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1. INTRODUCTION. Discussions that led to this paper began during an electronic version of the Secondary School Teachers Program, a part of the Park City Mathematics Institute (PCMI), offered at the University of Cincinnati in the summer of 2006 for local high school mathematics teachers. The authors were working with the teachers on daily problem sets provided by PCMI; complex numbers and polynomials were a theme throughout the problems. The second author raised the question of geometric connections between the triangle whose vertices are the roots of a cubic polynomial and the roots of the derivative of the polynomial. He had noticed some relationships, such as that the roots of the derivative were inside the triangle, by performing experiments using *Geometers Sketchpad*. The second author was able to justify some of the observed relations. We continued to investigate the matter for the three weeks of the program. Roughly speaking, we followed the trail of results described in the next section. The fundamental connection is that if z_1, z_2, z_3 are the vertices of a triangle and $p(z) = (z - z_1)(z - z_2)(z - z_3)$, then the roots of $p'(z)$ are the foci of the Steiner inellipse for the triangle. The Steiner inellipse is the unique ellipse that is inscribed in the triangle and tangent to the sides at their midpoints. The inellipse degenerates to a circle precisely when the triangle is equilateral, and this occurs if and only if $p'(z)$ has a repeated root. The Steiner inellipse for a triangle has the largest area among all ellipses contained in the triangle. A related result is that for a nonequilateral triangle the unique line of best fit for the vertices of the triangle is the line through the foci of the inellipse. These attractive results deserve to be better known; we offer an elementary approach using basic complex analysis and simple affine geometry.

2. MAIN RESULTS. There are a number of interesting connections between the roots of a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_n \neq 0$, and the roots of the derivative $p'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1$. We recall some of these.

Perhaps the best known result of this type is the Gauss-Lucas theorem, which asserts that the zeros of the derivative lie in the convex hull of the zeros of the polynomial [**1**, p. 29]. Another result of this type is that the centroid of the zeros of a polynomial equals the centroid of the roots of the derivative. This latter property is easily verified from a connection between the roots of a polynomial and the coefficients. If z_j, $1 \le j \le n$, are the roots of $p(z)$, then $a_{n-1}/a_n = -(z_1 + \cdots + z_n)$. Similarly, if z'_j, $1 \le j \le n-1$, are the roots of the derivative, then

$$\frac{(n-1)a_{n-1}}{n a_n} = -(z'_1 + \cdots + z'_{n-1}).$$

Therefore,

$$\frac{1}{n}(z_1 + \cdots + z_n) = \frac{1}{n-1}(z'_1 + \cdots + z'_{n-1}). \tag{2.1}$$

In particular, for a quadratic polynomial, the root of the derivative is the midpoint of the segment determined by the roots of the quadratic; for a real quadratic polynomial with distinct real roots this is the familiar result that the vertex of a parabola lies midway between the points at which the parabola crosses the x-axis.

The focus of this paper is on connections between the roots of a cubic polynomial and those of the quadratic derivative. We begin by establishing some elementary connections between these roots. We consider only the generic case in which the roots of the cubic are distinct and noncollinear, say the roots are z_1, z_2, z_3. We may assume the cubic is monic, $p(z) = (z - z_1)(z - z_2)(z - z_3)$, since division by a nonzero constant does not change the roots of a polynomial or its derivative. Note that

$$p(z) = z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_2z_3 + z_1z_3)z - z_1z_2z_3 \tag{2.2}$$

and

$$p'(z) = 3z^2 - 2(z_1 + z_2 + z_3)z + (z_1z_2 + z_2z_3 + z_1z_3). \tag{2.3}$$

It is convenient to let $g = \frac{1}{3}(z_1 + z_2 + z_3)$, which is the centroid of the triangle $\triangle z_1z_2z_3$ determined by the roots of the cubic. The roots of $p'(z)$ are

