Chapter 5. Lyapunov-Based Adaptive Control

1 General Nonlinear System

In the subsequent analysis, we will attack the control problem of the following nonlinear system

$$\dot{x} = -f(x,t,\theta) + g(x,t) \cdot u + \rho(t) \tag{1}$$

where $\theta \in \Re^n$ denotes unknown constant system parameters, u is the control input to the system, $f(x,t,\theta)$ represents some unknown nonlinear function, $\rho(t) \in \mathcal{L}_\infty$ represents some bounded known nonlinearity, while g is assumed to be known and satisfies the following condition

$$|g(x,t)| \geqslant g_0 > 0 \tag{2}$$

with g_0 being a positive constant.

Question: Why do we need the assumption of (2)?

Error

$$e = x_d - x \tag{3}$$

where x_d, $\dot{x}_d \in \mathcal{L}_\infty$, then

$$\begin{aligned} \dot{e} &= \dot{x}_d - \dot{x} \\ &= \dot{x}_d + f(x,t,\theta) - g(x,t) \cdot u - \rho(t) \end{aligned} \tag{4}$$

for the convenience of subsequent analysis, we define the transformed control u_1 as follows

$$u_1 = g(x,t) \cdot u - \dot{x}_d + \rho(t) \tag{5}$$

then

$$\dot{e} = f(x,t,\theta) - u_1 \tag{6}$$

It is easy to see from (5) that the real control u can be easily obtained from the transformed control u_1

$$u = \frac{u_1 + \dot{x}_d - \rho(t)}{g(x,t)} \tag{7}$$

Note that as the denominator $g(x,t) \geqslant g_0 > 0$ and $\rho(t) \in \mathcal{L}_\infty$, then $u_1 \in \mathcal{L}_\infty \Longleftrightarrow u \in \mathcal{L}_\infty$.

2 Adaptive Control

Assume that the system nonlinearity can be linear parameterized (LP) as follows

$$f(x,t,\theta) = Y(x,t)\,\theta \tag{8}$$

where $Y(x,t)$ is measurable and, if $x(t), \dot{x}(t) \in \mathcal{L}_\infty$ then $Y(x,t), \dot{Y}(x,\dot{x},t) \in \mathcal{L}_\infty$, θ is the unknown constant system parameters defined in (1).

Design the controller as follows

$$u_1 = Y(x,t)\,\hat{\theta} + ke \tag{9}$$

where $\hat{\theta}(t)$ represents the yet-to-generate on-line estimation of θ, k is a positive control gain. Substituting the controller $u_1(t)$ into the system dynamics (6) yields

$$\dot{e} = Y(x,t)\,\tilde{\theta} - ke \tag{10}$$

where $\tilde{\theta}(t)$ denotes the following parameter estimation error

$$\tilde{\theta} = \theta - \hat{\theta} \tag{11}$$

then

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}} \tag{12}$$

Define

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \geqslant 0 \tag{13}$$

Γ is some positive definite, diagonal parameter update matrix. Taking the time derivative of (13) yields

$$\begin{aligned} \dot{V} &= e\dot{e} - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &= -ke^2 + eY(x,t)\,\tilde{\theta} - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &= -ke^2 + \tilde{\theta}^T Y^T(x,t)\,e - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &= -ke^2 + \tilde{\theta}^T\left[Y^T(x,t)\,e - \Gamma^{-1}\dot{\hat{\theta}}\right] \end{aligned} \tag{14}$$

Question: In the second step above, we taken the transpose of the second term $eY(x,t)\,\tilde{\theta}$, **why** and **when** can we do this?

According to the final expression of $\dot{V}(t)$ presented in (14), we design the update law as follows

$$\dot{\hat{\theta}} = \Gamma Y^T(x,t)\,e \tag{15}$$

then

$$\dot{V} = -ke^2 \leq 0. \tag{16}$$

Question: Can you directly guess out the update law of the system parameters based on the dynamics $\dot{e}(t)$ of (10)?

