

High Dimensional Data Analysis With Dependency and Under Limited Memory

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Based on joint work with

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Outline

Multivariate Spectral Density Estimation Under Weak Sparsity

Low Rank Tucker Approximation of a Tensor from Streaming Data

A Small Example to Connect Two Researches

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Motivation

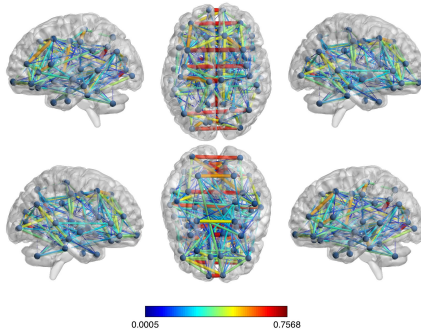


Figure: Interactions Between Regions in Brain

Weakly & strongly stationary time series

Definition (Weak Stationarity)

p -variate time series X is weakly stationary, if $\mathbb{E}X_t = \mathbb{E}X_s$ for any t, s and $\Gamma(\ell) := \mathbb{E}X_t X_{t-\ell}^\top$ only depends on the lag ℓ .

Definition (Strong Stationarity)

p -variate time series X is strongly stationary, if for any sequence t_1, \dots, t_n , $X_{t_1} \cdots X_{t_n}$ has the same distribution of $X_{t_1+\tau} \cdots X_{t_n+\tau}$ for any integer τ ,

Gaussian process

Definition (Gaussian Process)

p -variate time series X is Gaussian process if for any sequence t_1, \dots, t_n , $X_{t_1} \dots X_{t_n}$ are jointly Gaussian distributed.

For Gaussian process, weak stationarity is equivalent to strong stationarity.

Spectral density

Given a weakly stationary p -variate time series X , the spectral density at frequency $\omega \in [-\pi, \pi)$ is defined

$$f(\omega) = \sum_{\ell=-\infty}^{\infty} \Gamma(\ell) e^{-i\omega\ell}$$

where $\Gamma(\ell) = \mathbb{E}X_0X_{-\ell}^\top$. X_t is independent with X_s , $t \neq s$ iff $f_{rs}(\omega) = 0$ for any ω .

Thresholding estimator under weak sparsity- An example

Suppose that we have n observation of p -variate Gaussian distribution as follows.

$$y_i \stackrel{i.i.d}{\sim} \mathcal{N} \left(\mu, \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \sigma_p^2 \end{bmatrix} \right),$$

$$i = 1, \dots, n.$$

An example

The maximum likelihood estimator for μ_j is $\bar{y}_j = \frac{1}{n} \sum_{i=1}^n y_{ij}$.

Does not Work Well Under Weak Sparsity

$$\mu \in \left\{ \mu \in \mathbb{R}^p, \sum_{j=1}^p |\mu_j|^q \leq c_0(p) \right\}.$$

for some $0 \leq q < 1$ and $c_0(p)$ measures the weak sparsity.

Solution : Thresholding

Suppose $\sigma_i \leq B$, define element-wise thresholding operator

$$T_\lambda(x) = \begin{cases} x & |x| \geq \lambda \\ 0 & \text{else} \end{cases}$$

. Hard thresholding estimator $T_\lambda(\bar{y}_j)$ can be shown asymptotically consistent under weak sparsity where we set

$$\lambda \propto B \sqrt{\frac{\log p}{n}}$$

and assume $\lambda \rightarrow 0$.

Two key ingredients for thresholding

Two key ingredients under above example assuming weak sparsity. Cai & Liu 2011

- ▶ An element-wise concentration inequality :

$$\mathbb{P}(|\bar{y}_j - \mu_j| \geq \eta) \leq 2 \exp(-n\eta^2/2\sigma_j^2).$$

- ▶ σ_j are uniformly bounded.