$$g \pm \sqrt{g^2 - \frac{1}{3}(z_1z_2 + z_2z_3 + z_1z_3)}\,; \tag{2.4}$$

the roots are symmetric about the centroid. By the Gauss-Lucas theorem both roots of $p'(z)$ lie inside or on $\triangle z_1z_2z_3$. In fact, the roots of $p'(z)$ lie inside the triangle. To see this, suppose z' is a root of $p'(z)$. Observe that $z' \neq z_j$, $j = 1, 2, 3$, since $p(z)$ has no repeated roots. Then

$$\begin{aligned} 0 = \overline{\left(\frac{p'(z')}{p(z')}\right)} &= \frac{1}{\overline{z'} - \bar{z}_1} + \frac{1}{\overline{z'} - \bar{z}_2} + \frac{1}{\overline{z'} - \bar{z}_3} \\ &= \frac{z' - z_1}{|z' - z_1|^2} + \frac{z' - z_2}{|z' - z_2|^2} + \frac{z' - z_3}{|z' - z_3|^2}. \end{aligned}$$

This gives $z' = \alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3$, where

$$\alpha_k = \frac{|z' - z_k|^{-2}}{\sum_{j=1}^{3} |z' - z_j|^{-2}} > 0$$

and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Since z' is a proper convex combination of z_1, z_2, z_3, it lies inside $\triangle z_1z_2z_3$. Next, we show that $p'(z)$ has a single repeated root (at the centroid) if and only if $\triangle z_1z_2z_3$ is equilateral. From (2.4), $p'(z)$ has a repeated root if and only if

$$3g^2 = z_1z_2 + z_2z_3 + z_1z_3, \tag{2.5}$$

or equivalently,

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_1z_3. \tag{2.6}$$

It is a standard exercise in complex analysis ([**1**, p. 15] and [**5**, p. 32]) to show that the identity (2.6) holds if and only if z_1, z_2, z_3 are the vertices of an equilateral triangle. Here is a simple argument to establish this result. By making use of (2.5) we "complete the cube" and obtain

$$p(z) = z^3 - 3gz^2 + 3g^2z - z_1z_2z_3 = (z - g)^3 - z_1z_2z_3 + g^3.$$

If ζ denotes a cube root of $z_1z_2z_3 - g^3$, then the roots of $p(z)$ are $g + \zeta$, $g + \omega\zeta$, $g + \omega^2\zeta$, where $\omega = \exp(2\pi i/3)$. These points are the vertices of an equilateral triangle.

A more sophisticated connection between the roots of a cubic and those of the derivative follows from a lovely geometric result of Steiner. The incircle of a triangle is tangent to the sides and the three points of tangency are the midpoints of the sides if and only if the triangle is equilateral. Steiner showed that it is always possible to find an ellipse inscribed in a triangle that is tangent at the midpoints of the sides.

Theorem 2.1 (Steiner). *Given any triangle there is a unique ellipse inscribed in the triangle that passes through the midpoints of the sides of the triangle and is tangent to the sides of the triangle at these midpoints. If z_1, z_2, z_3 are the vertices of the triangle, then the foci of this ellipse are*

$$g \pm \sqrt{g^2 - \frac{1}{3}(z_1z_2 + z_2z_3 + z_1z_3)}, \tag{2.7}$$

where $g = \frac{1}{3}(z_1 + z_2 + z_3)$ is the centroid.

The ellipse that is inscribed in $\triangle z_1z_2z_3$ and tangent at the midpoints of the sides is called the *Steiner inellipse*. Note that the center of the Steiner inellipse is at the centroid of the triangle. The inellipse degenerates to a circle if and only if $\triangle z_1z_2z_3$ is equilateral. From (2.4) and (2.7) we immediately obtain the following result.

Corollary 2.2 (Siebeck). *Suppose z_1, z_2, z_3 are noncollinear points in $\mathbb{C}$ and $p(z) = (z - z_1)(z - z_2)(z - z_3)$. Then the roots of $p'(z)$ are the foci of the Steiner inellipse.*

This corollary is due to Siebeck [**6**]. His result has been reproved and extended by a number of authors; see [**4**, pp. 9–13] for details.

