Numerical Analysis Project

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1 Lagrange Polynomial

1.1 Problem (a)

Let $f(x) = \dfrac{1}{1+25x^2}$ be a function in $[-1,1]$, and let $p_n(x)$ be the polynomial of degree at most n that interpolates the function f at $n+1$ equally-spaced points $x_0, x_1, \cdots, x_n$ in the interval $[-1,1]$. Thus one gets

$$h = \frac{2.0}{n}, x_k = -1.0 + kh, k = 0, 1, \cdots, n \tag{1}$$

.

$p_n(x)$ is an approximation of $f(x)$ at these points. Choose $n = 5, 10, 20$ and compute $p_n(x)$ at

$$x = -0.95, -0.47, 0.1, 0.37, 0.93$$

and compare them with $f(x)$ at these points.

Solution A Python program was created to solve for the value of the polynomial obtained by Lagrangian interpolation at a given point, and a plot of the interpolated polynomial against the original function was drawn. The following outputs are the errors:

```
n = 5:
f(-0.95)-p_n(-0.95) = 0.05817799859084909
f(-0.47)-p_n(-0.47) = -0.09031533471876207
f(0.1)-p_n(0.1) = 0.2498798076923081
f(0.37)-p_n(0.37) = -0.12677490820134119
f(0.93)-p_n(0.93) = 0.07473742125006333

n = 10:
f(-0.95)-p_n(-0.95) = -1.8811908314168162
f(-0.47)-p_n(-0.47) = -0.0936037537592041
f(0.1)-p_n(0.1) = -0.04340742982890311
f(0.37)-p_n(0.37) = 0.03707696272967939
f(0.93)-p_n(0.93) = -1.8837448249769186
```

```
n = 20:
f(-0.95)-p_n(-0.95) = 39.99488935134396
f(-0.47)-p_n(-0.47) = -0.014037800024217284
f(0.1)-p_n(0.1) = -6.661338147750939e-16
f(0.37)-p_n(0.37) = 0.006985095986455858
f(0.93)-p_n(0.93) = 18.598125682561882
```

The plots generated are listed here:

From the errors outputed and Figure 1, it's clear that the value of the interpolating polynomial deviates significantly from the value of the original function near the boundaries of the range. And with the increase of n, the errors also increase. This is called Runge phenomenon.

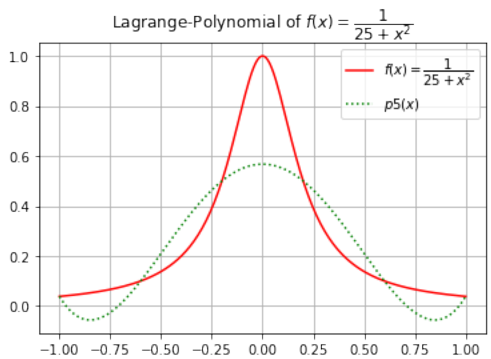
1.2 Problem (b)

Let $f(x) = e^x$ be a function in $[-1, 1]$ and $p_n(x)$ be the interpolating polynomia of $n + 1$ equally-spaced points. Choose $n = 5, 10, 20$ and compute $p_n(x)$ at $x = -0.95, -0.47, 0.1, 0.37, 0.93$ and compare them with $f(x)$ at these points.

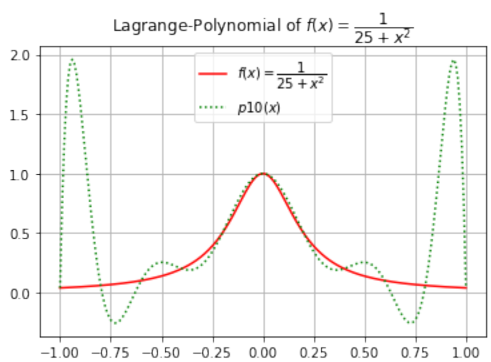
Discribe your observation. Based on your observation, do you think the higher the interpolating polynomial degree, the better?

Solution With the same method, the errors are calculated and displayed as follow (To make things clear, we replace $f(x) = e^x$ with $g(x) = e^x$):

