# Chapter 6. Lyapunov-Based Sliding Mode Control

## 1 General Nonlinear System

In the subsequent analysis, we will attack the control problem of the following nonlinear system

$$\dot{e} = f(x, t, \theta) - u_1 \tag{1}$$

where $f(x, t, \theta)$ represents an unknown nonlinear function, and it has the following property

$$x(t) \in \mathcal{L}_\infty \Longrightarrow f(x, t, \theta) \in \mathcal{L}_\infty. \tag{2}$$

## 2 Sliding Mode Control

Assume that the system nonlinearity $f(x, t, \theta)$ can be upper bounded as follows

$$|f(x, t, \theta)| \leq \rho(|x|, t) \tag{3}$$

where $\rho(x, t)$ denotes some known positive function which satisfies the property of

$$x(t) \in \mathcal{L}_\infty \Longrightarrow \rho(|x|, t) \in \mathcal{L}_\infty \tag{4}$$

**Question**: How strong the condition of (3) is? How to obtain such a function $\rho(|x|, t)$? Suppose that we know the structure of the function $f(x, t, \theta)$ and the bounds of the parameters $\theta$ in the sense of

$$\underline{\theta} \leq \theta \leq \bar{\theta} \tag{5}$$

in this case, is it possible to calculate the function $\rho(|x|, t)$?

**Example 1** *For the nonlinear function*

$$f(x, t, a, b) = ax^2 t + \sin(bt) - \frac{x}{b(t+2)} \tag{6}$$

*where $a, b$ represent system parameters with the following known bounds*

$$\underline{a} \leq a \leq \overline{a} \tag{7}$$

$$\underline{b} \leq b \leq \overline{b} \tag{8}$$

*it is pretty straightforward to show that*

$$f(x, t, a, b) \leq \overline{a}x^2 t + 1 + \frac{|x|}{\underline{b}(t+2)}$$

**Example 2** *For the nonlinear function*

$$f(x,t,a,b) = xe^{(b-a)t} + \frac{xt}{|a-\lambda| + \varepsilon} \tag{9}$$

*where $a, b$ represent system parameters with the following known bounds*

$$\underline{a} \leq a \leq \overline{a} \tag{10}$$

$$\underline{b} \leq b \leq \overline{b} \tag{11}$$

*$\lambda, \varepsilon$ denote known positive constants. After some mathematical analysis, the following upper bound of $f(x,t,a,b)$ can be obtained*

$$\begin{aligned} f(x,t,a,b) &= xe^{(b-a)t} + \frac{xt}{|a-\lambda| + \varepsilon} \\ &\leq |x| e^{(\overline{b}-\underline{a})t} + \frac{|x| t}{\min(|\underline{a} - \lambda|, |\overline{a} - \lambda|) + \varepsilon} \end{aligned} \tag{12}$$

**Remark 1** *In the sliding mode control or the next discussed robust control, the upper bound $\rho(|x|, t)$ for the uncertain nonlinear function $f(x,t,\theta)$ should be chosen as* ***small*** *as possible,* ***why****?*

Design the controller as follows

$$u_1 = sgn(e) \cdot \rho(|x|, t) + ke \tag{13}$$

where $k$ is a positive control gain, and $sgn(e)$ denotes the following sign function

$$sgn(e) = \begin{cases} 1; & \text{if } e > 0 \\ -1; & \text{if } e < 0 \\ 0; & \text{if } e = 0 \end{cases}.$$

Substituting the controller $u_1(t)$ into the system dynamics (1) yields

$$\dot{e} = f(x,t,\theta) - sgn(e) \cdot \rho(|x|, t) - ke \tag{14}$$

Define the Lyapunov function as follows

$$V = \frac{1}{2} e^2 \geqslant 0 \tag{15}$$

Taking the time derivative of (15) yields

$$\begin{aligned} \dot{V} &= e\dot{e} \\ &= -ke^2 + ef(x,t,\theta) - e \cdot sgn(e) \cdot \rho(|x|, t) \\ &= -ke^2 + ef(x,t,\theta) - |e| \cdot \rho(|x|, t) \\ &\leq -ke^2 + |e| |f(x,t,\theta)| - |e| \cdot \rho(|x|, t) \\ &= -ke^2 + |e| \cdot [|f(x,t,\theta)| - \rho(|x|, t)] \\ &\leq -ke^2 = -2kV \leq 0 \end{aligned} \tag{16}$$

## 2.1 Signal Chasing

Based on (15) and (16), we know that $e(t) \in \mathcal{L}_\infty \Longrightarrow x(t) \in \mathcal{L}_\infty \Longrightarrow \rho(x,t) \in \mathcal{L}_\infty \Longrightarrow u_1(t), u(t) \in \mathcal{L}_\infty$; hence $\dot{e}(t), \dot{x}(t) \in \mathcal{L}_\infty \Longrightarrow$all the signals in the closed-loop operation remain bounded.

## 2.2 Error Tracking

Solving the inequality of (16) yields

$$V(t) \leq V_0 e^{-2kt} \tag{17}$$

where $V_0$ is the following initial condition of $V(t)$

$$V_0 = \frac{1}{2} e_0^2 \tag{18}$$

with $e_0$ being the initial error. It can then be obtaind from (15) and (17) that

$$|e(t)| \leq |e_0| e^{-kt}$$

Hence, the tracking error $e(t)$ converges to zero exponentially fast.

**Result:** Exponentially Tracking

## 2.3 Chattering of the Sliding Mode Cotrol

The sliding mode control of (13) contains a sliding term $sgn(e) \cdot \rho(|x|, t)$ with $\rho(|x|, t)$ being defined as a bounding function of the uncertain nonlinear term $f(x, t, \theta)$, when the tracking error $e(t)$ is around the zero point, this term dominates in the controller of (13) (Note that when $e(t) \to 0$, the feedback term $ke$ in the controller goes to zero as well). Subsequently, when $e(t)$ oscillates around the zero point, the controller $u_1(t)$ will make changes between $\rho(|x_d(t)|, t)$and $-\rho(|x_d(t)|, t)$ frequently. To make it worse, the bounding function of $\rho(|x|, t)$ is usually of large amplitude. This phenominon is often referred as chattering. This drawback prevents the sliding mode controller from being applied to practical problems. To solve or weaken this problem, the high frequency feedback of robust control is proposed.

# 3 Camparison of Adaptive Control vs Sliding Mode Control

## 3.1 Adaptive Control

Assumption Needed: LP;

Control Result: Asymptotic Tracking;

Other Factors: System Parameters Identification: PE Condition Needed.

## 3.2 Sliding Mode Control

Assumption Needed: Uncertain Nonlinear Term Be Upper Bounded by a Known Function;

Control Result: Exponential Tracking;

Other Factors: Chattering of the Sliding Mode Contol;

Choose $\rho(|x|, t)$ as small as possible to decrease control input.