Another appealing connection was noted by Coolidge [**2**]. For a set of n points z_j in $\mathbb{C}$, $1 \le j \le n$, consider the line(s) that best fit these points in the following sense. Given a line ℓ, let $\operatorname{dist}(z_j, \ell)$ denote the distance from z_j to ℓ. A line ℓ for which

$$D = \sum_{j=1}^{n} \operatorname{dist}^2(z_j, \ell) \tag{2.8}$$

is minimized is called a *line of best fit* for the points z_j, $1 \le j \le n$. For $n = 2$ the unique line of best fit is the line through the two points. Note that the sum of the squares of orthogonal distances is being minimized, not the sum of the squares of the vertical separations used in determining a regression line. Also, if f is any Euclidean similarity and ℓ is a line of best fit for $z_1, \dots, z_n$, then $f(\ell)$ is a line of best fit for $f(z_1), \dots, f(z_n)$, a property not shared by the regression line.

Theorem 2.3. *Suppose z_j, $1 \le j \le n$, are points in $\mathbb{C}$, $g = \frac{1}{n}\sum_{j=1}^{n} z_j$ is the centroid, and $Z = \sum_{j=1}^{n}(z_j - g)^2 = \sum_{j=1}^{n} z_j^2 - ng^2$.*

(a) *If $Z = 0$, then every line through g is a line of best fit for the points $z_1, \dots, z_n$.*
(b) *If $Z \ne 0$, then the line through g that is parallel to the vector from 0 to $\sqrt{Z}$ is the unique line of best fit for $z_1, \dots, z_n$.*

For three noncollinear points z_1, z_2, z_3,

$$Z = \sum_{j=1}^{3} z_j^2 - 3g^2 = 6\left(g^2 - \frac{1}{3}(z_1z_2 + z_2z_3 + z_1z_3)\right).$$

From (2.5) $Z = 0$ if and only if z_1, z_2, z_3 are the vertices of an equilateral triangle. Thus, if z_j, $1 \leq j \leq 3$, are the vertices of an equilateral triangle, then *every* line through the centroid is a line of best fit. If $\triangle z_1z_2z_3$ is not equilateral, then the unique line of best fit is the line through g that is parallel to $\sqrt{g^2 - \frac{1}{3}(z_1z_2 + z_2z_3 + z_1z_3)}$.

Corollary 2.4 (Coolidge). *Suppose $\triangle z_1z_2z_3$ is nonequilateral. The line through the foci of the Steiner inellipse for $\triangle z_1z_2z_3$ is the line of best fit for z_1, z_2, z_3. Equivalently, if $p(z) = (z - z_1)(z - z_2)(z - z_3)$, then the line of best fit is the line through the roots of $p'(z)$.*

Proof. The line of best fit is the line through the centroid g that is parallel to $\sqrt{g^2 - \frac{1}{3}(z_1z_2 + z_2z_3 + z_1z_3)}$. From (2.4) this vector is parallel to the vector joining the roots of $p'(z)$, so the line of best fit is the line through the roots of $p'(z)$. ■

Let $p(z)$ be a given cubic polynomial with noncollinear roots, z_1, z_2, z_3. For any $\lambda \in \mathbb{C}$, let $p_\lambda(z) = p(z) + \lambda$. This specifies a one-parameter family of cubic polynomials and most of the $p_\lambda(z)$ will have noncollinear roots, say $z_1(\lambda), z_2(\lambda), z_3(\lambda)$. Since $p'_\lambda(z) = p'(z)$, the inellipses for the triangles $\triangle z_1(\lambda)z_2(\lambda)z_3(\lambda)$ are all confocal and the vertices of the triangles have the same line of best fit, independent of λ. Also, all of the triangles $\triangle z_1(\lambda)z_2(\lambda)z_3(\lambda)$ have the same centroid. This is a very remarkable situation.

The remainder of the paper is devoted to establishing Theorems 2.1 and 2.3.