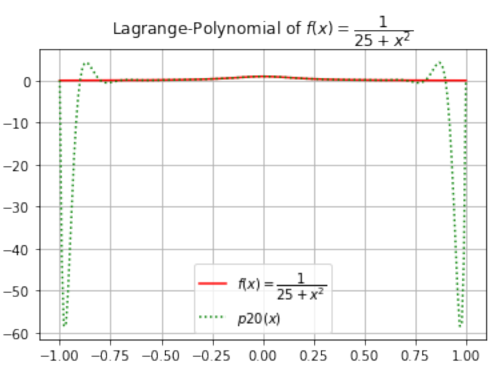
```
n = 5:
g(-0.95)-p_n(-0.95) = -5.713540349228108e-05
g(-0.47)-p_n(-0.47) = 2.6157349590438805e-05
g(0.1)-p_n(0.1) = -1.5015848680688393e-05
g(0.37)-p_n(0.37) = 2.8034715736424687e-05
g(0.93)-p_n(0.93) = -9.20797823367181e-05

n = 10:
g(-0.95)-p_n(-0.95) = 1.9882628876644048e-10
g(-0.47)-p_n(-0.47) = 5.749622999928761e-12
g(0.1)-p_n(0.1) = -2.517097641430155e-12
g(0.37)-p_n(0.37) = 2.1027624086400465e-12
g(0.93)-p_n(0.93) = -2.283302436012491e-10

n = 20:
g(-0.95)-p_n(-0.95) = -6.306066779870889e-14
g(-0.47)-p_n(-0.47) = 1.1102230246251565e-16
g(0.1)-p_n(0.1) = -8.881784197001252e-16
g(0.37)-p_n(0.37) = 0.0
g(0.93)-p_n(0.93) = 4.929390229335695e-14
```



(a) $n = 5$

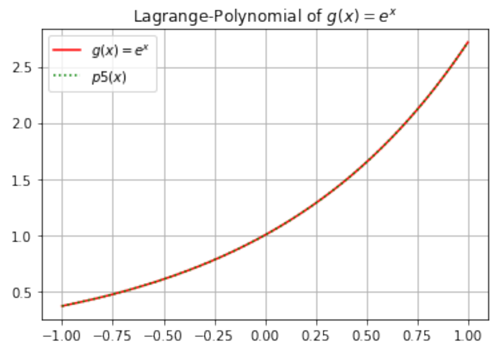


(b) $n = 10$

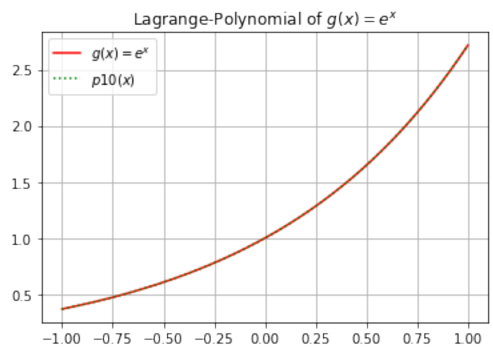


(c) $n = 20$

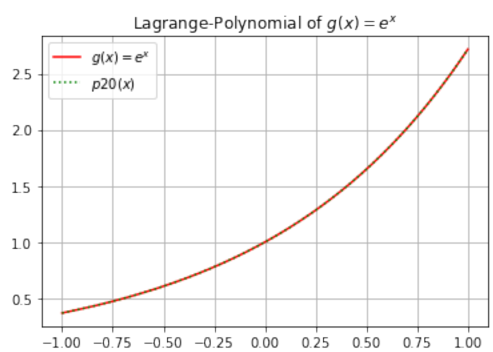
Figure 1: Comparison of function $f(x) = \dfrac{1}{1 + 25x^2}$ and interpolated polynomial



(a) $n = 5$



(b) $n = 10$



(c) $n = 20$

Figure 2: Comparison of function $g(x) = e^x$ and interpolated polynomial

Things are completely different with $g(x) = e^x$. The errors were extremely small, and with the increase of n, the errors go down. From Figure 2, it shows clearly that even for a small n, the interpolated polynomial fits the original function perfectly.

For diffetent functions, it's not the larger n, the more perfect the interpolated polynomial.

1.3 Problem (c)

Let $f(x) = \dfrac{1}{25 + x^2}$ be a function in $[-1, 1]$ and consider the interpolating polynomial with respect to thr distinct points $x_i = \cos\dfrac{(2k+1)\pi}{2(n+1)}$, $k = 0, 1, \cdots, n$. Choose $n = 5, 10, 20$ and compare them with $f(x)$ at these points.

Describe the observation. Read reference [1], p.315-323 and explain this phenomenon.

