Trading algorithms with learning in latent alpha models[☆]

Work In Progress

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Abstract

Alpha signals for statistical arbitrage strategies are often driven by latent factors. This paper analyses how to optimally trade with latent factors that cause prices to jump and diffuse. Moreover, we account for the effect of the trader's actions on quoted prices and the prices they receive from trading. Under fairly general assumptions, we demonstrate how the trader can learn the posterior distribution over the latent states, and explicitly solve the latent optimal trading problem. To illustrate the efficacy of the optimal strategy, we demonstrate its performance through simulations and compare it to strategies which ignore learning in the latent factors.

1. Introduction

The phrase "All models are wrong, but some are useful" (Box, 1978) rings true across all areas in finance, and intraday trading is no exception. If an investor wishes to efficiently trade assets, she must use a strategy that can anticipate the asset's price trajectory while simultaneously being mindful of the flaws in her model, as well as the costs borne from transaction fees and her own impact on prices. With all of the complexities in intraday markets, it is no surprise that strategies differ substantially based on what assumptions are made about asset price dynamics. Trading with an incorrect model can be very costly to an investor, and therefore being able to mitigate model risk is valuable.

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The availability of information at very high frequencies can help a trader partially overcome the problem of model selection. By making use of the information provided by realized trajectories of the asset price and the incoming flow of orders of other traders, she can infer which model best fits the observed data and in turn use it to predict future movements in asset prices. Ideally, the trader should be able to do incorporate this information in an on-line manner. In other words, the trader should be continuously updating her model as she observes new information, keeping in mind that the market may switch between a number of regimes over the course of the trading period. Furthermore, the trader would like to have some means of incorporating a-priori knowledge about markets into her trading strategy before beginning to trade.

This paper studies the optimal trading strategy for a single asset when there are latent alpha components to the asset price dynamics, and where the trader uses price information to learn about the latent factor. Prices can diffusive as well as jump. The trader's goal is to optimally trade subject to this model uncertainty, and end the trading horizon with zero inventory. By treating the trader's problem as a continuous time control problem where information is partially obscured, we succeed in obtaining a closed form strategy, up to the computation of an expectation that is specific the trader's prior assumptions on the model dynamics. The optimal trading strategy we find can be computed with ease for a wide variety of models, and we demonstrate its performance by comparing, in simulation, with approaches that that do not make use of learning.

A number of papers have studied some related problems with partial information. For example, Bäuerle and Rieder (2005), Bäuerle and Rieder (2007) and Frey et al. (2012) study model uncertainty in the context of portfolio optimization and the optimal allocation of assets. Ekstrom and Vaicenavicius (2016) investigates the optimal timing problem associated with liquidating a single unit of an asset when the asset price is a geometric Brownian motion with random (unobserved) drift. Colaneri et al. (2016) studies the optimal liquidation problem when the asset midprice is driven by a Poisson random measure with unknown mean-measure. Gârleanu and Pedersen (2013) study the optimal trading strategy for maximizing the discounted, and penalized, future expected excess returns in a discrete-time, infinite-time horizon problem. In their model, prices contain an unpredictable martingale component, and an independent stationary (visible) predictable component – the alpha component.

The structure of the remainder of this paper is as follows. Section 2 outlines our modelling assumptions, as well as providing the optimization problem with partial information that the trader wishes to solve. Section 3 provides the filter which the trader uses to make proper inference on the underlying model driving the data she is observing. Section 4 shows that the original optimization problem presented in section 2 can be simplified to an optimization problem with complete information using the filter presented in Section 3 . Section 5, shows how to solve the reduced optimization problem from section 4 and verifies that the resulting strategy indeed solves the original optimization problem. Lastly, section 6 provides some numerical examples by applying the theory to a few specific models, and compares the resulting strategy, using simulations, to an alternative which does not learn from price dynamics.

2. The Model

Consider the filtered probability space $(\Omega, \mathcal{F}, \mathfrak{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}, \mathbb{P})$, where $T > 0$ is some fixed time horizon. The filtration $\mathcal{F}_t$ is said to be the natural filtration generated by the paths of the un-impacted asset midprice process F_t and the counting processes for the number of market orders coming into buy and sell sides of the limit order book, N_t^+ and N_t^-. These processes will be defined in more detail in the remainder of the section.

An investor must decide how she wishes to either buy or sell a given asset over the course of some fixed time period, ending at a time horizon T. The speed at which the investor buys or sells the asset at a given time $t \in [0, T]$ is denoted by the process $\nu = (\nu_t)_{\{0 \leq t \leq T\}}$. The trader's remaining inventory, given some strategy ν, is denoted as $Q^\nu = (Q_t^\nu)_{\{0 \leq t \leq T\}}$. It is assumed that the trader begins the trading period with $\mathfrak{N}$ units of the asset already present in her inventory. The relationship between her inventory and her trading decision it is defined by

$$dQ_t^\nu = \nu_t \, dt, \qquad Q_0^\nu = \mathfrak{N} \, . \tag{2.1}$$

The above relationship is telling us that for a fixed strategy ν, the investor will choose to buy $\nu_t \epsilon$ units of the asset over the period $[t, t+\epsilon]$. A negative value for ν_t implies that the trader is selling the asset and vice versa. The rate at which the investor buys or sells the asset also affects prices through two mechanisms. The first is a temporary price impact, which has a negative effect on the sale or purchase price of her assets proportional to the rate at which she trades. Secondly, the investor can also permanently impact the asset midprice through her trading.

It is assumed that other market participants also have a permanent impact on the asset midprice through their own buy and sell market orders (MOs). The processes N_t^+ and N_t^- are doubly stochastic Poisson processes with respective intensity processes $\lambda^+ = (\lambda_t^+)_{\{0\le t\le T\}}$ and $\lambda^- = (\lambda_t^-)_{\{0\le t\le T\}}$, which count the number of market orders that have arrived on to the buy and sell side of the limit order book, respectively. In the rest of the paper, we will use the notation $\boldsymbol{N}_t = (N_t^+, N_t^-)$.

2.1. Asset Midprice Dynamics

As previously mentioned, we assume that the investor's rate of purchase will have a permanent impact on the asset midprice. To enable this, we must define the two processes $S = (S_t)_{0\le t\le T}$ and $F = (F_t)_{0\le t\le T}$ to represent the asset midprice and the asset midprice without the trader's impact, respectively. It is assumed that the trader has an impact on the asset midprice that is linearly proportional to her rate of trading. More specifically, the relationship between the trader's decision, ν, and the processes S_t and F_t is defined as

$$S_t^\nu = F_t + \beta \int_0^t \nu_u du \ , \tag{2.2}$$

where $\beta > 0$ controls the strength of the trader's impact on the asset midprice.

We assume that the investor does not have a complete knowledge of the dynamics of the asset midprice, nor the rates of arrival for market orders on either side of the limit order book. This uncertainty is introduced with the process $\Theta = (\Theta_t)_{0\le t\le T}$, which is defined to be an M-state, continuous time Markov Chain which takes values in the set $\{\theta_j\}_{j=1}^M$, whose path is invisible to the trader. The Markov chain Θ is assumed to have a known generator matrix $\boldsymbol{Q}$. Moreover, it is known that $\mathbb{P}[\Theta_0 = \theta_j] = \pi_0^j$ for all $j = 1 \ldots M$.

Conditional on a path of Θ_t and on a fixed control ν, the unaffected midprice F_t

satisfies the SDE

$$dF_t = A(t, F_t, \boldsymbol{N}_t, \Theta_t)\, dt + b(dN_t^+ - dN_t^-) + \sigma dW_t \; , F_0 = F \; ,$$

where the intensity processes for N_t^+ and N_t^- are defined as

$$\lambda_t^{\pm} = \sum_{j=1}^{M} \mathbb{1}_{\{\Theta_t = \theta_j\}} \lambda_t^{\pm,j} \tag{2.3}$$

and $W = (W_t)_{0 \le t \le T}$ is a $\mathbb{P}$-Brownian Motion. Each term $\lambda_t^{\pm,j}$, is an $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted process, where $\mathcal{F}^{F,\boldsymbol{N}} \subseteq \mathcal{F}$ is the natural filtration generated by the paths of the processes F and $\boldsymbol{N}$. Furthermore, for each $j = 1 \dots M$, $\lambda_t^{\pm,j}$ must satisfy the condition that $(\Theta_t, F_t, \lambda_t^{+,j}, \lambda_t^{-,j}, \boldsymbol{N}_t)$ is a Markov process. The Markovian condition will guarantee later on that the infinitesimal generator of $(\Theta_t, F_t, \lambda_t^{+,j}, \lambda_t^{-,j}, \boldsymbol{N}_t)$ exists. It is also assumed here that $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}$, where we use the definition $\boldsymbol{\lambda}_t = \left(\lambda_t^{+,1}, \; \dots \; , \lambda_t^{+,M}, \lambda_t^{-,1}, \; \dots \; , \lambda_t^{-,M}\right)$. Some examples of models for $\lambda_t^{j,\pm}$ that satisfy the above requirements can range from self-exciting Hawkes processes to constant intensity processes. Some examples of suitable models will be displayed in the numerical examples section of the paper. It is also assumed here that the function A and the processes $\lambda_t^{\pm}$ each satisfy

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (A(u, S_u, \boldsymbol{N}_u, \Theta_u))^2 + \left(\lambda_u^+\right)^2 + \left(\lambda_u^-\right)^2 \; du\right] < \infty \; . \tag{2.4}$$

We can regard Θ_t as a random variable that indexes M possible models for the asset price drift and the rates at which other market participants' market orders arrive into the limit order book. Θ_t is invisible to the investor, and determines at any instant in time which model drives the asset price dynamics and the rate of incoming market orders. Since Θ_t may change values over time, so will the underlying model. Furthermore, because Θ_t is invisible to the investor, it will be up to her to guess at any time what the underlying model driving asset prices will be.

2.1.1. A note on the price impact structure of equation (2.2)

The way the linear permanent impact is defined in equation (2.2) slightly differs from what exists already in the algorithmic trading literature. The model defined in equation (2.2) in fact defines an even more persistent permanent impact model than the classical linear impact model of Almgren and Chriss (2001), and is defined in a similar manner to the one present in Gârleanu and Pedersen (2013). The reason for this increased persistence can be seen in the following example. Consider the asset midprice process $\tilde{S} = (\tilde{S}_t)_{0 \le t \le T}$ with the standard linear permanent impact model to be the unique solution to the SDE

$$d\tilde{S}_t^\nu = \Big(\kappa(\theta + \tilde{S}_t) + \beta\nu_t\Big)\ dt + \sigma dW_t\ ,\ \tilde{S}_0^\nu = S\ . \tag{2.5}$$

As the investor trades, she will either drive up or down the value of $\tilde{S}_t$ through the linear impact term that instantaneously affects the drift in the dynamics (2.5). The mean reversion level of the midprice, however, does not change. This means that no matter how the trader chooses to trade the asset, its price will still revert towards the very same value θ. This implies that the impact that the trader will have on the midprice at any instant in our example will exponentially decay over time. This behaviour means that the classical linear impact term as defined in equation (2.5) is not permanently affecting the asset midprice, but is instead impacting it in a transient manner.