3. LINEAR AND AFFINE TRANSFORMATIONS. Our proof of Steiner's theorem makes use of affine transformations, so we recall the necessary facts about linear and affine transformations of the Cartesian plane $\mathbb{R}^2$. We regard the plane as the set $\mathbb{C}$ of complex numbers and write transformations in terms of complex numbers by identifying the point $\binom{x}{y}$ of $\mathbb{R}^2$ with the complex number $z = x + iy$. In terms of complex numbers, a real linear transformation f has the form

$$f(z) = Az + B\bar{z},$$

where $A = a_1 + ia_2$ and $B = b_1 + ib_2$. If $A \neq 0$ and $B = 0$, then $f(z) = Az$ is an orientation preserving Euclidean similarity. If $A = 0$ and $B \neq 0$, then $f(z) = B\bar{z}$ is an orientation reversing Euclidean similarity. The matrix representation of f is $f\binom{x}{y} = M\binom{x}{y}$, where

$$M = \begin{pmatrix} a_1 + b_1 & -a_2 + b_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix}.$$

The linear transformation f is nonsingular if and only if $0 \neq \det M = |A|^2 - |B|^2$. Thus, f is bijective if and only if $|A| \neq |B|$. If f is nonsingular, then the image of any circle about the origin is an ellipse or a circle with center at the origin. It is convenient to let $\mathbb{T} = \{z : |z| = 1\}$ denote the unit circle with center at the origin and $r\mathbb{T}$ the circle about the origin with radius $r > 0$. Conversely, given an ellipse E with center at the origin and any circle $r\mathbb{T}$, there is a nonsingular linear transformation that maps the circle $r\mathbb{T}$ onto the ellipse E.

In complex notation, an affine transformation has the form $f(z) = Az + B\bar{z} + C$, where C is a complex number; an affine transformation is a linear transformation followed by a translation. The defining property of an affine transformation is that for all $z, w \in \mathbb{C}$ and any $t \in \mathbb{R}$,

$$f\big((1-t)z + tw\big) = (1-t)f(z) + tf(w). \tag{3.1}$$

Affine geometry is the study of properties that are invariant under affine transformations. Affine transformations do not preserve length or angle. On the other hand, an affine transformation maps lines to lines, parallel lines to parallel lines, midpoints to midpoints, and centroids to centroids. That midpoints are preserved follows from (3.1) with $t = 1/2$. Affine transformations also preserve tangency. Any pair of triangles are affinely equivalent.

Theorem 3.1. *If $f(z) = Az + B\bar{z} + C$ is a bijective affine transformation, then the image of the circle $r\mathbb{T}$ is the ellipse with foci $C \pm 2r\sqrt{AB}$, semi-major axis with length $(|A| + |B|)r$, and semi-minor axis with length $\big||A| - |B|\big|r$.*

Proof. It suffices to consider the case in which $C = 0$, $f(z) = Az + B\bar{z}$ is a bijective linear transformation, and $r = 1$ since $f(rz) = rf(z)$. If $A = 0$ or $B = 0$, then f is a Euclidean similarity and the image of the unit circle is a circle with center 0 and radius $|B|$ or $|A|$, respectively. Now we assume A and B are both nonzero. From linear algebra we know that the image $f(\mathbb{T})$ of the unit circle is an ellipse. We show the ellipse $f(\mathbb{T})$ has foci $\pm 2\sqrt{AB}$, semi-major axis with length $|A| + |B|$, and semi-minor axis with length $\big||A| - |B|\big|$. A parametrization of the unit circle is $t \mapsto e^{it}$, so the image of the unit circle is $f(e^{it}) = Ae^{it} + Be^{-it} = |A|e^{i(\theta+t)} + |B|e^{i(\varphi-t)}$, where $A = |A|e^{i\theta}$ and $B = |B|e^{i\varphi}$. The elementary inequalities

$$\big||A| - |B|\big| \le \big||A|e^{i(\theta+t)} + |B|e^{i(\varphi-t)}\big| \le |A| + |B|,$$