2.1 Signal Chasing

Based on (13) and (16), we know that $e(t), \tilde{\theta}(t) \in \mathcal{L}_\infty \Longrightarrow x(t), \hat{\theta}(t) \in \mathcal{L}_\infty \Longrightarrow Y(x,t) \in \mathcal{L}_\infty \Longrightarrow u_1(t), u(t) \in \mathcal{L}_\infty$; hence, $\dot{e}(t), \dot{x}(t) \in \mathcal{L}_\infty \Longrightarrow \dot{Y}(x,\dot{x},t) \in \mathcal{L}_\infty \Longrightarrow$all the signals in the closed-loop operation remain bounded.

2.2 Error Tracking

Define

$$f(t) = ke^2 \tag{17}$$

then based on the facts that

$$V \geqslant 0 \tag{18}$$

and

$$\dot{V} \leq -f(t),\ f(t) \geqslant 0$$

and

$$\dot{f}(t) = 2ke\dot{e} \in \mathcal{L}_\infty \Longrightarrow f(t) \text{ is UC}$$

Lemma 14 can be utilized to conclude that

$$f(t) \to 0, \text{ i.e., } e(t) \to 0 \tag{19}$$

After some mathematical manipulation, we can obtain

$$\begin{aligned}
\ddot{e} &= \dot{Y}(x,\dot{x},t)\,\tilde{\theta} + Y(x,t)\,\dot{\tilde{\theta}} - k\dot{e} \\
&= \dot{Y}(x,\dot{x},t)\,\tilde{\theta} - Y(x,t)\,\Gamma Y^T(x,t)\,e - k\left[Y(x,t)\,\tilde{\theta} - ke\right]
\end{aligned}$$

thus $\ddot{e}(t) \in \mathcal{L}_\infty \Longrightarrow \dot{e}(t)$ is UC. Based on the previous fact and the equation of (19), we can then conclude that

$$\dot{e} \to 0. \tag{20}$$

Question: Why? Which Lemma do we utilize to make the previous conclusion? Extended Barbalat's Lemma.

It can then be easily seen from (10), (19) and (20) that

$$Y(x,t)\,\tilde{\theta} \to 0 \tag{21}$$

2.3 PE (Persistent Excitation) Condition

As we explained before, there is no guarantee that the parameter estimation error $\tilde{\theta}(t)$ will converge to zero! However, under the following PE condition, the fact is true.

Definition 1 *PE Condition: A signal $\omega(t) \in \Re^n$ is said to be of persistent excitation if there exists positive constants α_1, α_2 and δ, such that*

$$\alpha_2 I_n \geqslant \int_{t_0}^{t_0+\delta} \omega(\tau)\,\omega^T(\tau)\,d\tau \geqslant \alpha_1 I_n, \text{ for } \forall t_0 \geqslant 0 \tag{22}$$

where I_n denotes an $n \times n$ matrix.

Remark 1 *For square matrices A and B, if $A - B$ is positive definite, then we say $A \geqslant B$.*

Suppose $Y(x,t)$ satisfies the PE condition as follows

$$\alpha_2 I_n \geqslant \int_{t_0}^{t_0+\delta} Y(\tau) Y^T(\tau)\, d\tau \geqslant \alpha_1 I_n, \text{ for } \forall t_0 \geqslant 0, \tag{23}$$

then based on (21), we know that

$$\left[Y(x,t)\tilde{\theta}\right]^T \left[Y(x,t)\tilde{\theta}\right] \to 0 \Longrightarrow \tilde{\theta}^T \left[Y^T(x,t) Y(x,t)\right] \tilde{\theta} \to 0 \tag{24}$$

from (23) and (24), and after some mathematical analysis (beyond the scope of this course), we can prove that

$$\tilde{\theta} \to 0. \tag{25}$$

Remark 2 *Note that $Y(x,t)$ depends on the system state x which makes the condition of (23) hard to testify. Based on the thought that if the tracking error converges, then $x \to x_d$, a solution to make the PE condition possible to testify is to redesign the controller (9) as follows*

$$u_1 = Y(x_d,t)\hat{\theta} + ke + \chi(e)$$

where in the feedforward term $Y(x_d,t)\hat{\theta}$, x_d has taken the place of x. Therefore, the corresponding PE condition of (23) depends on the desired trajectory x_d, instead of x, it is thus possible to check whether the PE condition holds or not.

