High Dimensional Data Analysis With Dependency and Under Limited Memory

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Outline

Multivariate Spectral Density Estimation Under Weak Sparsity

Low Rank Tucker Approximation of a Tensor from Streaming Data
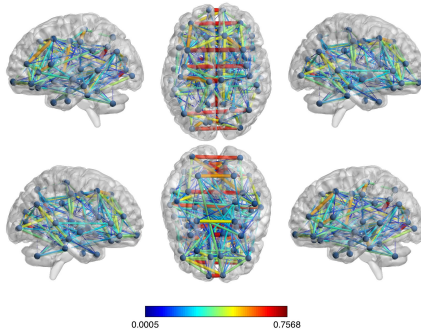
A Connection of These Two

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Motivation

Figure: Interactions Between Regions in Brain

Weakly & strongly stationary time series

Definition (Weak Stationarity)

p-variate time series X is weakly stationary, if $\mathbb{E}X_t = \mathbb{E}X_s$ for any t, s and $\Gamma(\ell) := \mathbb{E}X_t X_{t-\ell}^\top$ only depends on the lag ℓ.

Definition (Strong Stationarity)

p-variate time series X is strongly stationary, if for any sequence $t_1, \cdots, t_n$, $X_{t_1} \cdots X_{t_n}$ has the same distribution of $X_{t_1+\tau} \cdots X_{t_n+\tau}$ for any integer τ,

Gaussian process

Definition (Gaussian Process)

p-variate time series X is Gaussian process if for any sequence $t_1, \cdots, t_n$, $X_{t_1} \cdots X_{t_n}$ are jointly Gaussian distributed.

For Gaussian process, weak stationarity is equivalent to strong stationarity.

Spectral density

Given a weakly stationary p-variate time series X, the spectral density at frequency $\omega \in [-\pi, \pi)$ is defined

$$f(\omega) = \sum_{\ell=-\infty}^{\infty} \Gamma(\ell) e^{-i\omega\ell}$$

where $\Gamma(\ell) = \mathbb{E} X_0 X_{-\ell}^{\top}$. X_t is independent with X_s, $t \neq s$ iff $f_{rs}(\omega) = 0$ for any ω.

Thresholding estimator under weak sparsity- An example

Suppose that we have n observation of p-variate Gaussian distribution as follows.

$$
y_i \overset{i.i.d}{\sim} \mathcal{N} \left(\mu, \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \sigma_p^2 \end{bmatrix} \right),
$$

$i = 1, \cdots, n.$

An example

The maximum likelihood estimator for μ_j is $\bar{y}_j = \frac{1}{n} \sum_{i=1}^{n} y_{ij}$.
Does not Work Well Under Weak Sparsity

$$\mu \in \left\{ \mu \in \mathbb{R}^p, \sum_{j=1}^{p} |\mu_j|^q \le c_0(p) \right\}.$$

for some $0 \le q < 1$ and $c_0(p)$ measures the weak sparsity.

Solution : Thresholding

Suppose $\sigma_i \leq B$, define element-wise thresholding operator

$$T_\lambda(x) = \begin{cases} x & |x| \geq \lambda \\ 0 & \text{else} \end{cases}$$

. Hard thresholding estimator $T_\lambda(\bar{y}_j)$ can be shown asymptotically consistent under weak sparsity where we set

$$\lambda \propto B\sqrt{\frac{\log p}{n}}$$

and assume $\lambda \to 0$.

Two key ingredients for thresholding

Two key ingredients under above example assuming weak sparsity. Cai & Liu 2011

▶ An element-wise concentration inequality :

$$\mathbb{P}(|\bar{y}_j - \mu_j| \geq \eta) \leq 2\exp(-n\eta^2/2\sigma_j^2).$$

▶ σ_j are uniformly bounded.

Shortcomings for hard thresholding

- σ_j may variate much
- B will appear in the thresholding value making convergence rate slow

Solution: adaptive thresholding

Simply estimate σ_j, say with sample standard deviation:

$$\hat{\sigma}_j = \sqrt{1/(n-1)\sum_{i=1}^{n}(y_{ij} - \bar{y}_j)^2}$$

and replace B: $\lambda_j \propto \hat{\sigma}_j\sqrt{\frac{\log p}{n}}$. Now we can relax constraint in upper bound for σ_j and upper bound will not appear in rate of convergence.

