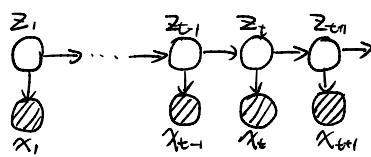


粒子滤波 (Particle Filter)

一. Background

Dynamic Model $\left\{ \begin{array}{l} \text{HMM} \rightarrow \text{关注 reading} \\ \text{Linear Dynamic System} \rightarrow \text{关注 filtering} \\ \text{Non-Linear, Non-Gaussian} \rightarrow \text{关注 filtering} \end{array} \right.$



$\left\{ \begin{array}{l} \text{齐次 Markov 假设} \\ \text{观测独立假设} \\ \left\{ \begin{array}{l} z_t = g(z_{t-1}, u, \varepsilon) \quad \text{相当于 HMM 中的 A} \\ x_t = h(z_t, u, \delta) \quad \text{相当于 HMM 中的 B} \end{array} \right. \end{array} \right.$

对于 Kalman Filter, 由于它服从 Gaussian 分布, 它是可以通过上节中的 step1, step2 一步求得其解析解的, 但对于 Particle Filter, 由于其 z_t 和 x_t , z_{t-1} 和 z_t 之间的关系是任意的, 因此无法求出其解析解, 只能通过采样的方法求解。

二. Importance Sampling & SIS

$P(z|x)$, 即 $E_{z|x}[f(z)]$, 我们可以从 $p(z|x)$ 中采样 N 个样本, $z^{(1)}, z^{(2)}, \dots, z^{(N)} \sim p^{(N)} \sim p(z|x)$, 则 $E_{z|x}[f(z)] = \int f(z) \cdot p(z|x) dz = \frac{1}{N} \sum f(z^{(i)})$

但是通常来说, 由于 $p(z|x)$ 的维度过高, 我们无法直接从 $p(z|x)$ 这个分布中采样, 因此, 我们会引入一个易于采样的分布 $q(z)$, 从其中采取 N 个样本:

$$z^{(1)}, z^{(2)}, \dots, z^{(N)} \sim q(z), q(z) \text{ 称为提议分布.}$$

$$\text{则 } E_{z|x}[f(z)] = \int f(z) \cdot p(z) dx = \int f(z) \cdot \frac{p(z)}{q(z)} \cdot q(z) dx = \frac{1}{N} \sum_{i=1}^N f(z^{(i)}) \cdot \frac{p(z^{(i)})}{q(z^{(i)})}$$

回到 Filtering 问题:

设 $x_1, \dots, x_t = X_{1:t}$, Filtering 问题即求解 $P(z_t|x_{1:t})$

$$w_t^{(i)} = \frac{p(z_t^{(i)}|X_{1:t})}{q(z_t^{(i)}|X_{1:t})}$$

对不同的 t : $t=1$: $w_1^{(1)}, w_1^{(2)}, \dots, w_1^{(N)}$

$$t \geq 2: w_2^{(1)}, \dots, w_2^{(N)}$$

而 $P(z_t^{(i)}|X_{1:t})$ 不好求, 如果我们按 Importance Sampling 采样, 我们要计算许多次 w , 计算量过大, 因此我们希望减少 w 的计算量。

寻找不同时间步之间的递推关系从而减少W的计算量.

Sequential Importance Sampling (SIS)

SIS关注的对象是 $P(z_{1:t} | x_{1:t})$, 而 Filtering 问题中的 $P(z_t | x_{1:t})$ 是其边缘概率.

(更确切地来讲, 在每个时间步 weight 都应该做归一化, 即 $\sum_t w_t^{(i)} = 1$, 之前为了简化运算并未提及)

因此, $w_t^{(i)} \propto \frac{P(z_{1:t}^{(i)} | x_{1:t})}{q(z_{1:t}^{(i)} | x_{1:t})}$, 我们要求 $w_{t-1}^{(i)}$ 和 $w_t^{(i)}$ 之间的关系.

推导:

$$\begin{aligned} \text{分子: } P(z_{1:t} | x_{1:t}) &= \underbrace{\frac{P(z_{1:t}, x_{1:t})}{P(x_{1:t})}}_{C} = \frac{1}{C} P(z_{1:t}, x_{1:t}) \\ &= \frac{1}{C} \cdot P(x_t | z_{1:t}, x_{1:t-1}) \cdot P(z_{1:t}, x_{1:t-1}) \\ &= \frac{1}{C} P(x_t | z_t) \cdot P(z_t | z_{1:t-1}, x_{1:t-1}) \\ &= \frac{1}{C} P(x_t | z_t) \cdot P(z_t | z_{t-1}) \cdot P(z_{1:t-1}, x_{1:t-1}) \\ &= \frac{1}{C} P(x_t | z_t) \cdot P(z_t | z_{t-1}) \cdot P(z_{1:t-1} | x_{1:t-1}) \cdot \underline{P(x_{1:t-1})} \\ &= \frac{D}{C} P(x_t | z_t) \cdot P(z_t | z_{t-1}) \cdot P(z_{1:t-1} | x_{1:t-1}) \end{aligned}$$