show that the image ellipse contains the circle $|z| = \big||A| - |B|\big|$ and lies within the circle $|z| = |A| + |B|$. Equality holds in the upper bound if and only if $e^{i(\theta+t)} = e^{i(\varphi-t)}$. This occurs when $\theta + t = \varphi - t + 2n\pi$, or $t = \frac{1}{2}(\varphi - \theta) + n\pi$, for some $n \in \mathbb{Z}$. Thus, for $t = \frac{1}{2}(\varphi - \theta)$ and $t = \frac{1}{2}(\varphi - \theta) + \pi$, $|f(e^{it})| = |A| + |B|$, so the length of the semi-major axis is $a = |A| + |B|$. Since $e^{i\varphi/2} = \sqrt{B}/|B|^{1/2}$ and $e^{i\theta/2} = \sqrt{A}/|A|^{1/2}$, we obtain

$$f\left(e^{i(\varphi-\theta)/2}\right) = \frac{|A| + |B|}{|AB|^{1/2}}\sqrt{AB}\,.$$

Therefore, a direction vector for the major axis is $\sqrt{AB}$. Likewise, equality holds in the lower bound when $e^{i(\theta+t)} = -e^{i(\varphi-t)}$; that is, $\theta + t = \varphi - t + \pi + 2n\pi$, or $t = \frac{1}{2}(\varphi - \theta) + \frac{\pi}{2} + n\pi$, for some $n \in \mathbb{Z}$. Hence, the length of the semi-minor axis is $b = \big||A| - |B|\big|$. Then c, the distance from the center of the ellipse to the foci, is determined from $c^2 = a^2 - b^2 = 4|A||B|$, so that $c = 2|A|^{1/2}|B|^{1/2}$. Consequently the foci are $\pm 2\sqrt{AB}$. ■

4. THE STEINER INELLIPSE.

For an equilateral triangle the inellipse is simply the inscribed circle. The general case follows from this special case by making use of affine transformations.

Proof of Steiner's theorem. We begin by establishing the existence of an ellipse inscribed in a triangle and tangent at the midpoints of the sides. Let $\triangle$ be the equilateral triangle with vertices at 1, $\omega = \exp(2\pi i/3)$ and ω^2. Given any triangle $\triangle z_1 z_2 z_3$ in $\mathbb{C}$, there is a unique affine transformation $f(z) = Az + B\bar{z} + C$ with $f(1) = z_1$, $f(\omega) = z_2$ and $f(\omega^2) = z_3$. Straightforward calculations give

$$A = \frac{1}{3}(z_1 + \omega^2 z_2 + \omega z_3),$$

$$B = \frac{1}{3}(z_1 + \omega z_2 + \omega^2 z_3),$$

$$C = \frac{1}{3}(z_1 + z_2 + z_3) = g.$$

Because the points z_1, z_2, z_3 are noncollinear, f is a bijection of $\mathbb{C}$. The incircle for $\triangle$ is $\frac{1}{2}\mathbb{T}$ and is tangent at the midpoints of the sides, so $f(\frac{1}{2}\mathbb{T})$ is an ellipse tangent to the sides of $\triangle z_1 z_2 z_3$ at the midpoints since affine transformations preserve midpoints and tangency. Note that

$$\begin{aligned} AB &= \frac{1}{9}\big(z_1^2 + z_2^2 + z_3^2 + (\omega + \omega^2)(z_1 z_2 + z_2 z_3 + z_1 z_3)\big) \\ &= \frac{1}{9}\big(z_1^2 + z_2^2 + z_3^2 - (z_1 z_2 + z_2 z_3 + z_1 z_3)\big) \\ &= \frac{1}{9}\big((z_1 + z_2 + z_3)^2 - 3(z_1 z_2 + z_2 z_3 + z_1 z_3)\big) \\ &= g^2 - \frac{1}{3}(z_1 z_2 + z_2 z_3 + z_1 z_3), \end{aligned}$$

since $1 + \omega + \omega^2 = 0$. By Theorem 3.1, f maps $\frac{1}{2}\mathbb{T}$ onto an ellipse with foci given by (2.7).