In contrast, the linear price impact structure of equation (2.2) would in fact admit a true permanent impact in this last example. This is easy to see since an Ornstein-Uhlenbeck model for $\tilde{F}_t$ and the price impact as defined in equation (2.2) implies that $\tilde{S}_t^\nu$ satisfies the SDE

$$d\tilde{S}_t^\nu = \left[\kappa\left((\theta + \beta\int_0^t \nu_u du) - \tilde{S}_t\right) + \beta\nu_t\right]\ dt + \sigma dW_t\ ,\ \tilde{S}_0^\nu = S\ . \tag{2.6}$$

It's clear that in the above model for the midprice, not only are we instantaneously affecting the midprice drift just like in (2.5), but we are also simultaneously changing the mean reversion level, meaning that the trader is indeed affecting the asset price in a permanent way.

This logic will of course extend to a wider range of models, but this last example was shown to illustrate how the price impact model of equation (2.2) interacts with the asset midprice process, and how this interaction is in fact persistent rather than transient.

2.2. Cash Process

The price at which the the trader either buys or sells each unit of the asset will be denoted as $\hat{S}^\nu = (\hat{S}_t^\nu)_{0 \le t \le T}$. Since only a limited quantity of the asset can be purchased at exactly the bid or ask price, the investor must 'walk the book' starting at the bid (ask) and buy (sell) her assets at higher (lower) prices as she increases the size of each of her market orders. For this reason, the price the investor pays will be

$$\hat{S}_t = S_t + a\nu_t \ , \tag{2.7}$$

where the $a > 0$ controls the size of the cost of having to 'walk the book'. This cost effect is also referred to as the temporary price impact, since it will only affect the investor's instantaneous price of purchase/sale and not the midprice. For simplicity, we are assuming that the bid/ask spread for this particular asset if 0.

The amount of cash that the investor holds at any one time is denoted as $X^\nu = (X_t^\nu)_{\{0\le t\le T\}}$, which satisfies the SDE

$$dX_t^\nu = -\nu_t \hat{S}_t^\nu dt \ , \qquad X_0^\nu = X \ , \tag{2.8}$$

which records the total amount of funds accumulated from trading until some time t for some fixed strategy ν.

2.3. Objective Criterion

Over the course of the trading window $t \in [0, T]$, the trader wishes to find some trading strategy $\nu \in \mathcal{A}$ which maximizes the objective criterion

$$\mathbb{E}^{\mathbb{P}}\left[X_T^\nu + Q_T^\nu \left(S_T^\nu - \alpha\, Q_T^\nu\right) - \phi \int_0^T \left(Q_u^\nu\right)^2 du\right] \ , \tag{2.9}$$

where $\mathcal{A}$ is said to be the set of admissible trading strategies, which consists of the collection of all $\mathcal{F}_t^{S,\boldsymbol{N}}$-predictable processes that satisfy $\mathbb{E}\left[\int_0^T \nu_u^2\, du\right] < \infty$, where $\mathcal{F}^{S,N} \subseteq \mathcal{F}$ is the natural filtration generated by S and $\boldsymbol{N}$.

The objective criterion (2.9) consists of three different parts. The first is X_T^ν, which represents the amount of cash that the trader has accumulated from her trading over the period $[0, T)$. Next is the amount of cash received from getting rid of all leftover exposure that the trader may have at time T (Q_T^ν). The amount of money obtained from selling/buying remaining assets is penalized by an amount αQ_T^ν, where it is assumed that $\alpha \ge 0$. The amount αQ_T^ν aims to represent the liquidity penalty taken by the trader if she chooses to sell or buy an amount of assets Q_T^ν all at once. The last term $-\phi \int_0^T (Q_u^\nu)^2\, du$

represents a running penalty that penalizes the trader for having a non-zero inventory and which encourages her to keep the amount of exposure she has lower than usual.

One important point to make note of is the fact that the trading strategy must be a $\mathcal{F}_t^{S,\boldsymbol{N}}$-predictable process. This means that at any point in time, the investor will only be able to make her trading decision based on the information obtained from the path of the process $(S_t^\nu, F_t, \boldsymbol{N}_t, X_t^\nu, Q_t^\nu)$. This predictability requirement will ensure that the trader does not have access to any information regarding the path of the process Θ_t, which governs the model driving the asset midprice drift and the intensities of the processes N_t^+ and N_t^-. The problem of maximizing the functional (2.9) with a control that is only restricted to the space of admissible controls $\mathcal{A}$ described above is a dynamic programming problem with partial information due to the fact that the information available to the trader at any time is restricted solely to the filtration $\mathcal{F}^{S,\boldsymbol{N}}$ rather than the complete filtration $\mathcal{F}$.

Solving the dynamic programming problem with partial information is very difficult to do directly, since most tools that are used to work with the case of complete information no longer work. The former will require an indirect approach in which we will first find an alternate $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted representation for the dynamics of the state variable process, and then extend the state variable process so that it becomes Markov when using Markov controls. The key to doing this will be to find out what the investor's best guess for Θ_t will be at any time, conditional on the information available to her.

3. Filtering

Since the investor cannot observe Θ_t, she wishes to formulate a guess for its value. The best possible guess for the distribution of Θ_t, will be the distribution of Θ_t conditional on the total amount of information accumulated up until that time. By the definition of $\mathcal{A}$ in section 2.3, we know that if $\nu \in \mathcal{A}$, then $\mathcal{F}_t^{S,\boldsymbol{N}} = \mathcal{F}_t^{F,\boldsymbol{N}}$. In other words, if our trading strategy is admissible, knowing the path of the process S_t allows us to reconstruct the path of F_t. Therefore, it is enough for the trader to compute

$$\pi_t^j = \mathbb{E}^{\mathbb{P}}\left[\ \mathbb{1}_{\{\Theta_t=\theta_j\}}\ \middle|\ \mathcal{F}_t^{F,\boldsymbol{N}}\right], \quad \forall j \in \{1 \dots M\}\ ,$$

which is an $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted process with initial value equal to π_0^j which represents the posterior distribution of the process Θ_t given all of the information accumulated by the investor up until that point. Methods for finding an expression for this type of filter were first developed in Wonham (1964) and can also be found in Björk (1980) or Elliott et al. (2008). Based on our specific set-up, the solution to the filter is stated in the theorem below.

Theorem 3.1. *The filter* $\{\pi_t^j\}_{j=1}^M$ *admits the solution*

$$\pi_t^j = \frac{\Lambda_t^j}{\sum_{i=1}^M \Lambda_t^i} \,, \tag{3.1}$$

where each $\{\Lambda_t^j\}_{j=1}^M$ *is the solution to the the SDE*

$$\begin{aligned} \frac{d\Lambda_t^j}{\Lambda_{t-}^j} &= \sigma^{-2} A(t, F_{t-}, \boldsymbol{N}_{t-}, \theta_j) \left(dF_t - b(dN_t^+ - dN_t^-)\right) \\ &\quad + (\lambda_{t-}^{+,j} - 1)(dN_t^+ - dt) + (\lambda_{t-}^{-,j} - 1)(dN_t^- - dt) + \sum_{i=1}^N \left(\frac{\Lambda_t^i}{\Lambda_t^j}\right) C_{j,i}\, dt \,, \end{aligned} \tag{3.2}$$

with initial condition $\Lambda_0^j = \pi_0^j$.

Proof. The proof can be found in Appendix A.1 $\square$

One thing to note about the above theorem, is that $\{\Lambda_t^j\}_{j=1}^M$ admits a very easy closed form solution in the case when $\boldsymbol{C} = 0$, which would correspond to the case where Θ_t is guaranteed to remain constant over the trading period $[0, T]$. In the more general case, solutions to the filter can be approximated reasonably well for most purposes by using methods outlined in George et al. (2004), which will be discussed further in section 6.

An SDE also exists for the normalized version of the posterior density $\{\pi_t^j\}_{j=1}^M$, but for simplicity, we will only keep track of the Λ_t^j terms, and define

$$\pi_t^j = \pi^j(\boldsymbol{\Lambda}_t) = \frac{\Lambda_t^j}{\sum_{i=1}^M \Lambda_t^i} \,, \tag{3.3}$$

since this form will always guarantee that the π_t^js always add up to 1, even when numerically approximating the solutions to the Λ_t^j terms. Furthermore, the notation $\boldsymbol{\Lambda}_t = \left(\Lambda_t^1 \ldots \Lambda_t^M\right)$ will be used from this point on.

4. Reducing the Problem

The task of finding a control $\nu \in \mathcal{A}$ which maximizes the functional (2.9) presents some difficulties surrounding the fact that $\mathcal{A}$ is restricts the information available to the investor to the filtration $\mathcal{F}_t^{S,\boldsymbol{N}}$. We will show that there exist a representation for the dynamics of F_t and for the intensity of $N_t^{\pm}$ that is $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted. This allows the dynamics for the the problem's state variables to be written independently of Θ_t. The sequence of arguments we will make resemble those found in (Bäuerle and Rieder, 2007, Section 3).

Let us define the $\mathcal{F}_t$-adapted martingales,

$$M_t^{\pm} = N_t^{\pm} - \int_0^t \lambda_u^{\pm} du \ , \tag{4.1}$$

to be the compensated versions of the Poisson processes $N_t^{\pm}$. The theorem that follows will provide the necessary ingredients so that we can write down the $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted representations of the state processes.

Theorem 4.1. *Define the processes $\widehat{W} = (\widehat{W}_t)_{t\geq 0}$, $(\widehat{M}_t^{\pm})_{t\geq 0}$ by the following relations*

$$\widehat{W}_t = W_t + \sigma^{-1} \int_0^t \left(\hat{A}(u, F_u, \boldsymbol{N}_u, \Theta_u) - \hat{A}(u, F_u, \boldsymbol{N}_u, \boldsymbol{\Lambda}_u) \right) \, du \ , \tag{4.2a}$$

$$\widehat{M}_t^{\pm} = M_t^{\pm} + \int_0^t \left(\lambda_u^{\pm} - \hat{\lambda}_u^{\pm} \right) \, du \ , \tag{4.2b}$$

where $\hat{A}(t, F_t, \boldsymbol{N}_t, \boldsymbol{\Lambda}_t) = \sum_{j=1}^{M} \pi^j(\boldsymbol{\Lambda}_t) \ A(t, F_t, \boldsymbol{N}_t, \theta_j)$ and $\hat{\lambda}_t^{\pm} = \sum_{j=1}^{M} \pi^j(\boldsymbol{\Lambda}_t) \ \lambda_t^{\pm,j}$. Then,

(A) The process $\widehat{W}$ is an $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted $\mathbb{P}$-Brownian Motion

(B) The processes $\widehat{M}^{\pm}$ are $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted $\mathbb{P}$-martingales

(C) $[\widehat{W}, \widehat{M}^{\pm}]_t = 0$, $\mathbb{P}$-almost surely.