Shortcomings for hard thresholding

- ▶ σ_j may vary much
- ▶ B will appear in the thresholding value making convergence rate slow

Solution: adaptive thresholding

Simply estimate σ_j , say with sample standard deviation:

$$\hat{\sigma}_j = \sqrt{1/(n-1) \sum_{i=1}^n (y_{ij} - \bar{y}_j)^2}$$

and replace B : $\lambda_j \propto \hat{\sigma}_j \sqrt{\frac{\log p}{n}}$. Now we can relax constraint in upper bound for σ_j and upper bound will not appear in rate of convergence.

An Similar Example: Covariance Matrix

$$y_i \stackrel{i.i.d}{\sim} \mathcal{N}(0, \Sigma_{p \times p})$$

Goal: Estimate Σ assuming weak sparsity, $\|\Sigma\|_1 \leq c_0(p)$. Bickel & Levina 2008

A similar example: covariance matrix

Estimate the expectation of a vector of length p^2 :

$$[(y_1 y_1^\top)_{rs}, 1 \leq r, s \leq p].$$

MLE: $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n y_i y_i^\top$. But we need to perform thresholding.
Remember two ingredients:

- ▶ $\mathbb{P}(|\hat{\Sigma}_{rs} - \Sigma_{rs}| \geq \eta) \leq c_1 \exp(-c_2 n \eta^2)$
- ▶ $\text{var}((y_1 y_1^\top)_{rs}) = \Sigma_{rr} \Sigma_{ss} + \Sigma_{rs}^2 \leq 2 \max_{r=1}^p \Sigma_{rr}^2$

Thus Bickel & Levina 2008 presents an assumption $\max_{r=1}^p \Sigma_{rr}$ is bounded.

A similar example: covariance matrix

hard thresholding: $\lambda_{rs} \propto (\max_{r=1}^p \Sigma_{rr}) \sqrt{\frac{\log p}{n}}$

adaptive thresholding: $\lambda_{rs} \propto \sqrt{\widehat{\mathbf{var}}(y_1 y_1^\top)_{rs}} \sqrt{\frac{\log p}{n}}$ where

$$\widehat{\mathbf{var}}(y_1 y_1^\top)_{rs} = \frac{1}{n-1} \sum_{i=1}^n \left[(y_i y_i^\top)_{rs} - \frac{1}{n} \sum_{i=1}^n (y_i y_i^\top)_{rs} \right]^2$$

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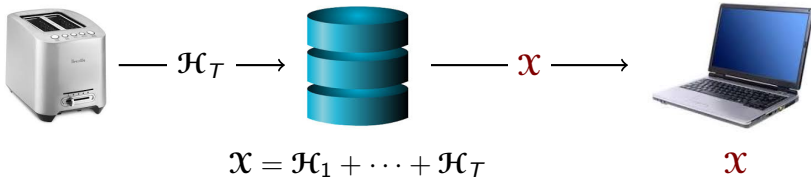
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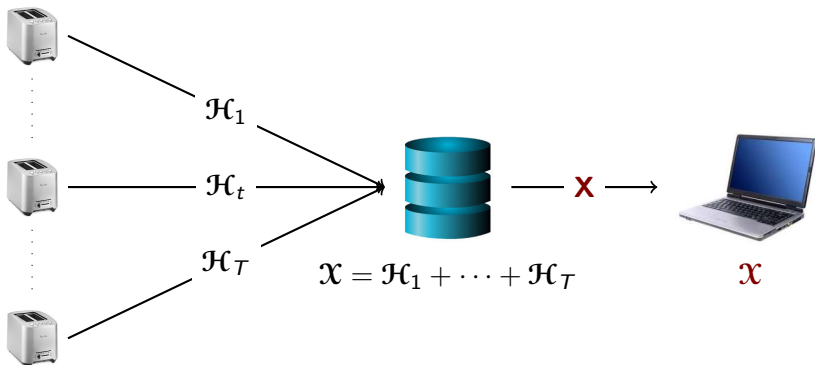
Motivation