Next, we establish uniqueness. Suppose E is an ellipse inscribed in $\triangle z_1 z_2 z_3$ and tangent at the midpoints of the sides. There is a bijective affine transformation g that maps the circle $\frac{1}{2}\mathbb{T}$ onto the ellipse E. Then the preimage of $\triangle z_1 z_2 z_3$ is a triangle, say $\triangle Z_1 Z_2 Z_3$, with incircle $\frac{1}{2}\mathbb{T}$ that is tangent to the sides at the midpoints. Because both tangent segments to a circle from an exterior point have the same length and the points of tangency are the midpoints of the sides, $\triangle Z_1 Z_2 Z_3$ is equilateral. The circumcircle of an equilateral triangle has double the radius of the incircle, so the vertices of $\triangle Z_1 Z_2 Z_3$ lie on $\mathbb{T}$. If necessary, we may precompose g with a rotation to insure that $Z_1 = 1$. Then $\{Z_2, Z_3\} = \{\omega, \omega^2\}$; if necessary, we also precompose g with a reflection over the real axis to guarantee that $Z_2 = \omega$ and $Z_3 = \omega^2$. Then $g = f$ and E is the ellipse constructed in the first part of the argument. ■

In passing we note that the ellipse $f(\mathbb{T})$ passes through the vertices of $\triangle z_1 z_2 z_3$ and has center at the centroid of the triangle; it is the unique ellipse with these properties and is called the *Steiner circumellipse*.

An appealing byproduct of the proof of Steiner's theorem is a known formula for the area of a triangle in terms of the complex numbers representing its vertices. From the formulas for A and B we obtain

$$\left||A|^2 - |B|^2\right| = \frac{2}{3\sqrt{3}}\left|\operatorname{Im}\left(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right)\right|.$$

The geometric interpretation of the determinant as a ratio of areas gives

$$\left||A|^2 - |B|^2\right| = |\det(f)| = \frac{\text{area}\,\triangle z_1z_2z_3}{\text{area}\,\triangle},$$

where $\det(f)$ denotes the determinant of the linear part $Az + B\bar{z}$ of f. From area $\triangle = 3\sqrt{3}/4$, we obtain the known formula

$$\text{area}\,\triangle z_1z_2z_3 = \frac{1}{2}\left|\operatorname{Im}\left(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right)\right|.$$

The incircle of a triangle has the largest radius, and so the largest area, among all circles contained in the triangle. The analog of this property holds for the inellipse.

Theorem 4.1. *Let C be any circle contained in $\triangle z_1z_2z_3$. Then*

$$\frac{\text{area}\,C}{\text{area}\,\triangle z_1z_2z_3} \le \frac{\pi}{3\sqrt{3}} \tag{4.1}$$

and equality holds if and only if $\triangle z_1z_2z_3$ is equilateral and C is the incircle.

Proof. Let r denote the inradius of $\triangle z_1z_2z_3$. Then area $C \le \pi r^2$ and equality holds if and only if C is the incircle. Therefore, it suffices to prove that

$$\frac{\pi r^2}{\text{area}\,\triangle z_1z_2z_3} \le \frac{\pi}{3\sqrt{3}}$$

with equality if and only if $\triangle z_1z_2z_3$ is equilateral. Let a, b, c denote the lengths of the sides of $\triangle z_1z_2z_3$ and $s = \frac{1}{2}(a + b + c)$ the semiperimeter. By Heron's formula $\sqrt{s(s-a)(s-b)(s-c)}$ is the area of $\triangle z_1z_2z_3$; the product rs also gives the area. Therefore,

$$r^2 s = \frac{\text{area}^2\,\triangle z_1z_2z_3}{s} = (s-a)(s-b)(s-c).$$

The arithmetic mean–geometric mean inequality gives

$$\left((s-a)(s-b)(s-c)\right)^{1/3} \le \frac{(s-a)+(s-b)+(s-c)}{3} = \frac{s}{3}$$

with equality if and only if $a = b = c$. Hence, $r^2 \le s^2/27$, or $r \le s/(3\sqrt{3})$, and equality holds if and only if $\triangle z_1z_2z_3$ is equilateral. Then

$$\frac{\pi r^2}{\text{area}\,\triangle z_1z_2z_3} = \frac{\pi r}{s} \le \frac{\pi}{3\sqrt{3}}$$

with equality if and only if $\triangle z_1z_2z_3$ is equilateral. ∎

Corollary 4.2 (Areal maximality of the inellipse). *Among all ellipses contained in a triangle the inellipse has the largest area. Precisely, for any ellipse E contained in* $\triangle z_1 z_2 z_3$,

$$\frac{\text{area } E}{\text{area } \triangle z_1 z_2 z_3} \leq \frac{\pi}{3\sqrt{3}} \tag{4.2}$$

and equality holds if and only if E is the inellipse.