3 An Adaptive Control Design Example (Brush DC Motor)

3.1 System Dynamics

The mechanical subsystem dynamics for a brush dc motor are assumed to be of the form

$$M\ddot{q} + B\dot{q} + N\sin(q) = I \tag{26}$$

while the electrical subsystem dynamics are assumed to be

$$L\dot{I} = v - RI - K_B\dot{q} \tag{27}$$

where M, B, N, R, L, K_B denote the system parameters, $q(t), \dot{q}(t), \ddot{q}(t)$ represent angular position, angular velocity and angular acceleration of the motor, respectively, $I(t)$ is the rotor current, and $v(t)$ is the input control voltage.

3.2 Adaptive Controller Design and Analysis

Given full state measurement (*i.e.*, q, $\dot{q}$, and I), the control objective is to design $v(t)$ to make the following position tracking error $e(t)$ converges to zero

$$e = q_d - q \tag{28}$$

where $q_d(t)$ represents the desired position trajectory, and $q_d(t)$, $\dot{q}_d(t), \ddot{q}_d(t) \in \mathcal{L}_\infty$. To simplify the control formulation and the stability analysis, we define the filtered tracking error $r(t)$ as

$$r = \dot{e} + \alpha e \tag{29}$$

where α is a positive, constant controller gain.

Remark 3 *Using of the filtered tracking error $r(t)$ allows us to analyze the second-order dynamics of (26) as though it is a first-order system.*

Why?It is easy to show that:

$$\text{if } r(t) \to 0, \text{then } e\left(t\right), \dot{e}\left(t\right) \to 0; \tag{30}$$

$$\text{if } r(t) \to 0 \text{ exp fast}, \text{then } e\left(t\right), \dot{e}\left(t\right) \to 0 \text{ exp fast} \tag{31}$$

$$\text{if } r(t) \text{ GUUB}, \text{then } e\left(t\right), \dot{e}\left(t\right) \text{ GUUB} \tag{32}$$

prove these facts yourself.

We differentiate (29) with respect to time and rearrange terms to yield

$$\dot{r} = \ddot{e} + \alpha\dot{e} = (\ddot{q}_d + \alpha\dot{e}) - \ddot{q}. \tag{33}$$

Multiplying (33) by M and substituting the mechanical subsystem dynamics of (26) yields the filtered tracking error dynamics as shown

$$M\dot{r} = M(\ddot{q}_d + \alpha\dot{e}) + B\dot{q} + N\sin(q) - I. \tag{34}$$

And the right-hand side of (34) can be rewritten as

$$M\dot{r} = W_\tau\theta_\tau \;\; - \;\; I \tag{35}$$

where the known matrix $W_\tau(q, \dot{q}, t) \in \Re^{1\times 3}$ is given by

$$W_\tau = \left[\begin{array}{ccc} \ddot{q}_d + \alpha\dot{e} & \dot{q} & \sin(q) \end{array}\right] \tag{36}$$

and the parameter vector $\theta_\tau \in \Re^3$ is defined as follows given

$$\theta_\tau = \left[\begin{array}{ccc} M & B & N \end{array}\right]^T. \tag{37}$$

Considering the structure of the electromechanical systems given by (26) and (27), we are only free to specify the rotor voltage $v(t)$. In other words, the mechanical subsystem error dynamics lack a true current (torque) level control input. For this reason, we shall add and subtract the desired current trajectory $I_d(t)$ to the right-hand side of (35), as shown

$$M\dot{r} = W_\tau\theta_\tau - I_d + \eta_I \tag{38}$$

where $\eta_I(t)$ represents the current tracking error perturbation to the mechanical subsystem dynamics of the form

$$\eta_I = I_d \;\; - \;\; I. \tag{39}$$

We then devide the design process into two steps. The first step in the procedure is to design an adaptive desired current trajectory $I_d(t)$ for the mechanical dynamics of (38) to make $r\left(t\right) \to 0$ when the current mismatch η_I does not exist (i.e., assume I_d as the virtual control input to (38) and the current mismatch η_I is assumed to be neglectable in this step). The second step is to design the real control voltage $v(t)$ for the electrical subsystem (27) so that $\eta_I \to 0$. This design process is often referred as backstepping.