An Similar Example: Covariance Matrix

$$y_i \overset{i.i.d}{\sim} \mathcal{N}(0, \Sigma_{p \times p})$$

Goal: Estimate Σ assuming weak sparsity, $\|\Sigma\|_1 \leq c_0(p)$. Bickel & Levina 2008

A similar example: covariance matrix

Estimate the expectation of a vector of length p^2:

$$[(y_1 y_1^\top)_{rs}, 1 \leq r, s \leq p].$$

MLE: $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} y_i y_i^\top$. But we need to perform thresholding. Remember two ingredients:

- $\mathbb{P}(|\hat{\Sigma}_{rs} - \Sigma_{rs}| \geq \eta) \leq c_1 \exp(-c_2 n \eta^2)$
- $\textbf{var}((y_1 y_1^\top)_{rs}) = \Sigma_{rr} \Sigma_{ss} + \Sigma_{rs}^2 \leq 2 \max_{r=1}^{p} \Sigma_{rr}^2$

Thus Bickel & Levina 2008 presents an assumption $\max_{r=1}^{p} \Sigma_{rr}$ is bounded.

A similar example: covariance matrix

hard thresholding: $\lambda_{rs} \propto (\max_{r=1}^{p} \Sigma_{rr}) \sqrt{\frac{\log p}{n}}$

adaptive thresholding: $\lambda_{rs} \propto \sqrt{\widehat{\mathbf{var}(y_1 y_1^\top)_{rs}}} \sqrt{\frac{\log p}{n}}$ where

$$\widehat{\mathbf{var}(y_1 y_1^\top)_{rs}} = \frac{1}{n-1} \sum_{i=1}^{n} \left[(y_i y_i^\top)_{rs} - \frac{1}{n} \sum_{i=1}^{n} (y_i y_i^\top)_{rs} \right]^2$$

,

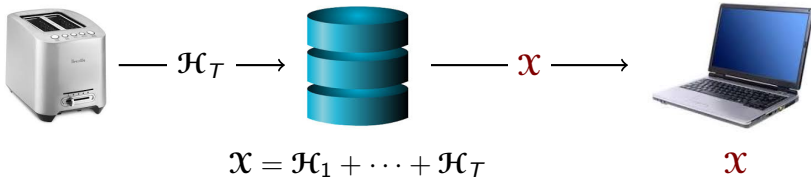# **Outline**

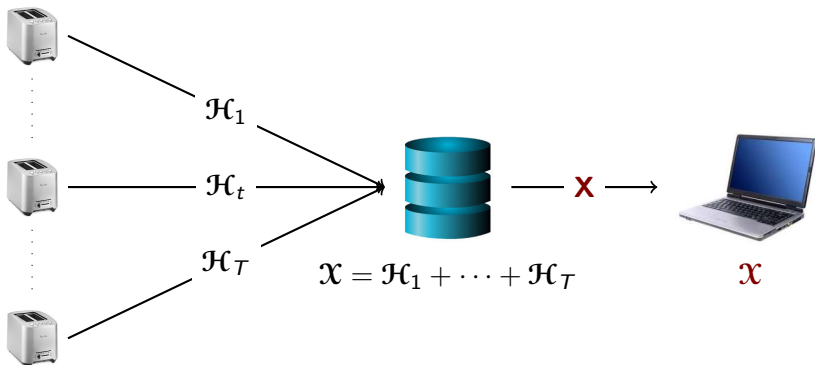## Motivation

We listed three scenarios for Motivation Borrowed from Professor Udell's Recent Talk

Big data, small laptop

$$\mathfrak{X} = \mathcal{H}_1 + \cdots + \mathcal{H}_T$$

Distributed data

$$\mathcal{X} = \mathcal{H}_1 + \cdots + \mathcal{H}_T$$

$\mathcal{H}_1$

$\mathcal{H}_t$

$\mathcal{H}_T$

$\mathbf{X} \longrightarrow$

$\mathcal{X}$

Streaming data

$$\mathfrak{X}^{(t)} = \mathcal{H}_1 + \cdots + \mathcal{H}_t$$

Notation

tensor to compress:

- ▶ tensor $\mathcal{X} \in \mathbf{R}^{I_1 \times \cdots \times I_N}$ with N modes
- ▶ sometimes assume $I_1 = \cdots = I_N = I$ for simplicity

indexing:

- ▶ $[N] = 1, \ldots, N$
- ▶ $I_{(-n)} = I_1 \times \cdots \times I_{n-1} \times I_{n+1} \times I_N$

tensor operations:

- ▶ mode n product: for $\mathcal{A} \in \mathbf{R}^{k \times I_n}$, $\mathcal{X} \times_n \mathbf{A} \in \mathbf{R}^{I_1 \times \cdots \times I_{n-1} \times k \times I_{n+1} \times \cdots \times I_N}$
- ▶ unfolding $\mathbf{X}^{(n)} \in \mathbf{R}^{I_n \times I_{(-n)}}$ stacks mode-n fibers of $\mathcal{X}$ as columns of matrix

Review of Our Tool: Linear Sketch

A linear random projection can be represented as a random matrix $\mathbf{\Omega} \in \mathbf{R}^{d \times k}$, operating on a vector $\mathbf{x} \in \mathbf{R}^d$ or a matrix $\mathbf{X} \in \mathbf{R}^{m \times d}$ to reduce the dimension:

$$\begin{aligned}
\mathbf{x} \in \mathbf{R}^n &\rightarrow \mathbf{\Omega}^\top \mathbf{x} \in \mathbf{R}^k \\
\mathbf{X} \in \mathbf{R}^{m \times d} &\rightarrow \mathbf{X}\mathbf{\Omega} \in \mathbf{R}^{m \times k}.
\end{aligned} \tag{1}$$

Properties Preserved after Projection

Lemma (Arriaga & Vempala 2006)

Let $\mathbf{x} \in \mathbf{R}^d$, assume that the entries in $\mathbf{\Omega} \in \mathbf{R}^{d \times k}$ are sampled independently from $\mathcal{N}(0, 1)$. Then

$$\mathbf{Prob}\left((1 - \epsilon)\|\mathbf{x}\|^2 \leq \left\|\frac{1}{\sqrt{k}}\mathbf{\Omega}^{\top}\mathbf{x}\right\| \leq (1 + \epsilon)\|\mathbf{x}\|^2\right) \leq 1 - 2e^{-(\epsilon^2 - \epsilon^3)k/4}. \tag{2}$$

Lemma (Halko, Martinsson & Tropp 2011)

Let $\mathbf{X} \in \mathbf{R}^{m \times d}$, assume that the entries in $\mathbf{\Omega} \in \mathbf{R}^{d \times (k+p)}$ are sampled independently from $\mathcal{N}(0, 1)$. Then let $\mathbf{Q}$ be the orthonormal matrix from QR factorization $\mathbf{X}\mathbf{\Omega} = \mathbf{Q}\mathbf{R}$, then

$$\|\mathbf{X} - \mathbf{Q}\mathbf{Q}^{\top}\mathbf{X}\|_F \leq \left(1 + \frac{k}{p-1}\right)^{1/2} \left(\sum_{j>k} \sigma_j^2\right)^{1/2}. \tag{3}$$

Tucker factorization

rank $\mathbf{r} = (r_1, \ldots, r_N)$ **Tucker factorization** of $\mathfrak{X} \in \mathbf{R}^{I_1 \times \cdots \times I_N}$:

$$\mathfrak{X} = \mathfrak{G} \times_1 \mathbf{U}_1 \cdots \times_N \mathbf{U}_N =: [\![\mathfrak{G}; \mathbf{U}_1, \ldots, \mathbf{U}_N]\!]$$

where

▶ $\mathfrak{G} \in \mathbf{R}^{r_1 \times \cdots \times r_N}$ is the **core matrix**

▶ $\mathbf{U}_n \in \mathbf{R}^{I_n \times r_n}$ is the **factor matrix** for each mode $n \in [N]$

(sometimes assume $r_1 = \cdots = r_N = r$ for simplicity)

Tucker is useful for compression: when N is small,

▶ Tucker stores $O(rNI)$ numbers for rank r^3 approximation

▶ CP stores $O(rNI)$ numbers for rank r approximation

The sketch

approximate factor matrices and core:

▶ **Factor sketch ($\mathbf{k}$).** For each $n \in [N]$,
fix random DRM $\mathbf{\Omega}_n \in \mathbb{R}^{I_{(-n)} \times k_n}$ and compute the sketch

$$\mathbf{V}_n = \mathbf{X}^{(n)} \mathbf{\Omega}_n \quad \in \mathbf{R}^{I_n \times k_n}.$$