Proof. The proof can be found in Appendix B.1. □

Theorem 4.1 tells us that we can write $N_t^{\pm}$ in terms of $\widehat{M}_t^{\pm}$ as

$$N_t^{\pm} = \widehat{M}_t^{\pm} + \int_0^t \widehat{\lambda}_u du \ , \tag{4.3}$$

which implies by Watanabe's characterization theorem that the $N_t^{\pm}$ are doubly stochastic $\mathcal{F}_t^{F,\boldsymbol{N}}$ adapted Poisson processes with respective intensity processes $\widehat{\lambda}_t^{\pm}$. This last remark and part (A) of theorem 4.1 allows us to represent the dynamics of F_t in their $\mathcal{F}_t^{F,\boldsymbol{N}}$-predictable form as

$$dF_t = \left(\widehat{A}(t, F_t, \boldsymbol{N}_t, \Lambda_t) + b(\widehat{\lambda}_t^+ - \widehat{\lambda}_t^-)\right) dt + b\left(d\widehat{M}_t^+ - d\widehat{M}_t^-\right) + \sigma\, d\widehat{W}_t\ . \tag{4.4}$$

The predictable representation for the dynamics of F_t, $\boldsymbol{\Lambda}_t$ and $\boldsymbol{\lambda}_t$ tell us that the process $(F_t, \boldsymbol{N}_t, \boldsymbol{\lambda}_t, \boldsymbol{\Lambda}_t)$ is in fact a $\mathcal{F}^{S,\boldsymbol{N}}$-adapted process whenever $\nu \in \mathcal{A}$. Hence, the dynamic programing problem consisting of finding an admissible control to maximize the objective criterion (2.9) can be regarded as a problem with complete information with respect to the extended state variable process $(S_t^\nu, F_t, \boldsymbol{N}_t, X_t^\nu, Q_t^\nu, \boldsymbol{\lambda}_t, \boldsymbol{\Lambda}_t)$ where the joint dynamics of this state process are all $\mathcal{F}_t^{S,\boldsymbol{N}}$-adapted and do not depend on the process Θ. This implies that the dynamics of the extended state process are completely visible to the investor, which reduces the optimization problem with partial information, in which we did not know the dynamics of the state variables, into a classical dynamic programing problem for which information is no longer hidden from her.

The next section will consist of solving this problem by using the fact that since the extended state variable dynamics are now $\mathcal{F}_t^{S,\boldsymbol{N}}$-adapted for each $\nu \in \mathcal{A}$, we will be able to apply the dynamic programming principle to the optimization problem (2.9) and derive a dynamic programming equation for this problem.

5. Solving the Dynamic Programming Problem

5.1. The Dynamic Programming Equation

To solve the optimization problem 2.9, we must make use of the fact that for each $\nu \in \mathcal{A}$, the $(2M+4)$-dimensional state variable process $Z_t^\nu = (S_t^\nu, F_t, \boldsymbol{N}_t, X_t^\nu, Q_t^\nu, \boldsymbol{\lambda}_t, \boldsymbol{\Lambda}_t)$ has dynamics that are visible to the trader. First, however, let us define the functionals

$$H^\nu(t, S, F, \boldsymbol{N}, X, Q, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = \mathop{\mathbb{E}^{\mathbb{P}}}_{t,Z}\left[X_T^\nu + Q_T^\nu\,(S_T^\nu - \alpha Q_T^\nu) - \phi \int_t^T (Q_u^\nu)^2\, du\right] \tag{5.1}$$

and

$$H(t, S, F, \boldsymbol{N}, X, Q, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = \sup_{\nu \in \mathcal{A}} H^\nu(t, S, F, \boldsymbol{N}, X, Q, \boldsymbol{\lambda}, \boldsymbol{\Lambda})\ , \tag{5.2}$$

where we use $\mathop{\mathbb{E}}_{t,Z}[\,\cdot\,]$ to represent the expected value conditional on $Z_t^\nu = Z$. The definition of H^ν implies that $H^\nu(0, F, F, \boldsymbol{0}, X, \mathfrak{N}, \boldsymbol{\lambda}, \boldsymbol{\pi}_0)$ is exactly equal to the objective criterion defined in equation (2.9). Furthermore, let us note that a control $\nu^\star \in \mathcal{A}$ is optimal and solves the optimization problem described in section 2.3 if it satisfies

$$H^{\nu^\star}(0, F, F, \mathbf{0}, X, \mathfrak{N}, \boldsymbol{\lambda}, \boldsymbol{\pi}_0) = H(0, F, F, \mathbf{0}, X, \mathfrak{N}, \boldsymbol{\lambda}, \boldsymbol{\pi}_0) \ . \tag{5.3}$$

The $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted version of the dynamics for the state variable process Z_t^ν make it clear that it is Markov for any Markov admissible control $\widehat{\nu}_t = \widehat{\nu}(t, Z_t^\nu) \in \mathcal{A}$. This implies that for such controls, the function H must satisfy the Dynamic Programming Principle and the Dynamic Programming Equation (DPE) (see e.g. (Pham, 2009, Chapter 3)). The DPE for our specific problem is

$$\begin{cases} -\phi q^2 + \sup\limits_{\tilde{\nu}\in\mathbb{R}} \left\{ (\partial_t + \mathcal{L}^{\tilde{\nu}}) J(t, S, F, \boldsymbol{N}, X, Q, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) \right\} = 0 \ , \\ \qquad\qquad H(T, S, F, Q, X, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = X + q\,(S - \alpha\, q) \ , \end{cases} \tag{5.4}$$

where $\mathcal{L}^\nu$ is the infinitesimal generator for the state process Z_t^ν using the predictable representation for the dynamics of F_t and the intensity of $N_t^\pm$, given a fixed control ν. Furthermore, we can write the operator $\mathcal{L}^\nu$ as

$$\mathcal{L}^\nu f = \nu\, \partial_Q f - \nu\,(S + a\nu)\partial_X f + \beta\, \nu\, \partial_S f + \bar{\mathcal{L}} f \ ,$$

where $\bar{\mathcal{L}} f$ is the joint infinitesimal generator for $(S_t, F_t, \boldsymbol{N}_t, \boldsymbol{\lambda}_t, \boldsymbol{\Lambda}_t)$ under the measure $\bar{\mathbb{P}}$, where under $\bar{\mathbb{P}}$

$$dS_t = dF_t \ , \tag{5.5}$$

and the dynamics of $(F_t, \boldsymbol{N}_t, \boldsymbol{\lambda}_t, \boldsymbol{\Lambda}_t)$ remain exactly as they are in the measure $\mathbb{P}$. Under the $\bar{\mathbb{P}}$-measure, there is no permanent impact from our trading strategy ν on the midprice, and therefore the operator $\bar{\mathcal{L}}$ does not depend on the value of ν.

The Dynamic Programming Equation (5.4) can be simplified by introducing the ansatz

$$H(t, S, F, Q, X, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = X + QS + h(t, Q, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) \ ,$$

which yields the PDE for h,

$$\begin{cases} 0 = -\phi Q^2 + \left(\partial_t + \bar{\mathcal{L}}\right) h + Q\, \left(\widehat{A}(t, F, \boldsymbol{N}_t, \boldsymbol{\Lambda}) + b\left(\widehat{\lambda}^+(\boldsymbol{\lambda}, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\boldsymbol{\lambda}, \boldsymbol{\Lambda}) \right) \right) \\ \qquad + \sup_{\tilde{\nu}\in\mathbb{R}} \left\{ (\beta Q + \partial_Q h)\tilde{\nu} - a\tilde{\nu}^2 \right\} \\ h(T, \dots) = -\alpha Q^2 \ . \end{cases} \tag{5.6}$$

This PDE implies that the feedback control for this problem should be

$$\tilde{\nu}^\star = \frac{\beta Q + \partial_q h}{2a} \,. \tag{5.7}$$

In other words, the PDE (5.6) attains its supremum when $\tilde{\nu}$ is set to $\tilde{\nu}^\star$ defined above. The ansatz provided above permits us to indeed find a solution to the PDE (5.4) which is presented in the proposition that follows.

Proposition 5.1 (Candidate Solution)**.** *The PDE* (5.4)*, admits a classical solution J, which can be written as*

$$H(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) = X + QS + h_0(t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) + Q\, h_1(t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) + Q^2\, h_2(t) \;,$$

where

$$\begin{aligned}
h_2(t) &= -a\,\gamma \tanh\left(\zeta + \gamma(T-t)\right) - \frac{\beta}{2} \\
h_1(t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) &= \int_t^T \mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}} \left[\widehat{A}(u,F_u,\boldsymbol{N}_u,\boldsymbol{\Lambda}_u) + b(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-)\right] \left(\frac{\cosh\left(\zeta+\gamma(T-u)\right)}{\cosh\left(\zeta+\gamma(T-t)\right)}\right) du \\
h_0(t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) &= \left(\frac{1}{4a}\right) \mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}} \left[\int_t^T \left(h_1(u,F_u,\boldsymbol{N}_u,\boldsymbol{\lambda}_u,\boldsymbol{\Lambda}_u)\right)^2\, du\right]
\end{aligned}$$

with the constants $\gamma = \sqrt{\frac{\phi}{a}}$, *and* $\zeta = \tanh^{-1}\left(\frac{\alpha - \frac{1}{2}\beta}{a\gamma}\right)$.

Proof. The proof can be found in Appendix C.1

The above proposition and equation (5.7) suggests that the trading speed employed by the investor should be

$$\nu_t^\star = \tfrac{1}{a}\, h_2(t)\, Q_t^{\nu^\star} + \tfrac{1}{2a}\, h_1(t,F_t,\boldsymbol{N}_t,\boldsymbol{\lambda}_t,\boldsymbol{\Lambda}_t) \;. \tag{5.8}$$

The above solution for the optimal trading strategy is a combination of the classical Almgren-Chriss (AC) liquidation strategy, $\frac{Qh_2}{a}$, with another term, $\frac{h_1}{2a}$, that adjusts the trading speed based on how the investor expects the asset midprice to behave in the future. The latter term, as can be seen from the solution to h_1, is in fact a weighted average of the expected future drift of the asset midprice. Therefore if, based on her current information, the trader believes that the asset midprice drift will remain largely positive for the remainder of the trading period, $\frac{h_1}{2a}$ will be positive, and she will buy more of the asset relative to the AC strategy, since she knows she will be able to sell the asset at a higher price once asset prices have risen. The exact opposite effect occurs when she expects the asset price drift to remain mostly negative over the rest of the trading period. This behaviour illustrates how the investor uses the filter for Θ_t to consistently update her predictions of the future path of the asset midprice so that she can adjust her strategy accordingly. Moreover, the solution provided in proposition 5.1 very closely resembles the result obtained by Cartea and Jaimungal (2016).

Let us also note that we can write for $u \geq t$,

$$\begin{aligned}&\mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}}\left[\widehat{A}(u,F_u,\boldsymbol{N}_u,\boldsymbol{\Lambda}_u)+b(\widehat{\lambda}_t^{+}-\widehat{\lambda}_t^{-})\right]\\&\qquad=\sum_{j=1}^{M}\pi^j(\boldsymbol{\Lambda})\mathop{\mathbb{E}^{\mathbb{P}}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda},\theta_j}\left[A(u,F_u,\boldsymbol{N}_u,\Theta_u)+b(\lambda_t^{+}-\lambda_t^{-})\right]\ ,\end{aligned} \tag{5.9}$$

in which we are also conditioning on the event $\Theta_t = \theta_j$ on the right. The above expression provides an alternative means of computing the expected value present in the solution of h_1. This alternative method is almost always easier to compute than a direct computation of the expected value present in 5.1.

What follows next is a theorem that verifies that the candidate solution found in Proposition (Appendix C.1) is exactly equal to the value function H defined in equation (5.2).

