# Matrix Function[1][1]

**Definition 3**

Let $F : S \to \mathbf{R}^{m\times p}$ be a matrix function defined on a set $S$ in $\mathbf{R}^{n\times q}$. Let $C$ be an interior point of $S$, and let $B(C;r) \subset S$ be a ball with centre $C$ and radius $r$ (also called a neighbourhood of $C$ and denoted $N(C)$. Let $U$ be a point in $\mathbf{R}^{n\times q}$ with $\|U\| < r$, so that $C + U \in B(C;r)$. If there exists a real $mp \times nq$ matrix $A$, depending on $C$ but not on $U$, such that

$$\operatorname{vec} F(C+U) = \operatorname{vec} F(C) + A(C)\operatorname{vec} U + \operatorname{vec} R_C(U) \tag{2}$$

for all $U \in \mathbf{R}^{n\times q}$ with $\|U\| < r$ and

$$\lim_{U\to 0} \frac{R_C(U)}{\|U\|} = 0, \tag{3}$$

then the function $F$ is said to be *differentiable at* $C$. The $m \times p$ matrix $\mathrm{d}F(C;U)$ defined by

$$\operatorname{vec} \mathrm{d}F(C;U) = A(C)\operatorname{vec} U \tag{4}$$

is then called the *(first) differential of* $F$ *at* $C$ *with increment* $U$ and the $mp \times nq$ matrix $A(C)$ is called the *(first) derivative of* $F$ *at* $C$.

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[1] Magnus, Jan R.; Neudecker, Heinz (1999). Matrix differential calculus with applications in statistics and econometrics

# Matrix Function[2]

In view of Definition 3, all calculus properties of matrix functions follow immediately from the corresponding properties of vector functions because, instead of the matrix function $F$, we can consider the vector function $f : \operatorname{vec} S \to \mathbf{R}^{mp}$ defined by

$$f(\operatorname{vec} X) = \operatorname{vec} F(X). \tag{7}$$

It is easy to see from (2) and (3) that the differentials of $F$ and $f$ are related by

$$\operatorname{vec} \mathsf{d}F(C; U) = \mathsf{d}f(\operatorname{vec} C; \operatorname{vec} U). \tag{8}$$

We then define the *Jacobian matrix of $F$ at $C$* as

$$\mathsf{D}F(C) = \mathsf{D}f(\operatorname{vec} C). \tag{9}$$

This is an $mp \times nq$ matrix, whose $ij$-th element is the partial derivative of the $i$-th component of $\operatorname{vec} F(X)$ with respect to the $j$-th element of $\operatorname{vec} X$, evaluated at $X = C$.

[2] Magnus, Jan R.; Neudecker, Heinz (1999). Matrix differential calculus with applications in statistics and econometrics

$$f(\xi) = \boldsymbol{x}(\xi)^T \boldsymbol{C}(\xi)^{-1} \boldsymbol{x}(\xi)$$

$$\xi \in \mathbb{R}^6, \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{C} \in \mathbb{R}^{3\times 3}$$

$$\frac{\partial f(\xi)}{\partial \xi} = \left(\mathrm{D}\boldsymbol{C}^{-1}\right)^T \cdot \operatorname{vec} \boldsymbol{x}\boldsymbol{x}^T + \frac{\partial \boldsymbol{x}}{\partial \xi}^T \left(\boldsymbol{C} + \boldsymbol{C}^T\right) \boldsymbol{x}$$

## 边缘点协方差${}^{c}\boldsymbol{C}_{pk}$的估计

取${}^{c}\boldsymbol{p}_k$邻域内的边缘点，拟合协方差${}^{c}\boldsymbol{C}_{pk}$，假设其特征值$\lambda_{p1},\lambda_{p2},\lambda_{p2}$，及对应特征向量$\boldsymbol{u}_{p1},\boldsymbol{u}_{p2},\boldsymbol{u}_{p2}$。

$$
{}^{c}\boldsymbol{C}_{pk}^{-1}=\frac{1}{\lambda_{p1}}\boldsymbol{u}_{p1}\boldsymbol{u}_{p1}^{T}+\frac{1}{\lambda_{p2}}\boldsymbol{u}_{p2}\boldsymbol{u}_{p2}^{T}+\frac{1}{\lambda_{p3}}\boldsymbol{u}_{p3}\boldsymbol{u}_{p3}^{T} \tag{1}
$$

对于边缘上的点，有$\lambda_{p1}\gg\lambda_{p2}\geq\lambda_{p2}$，即$\frac{1}{\lambda_{p3}}\geq\frac{1}{\lambda_{p2}}\gg\frac{1}{\lambda_{p1}}$。则

$$
\begin{aligned}
J_{pk}(\xi)&=\frac{1}{2}\left({}^{c}\boldsymbol{p}_k-T_{cr}\left({}^{r}\boldsymbol{p}_k,\xi\right)\right)^{T}{}^{c}\boldsymbol{C}_{pk}^{-1}\left({}^{c}\boldsymbol{p}_k-T_{cr}\left({}^{r}\boldsymbol{p}_k,\xi\right)\right)\\
&\approx\frac{1}{2}\left({}^{c}\boldsymbol{p}_k-T_{cr}\left({}^{r}\boldsymbol{p}_k,\xi\right)\right)^{T}\left(\frac{1}{\lambda_{p2}}\boldsymbol{u}_{p2}\boldsymbol{u}_{p2}^{T}+\frac{1}{\lambda_{p3}}\boldsymbol{u}_{p3}\boldsymbol{u}_{p3}^{T}\right)\left({}^{c}\boldsymbol{p}_k-T_{cr}\left({}^{r}\boldsymbol{p}_k,\xi\right)\right)
\end{aligned} \tag{2}
$$

只有沿${}^{c}\boldsymbol{p}_k$所在边缘的垂直方向上的运动才会导致$J_{pk}(\xi)$变化。

$$J_{pk}(\xi) = \frac{1}{2}\left({}^{c}\boldsymbol{p}_k - {}^{r}\boldsymbol{p}_k\right)^T \boldsymbol{C}_{\nu pk}^{-1}\left({}^{c}\boldsymbol{p}_k - {}^{r}\boldsymbol{p}_k\right))$$

$$\begin{aligned}
\boldsymbol{C}_{\nu pk} &= {}^{c}\boldsymbol{C}_{pk} + {}^{r}\boldsymbol{C}_{pk} \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left({}^{c}\boldsymbol{p}_{pi} - {}^{c}\mu_p\right)\left({}^{c}\boldsymbol{p}_{pi} - {}^{c}\mu_p\right)^T + \frac{1}{N-1}\sum_{j=1}^{N}\left({}^{r}\boldsymbol{p}_{pj} - {}^{r}\mu_p\right)\left({}^{r}\boldsymbol{p}_{pj} - {}^{r}\mu_p\right)^T \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N} {}^{c}\bar{\boldsymbol{p}}_{pi}\,{}^{c}\bar{\boldsymbol{p}}_{pi}^T + \sum_{j=1}^{N} {}^{r}\bar{\boldsymbol{p}}_{pj}\,{}^{r}\bar{\boldsymbol{p}}_{pj}^T\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{2N}\bar{\boldsymbol{p}}_{pi}\bar{\boldsymbol{p}}_{pi}^T
\end{aligned}$$

[2]

J. R. Magnus and H. Neudecker, *Matrix differential calculus with applications in statistics and econometrics /*. Wiley,, 1988.

A. Knutson and T. Tao, “Honeycombs and sums of hermitian matrices,” *Notices of the American Mathematical Society*, vol. 48, no. 2, pp. 175–186, 2000.