For the first step, we select $I_d(t)$ as follows

$$I_d = W_\tau \hat{\theta}_\tau + k_s r \tag{40}$$

where $W_\tau(q, \dot{q}, t)$ was defined in (36), $\hat{\theta}_\tau(t) \in \Re^3$ represents a dynamic estimate for the unknown parameter vector θ_τ defined in (37), and k_s is a positive, constant controller gain. The parameter estimate $\hat{\theta}_\tau(t)$ defined in (40) is updated online according to the following adaptation law

$$\dot{\hat{\theta}}_\tau = \Gamma_\tau W_\tau^T r \tag{41}$$

where $\Gamma_\tau \in \Re^{3\times 3}$ is a constant, positive definite, diagonal adaptive gain matrix. Defining the mismatch between $\hat{\theta}_\tau(t)$ and θ_τ as

$$\tilde{\theta}_\tau = \theta_\tau - \hat{\theta}_\tau, \tag{42}$$

allows the time derivative of the parameter observation error to be written in terms of the adaptation law of (41) as

$$\dot{\tilde{\theta}}_\tau = -\Gamma_\tau W_\tau^T r. \tag{43}$$

Substituting (40) into the open-loop dynamics of (38) yields the closed-loop filtered tracking error dynamics, as shown

$$M\dot{r} = W_\tau \tilde{\theta}_\tau - k_s r + \eta_I. \tag{44}$$

From (44) and (43), we know that if η_I disappears in (44), then $r(t) \to 0$. To show that, we choose the following Lyapunov function

$$V_1 = \frac{1}{2} M r^2 + \frac{1}{2} \tilde{\theta}_\tau^T \Gamma_\tau^{-1} \tilde{\theta}_\tau$$

then

$$\begin{aligned} \dot{V}_1 &= M r \dot{r} - \tilde{\theta}_\tau^T \Gamma_\tau^{-1} \dot{\hat{\theta}}_\tau \\ &= \left[-k_s r^2 + r W_\tau \tilde{\theta}_\tau + r\eta_I \right] - \tilde{\theta}_\tau^T W_\tau^T r \\ &= -k_s r^2 + r\eta_I \end{aligned}$$

Now that we have designed the adaptive desired current trajectory $I_d(t)$, our next step is to design the control voltage $v(t)$ for the electrical subsystem to make $\eta_I(t) \to 0$. To do that, we first need to obtain the open-loop dynamics of current tracking error $\eta_I(t)$

$$\begin{aligned} L\dot{\eta}_I &= L\dot{I}_d - L\dot{I} \\ &= L\dot{I}_d + RI + K_B \dot{q} - v \end{aligned} \tag{45}$$

The term $\dot{I}_d(t)$ is calculated as follows

$$\dot{I}_d = \dot{W}_\tau \hat{\theta}_\tau + W_\tau \dot{\hat{\theta}}_\tau + k_s \dot{r}. \tag{46}$$

Substituting (33), and the time derivatives of (36) and (41) into the right-hand side of (46) yields

$$\begin{aligned} \dot{I}_d = \; & \hat{M}\left(\dddot{q}_d + \alpha\left(\ddot{q}_d - \ddot{q}\right)\right) + \hat{B}\ddot{q} + \hat{N}\dot{q}\cos(q) \\ & + W_\tau \Gamma_\tau W_\tau^T r + k_s\left(\ddot{q}_d - \ddot{q} + \alpha\dot{e}\right) \end{aligned} \tag{47}$$

where $\hat{M}(t)$, $\hat{B}(t)$, and $\hat{N}(t)$ denote the scalar components of the vector $\hat{\theta}_\tau(t)$ (*i.e.*, $\hat{\theta}_\tau = \begin{bmatrix} \hat{M} & \hat{B} & \hat{N} \end{bmatrix}^T$). Note that $\dot{I}_d(t)$ of (47) is in terms of measurable states (*i.e.*, $q(t)$ and $\dot{q}(t)$), known functions, and the unmeasurable quantity $\ddot{q}(t)$.