Theorem 5.2 (Verification Theorem)**.** *Suppose that h is the solution to the PDE* (5.6)*, and that $\alpha - \frac{1}{2}\beta \neq a\gamma$. Let $\widehat{H}(t,S,F,\boldsymbol{N},X,Q,\boldsymbol{\lambda},\boldsymbol{\Lambda}) = X + QS + h(t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda})$.*

Then $\widehat{H}$ is equal to the value function H defined in (5.2)*. Furthermore the control*

$$\nu_t^{\star} = \tfrac{1}{a}\, h_2(t)\, Q_t^{\nu^{\star}} + \tfrac{1}{2a}\, h_1(t,F_t,\boldsymbol{N}_t,\boldsymbol{\lambda}_t,\boldsymbol{\Lambda}_t) \tag{5.10}$$

is optimal and satisfies

$$H(t,S,F,\boldsymbol{N},X,Q,\boldsymbol{\lambda},\boldsymbol{\Lambda}) = H^{\nu^{\star}}(t,S,F,\boldsymbol{N},X,Q,\boldsymbol{\lambda},\boldsymbol{\Lambda})\ . \tag{5.11}$$

Proof. The proof can be found in Appendix C.2 □

The theorem above now guarantees that the control provided above indeed solves the optimization problem presented in section 2.3. In retrospect, we have found that the optimal control to an optimization problem with partial information has turned out to be a Markov control. The key to obtaining a solution to this problem was achieved mainly through the extension of the original state process to include $\boldsymbol{\Lambda}_t$, as well as the introduction of the predictable representation for the dynamics of the process F_t.

A useful case to consider is when the investor is forced to eliminate her market exposure before time T. This corresponds to the case when we take the limit $\alpha \to \infty$ which forces the terminal penalty in the term (2.9) to tend towards infinity. The resulting optimal control in this case then becomes

$$\begin{aligned}\lim_{\alpha\to\infty} \nu_t^\star =& -\gamma \tanh\left(\gamma(T-t)\right)\; Q_t^{\nu^\star} \\ &+ \tfrac{1}{2a}\sum_{j=1}^{M} \pi^j(\boldsymbol{\Lambda}) \int_t^T \mathbb{E}^{\mathbb{P}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda},\theta_j}\left[A_u + b(\lambda_t^+ - \lambda_t^-)\right] \left(\frac{\sinh\left(\gamma(T-u)\right)}{\sinh\left(\gamma(T-t)\right)}\right) du\,,\end{aligned}$$

where $A_u = A(u, F_u, \boldsymbol{N}_u, \Theta_u)$. A second interesting case is when there is no running penalty to the investor and the investor is forced to eliminater her inventory before time T in which the optimal control will be

$$\begin{aligned}\lim_{\phi\to 0}\lim_{\alpha\to\infty} \nu_t^\star =& -\frac{1}{T-t}\, Q_t^{\nu^\star} \\ & \frac{1}{2a}\sum_{j=1}^{M} \pi^j(\boldsymbol{\Lambda}) \int_t^T \mathbb{E}^{\mathbb{P}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda},\theta_j}\left[A_u + b(\lambda_t^+ - \lambda_t^-)\right] \left(\frac{T-u}{T-t}\right) du\end{aligned}$$

where the resulting optimal control is a TWAP strategy plus another weighting factor that adjusts the trading strategy based on the expected future drift of the asset midprice.

One thing to note is that the optimal control found in this section can be computed in closed form for a large variety of possible models. Some examples in the numerical examples section demonstrate the closed form solutions that are obtained for some non-trivial models.

6. Numerical Examples

In this section we will carry out some numerical experiments to test the performance of the optimal trading algorithms developed in section 5. The examples will show how the optimal trading performs using situations for a variety of model set-ups.

6.1. Mean-Reverting Diffusion

This section will look at the case in which the trader wishes to liquidate her inventory before some specified time T. The asset price is assumed to be a pure diffusive Ornstein-Uhlenbeck process. The trader knows the volatility and rate of mean reversion, but does not know the level at which the price will mean-revert. It is also assumed that the mean reversion level will remain constant over the course of the trading period $[0, T]$. More specifically, we assume that the asset midprice has the dynamics

$$dS_t = \kappa(\Theta - S_t)\, dt + \sigma dW_t \ , \tag{6.1}$$

where Θ is a random variable that takes values in the set $\{\theta_j\}_{j=1}^M$ with probabilities $\{\pi_0^j\}_{j=1}^M$, and that remains constant over time but is hidden from the trader. This model does not contain any jumps so we can ignore the variables $\boldsymbol{N}$ and $\boldsymbol{\lambda}$.

As mentioned in section 3, there exists an exact closed form for the filter when we assume that Θ_t does not depend on time. For the above model, the exact solution for the un-normalized filter is

$$\Lambda_t^j = \pi_0^j \, e^{\sigma^{-2}\left(\int_0^t \kappa(\theta_j - S_u)\, dS_u - \frac{1}{2}\int_0^t \kappa^2(\theta_j - S_u)^2\, du\right)} \ , \qquad \forall j = 1 \dots M \ . \tag{6.2}$$

Since only discrete data on S_t is ever available to the investor, we can approximate this last expression using the appropriate Riemann sums. In the event where the trader has access values of S_t at a very high frequency, the filter is very well approximated.

The solution to the optimal control in the case when $\alpha \to \infty$ can be computed exactly as,

$$\begin{aligned}\nu^\star(t, S, Q, \boldsymbol{\Lambda}) = &-\gamma \tanh\left(\gamma(T-t)\right)\, Q \\ &+ \sum_{j=1}^M \pi^j(\boldsymbol{\Lambda}) \frac{(S - \theta_j)}{\sinh\left(\gamma(T-t)\right)} \int_t^T \kappa e^{-\kappa(u-t)} \sinh\left(\gamma(T-u)\right) du \ .\end{aligned}$$

For this simulation, we assume that there are two possible values that the asset price can mean revert to, so we set $M = 2$ with $\theta_1 = 5$ and $\theta_2 = 5.5$. Furthermore, we assume that the investor has no particular belief that either event is more likely than another, so we set the probability of each to be equal, so $\pi_0^1 = \pi_0^2 = 0.5$. Some other parameters used in the simulation are

$$\mathfrak{N} = 10^4, \quad S_0 = 5, \qquad \sigma = 0.2, \qquad\qquad \beta = 0.01,$$
$$\kappa = 2, \qquad a = 10^{-5}, \quad \phi = 2 \times 10^{-5}.$$

Furthermore, in the simulation we assume that the true model governing the asset price dynamics is that with $\Theta = 5.5$. It will be up to the trader to detect this as she observes price paths.

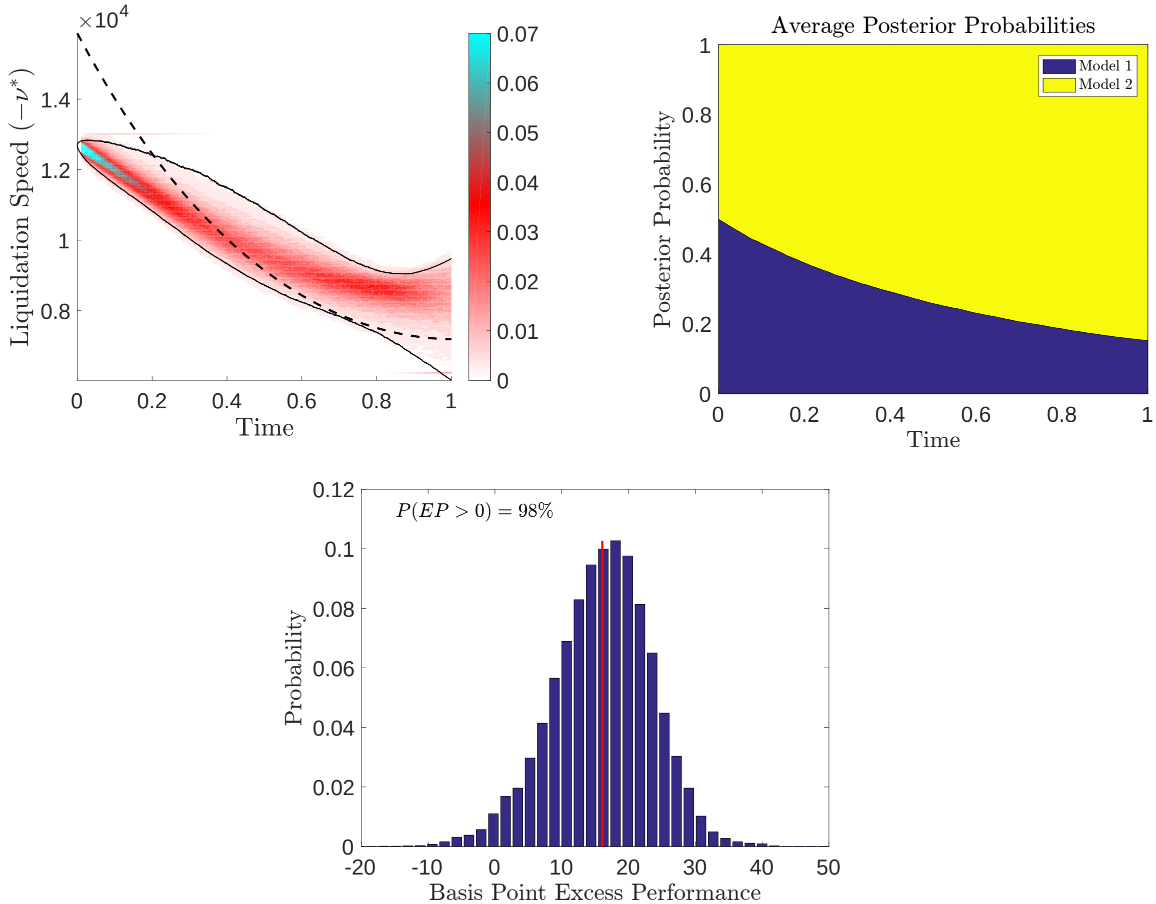

Figure 1: Simulation Results with an Ornstein-Uhlenbeck process

The simulation results can be found in figure 1. The top right plot demonstrates that as time advances, the trader on average will detect that the true rate of mean reversion is $\theta_2 = 5.5$. Moreover, by the end of the trading period, she is on average at least 80% confident that model 2 is the true model governing asset prices. The top left graph in figure 1 shows a heat-map of the trading speed for the investor, in which blue and white areas represent areas of high and low probability respectively, and where the dotted line represents the classical AC strategy. We can see from the heat-map that the trader adjusts her positions in a manner consistent with her predictions: since the investor expects the asset price to rise over time, she slows down her trading initially, so that she can sell her assets at a higher price towards the end of the trading period. The bottom part of figure 1 displays the histogram of the excess return of the optimal control over the AC control. This excess return is defined as

$$\frac{X_T^{\nu^\star} - X_T^{\nu^{\text{AC}}}}{X_T^{\nu^{\text{AC}}}} \times 10^4$$

where $X_T^{\nu^{\text{AC}}}$ is the total amount of cash that the trader would have earned given that she had used the AC liquidation strategy. It's clear from the histogram that the excess return of the optimal trading strategy almost always exceeds that of the AC strategy by a fairly large amount.