Proof. First, let f be the affine transformation with $f(1) = z_1$, $f(\omega) = z_2$ and $f(\omega^2) = z_3$. Then $f(\frac{1}{2}\mathbb{T}) = E_0$ is the inellipse. Since an affine transformation scales all areas by the same multiplicative factor,

$$\frac{\text{area } E_0}{\text{area } \triangle z_1 z_2 z_3} = \frac{\text{area } \frac{1}{2}\mathbb{T}}{\text{area } \triangle} = \frac{\pi}{3\sqrt{3}}. \tag{4.3}$$

Now, consider any ellipse E contained in $\triangle z_1 z_2 z_3$. There is an affine transformation g that maps the circle $\frac{1}{2}\mathbb{T}$ onto the ellipse E. The preimage of $\triangle z_1 z_2 z_3$ is a triangle, say $\triangle Z_1 Z_2 Z_3$, that contains $\frac{1}{2}\mathbb{T}$. Then by Theorem 4.1

$$\frac{\text{area } E}{\text{area } \triangle z_1 z_2 z_3} = \frac{\text{area } \frac{1}{2}\mathbb{T}}{\text{area } \triangle Z_1 Z_2 Z_3} \leq \frac{\pi}{3\sqrt{3}} \tag{4.4}$$

with equality if and only if $\triangle Z_1 Z_2 Z_3$ is equilateral and $\frac{1}{2}\mathbb{T}$ is the incircle. As in the proof of Steiner's theorem, the conditions for equality imply E is the inellipse. ∎

5. THE PROOF OF THEOREM 2.3. A proof of Theorem 2.3 is given in [**2**]; we present a proof using complex numbers.

Proof of Theorem 2.3. Any line ℓ normal to the unit vector $e^{i\theta}$ has an equation of the form

$$x \cos\theta + y \sin\theta = \text{Re}\,(e^{-i\theta} z) = c$$

for some $c \in \mathbb{R}$. Since $\text{dist}\,(z_j, \ell) = \left|\text{Re}\left(e^{-i\theta} z_j\right) - c\right|$,

$$D = \sum_{j=1}^{n} \left(\text{Re}\left(e^{-i\theta} z_j\right) - c\right)^2. \tag{5.1}$$

The goal is to determine c and θ so that D is minimized. From

$$0 = \frac{\partial D}{\partial c} = -\sum_{j=1}^{n} 2\left(\text{Re}\left(e^{-i\theta} z_j\right) - c\right),$$

we obtain

$$c = \text{Re}\, e^{-i\theta} \left(\frac{1}{n} \sum_{j=1}^{n} z_j\right). \tag{5.2}$$

Thus, if a line ℓ produces an extreme value for D, then the centroid g lies on ℓ. If we let $w_j = z_j - g$, then from (5.1) and (5.2) we obtain

$$D = \sum_{j=1}^{n} \mathrm{Re}^2\left(e^{-i\theta} w_j\right).$$

By using the elementary identities $\mathrm{Re}(-iz) = \mathrm{Im}\, z$ and $\mathrm{Im}(z^2) = 2(\mathrm{Re}\, z)(\mathrm{Im}\, z)$, we find that

$$\begin{aligned} 0 = \frac{\partial D}{\partial \theta} &= 2\sum_{j=1}^{n} \mathrm{Re}\big(e^{-i\theta} w_j\big)\mathrm{Re}\big(- ie^{-i\theta} w_j\big) \\ &= 2\sum_{j=1}^{n} \mathrm{Re}\big(e^{-i\theta} w_j\big)\mathrm{Im}\big(e^{-i\theta} w_j\big) \\ &= \sum_{j=1}^{n} \mathrm{Im}\big(e^{-2i\theta} w_j^2\big) \\ &= \mathrm{Im}\left(e^{-i\theta}\sqrt{\sum_{j=1}^{n} w_j^2}\right)^2. \end{aligned}$$