▶ **Core sketch ($\mathbf{s}$).** For each $n \in [N]$,
fix random DRM $\mathbf{\Phi}_n \in \mathbf{R}^{I_n \times s_n}$. Compute the sketch

$$\mathcal{H} = \mathcal{X} \times_1 \mathbf{\Phi}_1^\top \cdots \times_N \mathbf{\Phi}_N^\top \quad \in \mathbf{R}^{s_1 \times \cdots \times s_N}.$$

▶ *Rule of thumb.* Pick $\mathbf{k}$ as big as you can afford, pick $\mathbf{s} = 2\mathbf{k}$.

▶ define $(\mathcal{H}, \mathbf{V}_1, \ldots, \mathbf{V}_N) = \text{SKETCH}\left(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n \in [N]}\right)$

Low memory DRMs

factor sketch DRMs are big! Same size of the tensor

- $I_{(-n)} \times k_n$ for each $n \in [N]$

- Solution: Generate random matrix $\mathbf{A}_n \in \mathbf{R}^{I_n \times k}$ [Sun, Guo, Luo, Tropp & Udell 2019]

$$\mathbf{\Omega} := (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} A_{11}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & \cdots & A_{mn}\mathbf{B} \end{bmatrix}.$$

We let $\mathbf{X} \odot \mathbf{Y}$ denotes the *Khatri-Rao product*, $\mathbf{A} \in \mathbb{R}^{I \times K}, \mathbf{B} \in \mathbb{R}^{J \times K}$, i.e. the "matching column-wise" Kronecker product. The resulting matrix of size $(IJ) \times K$ is given by:

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{A}_{(1,.)} \otimes \mathbf{B}_{(1,.)}, \dots, \mathbf{A}_{(K,.)} \otimes \mathbf{B}_{(K,.)}]. \qquad (4)$$

Two pass algorithm

Algorithm Two Pass Sketch and Low Rank Recovery

Given: tensor $\mathcal{X}$, DRMs $\{\boldsymbol{\Phi}_n, \boldsymbol{\Omega}_n\}_{n \in [N]}$ with parameters $\mathbf{k}$ and $\mathbf{s} \geq \mathbf{k}$

1. *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \ldots, \mathbf{V}_N) = \text{SKETCH}\left(\mathcal{X}; \{\boldsymbol{\Phi}_n, \boldsymbol{\Omega}_n\}_{n \in [N]}\right)$

2. *Recover factor matrices.* For $n \in [N]$,

$$(\mathbf{Q}_n, \sim) \leftarrow \text{QR}(\mathbf{V}_n)$$

3. *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{X} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N$$

Return: Tucker approximation $\tilde{\mathcal{X}} = [\![\mathcal{W}; \mathbf{Q}_1, \ldots, \mathbf{Q}_N]\!]$ with rank $\leq \mathbf{k}$

accesses $\mathcal{X}$ twice: 1) to sketch 2) to recover core

Intuition: one pass core recovery

▶ we want to know $\mathcal{W}$:
  compression of $\mathcal{X}$ using factor range approximations $\mathbf{Q}_n$
▶ we observe $\mathcal{H}$:
  compression of $\mathcal{X}$ using random projections $\mathbf{\Phi}_n$

how to approximate $\mathcal{W}$?

$$\begin{aligned}
\mathcal{X} &\approx \mathcal{X} \times_1 \mathbf{Q}_1\mathbf{Q}_1^\top \times \cdots \times_N \mathbf{Q}_N\mathbf{Q}_N^\top \\
&= \left(\mathcal{X} \times_1 \mathbf{Q}_1^\top \times_N \cdots \times \mathbf{Q}_N^\top \right) \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \\
&= \mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \\
\underbrace{\mathcal{X} \times_1 \mathbf{\Phi}_1^\top \cdots \times_N \mathbf{\Phi}_N^\top}_{\mathcal{H}} &\approx \mathcal{W} \times_1 \mathbf{\Phi}_1^\top\mathbf{Q}_1 \times \cdots \times_N \mathbf{\Phi}_N^\top\mathbf{Q}_N
\end{aligned}$$

we can solve for $\mathcal{W}$: $\mathbf{s} > \mathbf{k}$, so each $\mathbf{\Phi}_n^\top \mathbf{Q}_n$ has a left inverse (whp):

$$\mathcal{W} \approx \mathcal{H} \times_1 (\mathbf{\Phi}_1^\top\mathbf{Q}_1)^\dagger \times \cdots \times_N (\mathbf{\Phi}_N^\top\mathbf{Q}_N)^\dagger$$

