper bound of the limit of $e(t)$ can be made arbitrary small by choosing the control gain k large enough or ε small enough.*

Result: GUUB

2.2 High Gain Feedback Robust Control

Design the controller as follows

$$u_1 = k_n \rho^2 \left(|x|, t\right) e + ke \tag{12}$$

where k, k_n denote positive control gains.

Question: Why do we call it high gain feedback control? In the first term of the controller (12), the coefficient before the error $e(t)$ is very large, thus this term is very similar to a standard feedback control except that the control gain is variant and high.

Substituting the controller $u_1(t)$ into the system dynamics (1) yields

$$\dot{e} = f(x, t, \theta) - \rho^2 \left(|x|, t\right) e - ke \tag{13}$$

Define the Lyapunov function as follows

$$V = \frac{1}{2} e^2 \geqslant 0 \tag{14}$$

Taking the time derivative of (14) yields

$$\begin{aligned} \dot{V} &= e\dot{e} \\ &= -ke^2 + ef(x, t, \theta) - k_n e \cdot \rho^2 \left(|x|, t\right) e \\ &= -ke^2 + ef(x, t, \theta) - k_n e^2 \cdot \rho^2 \left(|x|, t\right) \\ &\leq -ke^2 + \left[|e| \rho\left(|x|, t\right) - k_n e^2 \cdot \rho^2 \left(|x|, t\right)\right] \\ &\leq -ke^2 + \frac{1}{k_n} \\ &= -2kV + \frac{1}{k_n} \end{aligned} \tag{15}$$

where nonlinear damping has been utilized to the bracketed terms. Lemma 8 can then be applied to (15) to obtain the upper bound of $V(t)$ as follows

$$V(t) \leq V_0 e^{-2kt} + \frac{1}{2kk_n}\left(1 - e^{-2kt}\right) \tag{16}$$

therefore, $V(t) \in \mathcal{L}_\infty$. In fact,

$$\lim_{t\to\infty} V(t) \leq \frac{1}{2kk_n} \tag{17}$$

2.2.1 Signal Chasing

Based on (14) and (16), we know that $e(t) \in \mathcal{L}_\infty \Longrightarrow x(t) \in \mathcal{L}_\infty \Longrightarrow \rho(|x|, t) \in \mathcal{L}_\infty \Longrightarrow u_1(t), u(t) \in \mathcal{L}_\infty$; hence $\dot{e}(t), \dot{x}(t) \in \mathcal{L}_\infty \Longrightarrow$all the signals in the closed-loop operation remain bounded.

2.2.2 Error Tracking

From the definition of $V(t)$ and the fact of (16), after some mathematical manipulation, we know

$$|e(t)| \leq \sqrt{e_0^2 e^{-2kt} + \frac{1}{kk_n}\left(1 - e^{-2kt}\right)} \tag{18}$$

with e_0 being the initial error. It can then be obtaind from (18) that

$$\lim_{t\to\infty} |e(t)| \leq \sqrt{\frac{1}{kk_n}}$$

Hence, the tracking error $e(t)$ is global uniformly ultimately bounded (GUUB).

Remark 2 *The upper bound of the limit of $e(t)$ can be made arbitrary small by choosing the control gains k or k_n large enough.*

Result: GUUB