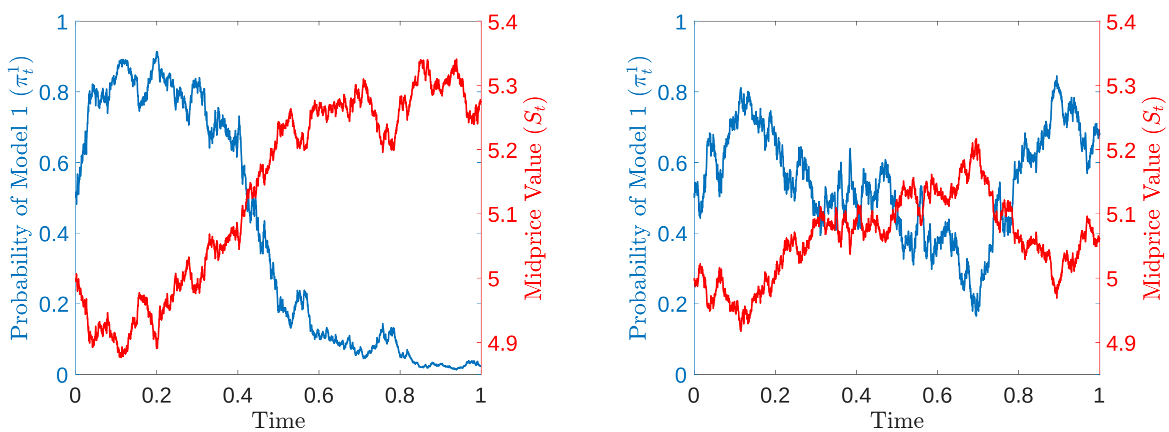

Figure 2: Sample Simulation Paths with and Ornstein-Uhlenbeck process

Figure 2 shows two sample paths of the asset price and the filter. The graph to the left demonstrates how the trader quickly detects the correct model based on the asset trajectory path. Since the asset price is quickly increasing, the probability of model 2 becomes much higher in proportion to model 1, and we see the filter gradually increase. In the simulated path on the right of figure 2, the path of the midprice stays close to \$5, so the trader remains unsure about which model is in fact driving prices and so the filter for model 1 floats around 0.5. This actually ends up being to the advantage of the trader since the filter values more accurately reflects the actual behaviour of the asset price path.

6.2. Mean-Reverting Pure Jump Process

This section outlines the case in which the trader begins with no inventory and aims to trade the asset over the course of the trading period. The asset price is assumed to be completely driven by the market orders of other market participants. It is also assumed here that the asset price mean reverts to some unknown level Θ, that the trader will aim to detect. More specifically, the asset midprice satisfies the SDE

$$dS_t = b\left(dN_t^+ - dN_t^-\right), \tag{6.3}$$

where N_t^+ and N_t^- are doubly stochastic Poisson processes with intensities λ_t^+ and λ_t^- defined by

$$\lambda_t^+ = \mu + \kappa\left(\Theta - S_t\right)_+ \text{ and } \lambda_t^- = \mu + \kappa\left(\Theta - S_t\right)_- , \tag{6.4}$$

where x_+ and x_- denote the positive and negative parts of x, respectively.

In contrast to the previous example, this case assumes that Θ_t is a Markov chain with generator matrix $\boldsymbol{C}$. Over the course of the trading period, the underlying model driving the asset price dynamics may switch from one to the other. The filter for Θ_t cannot be computed directly, but can be approximated via a Euler-Maruyama scheme of the SDE for the logarithm of the filter, whose SDE was given in Theorem 3.1. The resulting approximation for the value of the filter, given that the values of $\boldsymbol{N}$ have been observed at times $\{t_k\}_{k=1}^K$, where $t_0 = 0$ and $t_K = T$ can be found via the recursive formula

$$\begin{cases} \Lambda_{t_0}^j = \pi_0^j \\ \Lambda_{t_{k+1}}^j = \Lambda_{t_k}^j \times \exp\left\{2\left(1 - \mu - \frac{\kappa}{2}\left|\theta_j - S_{t_k}\right|\right)\Delta_{k+1} + \sum_{i=1}^N \left(\frac{\Lambda_{t_k}^i}{\Lambda_{t_k}^j}\right) C_{j,i}\,\Delta_{k+1}\right\} \\ \qquad\qquad \times\left(\mu + \kappa\left(\theta_j - S_{t_k}\right)_+\right)^{\Delta N_{t_{k+1}}^+} \times \left(\mu + \kappa\left(\theta_j - S_{t_k}\right)_-\right)^{\Delta N_{t_{k+1}}^-} & \text{, for } k \geq 1 \end{cases}, \tag{6.5}$$

where $\Delta N_{t_k}^{\pm} = N_{t_k}^{\pm} - N_{t_{k-1}}^{\pm}$ ans $\Delta_k = t_k = t_{k-1}$. The optimal control in this set-up can be found in closed form for this example. We assume here, just as in the last example, that there are two possible values for Θ_t, $\theta_1 = 5$ and $\theta_2 = 5.5$, and that the investor is again using the starting filter values of $\pi_0^1 = \pi_0^2 = 0.5$. The parameters

$$\begin{aligned} &\mathfrak{N} = 10^4, \quad S_0 = 5, \quad \sigma = 0.2, \quad b = 0.01 \quad \phi = 2.5\times 10^{-5}, \\ &\kappa = 25, \quad \mu = 50, \quad a = 10^{-5}, \quad \beta = 0.01, \quad Q = \left[\begin{smallmatrix} -0.1 & 0.1 \\ 1 & -1 \end{smallmatrix}\right]. \end{aligned}$$

will be used for this round of simulations. It is assumed that the value of Θ_t stays at 5.25 over the period $t \in [0, 0.55]$ after which it jumps down to 5 and remains there until the end of the trading period.

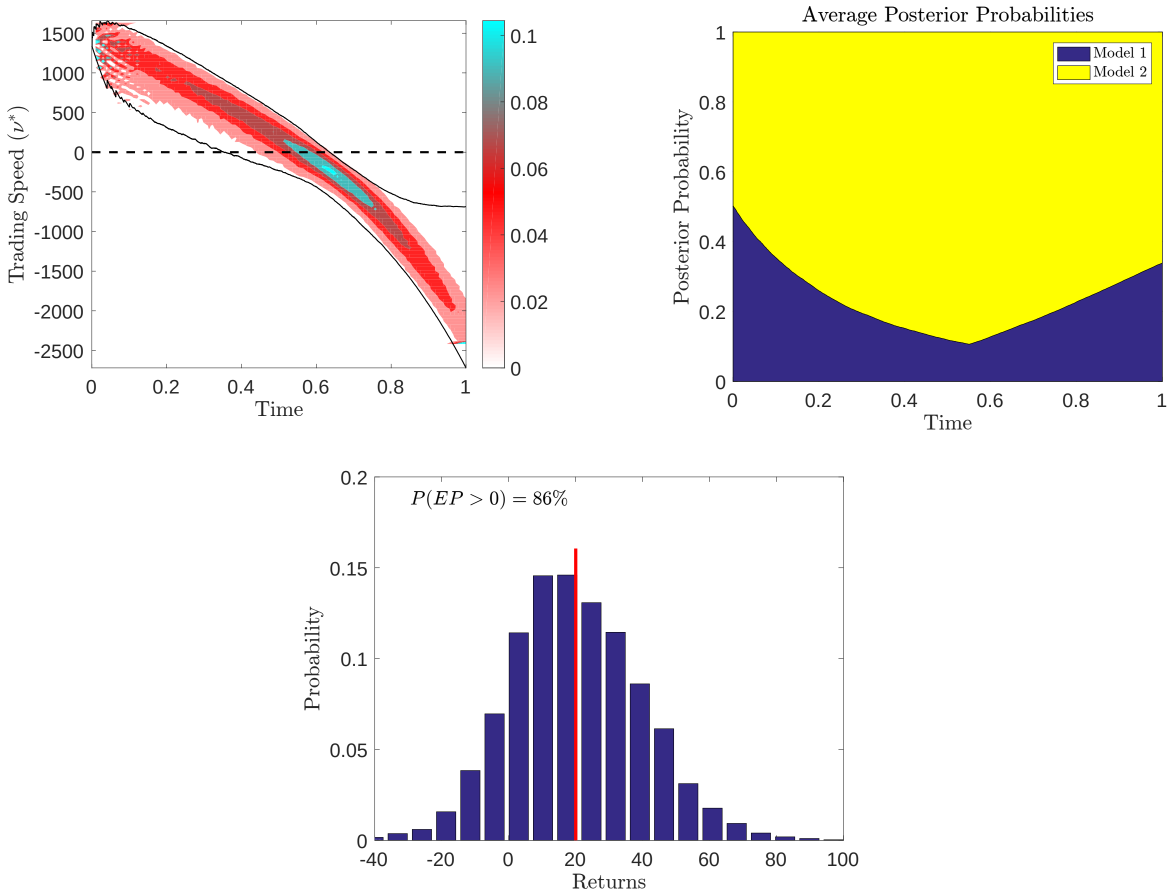

Figure 3: Simulation Results with a Pure-Jump Mean-Reverting Process

The top right portion of figure 3 demonstrates that on average, the filter does in fact detect the jump from $\Theta_t = 5.25$ to $\Theta_t = 5$. The optimal strategy's behaviour can be seen in the top left portion of figure 3. Since the trader believes that there is a chance for the asset price to increase, she begins to purchase the asset during the first half of the trading window. When she begins to approach the middle of the trading window she gradually liquidates her inventory so that she can eliminate her exposure before time T. Moreover, the histogram for the investor's returns ($X_T^{\nu^\star}$) is displayed in the bottom part of figure 3. The histogram shows that the strategy yields positive returns during 86% of the simulated scenarios.

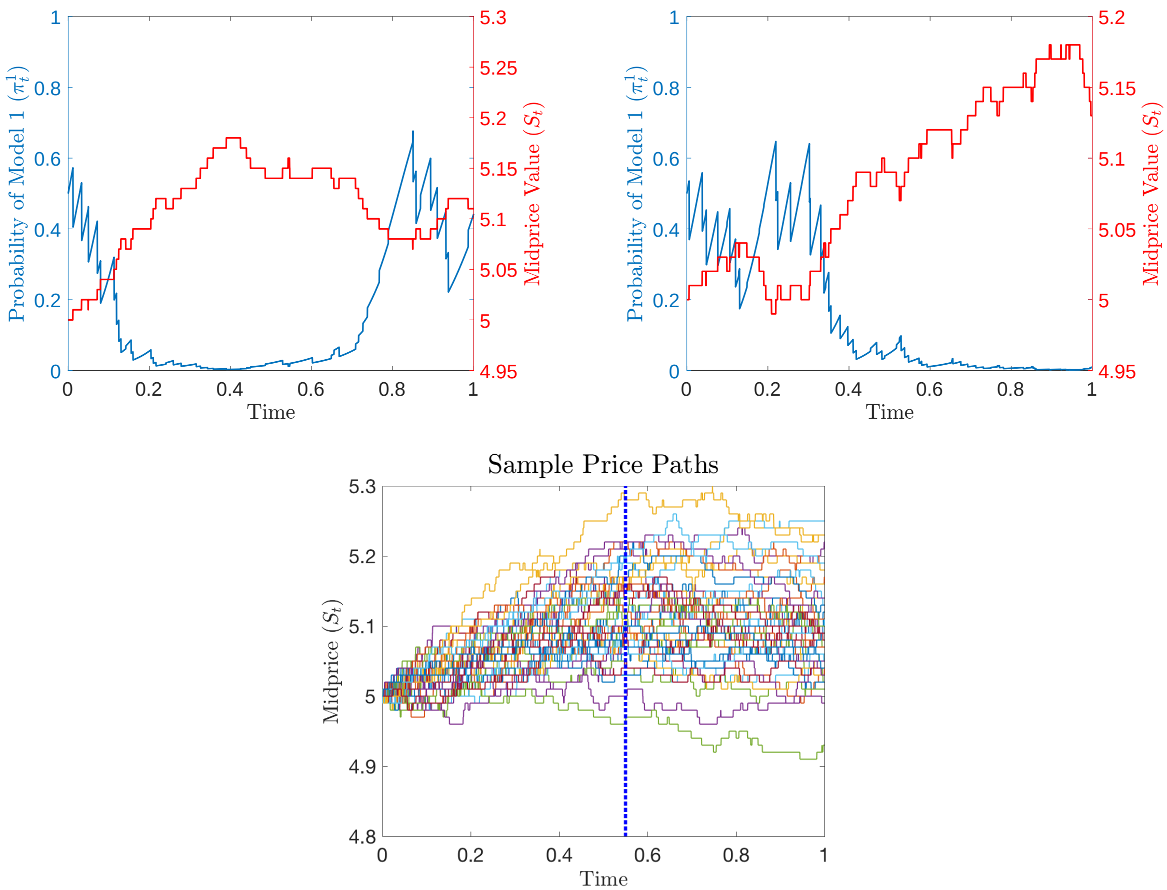

Figure 4: Sample Simulation Paths with a Pure-Jump Mean-Reverting Process

Figure 4 show sample paths from the simulations performed. The graph on the top left shows a case where the trader accurately detects Θ_t's switch from state 2 to state 1 over the course of the trading period. The second example in the center demonstrates a case where the asset price keeps on drifting upwards even though Θ_t switches to state 1 halfway though. The trader just assumes that Θ_t is still in state 1 during most of the trading period. This is in fact an assumption that better suits the price data and allows the trader to perform better than if he had indeed realized that Θ_t had changed values half-way through. The rightmost figure shows more price path examples and shows that the mean-reversion level indeed drops at time $t = 0.55$.