Thus, at a critical point of D, $\left(e^{-i\theta}\sqrt{\sum_{j=1}^{n} w_j^2}\right)^2$ is a real number, say t, and so

$$e^{-i\theta}\sqrt{\sum_{j=1}^{n} w_j^2} = \begin{cases} \sqrt{t} & \text{if } t \geq 0, \\ i\sqrt{|t|} & \text{if } t < 0. \end{cases} \tag{5.3}$$

If $t = 0$, or equivalently, $\sum_{j=1}^{n} w_j^2 = 0$, then the function D is constant and every line through g is a line of best fit. The Second Derivative Test shows that $t > 0$ corresponds to a maximum value and $t < 0$ to a minimum value. For $t < 0$

$$ie^{i\theta} = \sqrt{\frac{\sum_{j=1}^{n} w_j^2}{\left|\sum_{j=1}^{n} w_j^2\right|}}. \tag{5.4}$$

Thus, when $\sum_{j=1}^{n} w_j^2 \neq 0$, the minimum value of D occurs when ℓ is a line through g that is parallel to the vector from 0 to $\sqrt{\sum_{j=1}^{n} w_j^2}$. ■

Example 5.1. If z_j, $1 \leq j \leq 3$, are the vertices of an equilateral triangle, then *every* line through the centroid is a line of best fit because D is constant for lines through the centroid. We determine the constant for the equilateral triangle $\triangle$ with vertices $z_1 = 1$, $z_2 = \omega = \exp(2\pi i/3)$ and $z_3 = \omega^2$. If ℓ is a line through the origin that is perpendicular to $e^{i\theta}$, then

$$\sum_{j=1}^{3} \mathrm{dist}^2(z_j, \ell) = \sum_{j=1}^{3} \mathrm{Re}^2\left(e^{-i\theta} e^{2\pi i j/n}\right)$$
$$= \frac{3}{2} + \mathrm{Re}\left(e^{-2i\theta} \sum_{j=1}^{3} e^{4\pi i j/n}\right)$$
$$= \frac{3}{2}$$

because $\mathrm{Re}^2(z) = \frac{1}{2}\left(|z|^2 + \mathrm{Re}(z^2)\right)$. Thus, $D = \frac{3}{2}$. It would be useful to have better geometric insight as to why every line through the centroid of an equilateral triangle is a line of best fit.

More generally, for the vertices of a regular polygon, any line through the centroid is a line of best fit.

Example 5.2. It is not difficult to determine the line of best fit for the vertices of an isosceles triangle; we state the result and leave the details to the reader. Let $\theta \in (0, \pi)$ denote the vertex angle of an isosceles triangle. For narrow isosceles triangles ($\theta \in (0, \pi/3)$) the line of best fit is the median joining the vertex to the midpoint of the base, while for wide isosceles triangles ($\theta \in (\pi/3, \pi)$) it is the line through the centroid that is parallel to the base. When $\theta = \pi/3$ the isosceles triangle is equilateral and every line through the center is a line of best fit.

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Added in proof: An interesting, closely related article that employs a different method recently appeared in the MONTHLY [**3**].

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An Amendatory Limerick

Following the pattern of two previous communications to this MONTHLY [**3, 1**], I submit the following claim:

$-1 = e^{i\pi}$
Proves that Euler was doubtless a sly guy.
  But $\zeta(2)$
  Was totally new
And raised the respect for him sky-high.

This claim is confirmed by a letter [**2**, p. 194] that James Stirling wrote to Euler on April 13, 1738. It says:

> "But most pleasing of all for me was your method for summing certain series by means of powers of the circumference of the circle. I acknowledge this to be quite ingenious and entirely new and I do not see that it has anything in common with the accepted methods, so that I readily believe that you have drawn it from a new source."

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