One pass algorithm

Algorithm One Pass Sketch and Low Rank Recovery

Given: tensor $\mathcal{X}$, rank $\mathbf{r} = (r_1, \ldots, r_N)$, DRMs $\{\boldsymbol{\Phi}_n, \boldsymbol{\Omega}_n\}_{n \in [N]}$

▶ *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \ldots, \mathbf{V}_N) = \mathrm{SKETCH}\left(\mathcal{X}; \{\boldsymbol{\Phi}_n, \boldsymbol{\Omega}_n\}_{n \in [N]}\right)$

▶ *Recover factor matrices.* For $n \in [N]$,

$$(\mathbf{Q}_n, \sim) \leftarrow \mathrm{QR}(\mathbf{V}_n)$$

▶ *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{H} \times_1 (\boldsymbol{\Phi}_1^\top \mathbf{Q}_1)^\dagger \times \cdots \times_N (\boldsymbol{\Phi}_N^\top \mathbf{Q}_N)^\dagger$$

Return: Tucker approximation $\hat{\mathcal{X}} = [\![\mathcal{W}; \mathbf{Q}_1, \ldots, \mathbf{Q}_N]\!]$

accesses $\mathcal{X}$ only once, to sketch

Source: [Sun et al. 2019]

Fixed rank approximation

to truncate reconstruction to rank $\mathbf{r}$, truncate core:

Lemma
For a tensor $\mathcal{W} \in \mathbb{R}^{k_1 \times \cdots \times k_N}$, orthogonal matrices $\mathbf{Q}_n \in \mathbb{R}^{k_n \times r_n}$,

$$\llbracket \mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \rrbracket_{\mathbf{r}} = \llbracket \mathcal{W} \rrbracket_{\mathbf{r}} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N,$$

where $\llbracket \cdot \rrbracket$ denotes the best rank $\mathbf{r}$ Tucker approximation.

$\implies$ compute fixed rank approximation using, e.g., HOOI on (small) core approximation $\mathcal{W}$

Tail Energy

For each unfolding $\mathbf{X}^{(n)}$, define its ρth *tail energy* as

$$(\tau_\rho^{(n)})^2 := \sum_{k>\rho}^{\min(I_n, I_{(-n)})} \sigma_k^2(\mathbf{X}^{(n)}),$$

where $\sigma_k(\mathbf{X}^{(n)})$ is the kth largest singular value of $\mathbf{X}^{(n)}$.

Guarantees for two pass

Theorem ([Sun, Guo, Tropp & Udell 2018])

Sketch the tensor $\mathcal{X}$ using a Tucker sketch with parameters **k** using DRMs with i.i.d. Gaussian $\mathcal{N}(0,1)$ entries. Then the approximation $\hat{\mathcal{X}}_2$ computed with the two pass method satisfies

$$\mathbb{E}\|\mathcal{X} - \hat{\mathcal{X}}_2\|_F^2 \leq \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^{N} \left(1 + \frac{\rho_n}{k_n - \rho_n - 1}\right)(\tau_{\rho_n}^{(n)})^2.$$

Guarantees for one pass

Theorem ([Sun et al. 2018])

Sketch $\mathcal{X}$ with Gaussian DRMs of parameters $\mathbf{k}$, $\mathbf{s} \geq 2\mathbf{k} + 1$. Form a rank $\mathbf{r}$ Tucker approximation $\hat{\mathcal{X}}$ using the one pass algorithm. Then

$$\mathbb{E}\|\mathcal{X} - \hat{\mathcal{X}}\|_F^2 \leq (1 + \Delta) \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^{N} \left(1 + \frac{\rho_n}{k_n - \rho_n - 1} \right) (\tau_{\rho_n}^{(n)})^2$$