We listed three scenarios for Motivation Borrowed from Professor Udell's Recent Talk

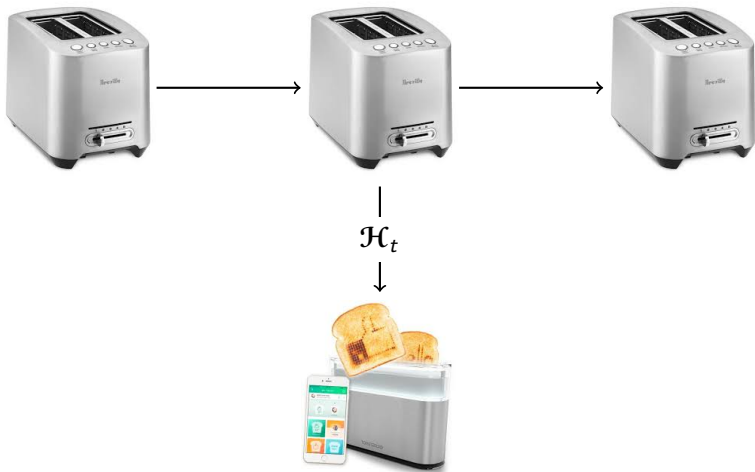
Big data, small laptop



Distributed data



Streaming data



$$\mathcal{X}^{(t)} = \mathcal{H}_1 + \cdots + \mathcal{H}_t$$

Notation

tensor to compress:

- ▶ tensor $\mathcal{X} \in \mathbf{R}^{l_1 \times \cdots \times l_N}$ with N modes
- ▶ sometimes assume $l_1 = \cdots = l_N = l$ for simplicity

indexing:

- ▶ $[N] = 1, \dots, N$
- ▶ $l_{(-n)} = l_1 \times \cdots \times l_{n-1} \times l_{n+1} \times \cdots \times l_N$

tensor operations:

- ▶ mode n product: for $\mathcal{A} \in \mathbf{R}^{k \times l_n}$,
 $\mathcal{X} \times_n \mathbf{A} \in \mathbf{R}^{l_1 \times \cdots \times l_{n-1} \times k \times l_{n+1} \times \cdots \times l_N}$
- ▶ unfolding $\mathbf{X}^{(n)} \in \mathbf{R}^{l_n \times l_{(-n)}}$ stacks mode- n fibers of \mathcal{X} as columns of matrix

Review of Our Tool: Linear Sketch

A linear random projection can be represented as a random matrix $\mathbf{\Omega} \in \mathbf{R}^{d \times k}$, operating on a vector $\mathbf{x} \in \mathbf{R}^d$ or a matrix $\mathbf{X} \in \mathbf{R}^{m \times d}$ to reduce the dimension:

$$\begin{aligned}\mathbf{x} \in \mathbf{R}^n &\rightarrow \mathbf{\Omega}^\top \mathbf{x} \in \mathbf{R}^k \\ \mathbf{X} \in \mathbf{R}^{m \times d} &\rightarrow \mathbf{X}\mathbf{\Omega} \in \mathbf{R}^{m \times k}.\end{aligned}\tag{1}$$

Properties Preserved after Projection

Lemma (Arriaga & Vempala 2006)

Let $\mathbf{x} \in \mathbf{R}^d$, assume that the entries in $\mathbf{\Omega} \in \mathbf{R}^{d \times k}$ are sampled independently from $\mathcal{N}(0, 1)$. Then

$$\mathbf{Prob} \left((1 - \epsilon) \|\mathbf{x}\|^2 \leq \left\| \frac{1}{\sqrt{k}} \mathbf{\Omega}^\top \mathbf{x} \right\|^2 \leq (1 + \epsilon) \|\mathbf{x}\|^2 \right) \leq 1 - 2e^{-(\epsilon^2 - \epsilon^3)k/4}. \quad (2)$$