Question: In the subsequent design for the control voltage $v(t)$, we need to cancel out the dynamics of $L\dot{I}_d$. How to deal with the terms containing the unmeasurable $\ddot{q}(t)$? Can we possibly rewrite them in terms of measurable quatities?

After substituting the following relationship for $\ddot{q}(t)$ into (47)

$$\ddot{q} = \frac{1}{M}I - \frac{B}{M}\dot{q} - \frac{N}{M}\sin(q) \tag{48}$$

we can write $\dot{I}_d(t)$ in terms of measurable states (*i.e.*, $q(t)$, $\dot{q}(t)$, and $I(t)$), known functions, and unknown constant parameters. Substituting this expression for $\dot{I}_d(t)$ into (45) and then performing the necessary algebra yields a linear parameterized open-loop model of the form

$$L\dot{\eta}_I = W_1\theta_1 - v \tag{49}$$

where the known regression matrix $W_1(q, \dot{q}, I, \hat{\theta}_\tau, t) \in \Re^{1\times 6}$ and the unknown constant parameter vector $\theta_1 \in \Re^6$ are explicitly defined as follows

$$\theta_1 = \begin{bmatrix} \frac{L}{M} & \frac{LB}{M} & R & K_B & \frac{LN}{M} & L \end{bmatrix}^T, \tag{50}$$

$$W_1 = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} \end{bmatrix}, \tag{51}$$

$$W_{11} = \hat{B}I - k_sI - \alpha\hat{M}I, \qquad W_{12} = k_s\dot{q} - \hat{B}\dot{q} + \alpha\hat{M}\dot{q},$$

$$W_{13} = I, \quad W_{14} = \dot{q}, \quad W_{15} = k_s\sin(q) - \hat{B}\sin(q) + \alpha\hat{M}\sin(q),$$

and

$$W_{16} = \hat{M}\dddot{q}_d + \alpha\hat{M}\ddot{q}_d + W_\tau\Gamma_\tau W_\tau^T r + k_s\ddot{q}_d + k_s\alpha\dot{e} + \hat{N}\dot{q}\cos(q).$$

The second step in the design procedure involves the design of the voltage control input $v(t)$ for the open-loop system of (49). Given the structure of (49) and (44), we define the input voltage controller as

$$v = W_1\hat{\theta}_1 + k_e\eta_I + r \tag{52}$$

where k_e is a positive, constant control gain, and $\hat{\theta}_1(t) \in \Re^6$ is a dynamic estimate of the unknown parameter vector θ_1, the third term $r\,(t)$ is an interconnecting term to cancel out the corresponding term in the dynamics of $\dot{r}\,(t)$ (specifically, the last term in (44)). The parameter estimates are updated on-line by the following adaptation law

$$\hat{\theta}_1 = \int_0^t \Gamma_e W_1^T(\sigma)\eta_I(\sigma)\, d\sigma \tag{53}$$

where $\Gamma_e \in \Re^{6\times 6}$ is a constant positive definite, diagonal adaptive gain matrix. If we define the mismatch between θ_1 and $\hat{\theta}_1(t)$ as

$$\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1, \tag{54}$$

then the adaptation law of (53) can be written in terms of the parameter error as

$$\dot{\tilde{\theta}}_1 = -\Gamma_e W_1^T \eta_I. \tag{55}$$

Substituting (52) into the open-loop dynamics of (49) yields the closed-loop current tracking error dynamics in the form

$$L\dot{\eta}_I = W_1\tilde{\theta}_1 - k_e\eta_I - r. \tag{56}$$

The dynamics given by (43), (44), (55), and (56) represent the electromechanical closed-loop system for which the stability analysis is performed while the adaptive controller given by (40), (41), (52), and (53) represents the controller which is implemented at the voltage terminals of the motor. The theorem given below delineates the performance of the closed-loop system under the proposed control.