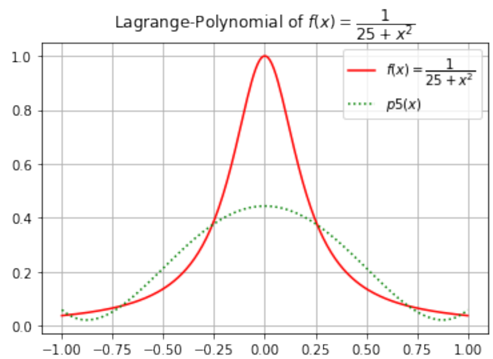
Solution Use the improved method, the results are outputed.

```
n = 5:
f(-0.95)-p_n(-0.95) = 0.007747003469783263
f(-0.47)-p_n(-0.47) = -0.08343042460406644
f(0.1)-p_n(0.1) = 0.3667983064716973
f(0.37)-p_n(0.37) = -0.08129139149124548
f(0.93)-p_n(0.93) = 0.015594529058462814

n = 10:
f(-0.95)-p_n(-0.95) = -0.043094613035724004
f(-0.47)-p_n(-0.47) = 0.06431306024108874
f(0.1)-p_n(0.1) = -0.08124703825599944
f(0.37)-p_n(0.37) = 0.08153549403081947
f(0.93)-p_n(0.93) = -0.025341946966374165

n = 20:
f(-0.95)-p_n(-0.95) = -0.0057596689619212466
f(-0.47)-p_n(-0.47) = 0.00834879653818646
f(0.1)-p_n(0.1) = -0.010624838022320393
f(0.37)-p_n(0.37) = -0.012827402039512492
f(0.93)-p_n(0.93) = 0.00031722267527464765
```

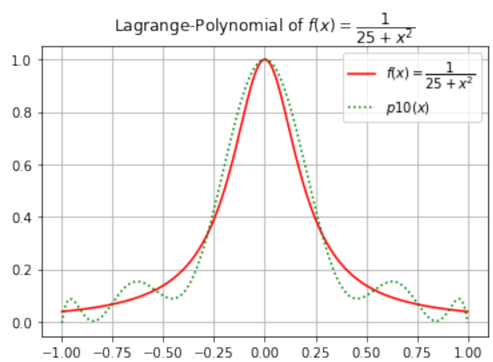
According to Figure 3, when replace equidistant nodes with Chebyshev nodes, the errors converge to 0 with the increase of n. Whether or not the Runge phenomenon appears is related to the selection of the interpolation node set.

In Probelm (a), the Runge phenomenon appears because of

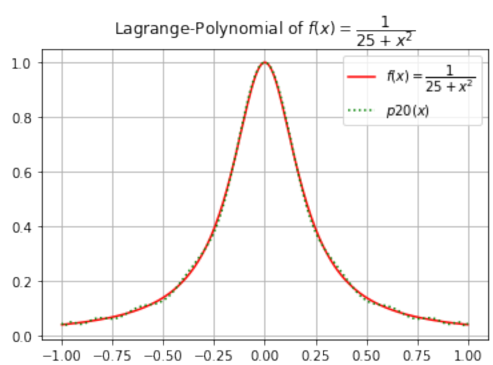
$$\lim_{n\to\infty} (\max_{-1\leq x\leq 1} |f(x) - P_n(x)|) = \infty \tag{2}$$



(a) $n = 5$



(b) $n = 10$



(c) $n = 20$

Figure 3: Comparison of function $f(x) = \dfrac{1}{1 + 25x^2}$ and interpolated polynomial

.

The errors of interpolated polynomial comes from higher order remainder term of Taylor expansion.

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i) \tag{3}$$

.

For some ξ in range $(-1, 1)$, there is:

$$\max_{-1\le x\le 1}|f(x) - P_n(x)| \le \max_{-1\le x\le 1}\frac{|f^{(n+1)}(x)|}{(n+1)!}\max_{-1\le x\le 1}\prod_{i=0}^{n}|x - x_i| \tag{4}$$

.

Define $w(x)$ to be the nodal function:

$$w_n(x) = \prod_{i=1}^{n}(x - x_i) \tag{5}$$

.

It's hard to solve $\max\limits_{-1\le x\le 1}|f^{(n+1)}(x)|$, consider Cauchy's integral formula:

$$f^{(n)}(\xi) = \frac{n!}{2\pi j}\int_C \frac{f(z)}{(z-\xi)^{n+1}}\mathrm{d}z \tag{6}$$

, now we can solve the errors:

$$f(z) - P_n(z) = \frac{1}{2\pi j}\int_C \frac{w_n(z)f(\xi)}{w_n(\xi)(\xi - z)}\mathrm{d}\xi \tag{7}$$

,C is the boundary of domain D. According to the conditions of Cauchy's integral formula, f is a holomorphic function in D, and all the interpolation nodes should be in D.

For a sufficiently large r, $w_n(z) = r^n$ is nearly a circle. Use $C^* : w_n(\xi) = r^n$ as the loop of integration. For all z in C^*, $\lim\limits_{n\to\infty}|\frac{w_n(z)}{w_n(\xi)}| = 0$. For all z out of C^*, $\lim\limits_{n\to\infty}|\frac{w_n(z)}{w_n(\xi)}| = \infty$. When the singularities of $f(z)$:$z_1 = \frac{j}{5}$, $z_2 = -\frac{j}{5}$ are in C^*, $f(z) - P_n(z)$ goes towards to infinity.