Lemma (Halko, Martinsson & Tropp 2011)

Let $\mathbf{X} \in \mathbf{R}^{m \times d}$, assume that the entries in $\mathbf{\Omega} \in \mathbf{R}^{d \times (k+p)}$ are sampled independently from $\mathcal{N}(0, 1)$. Then let \mathbf{Q} be the orthonormal matrix from QR factorization $\mathbf{X}\mathbf{\Omega} = \mathbf{Q}\mathbf{R}$, then

$$\|\mathbf{X} - \mathbf{Q}\mathbf{Q}^\top \mathbf{X}\|_F \leq \left(1 + \frac{k}{p-1}\right)^{1/2} \left(\sum_{j>k} \sigma_j^2\right)^{1/2}. \quad (3)$$

Tucker factorization

rank $\mathbf{r} = (r_1, \dots, r_N)$ **Tucker factorization** of $\mathcal{X} \in \mathbf{R}^{I_1 \times \dots \times I_N}$:

$$\mathcal{X} = \mathcal{G} \times_1 \mathbf{U}_1 \cdots \times_N \mathbf{U}_N =: \llbracket \mathcal{G}; \mathbf{U}_1, \dots, \mathbf{U}_N \rrbracket$$

where

- ▶ $\mathcal{G} \in \mathbf{R}^{r_1 \times \dots \times r_N}$ is the **core matrix**
- ▶ $\mathbf{U}_n \in \mathbf{R}^{I_n \times r_n}$ is the **factor matrix** for each mode $n \in [N]$

(sometimes assume $r_1 = \dots = r_N = r$ for simplicity)

Tucker is useful for compression: when N is small,

- ▶ Tucker stores $O(rNI)$ numbers for rank r^3 approximation
- ▶ CP stores $O(rNI)$ numbers for rank r approximation

The sketch

approximate factor matrices and core:

- ▶ **Factor sketch (k).** For each $n \in [N]$, fix random DRM $\mathbf{\Omega}_n \in \mathbb{R}^{l_{(-n)} \times k_n}$ and compute the sketch

$$\mathbf{V}_n = \mathbf{X}^{(n)} \mathbf{\Omega}_n \in \mathbb{R}^{l_n \times k_n}.$$

- ▶ **Core sketch (s).** For each $n \in [N]$, fix random DRM $\mathbf{\Phi}_n \in \mathbb{R}^{l_n \times s_n}$. Compute the sketch

$$\mathcal{H} = \mathcal{X} \times_1 \mathbf{\Phi}_1^\top \cdots \times_N \mathbf{\Phi}_N^\top \in \mathbb{R}^{s_1 \times \cdots \times s_N}.$$

- ▶ *Rule of thumb.* Pick \mathbf{k} as big as you can afford, pick $\mathbf{s} = 2\mathbf{k}$.
- ▶ define $(\mathcal{H}, \mathbf{V}_1, \dots, \mathbf{V}_N) = \text{SKETCH}(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n \in [N]})$

Low memory DRMs

factor sketch DRMs are big! Same size of the tensor

- ▶ $l_{(-n)} \times k_n$ for each $n \in [N]$
- ▶ **Solution:** Generate random matrix $\mathbf{A}_n \in \mathbb{R}^{l_n \times k}$ [Sun, Guo, Luo, Tropp & Udell 2019]

$$\mathbf{\Omega} := (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} A_{11}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & \cdots & A_{mn}\mathbf{B} \end{bmatrix}.$$

We let $\mathbf{X} \odot \mathbf{Y}$ denotes the *Khatri-Rao product*, $\mathbf{A} \in \mathbb{R}^{I \times K}$, $\mathbf{B} \in \mathbb{R}^{J \times K}$, i.e. the "matching column-wise" Kronecker product. The resulting matrix of size $(IJ) \times K$ is given by:

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{A}_{(1,\cdot)} \otimes \mathbf{B}_{(1,\cdot)}, \dots, \mathbf{A}_{(K,\cdot)} \otimes \mathbf{B}_{(K,\cdot)}]. \quad (4)$$

Two pass algorithm

Algorithm Two Pass Sketch and Low Rank Recovery

Given: tensor \mathcal{X} , DRMs $\{\Phi_n, \Omega_n\}_{n \in [N]}$ with parameters \mathbf{k} and $\mathbf{s} \geq \mathbf{k}$

1. *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \dots, \mathbf{V}_N) = \text{SKETCH}(\mathcal{X}; \{\Phi_n, \Omega_n\}_{n \in [N]})$
2. *Recover factor matrices.* For $n \in [N]$,

$$(\mathbf{Q}_n, \sim) \leftarrow \text{QR}(\mathbf{V}_n)$$

3. *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{X} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N$$

Return: Tucker approximation $\tilde{\mathcal{X}} = \llbracket \mathcal{W}; \mathbf{Q}_1, \dots, \mathbf{Q}_N \rrbracket$ with rank $\leq \mathbf{k}$

accesses \mathcal{X} twice: 1) to sketch 2) to recover core

Intuition: one pass core recovery

- ▶ we want to know \mathcal{W} :
compression of \mathcal{X} using factor range approximations \mathbf{Q}_n
- ▶ we observe \mathcal{H} :
compression of \mathcal{X} using random projections Φ_n

how to approximate \mathcal{W} ?

$$\begin{aligned}\mathcal{X} &\approx \mathcal{X} \times_1 \mathbf{Q}_1 \mathbf{Q}_1^\top \times \cdots \times_N \mathbf{Q}_N \mathbf{Q}_N^\top \\ &= \left(\mathcal{X} \times_1 \mathbf{Q}_1^\top \times_N \cdots \times \mathbf{Q}_N^\top \right) \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \\ &= \mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \\ \underbrace{\mathcal{X} \times_1 \Phi_1^\top \cdots \times_N \Phi_N^\top}_{\mathcal{H}} &\approx \mathcal{W} \times_1 \Phi_1^\top \mathbf{Q}_1 \times \cdots \times_N \Phi_N^\top \mathbf{Q}_N\end{aligned}$$

we can solve for \mathcal{W} : $s > k$, so each $\Phi_n^\top \mathbf{Q}_n$ has a left inverse (whp):

$$\mathcal{W} \approx \mathcal{H} \times_1 (\Phi_1^\top \mathbf{Q}_1)^\dagger \times \cdots \times_N (\Phi_N^\top \mathbf{Q}_N)^\dagger$$

One pass algorithm

Algorithm One Pass Sketch and Low Rank Recovery

Given: tensor \mathcal{X} , rank $\mathbf{r} = (r_1, \dots, r_N)$, DRMs $\{\Phi_n, \Omega_n\}_{n \in [N]}$

- ▶ *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \dots, \mathbf{V}_N) = \text{SKETCH}(\mathcal{X}; \{\Phi_n, \Omega_n\}_{n \in [N]})$
- ▶ *Recover factor matrices.* For $n \in [N]$,

$$(\mathbf{Q}_n, \sim) \leftarrow \text{QR}(\mathbf{V}_n)$$

- ▶ *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{H} \times_1 (\Phi_1^\top \mathbf{Q}_1)^\dagger \times \dots \times_N (\Phi_N^\top \mathbf{Q}_N)^\dagger$$

Return: Tucker approximation $\hat{\mathcal{X}} = \llbracket \mathcal{W}; \mathbf{Q}_1, \dots, \mathbf{Q}_N \rrbracket$

accesses \mathcal{X} only once, to sketch

Source: [Sun et al. 2019]

Fixed rank approximation

to truncate reconstruction to rank \mathbf{r} , truncate core:

Lemma

For a tensor $\mathcal{W} \in \mathbb{R}^{k_1 \times \cdots \times k_N}$, orthogonal matrices $\mathbf{Q}_n \in \mathbb{R}^{k_n \times r_n}$,

$$[\![\mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N]\!]_{\mathbf{r}} = [\![\mathcal{W}]\!]_{\mathbf{r}} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N,$$

where $[\![\cdot]\!]$ denotes the best rank \mathbf{r} Tucker approximation.