Theorem 2 *The proposed adaptive controller ensures that the filtered tracking error goes to zero asymptotically for the electromechanical dynamics of (26) and (27) as shown*

$$\lim_{t\to\infty} r(t) = 0. \tag{57}$$

Proof. *First, define the following non-negative function*

$$V = V_1 + \frac{1}{2}L\eta_I^2 + \frac{1}{2}\tilde{\theta}_1^T\Gamma_e^{-1}\tilde{\theta}_1 \geqslant 0. \tag{58}$$

Taking the time derivative of (58) with respect to time yields

$$\dot{V} = \dot{V}_1 + \eta_I L\dot{\eta}_I + \tilde{\theta}_1^T\Gamma_e^{-1}\dot{\tilde{\theta}}_1 \tag{59}$$

where the facts that: i) scalars can be transposed and ii) Γ_τ, Γ_e are diagonal matrices, have been used. Substituting the error dynamics of (43), (55), (56), and (44) into (59) yields

$$\begin{aligned} \dot{V} &= \left[-k_s r^2 + r\eta_I\right] + \eta_I\left[W_1\tilde{\theta}_1 - k_e\eta_I - r\right] - \tilde{\theta}_1^T W_1^T\eta_I \\ &= -k_s r^2 - k_e\eta_I^2 + \left[\eta_I W_1\tilde{\theta}_1 - \tilde{\theta}_1^T W_1^T\eta_I\right] \\ &= -k_s r^2 - k_e\eta_I^2 \leq 0. \end{aligned} \tag{60}$$

Therefore, $V(t)\in L_\infty \Longrightarrow V_1(t)\in L_\infty$, $r(t)\in L_\infty$, $\tilde{\theta}_\tau(t)\in L_\infty$, $\eta_I(t)\in L_\infty$, $\tilde{\theta}_1(t)\in L_\infty \Longrightarrow e(t)\in L_\infty$, $\dot{e}(t)\in L_\infty$, $\hat{\theta}_\tau(t)\in L_\infty$, $\hat{\theta}_1(t)\in L_\infty \Longrightarrow q(t)\in L_\infty$, $\dot{q}(t)\in L_\infty \Longrightarrow I_d(t)\in L_\infty, I(t)\in L_\infty \Longrightarrow v(t)\in L_\infty, W_1(q,\dot{q},I,\hat{\theta}_\tau,t)\in L_\infty, W_\tau(q,\dot{q},t)\in L_\infty \Longrightarrow \dot{I}_d(t)\in L_\infty, \dot{W}_1\in L_\infty, \dot{W}_\tau\in L_\infty \Longrightarrow \dot{r}(t)\in L_\infty$ and $\dot{\eta}_I(t)\in L_\infty \Longrightarrow$All the signals during the closed-loop operation are bounded.

Based on the fact that $\dot{r}(t)\in L_\infty$ and $\dot{\eta}_I(t)\in L_\infty$, together with the expressions of (58) and (60), it is then easy to show the conclusion of (57). **How? Which Lemma to utilize?**

3.3 Reduction of Overparameterization

From the dynamics given by (26) and (27), we can see that there are only six parameters; however, in the adaptive controller presented above, there are nine adaptive update laws. That is, $\hat{\theta}_\tau(t)\in\Re^3$ of (41) contains three dynamic estimates while $\hat{\theta}_1(t)\in\Re^6$ of (53) contains six dynamic estimates. This is a very common problem associated with adaptive control, it is often referred as overparameterization. We can make a transformation for the states and modify the corresponding adaptive controller to reduce this overparameterization problem (not in details here).

3.4 What have we learned from this example?

1. Introduce a filtered tracking error $r(t)$ to convert a second-order system into a first-order system to simplify the controller design;

2. Backstepping design proceduce;

3. Overparameterizaton problem.