7. Conclusion

In this paper, we have solved a finite horizon optimal trading problem in which the midprice of the asset contains a latent alpha component stemming from a diffusive drift as well as a pure jump component. We have obtained the solution in closed form, up to the computation of an expectation which depends on the class of potential models. The optimal trading speed was found to be a combination of the classical AC trading strategy plus an additional extra term that incorporates the trader's estimate of the latent factors and its forecast. The form of the solution is similar in spirit to the results in Cartea and Jaimungal (2016) where the authors have an alpha component stemming from market order-flow, but in that work viewed as visible. We presented two examples where the trader wishes to completely liquidate a large position – the optimal execution problem. Our resulting strategy with learning was then compared with the AC liquidation strategy. Both examples show that our algorithm significantly outperforms AC.

There are many potential future directions left open for applying the approach here. One direction which we have already been investigating is to generalize the analysis to trading multiple assets, as well as incorporating multiple latent factors. In this work, the trader is assumed to execute trades continuously and uses market orders which walk the limit order book (LOB) and hence obtain a temporary price impact. It would be interesting to analyse the case of executing market orders at discrete times, and hence recast the problem as an impulse control problem with latent alpha factors. Along similar lines, the agent may wish to use limit orders to squeeze even more profits out of the strategy. Combining market and limit orders, along the lines of Cartea and Jaimungal (2015) and Huitema (2013), but including latent alpha factors would be quite interesting.

Appendix A.

Appendix A.1. Proof of Theorem 3.1

Proof. Let us first define the measure $\mathbb{Q}$ through the Radon-Nikodym derivative

$$\begin{aligned}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t &= \exp\left\{-\sigma^{-1}\int_0^t A(u,F_{u-},\boldsymbol{N}_{u-},\Theta_{u-})dW_u - \frac{\sigma^{-2}}{2}\int_0^t \left(A(u,F_{u-},\boldsymbol{N}_{u-},\Theta_{u-})\right)^2du\right\}\\ &\times\exp\left\{\int_0^t(\lambda^+_{u-}-1)du - \int_0^t \ln(\lambda^+_{u-})dN^+_u\right\}\\ &\times\exp\left\{\int_0^t(\lambda^-_{u-}-1)du - \int_0^t \ln(\lambda^-_{u-})dN^-_u\right\}\,,\end{aligned}$$

which is defined so that under the measure $\mathbb{Q}$, the process $\sigma^{-1}(F_t - b(N_t^+ - N_t^-))$ is a Brownian Motion and both N_t^+ and N_t^- have intensity process equal to 1. This measure is deliberately chosen so that F_t, N_t^+ and N_t^- are $\mathbb{Q}$-independent of Θ_t. Additionally, the inverse of this Radon-Nikodym derivative is

$$\begin{aligned}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_t = \zeta_t &= \exp\left\{\sigma^{-2}\int_0^t A(u,F_{u-},\boldsymbol{N}_{u-},\Theta_{u-})(dF_u - b\,d N_u) - \frac{\sigma^{-2}}{2}\int_0^t \left(A(u,F_{u-},\boldsymbol{N}_{u-},\Theta_{u-})\right)^2du\right\}\\ &\times\exp\left\{\int_0^t(1-\lambda^+_{u-})du + \int_0^t \ln(\lambda^+_{u-})dN^+_u\right\}\\ &\times\exp\left\{\int_0^t(1-\lambda^-_{u-})du + \int_0^t \ln(\lambda^-_{u-})dN^-_u\right\}\,.\end{aligned} \tag{A.1}$$

Using these last two expressions, we can re-write the filter in terms $\mathbb{Q}$ expected values to obtain

$$\pi_t^j = \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\Theta_t=\theta_j\}}\zeta_t \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\zeta_t \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right]} \tag{A.2}$$

$$= \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\Theta_t=\theta_j\}}\zeta_t \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right]}{\sum_{i=1}^M \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\Theta_t=\theta_i\}}\zeta_t \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right]} \tag{A.3}$$

$$= \frac{\Lambda_t^j}{\sum_{i=1}^M \Lambda_t^i}\,. \tag{A.4}$$

Next, we will attempt to find an SDE for each Λ_t^j term. This can be done by first defining the process $\delta_t^j = \mathbb{1}_{\{\Theta_t=\theta_j\}}$, which is the satisfies the SDE

$$d\delta_t^j = \sum_{i=1}^N \delta_{t-}^i C_{j,i}\,dt + d\tilde{\mathcal{M}}_t^j\,,$$

under the measure $\mathbb{Q}$, where $\tilde{\mathcal{M}}_t^j$ is a square-integrable, $\mathcal{F}_t$-adapted, $\mathbb{Q}$-martingale and $\boldsymbol{C}$ is the generator matrix for Θ_t. All that's left to compute the dynamics of the Λ_t^j is to figure out the dynamics of $\zeta_t\delta_t^j$ and then take the appropriate $\mathbb{Q}$ expected value while conditioning on $\mathcal{F}_t^{F,\boldsymbol{N}}$. The process $\delta_t^j\zeta_t$ satisfies the SDE

$$
\begin{aligned}
d\left(\zeta_t \delta_t^j\right) = \left(\zeta_t \delta_t^j\right) \Bigg(&\sigma^{-2} A(t, F_{t-}, \boldsymbol{N}_{t-}, \Theta_{t-})\ (dF_t - b(dN_t^+ - dN_t^-)) \\
&+ (\lambda_{t-}^+ - 1)(dN_t^+ - dt) + (\lambda_{t-}^- - 1)(dN_t^- - dt)\Bigg) \\
&+ \sum_{i=1}^N \zeta_{t-} \delta_{t-}^i C_{j,i}\ dt + d\mathcal{M}_t^j \\
= \left(\zeta_{t-} \delta_{t-}^j\right) \Bigg(&\sigma^{-2} A(t, F_{t-}, \boldsymbol{N}_{t-}, \theta_j)\ (dF_t - b(dN_t^+ - dN_t^-)) \\
&+ (\lambda_{t-}^{+,j} - 1)(dN_t^+ - dt) + (\lambda_{t-}^{-,j} - 1)(dN_t^- - dt)\Bigg) \\
&+ \sum_{i=1}^N \zeta_{t-} \delta_{t-}^i C_{j,i}\ dt + d\mathcal{M}_t^j\ ,
\end{aligned}
$$

where $\mathcal{M}_t^j$ is another square-integrable, $\mathcal{F}_t$-adapted, $\mathbb{Q}$-martingale.

Now we re-write the expression for Λ_t^j as the expected value of a stochastic integral

$$
\begin{aligned}
\Lambda_t^j &= \mathbb{E}^{\mathbb{Q}}\left[\delta_t^j \zeta_t \mid \mathcal{F}_t^{S,\boldsymbol{N}}\right] && \text{(A.5)} \\
&= \mathbb{E}^{\mathbb{Q}}\Bigg[\left(\zeta_0 \delta_0^j\right) + \int_0^t \left(\zeta_{u-} \delta_{u-}^j\right) \left(\sigma^{-2} A(u, F_{u-}, \boldsymbol{N}_{u-}, \theta_j)\ (dF_u - b(dN_u^+ - dN_u^-))\right) && \text{(A.6)} \\
&\quad + \int_0^t \left(\zeta_{u-} \delta_{u-}^j\right) (\lambda_{u-}^{+,j} - 1)(dN_u^+ - du) + \int_0^t \left(\zeta_{u-} \delta_{u-}^j\right) (\lambda_{u-}^{-,j} - 1)(dN_u^- - du) \\
&\quad + \int_0^t \left(\sum_{i=1}^N \zeta_{u-} \delta_{u-}^i C_{j,i}\ du + d\mathcal{M}_u^j\right) \Bigg| \mathcal{F}_t^{F,\boldsymbol{N}}\Bigg]\ .
\end{aligned}
$$

A this point we will need conditions guaranteeing that we can exchange the order of integration, in the expression above. First, by the definition of Λ_t^j,

$$
\mathbb{E}^{\mathbb{Q}}\left[\zeta_0 \delta_0^j \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] = \Lambda_0^j\ , \tag{A.7}
$$

which allows us to replace the first term in (A.6). We can use the fact that $F_t - F_0 - b(N_t^+ - N_t^-) = W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion to write

$$
\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left(\zeta_{u-} \delta_{u-}^j\right) \sigma^{-2} A(u, F_{u-}, \boldsymbol{N}_{u-}, \theta_j)\ dW_u^{\mathbb{Q}} \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right]\ . \tag{A.8}
$$

The conditions (2.4) guarantee that the conditions of (Liptser and Shiryaev, 2013, Theorem 5.15) are met, and thus we can apply the theorem and write the second term as

$$
\int_0^t \Lambda_t^j \sigma^{-2} A(u, F_{u-}, \boldsymbol{N}_{u-}, \theta_j)\ dW_u^{\mathbb{Q}}\ . \tag{A.9}
$$

For the third and fourth terms, let us note that if we let $\mathcal{U}_t^{\pm} = \{u \in [0,t] : N_u^{\pm} > N_{u-}^{\pm}\}$, then by condition (2.4) we know $U^{\pm}$ is almost surely finite and $\mathcal{F}_t^{F,\boldsymbol{N}}$-measurable. Therefore

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left(\zeta_{u-}\delta_{u-}^j\right)(\lambda_{u-}^{+,j}-1)dN_u^{\pm} \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{u\in\mathcal{U}_t^{\pm}} \left(\zeta_{u-}\delta_{u-}^j\right)(\lambda_{u-}^{+,j}-1)(N_u^{\pm}-N_{u-}^{\pm}) \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \tag{A.10}$$

$$= \sum_{u\in\mathcal{U}_t^{\pm}} \Lambda_{u-}^j(\lambda_{u-}^{+,j}-1)(N_u^{\pm}-N_{u-}^{\pm})\,. \tag{A.11}$$

We can apply Fubini's theorem on the remaining Riemann integral since the integrands are square integrable to exchange the order of integration and get

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left(\sum_{i=1}^N \zeta_{u-}\delta_{u-}^i C_{j,i} - \left(\zeta_{u-}\delta_{u-}^j\right)(\lambda_{u-}^{+,j}+\lambda_{u-}^{-,j}-2)\right) du \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right]$$
$$= \int_0^t \left(\sum_{i=1}^N \Lambda_{u-}^i C_{j,i} - \Lambda_{u-}^j(\lambda_{u-}^{+,j}+\lambda_{u-}^{-,j}-2)\right) du\,.$$

By using the last steps to exchange the order of integration for each part, we can let the martingale $(\mathcal{M}_t^j)$ portion vanish to obtain

$$\Lambda_t^j = \Lambda_0^j + \int_0^t \Lambda_{u-}^j\Big(\sigma^{-2}A(u,F_{u-},\boldsymbol{N}_{u-},\theta_j)\,(dF_u - b(dN_u^+ - dN_u^-))\Big)$$
$$+\int_0^t \Lambda_{u-}^j(\lambda_{u-}^{+,j}-1)(dN_u^+ - du) + \int_0^t \Lambda_{u-}^j(\lambda_{u-}^{i,j}-1)(dN_u^- - du) + \sum_{i=1}^M \int_0^t \Lambda_{u-}^i C_{j,i} du\,.$$

Noting that $\Lambda_0^j = \pi_0^j$, we obtain the desired result. □

Appendix B.