\implies compute fixed rank approximation using, e.g., HOOI on (small) core approximation \mathcal{W}

Tail Energy

For each unfolding $\mathbf{X}^{(n)}$, define its ρ th tail energy as

$$(\tau_{\rho}^{(n)})^2 := \sum_{k > \rho}^{\min(l_n, l_{(-n)})} \sigma_k^2(\mathbf{X}^{(n)}),$$

where $\sigma_k(\mathbf{X}^{(n)})$ is the k th largest singular value of $\mathbf{X}^{(n)}$.

Guarantees for two pass

Theorem ([Sun, Guo, Tropp & Udell 2018])

Sketch the tensor \mathcal{X} using a Tucker sketch with parameters \mathbf{k} using DRMs with i.i.d. Gaussian $\mathcal{N}(0, 1)$ entries. Then the approximation $\hat{\mathcal{X}}_2$ computed with the two pass method satisfies

$$\mathbb{E} \|\mathcal{X} - \hat{\mathcal{X}}_2\|_F^2 \leq \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^N \left(1 + \frac{\rho_n}{k_n - \rho_n - 1} \right) (\tau_{\rho_n}^{(n)})^2.$$

Guarantees for one pass

Theorem ([Sun et al. 2018])

Sketch \mathcal{X} with Gaussian DRMs of parameters \mathbf{k} , $\mathbf{s} \geq 2\mathbf{k} + 1$.
Form a rank \mathbf{r} Tucker approximation $\hat{\mathcal{X}}$ using the one pass algorithm. Then

$$\mathbb{E} \|\mathcal{X} - \hat{\mathcal{X}}\|_F^2 \leq (1 + \Delta) \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^N \left(1 + \frac{\rho_n}{k_n - \rho_n - 1} \right) (\tau_{\rho_n}^{(n)})^2$$

where $\Delta = \max_{n=1}^N k_n / (s_n - k_n - 1)$

Comparison to other methods in pseudo optimality

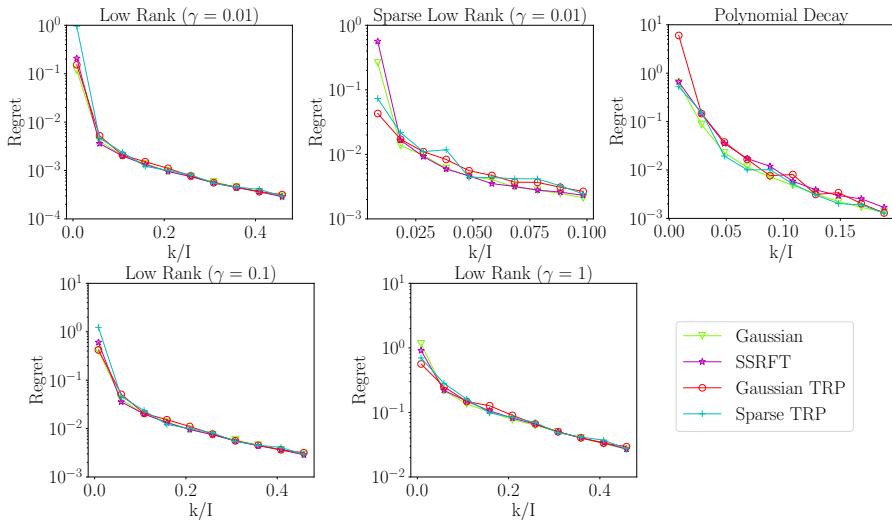
- ▶ HOSVD and ST-HOSVD is pseudo optimal with factor N :

$$\|\mathcal{X} - \llbracket \mathcal{X} \rrbracket_{\mathbf{ST-r}}\|_F \leq \sqrt{\sum_{n=1}^N (\tau_{r_n}^{(n)})^2} \leq \sqrt{N} \|\mathcal{X} - \llbracket \mathcal{X} \rrbracket_{\mathbf{r}}\|_F, \quad (5)$$