分母: 由于 q 是我们自己假设的, 因此:

$$\text{假定有: } q(z_{1:t} | x_{1:t}) = q(z_t | z_{1:t-1}, x_{1:t}) \cdot q(z_{1:t-1} | x_{1:t-1})$$

$$\begin{aligned} \text{则: } w_t^{(i)} &\propto \frac{P(z_{1:t} | x_{1:t})}{q(z_{1:t} | x_{1:t})} \propto \frac{P(x_t | z_t) \cdot P(z_t | z_{t-1}) \cdot P(z_{1:t-1} | x_{1:t-1})}{q(z_t | z_{1:t-1}, x_{1:t}) \cdot q(z_{1:t-1} | x_{1:t-1})} \\ &= \frac{P(x_t | z_t) \cdot P(z_t | z_{t-1})}{q(z_t | z_{1:t-1}, x_{1:t})} \cdot \boxed{w_{t-1}^{(i)}} \end{aligned}$$

Sequential Importance Sampling Filter (SIS)

$$w_t^{(i)} \propto \frac{p(z_{1:t} | x_{1:t})}{q(z_{1:t} | x_{1:t})} \propto \underbrace{\frac{p(x_t | z_t) \cdot p(z_t | z_{t-1})}{q(z_t | z_{t-1}, x_{1:t})}}_{\text{G } q(z_t | z_{t-1}, x_{1:t})} w_{t-1}^{(i)}$$

G $q(z_t | z_{t-1}, x_{1:t})$, 可以设为只和 z_t 有关.

Algorithm:

前提: $t-1$ 时刻采样已完成 $\Rightarrow w_{t-1}^{(i)}$ 已知

t 时刻: for $i=1, \dots, N$

① Sampling: $z_t^{(i)} \sim q(z_t | z_{t-1}, x_{1:t})$

$$w_t^{(i)} \propto w_{t-1}^{(i)} \cdot \frac{p(x_t | z_t) \cdot p(z_t | z_{t-1})}{q(z_t | z_{t-1}, x_{1:t})}$$

end

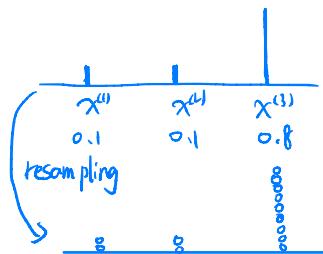
② Normalized: $w_t^{(i)}$ 归一化, 使 $\sum w_t^{(i)} = 1$. ③ Resampling

但 SIS 在实际运行时, 会出现权值退化的问题: 在运行一定步数后, $w_t^{(i)}$ 会变得越来越不平均, 这是由高维空间所引起的.

为了应对这一问题, 有 { Resampling 两种方法来解决.
选择一个更好的提议分布 $q(z)$

Resampling

假设得到 3 个样本 $x^{(1)}, x^{(2)}, x^{(3)}$, 它们的权重分别为 0.1, 0.1, 0.8. 重采样就是以它们的权重值作为概率分布进行采样, 用样本个数来代替权重.



SIS + Resampling \Rightarrow Basic Particle Filter

如何选择一个好的 $q(z)$?

一般选择状态转移概率:

$$q(z_t | z_{1:t-1}, x_{1:t}) = p(z_t | z_{t-1})$$

此时:

$$\begin{aligned} w_t^{(i)} &= w_{t-1}^{(i)} \cdot \frac{p(x_t | z_t^{(i)}) \cdot p(z_t^{(i)} | z_{t-1}^{(i)})}{q(z_t^{(i)} | z_{t-1}^{(i)}, x_{1:t})} \\ &= w_{t-1}^{(i)} \cdot \frac{p(x_t | z_t^{(i)}) \cdot p(z_t^{(i)} | z_{t-1}^{(i)})}{p(z_t^{(i)} | z_{t-1}^{(i)})} \\ &= w_{t-1}^{(i)} \cdot p(x_t | z_t^{(i)}) \end{aligned}$$

其中:

$$z_t^{(i)} \sim p(z_t, z_{t-1}^{(i)})$$

此时算法称为 Sampling - Importance - Resampling Filter (SIR Filter)

$$\hookrightarrow SIS + Resampling + q(z) = p(z_t | z_{t-1})$$