Appendix B.1. Proof of Theorem 4.1

Proof. We shall prove the claims of Theorem 4.1 in order. First of all, it is clear in the definition of the process $\widehat{W}_t$ is an $\mathbb{P}$-almost-surely continuous process satisfying $[\widehat{W}]_t = t$ since it is the sum of a $\mathbb{P}$-Brownian Motion and a process of finite variation. Moreover, by the definition of the process F_t, we can write W_t as

$$\sigma W_t = (F_t - F_0) - \int_0^t A(u,F_u,\boldsymbol{N}_u,\Theta_u)\,du - b(N_t^+ - N_t^-)\,. \tag{B.1}$$

Hence we can insert this last formula into the definition of $\widehat{W}_t$ to yield,

$$\sigma\widehat{W}_t = (F_t - F_0) - \int_0^t \widehat{A}(u,F_u,\boldsymbol{N}_u,\boldsymbol{\Lambda}_u)\,du - b(N_t^+ - N_t^-)\,, \tag{B.2}$$

which demonstrates that the process $\widehat{W}_t$ is $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted. Next, we will show that $\widehat{W}_t$ is a $\mathbb{P}$-martingale with respect to the filtration $\mathcal{F}_t^{F,\boldsymbol{N}}$. By taking the conditional expectation of $\widehat{W}_{t+h}$ for $h \geq 0$ and by using the properties of W, we get

$$\begin{aligned}\mathbb{E}\left[\widehat{W}_{t+h} \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] &= \widehat{W}_t + \sigma^{-1}\mathbb{E}\left[\widehat{W}_{t+h} - \widehat{W}_t \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \\ &= \widehat{W}_t + \sigma^{-1}\mathbb{E}\left[\int_t^{t+h} \left(\widehat{A}(u, F_u, \boldsymbol{N}_u, \boldsymbol{\Lambda}_u) - A(u, F_u, \boldsymbol{N}_u, \boldsymbol{\Theta}_u)\right) \, du \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \\ &\qquad + \mathbb{E}\left[\mathbb{E}\left[W_{t+h} - W_t \mid \mathcal{F}_t\right] \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \\ &= \widehat{W}_t + \sigma^{-1}\mathbb{E}\left[\int_t^{t+h} \mathbb{E}\left[\widehat{A}(u, F_u, \boldsymbol{N}_u, \boldsymbol{\Lambda}_u) - A(u, F_u, \boldsymbol{N}_u, \boldsymbol{\Theta}_u) \mid \mathcal{F}_u^{F,\boldsymbol{N}}\right] \, du \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \\ &= \widehat{W}_t \ ,\end{aligned}$$

where in the above, the use of Fubini's theorem is allowed due to equation (2.4). We have shown that $\widehat{W}$ is a $\mathbb{P}$-a.s. continuous martingale with quadratic variation equal to t. Therefore by the Lévy characterization of Brownian Motion, the process $\widehat{W}$ is an $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted $\mathbb{P}$-Brownian Motion.

Next we need to verify the claims made about the processes $\widehat{M}_t^{\pm}$. By the definitions of $\widehat{M}_t^{\pm}$ and of $M_t^{\pm}$,

$$\widehat{M}_t^{\pm} = N_t^{\pm} - \int_0^t \widehat{\lambda}_u^{\pm} du \ . \tag{B.3}$$

Since the processes $\widehat{\lambda}_t^{\pm}$ are $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted, we get that $\widehat{M}_t^{\pm}$ must also be $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted processes. The processes $\widehat{M}_t^{\pm}$ are $\mathcal{F}_t^{F,\boldsymbol{N}}$-martingales since for any $h > 0$,

$$\mathbb{E}\left[\widehat{M}_{t+h}^{\pm} \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] = \widehat{M}_t^{\pm} + \mathbb{E}\left[M_{t+h}^{\pm} - M_t^{\pm} + \int_t^{t+h} (\lambda_u^{\pm} - \widehat{\lambda}_u^{\pm}) \, du \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \tag{B.4}$$

$$= \widehat{M}_t^{\pm} + \mathbb{E}\left[\int_t^{t+h} \mathbb{E}\left[(\lambda_u^{\pm} - \widehat{\lambda}_u^{\pm}) \mid \mathcal{F}_u^{F,\boldsymbol{N}}\right] \, du \mid \mathcal{F}_t^{F,\boldsymbol{N}}\right] \tag{B.5}$$

$$= \widehat{M}_t^{\pm} \ . \tag{B.6}$$

Lastly, by the definition of $\widehat{M}^{\pm}$ in equation (B.3), $\widehat{M}^{\pm}$ is the sum of a process with an almost-surely finite number of jumps in the interval $[0, T]$ and a process of finite variation. From its definition, $\widehat{W}$ is the sum of a Brownian Motion and and a process of finite variation. The last two remarks imply that $[\widehat{W}, \widehat{M}^{\pm}] = 0$.

□

Appendix C.

Appendix C.1. Proof of Proposition 5.1

Proof. Let us begin with the PDE (5.4),

$$\begin{cases} 0 = -\phi Q^2 + \sup_{\nu\in\mathbb{R}} \left\{(\partial_t + \bar{\mathcal{L}})J + \nu \, \partial_Q J - \nu \, (S + a\nu)\partial_X J + \beta \, \nu \, \partial_S J\right\} \\ J(T, \dots) = X + Q(S - \alpha Q) \end{cases} . \tag{C.1}$$

Since the term inside of the curly brackets is quadratic in ν, we can complete the square and simplify the supremum expression. This yields

$$0 = -\phi Q^2 + (\partial_t + \bar{\mathcal{L}})J + \frac{1}{4a}\frac{(S\partial_X J - \partial_Q J - \beta\partial_S J)^2}{\partial_X J} \,. \tag{C.2}$$

Next, we can insert the ansatz

$$J(t, S, F, Q, X, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = X + QS + h(t, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) \tag{C.3}$$

into the last PDE to yield another PDE in terms of h,

$$\begin{cases} 0 = -\phi Q^2 + (\partial_t + \bar{\mathcal{L}})h + Q\left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\boldsymbol{\lambda}, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\boldsymbol{\lambda}, \boldsymbol{\Lambda})\right) + \frac{1}{4a}\left(\beta Q + \partial_Q h\right)^2 \\ h(T, \dots) = -\alpha Q^2 \end{cases} \,. \tag{C.4}$$

If we assume that h is quadratic in the variable Q so that

$$h(t, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = h_0(t, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) + Q\, h_1(t, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) + Q^2\, h_2(t) \,, \tag{C.5}$$

then the PDE further simplifies down to

$$\begin{cases} 0 = \quad \left\{(\partial_t + \bar{\mathcal{L}})h_0 + h_1^2\right\} \\ \quad +Q\left\{(\partial_t + \bar{\mathcal{L}})h_1 + \left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\boldsymbol{\lambda}, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\boldsymbol{\lambda}, \boldsymbol{\Lambda})\right) + \frac{1}{2a}(\beta + 2h_2)h_1\right\} \\ \quad +Q^2\left\{\partial_t h_2 - \phi + \frac{1}{4a}\left(\beta + 2h_2\right)^2\right\} \\ h(T, \dots) = -\alpha Q^2 \end{cases} \,. \tag{C.6}$$

The above PDE must be satisfied for all values of $Q \in \mathbb{R}$. Since h_0 h_1 and h_2 are independent of Q, each of the terms inside curly brackets in (C.6) must be equal to zero independently of Q. This yields the system of PDEs for h_0, h_1 and h_2

$$\begin{cases} 0 = \partial_t h_0 + \frac{1}{4a}h_1^2 \\ h_0(T, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = 0 \end{cases} \tag{C.7}$$

$$\begin{cases} (\partial_t + \bar{\mathcal{L}})h_1 + \left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\boldsymbol{\lambda}, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\boldsymbol{\lambda}, \boldsymbol{\Lambda})\right) + \frac{1}{2a}(\beta + 2h_2)h_1 \\ h_1(T, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = 0 \end{cases} \tag{C.8}$$

$$\begin{cases} \partial_t h_2 - \phi + \frac{1}{4a}\left(\beta + 2h_2\right)^2 = 0 \\ h_2(T) = -\alpha \end{cases} \tag{C.9}$$

which, due to their dependence on one another, can be solved in the order $h_2 \to h_1 \to h_0$. The ODE for h_2 is a standard Riccati-type ODE which admits the unique solution defined in the statement of the proposition. Next, the PDE for h_1 is linear and depends only on the solution for h_2. Therefore we can use the Feynman-Kac formula to write the solution PDE (C.8) as

$$h_1(t, F, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) = \mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}}\left[\int_t^T \left(\widehat{A}(u, F_u, \boldsymbol{N}_u, \boldsymbol{\lambda}_u, \boldsymbol{\Lambda}_u) + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right) e^{\frac{1}{2a}\int_t^u (\beta + 2h_2(\tau))\, d\tau}\, du\right] \,. \tag{C.10}$$

When plugging in the solution for h_2, we get the exact form presented in the statement of the proposition. Furthermore, condition (2.4) and the fact that the term $e^{\frac{1}{2a}\int_t^u(\beta+2h_2(\tau))\,d\tau}$ is bounded for all $0 \le t \le u \le T$ allow us to use Fubini's theorem to yield

$$h_1(t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) = \int_t^T \mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}}\left[\widehat{A}(u,F_u,\boldsymbol{N}_u,\boldsymbol{\lambda}_u,\boldsymbol{\Lambda}_u) + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right] e^{\frac{1}{2a}\int_t^u(\beta+2h_2(\tau))\,d\tau}\,du\ , \quad \text{(C.11)}$$

which in combination with the solution for h_2, gives us the form present in the statement of the proposition.