- ▶ Set $\mathbf{k} = 2\mathbf{r} + 1$ and $\mathbf{s} = 2\mathbf{k} + 1$, and use truncated QR factorization to get $\mathbf{Q} \in \mathbf{R}^{l_n \times r_n}$ from factor sketch.

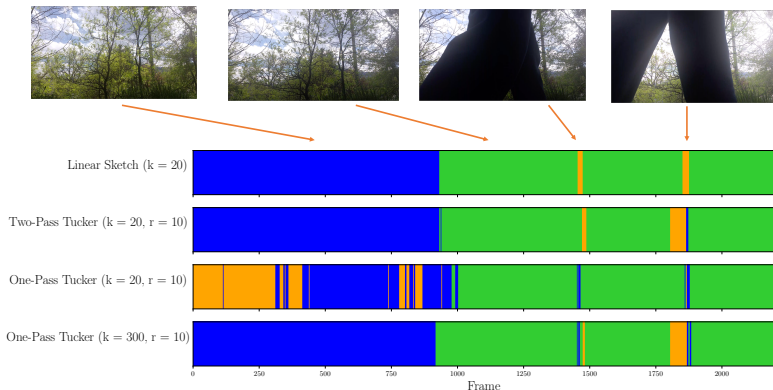
$$\|\mathcal{X} - \hat{\mathcal{X}}_2\|_F \leq \sqrt{\sum_{n=1}^N (\tau_{r_n}^{(n)})^2} \leq \sqrt{2N} \|\mathcal{X} - \llbracket \mathcal{X} \rrbracket_{\mathbf{r}}\|_F. \quad (6)$$

Different DRMs perform similarly



Comments: Synthetic data, $l = 600$ and $\mathbf{r} = (5, 5, 5)$. $k/I = .4 \implies 20\times$ compression.

Video scene classification



Comments: Video data $2200 \times 1080 \times 1980$. Classify scenes using k -means on: 1) linear sketch along the time dimension $k = 20$ (Row 1); 2) The Tucker factor along the time dimension, computed via our two pass (Row 2) and one pass (Row 3) sketching algorithm $(r, k, s) = (10, 20, 41)$. 3) The Tucker factor along the time dimension, computed via our one pass (Row 4) sketching algorithm $(r, k, s) = (10, 300, 601)$.

Property of tensor random projection

Preserve Pair-wise Distance: Fix $\mathbf{x} \in \mathbb{R}^{\prod_{n=1}^N d_n}$. Generate random matrix

$$\mathbf{\Omega} = (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$$

Theorem

Fix $\mathbf{x} \in \mathbb{R}^{\prod_{n=1}^N d_n}$. Form a TRP and TRP_T of order N with range k composed of independent matrices with independent columns whose entries are mean zero, variance one, fourth moment Δ , and within each column every pair of elements has covariance zero. Then

$$\begin{aligned} \mathbb{E} \left\| \frac{1}{\sqrt{k}} \mathbf{\Omega}^\top \mathbf{x} \right\|^2 &= \|\mathbf{x}\|^2 \\ \text{var}(\|\text{TRP}(\mathbf{x})\|^2) &= \frac{1}{k}(\Delta^N - 3)\|\mathbf{x}\|_4^4 + \frac{2}{k}\|\mathbf{x}\|_2^4 \end{aligned} \tag{7}$$

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