where $\Delta = \max_{n=1}^{N} k_n / (s_n - k_n - 1)$
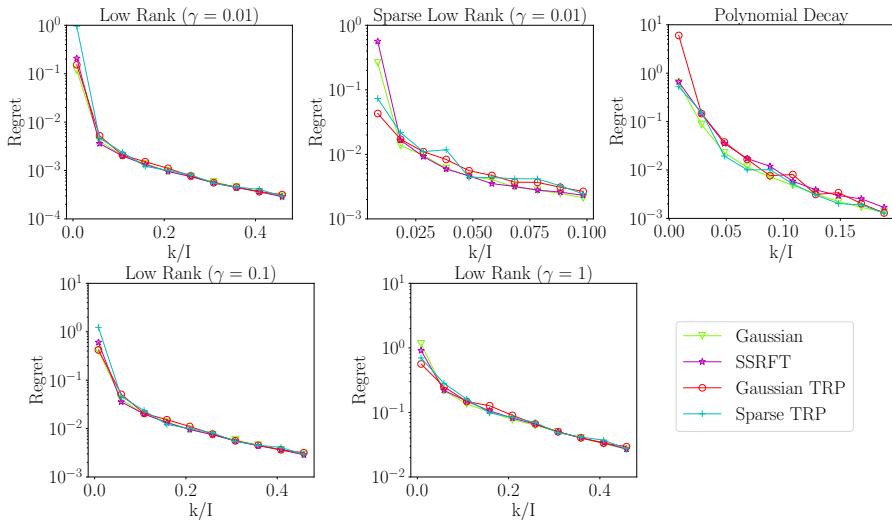
Comparison to other methods in pseudo optimality

▶ HOSVD and ST-HOSVD is pseudo optimal with factor N:

$$\|\mathcal{X} - [\![\mathcal{X}]\!]_{\mathbf{ST-r}}\|_F \leq \sqrt{\sum_{n=1}^{N}(\tau_{r_n}^{(n)})^2} \leq \sqrt{N}\|\mathcal{X} - [\![\mathcal{X}]\!]_{\mathbf{r}}\|_F, \tag{5}$$

▶ Set $\mathbf{k} = 2\mathbf{r} + 1$ and $\mathbf{s} = 2\mathbf{k} + 1$, and use truncated QR factorization to get $\mathbf{Q} \in \mathbf{R}^{I_n \times r_n}$ from factor sketch.
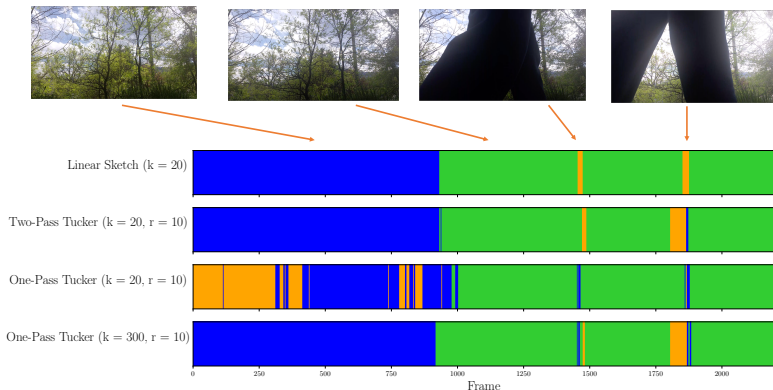
$$\|\mathcal{X} - \hat{\mathcal{X}}_2|_F \leq \sqrt{\sum_{n=1}^{N}(\tau_{r_n}^{(n)})^2} \leq \sqrt{2N}\|\mathcal{X} - [\![\mathcal{X}]\!]_{\mathbf{r}}\|_F. \tag{6}$$

Different DRMs perform similarly

Comments: Synthetic data, $l = 600$ and $\mathbf{r} = (5, 5, 5)$. $k/l = .4 \implies 20\times$ compression.

Video scene classification

Comments: Video data $2200 \times 1080 \times 1980$. Classify scenes using k-means on: 1) linear sketch along the time dimension $k = 20$ (Row 1); 2) The Tucker factor along the time dimension, computed via our two pass (Row 2) and one pass (Row 3) sketching algorithm $(r, k, s) = (10, 20, 41)$. 3) The Tucker factor along the time dimension, computed via our one pass (Row 4) sketching algorithm $(r, k, s) = (10, 300, 601)$.

Property of Tensor Random Projection

Outline

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References

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