Lastly, the PDE for h_0 is also linear, and we can therefore use the Feynman-Kac formula once more for a representation of the solution. This representation gives us the expression for h_0 that is present in the statement of the proposition. We can also guarantee that the solution for h_0 provided in proposition 5.1 is bounded by condition 2.4. □

Appendix C.2. Proof of Theorem 5.2

Proof. **Showing the control $\nu^\star$ is admissible.**

The candidate optimal control $\nu^\star$ is defined as

$$\nu_t^\star = \frac{1}{2a}\left(Q_t^{\nu^\star}(\beta+2h_2(t)) + h_1(t,F_t,\boldsymbol{N}_{t-},\boldsymbol{\lambda}_{t-},\boldsymbol{\Lambda}_{t-})\right)\ . \quad \text{(C.12)}$$

It is clear from the definition above that the control is $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted, since it is a continuous function of $\mathcal{F}_t^{F,\boldsymbol{N}}$-adapted processes. To guarantee that the control $\nu_t^\star$ is admissible we must show that

$$\mathbb{E}\left[\int_0^T(\nu_u^\star)^2du\right] < \infty\ . \quad \text{(C.13)}$$

By expanding the expression for $(\nu_u^\star)^2$ and by using Young's inequality twice, we can write an upper bound for $(\nu_u^\star)^2$ as

$$(\nu_u^\star)^2 \le \left(\frac{1}{2a^2}\right)\left((Q_u^{\nu^\star})^2 + (\beta+2h_2(u))^2 + (h_{1,u})^2\right)\ , \quad \text{(C.14)}$$

where $h_{1,u} = h_1(u,F_u,\boldsymbol{N}_{u-},\boldsymbol{\lambda}_{u-},\boldsymbol{\Lambda}_{u-})$. This last inequality shows that to show that equation (C.13) holds if each of $\mathbb{E}\left[\int_0^T(\beta+2h_2(u))^2du\right]$, $\mathbb{E}\left[\int_0^T(Q_u^{\nu^\star})^2du\right]$, and $\mathbb{E}\left[\int_0^T(h_{1,u})^2du\right]$ are bounded.

Using the definition of h_2 in proposition 5.1, we can integrate the first term directly to obtain

$$\mathbb{E}\left[\int_0^T(\beta+2h_2(u))^2du\right] = a^2\gamma^2\left(T+\frac{2}{\gamma}\right)\left(\frac{1}{1-\zeta' e^{2T\gamma}} - \frac{1}{1-\zeta'}\right)\ , \quad \text{(C.15)}$$

where $\zeta' = \frac{\alpha-\frac{1}{2}\beta}{a\gamma}$. This last expression is bounded since $\alpha - \frac{1}{2}\beta \ne a\gamma$.

Next, we can use the definition of $h_{1,u}$ provided in proposition 5.1 to write

$$\mathbb{E}\left[\int_0^T(h_{1,t})^2dt\right] = \tfrac{1}{16a^2}\mathbb{E}\left[\int_0^T\left(\int_t^T \mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F_t,\boldsymbol{N}_t,\boldsymbol{\lambda}_t,\boldsymbol{\Lambda}_T}\left[\widehat{A}(u,F_u,\boldsymbol{N}_u,\boldsymbol{\Lambda}_u) + b(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-)\right]\left(\tfrac{\cosh(\zeta+\gamma(T-u))}{\cosh(\zeta+\gamma(T-t))}\right)du\right)^2dt\right]\ .$$

Now if we notice that $\left(\frac{\cosh(\zeta+\gamma(T-u))}{\cosh(\zeta+\gamma(T-t))}\right)^2 \le 1$ and that

$$\mathop{\mathbb{E}^{\bar{\mathbb{P}}}}_{t,F_t,\boldsymbol{N}_t,\boldsymbol{\lambda}_t,\boldsymbol{\Lambda}_T}\left[\widehat{A}(u,F_u,\boldsymbol{N}_u,\boldsymbol{\Lambda}_u) + b(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-)\right] = \mathop{\mathbb{E}^{\mathbb{P}}}_{t,F_t,\boldsymbol{N}_t,\boldsymbol{\lambda}_t,\boldsymbol{\Lambda}_T}\left[A(u,F_u,\boldsymbol{N}_u,\Theta_u) + b(\lambda_u^+ - \lambda_u^-)\right]\ ,$$

then we can apply Jensen's inequality and Fubini's theorem, followed by Young's inequality to obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^T (h_{1,t})^2 dt\right] &\le \frac{1}{4a^2}\int_0^T\int_t^T \mathbb{E}\left[A_u^2 + b^2\left((\lambda_u^+)^2 - (\lambda_u^-)^2\right)\right] du\, dt \\ &\le \frac{T}{4a^2}\int_0^T \mathbb{E}\left[A_u^2 + b^2\left((\lambda_u^+)^2 - (\lambda_u^-)^2\right)\right] du\end{aligned}$$

where $A_u = A(u, F_u, \boldsymbol{N}_u, \Theta_u)$. By the condition of equation 2.4, this last term is bounded.

By the definition of $Q_t^{\nu^\star}$ and of $\nu^\star$, we have that

$$dQ_t^{\nu^\star} = \frac{1}{2a}\left(Q_t^{\nu^\star}(\beta + 2h_2(t)) + h_{1,t}\right) dt\, , Q_0^{\nu^\star} = \mathfrak{N}\, . \tag{C.16}$$

The above SDE has the solution

$$Q_t^{\nu^\star} = \mathfrak{N} + \frac{1}{2a}\int_0^t h_{1,u}\left(\frac{\cosh(\zeta + \gamma(T-u))}{\cosh(\zeta + \gamma(T-t))}\right) du\, . \tag{C.17}$$

By using Young's inequality and Jensen's inequality again, and by using the fact that $\left(\frac{\cosh(\zeta+\gamma(T-u))}{\cosh(\zeta+\gamma(T-t))}\right)^2 \le 1$, then we can write

$$(Q_t^{\nu^\star})^2 \le \frac{1}{a}\left(\mathfrak{N}^2 + \int_0^t (h_{1,u})^2 du\right)\, . \tag{C.18}$$

Now by taking the expectation and the integral of this last expression, we get

$$\mathbb{E}\left[\int_0^T (Q_u^{\nu^\star})^2\, du\right] \le \frac{1}{a}\left(T\,\mathfrak{N}^2 + \mathbb{E}\left[\int_0^T\int_0^t (h_{1,u})^2 du\, dt\right]\right) \tag{C.19}$$

$$\le \frac{1}{a}\left(T\,\mathfrak{N}^2 + T\mathbb{E}\left[\int_0^T (h_{1,u})^2 du\right]\right)\, . \tag{C.20}$$

Since the term $\mathbb{E}\left[\int_0^T (h_{1,u})^2 du\right]$ has already been shown to be bounded, we can conclude that $\mathbb{E}\left[\int_0^T (Q_u^{\nu^\star})^2\, du\right] < \infty$.

$\nu_t^\star$ is $\mathcal{F}_t^{S,\boldsymbol{N}}$-adapted and satisfies $\mathbb{E}\left[\int_0^T (\nu_u^\star)^2 du\right] < \infty$, therefore it is an admissible control.

Showing $H \le \widehat{H}$. By applying Itô's lemma to the function $\widehat{H} = X + QS + h$ with an arbitrary control $\nu_t \in \mathcal{A}$ and the $\mathcal{F}_t^{F,\boldsymbol{N}}$-predictable dynamics, we get

$$\begin{aligned}\widehat{H}_T^\nu = \widehat{H}(t, S, F, Q, X, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) &+ \int_t^T \left\{Q_u^\nu\left(\widehat{A}_u + b(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-)\right) - a\nu_u^2 + (\beta + \partial_Q h_u)\nu_u + \left(\partial_t + \bar{\mathcal{L}}\right) h_u\right\} du \\ &+ \int_t^T \eta_u^W\, d\widehat{W}_u + \int_t^T \eta_u^+\, d\widehat{M}_t^+ + \int_t^T \eta_u^-\, \widehat{H}_u\, d\widehat{M}_t^-\, ,\end{aligned}$$

where in the above, we use the notation $f_t = f(t, Y_t)$ and η_u^W, η_u^+ and η_u^- are square-integrable $\mathcal{F}_t^{F,\boldsymbol{N}}$-predictable processes obtained by the martingale representation theorem.

By taking the conditional expected value of both sides, the Martingale portions vanish and we are left with

$$\begin{aligned}\mathbb{E}_t\left[\widehat{H}_T^\nu\right] &= \widehat{H}(t, S, F, Q, X, \boldsymbol{N}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) \\ &+ \mathbb{E}_t\left[\int_t^T \left\{Q_u^\nu\left(\widehat{A}_u + b(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-)\right) - a\nu_u^2 + (\beta + \partial_Q h_u)\nu_u + \left(\partial_t + \bar{\mathcal{L}}\right) h_u\right\} du\right]\, ,\end{aligned} \tag{C.21}$$

where in the above we use the shorthand $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot \mid Z_t = (t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda})]$. From equation (5.6), we get that for all $\nu \in \mathbb{R}$,

$$0 \geq -\phi Q^2 + Q\,\left(\widehat{A}(t,F,\boldsymbol{N}_t,\boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\boldsymbol{\lambda},\boldsymbol{\Lambda}) - \widehat{\lambda}^-(\boldsymbol{\lambda},\boldsymbol{\Lambda}))\right) + (\beta Q + \partial_Q h)\nu - a\nu^2 + \left(\partial_t + \bar{\mathcal{L}}\right) h\;. \tag{C.22}$$

Therefore, by plugging in the boundary condition for $\widehat{H}_T$

$$\mathbb{E}_t\left[\widehat{H}^\nu_T - \phi\int_t^T Q^\nu_u du\right] = \mathbb{E}_t\left[X^\nu_T + Q^\nu_t(S^\nu_T - \alpha Q^\nu_T) - \phi\int_t^T Q^\nu_u du\right] \leq \widehat{H}(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda})\;. \tag{C.23}$$

Now since the above holds for an arbitrary $\nu_t \in \mathcal{A}$, we obtain

$$H(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) \leq \widehat{H}(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda})\;. \tag{C.24}$$

Showing $H \geq H^{\nu^\star} \geq \widehat{H}$. Next let us note that if we let $\nu^\star = \frac{\beta Q + \partial_Q h}{2a}$, then by equation (5.6), $\forall\varepsilon > 0$,

$$-\varepsilon < -\phi Q^2 + Q\,\left(\widehat{A}(t,F,\boldsymbol{N}_t,\boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\boldsymbol{\lambda},\boldsymbol{\Lambda}) - \widehat{\lambda}^-(\boldsymbol{\lambda},\boldsymbol{\Lambda}))\right) + (\beta Q + \partial_Q h)\nu^\star - a\nu^{\star 2} + \left(\partial_t + \bar{\mathcal{L}}\right) h\;. \tag{C.25}$$

Using this last inequality with equation (C.21) and the definition of H^ν gives

$$\begin{aligned} H^{\nu^\star}(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) &\geq \mathbb{E}_t\left[X^{\nu^\star}_T + Q^{\nu^\star}_t(S^{\nu^\star}_T - \alpha Q^{\nu^\star}_T) - \phi\int_t^T Q^{\nu^\star}_u du\right] - \varepsilon \\ &> \widehat{H}(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda})\;, \end{aligned}$$

Since $H \geq H^\nu$, $\forall \nu \in \mathcal{A}$ we get

$$H(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) \geq H^{\nu^\star}(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda}) \geq \widehat{H}(t,S,F,Q,X,\boldsymbol{N},\boldsymbol{\lambda},\boldsymbol{\Lambda})\;. \tag{C.26}$$

Therefore we obtain the desired result that

$$H = H^{\nu^\star} = \widehat{H} \tag{C.27}$$

□

Appendix D. Bibliography

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