When we replace equidistant nodes with Chebyshev nodes, $w_n(z) = r^n$ becomes a ellipse with focuses ± 1. Singularities $\pm\frac{j}{5}$ are out of C^*, so $f(z) - P_n(z)$ goes towards to zero. The Runge phenomenon disappeared.

2 Numerical Integration

2.1 Problem (a)

To compute the integral $\int_a^b f(x)\mathrm{d}x$ numerically, one can use the Romberg integration formula. To be more specific, let $R(n, 0)$ denote the trapezoid estimate with 2^n subintervals, we have the Romberg formula:

$$\begin{cases} R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)], \\ R(n,0) = \frac{1}{2}R(n-1,0) + h_n \sum_{i=1}^{2^{n-1}} f(a+(2i-1)h_n) \end{cases} \tag{8}$$

where

$$h_0 = b-a, h_n = \frac{h_{n-1}}{2} \tag{9}$$

.

Use Romberg integration to compute the following integrals with the tolerance of accuracy to be $\epsilon = 10^{-7}$, which means if $[R(n+1,0) - R(n,0)] \leq \epsilon$, then we use $R(n,0)$ as the desired numerical integration.

- (a) $\int_0^1 x^2 e^x \mathrm{d}x$
- (b) $\int_1^3 e^x \sin x \mathrm{d}x$
- (c) $\int_0^1 \frac{4}{1+x^2} \mathrm{d}x$
- (d) $\int_0^1 \frac{1}{x+1} \mathrm{d}x$

Solution First, solve for the parsing solutions:

$$\begin{aligned} \int_0^1 x^2 e^x \mathrm{d}x = e-2 \\ \int_1^3 e^x \sin x \mathrm{d}x = \frac{e}{2}(-\sin 1 + \cos 1 + e^2(\sin 3 - \cos 3)) \\ \int_0^1 \frac{4}{1+x^2} \mathrm{d}x = \pi \\ \int_0^1 \frac{1}{x+1} \mathrm{d}x = \ln 2 \end{aligned} \tag{10}$$

The Romberg integration algorithm is simple. There comes the results:

```
Romberg Integration

f(x) = x^2 e^x
a = 0, b = 1
I =  0.7182818385854369

f(x) = e^x sin(x)
```

```
a = 1, b = 3
I =  10.950170288849279

f(x) = 4/(1+x^2)
a = 0, b = 1
I =  3.1415926436556827

f(x) = 1/(1+x)
a = 0, b = 1
I =  0.6931471954611083
```

Errors are less than ϵ.

2.2 Problem (b)

Use Gauss-Legendre quadrature rule to compute the above integrals to $\epsilon = 10^{-7}$ and compare the number of quadrature points used.

Solution The code to generate the weights and Gauss nodes are not given. So it's necessary to write it by myself. There comes the results:

```
Gauss-Legendre Integration

f(x) = x^2 e^x
a = 0, b = 1
I =  0.718281828393355

f(x) = e^x sin(x)
a = 1, b = 3
I =  10.95017031533695

f(x) = 4/(1+x^2)
a = 0, b = 1
I =  3.1415926111875865

f(x) = 1/(1+x)
a = 0, b = 1
I =  0.6931471798865281
```

Errors are less than ϵ.

2.3 Problem (c)

Compute $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x$ to the tolerance of accuracy $\epsilon = 10^{-7}$. (Hint: although the integral is in unbounded domain, the integrand itself decays fast.)

Solution The parsing solution:

$$\int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi} \tag{11}$$

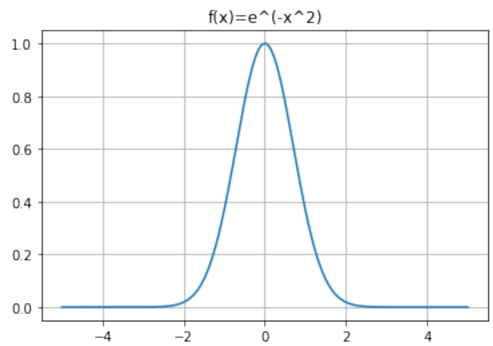


Figure 4: Function $f(x) = e^{-x^2}$

The function's plot is Figure 4. When $|x|$ increases, $f(x)$ converge to 0 rapidly. Calculate the integration from -5 to 5 using two methods:

```
Romberg Integration
I = 1.7724538509008183

Gauss-Legendre Integration
I = 1.772453844220363
```

Errors are less than ϵ.