$\mathbf{x}$
$\prod_{n=1}^{N} d_n$
$\mathbf{x}_{r_1,\cdots,r_N}, \forall r_n \in$
$[d_n]$
$(1+$
$\sum_{n=1}^{N}(r_n-$
$1)s_n)^{th}$
$s_n =$
$\prod_{n+1}^{N} d_n$
$n <$
N
1
$n =$
N
$\mathbf{r}_1$
$\mathbf{r}_2$
$\mathbf{r}_1 =$
$\mathbf{r}_2$
$\mathbf{vec}(\mathbf{A})$
$\mathbf{A} \in$
$R^{d\times k}$
$\mathbf{A}$
kd
$[\mathbf{A}(\cdot,1); \cdots; \mathbf{A}(\cdot,k);]$
$\mathbf{x}$
$\mathbf{y}$
$[\mathbf{x};\mathbf{y}]$
$[\mathbf{x}^\top, \mathbf{y}^\top]$
??
??
$\mathbf{A} \in$
$R^{k\times d}$
$\mathbf{x} \in$
R^d
$E\mathbf{A}^2(r,s) =$
$1, \forall r, s$
$E\mathbf{A}(r,s_1)\mathbf{A}(r,s_2) =$
$0, \forall r \in$
$[k], s_1 \neq$
$s_2 \in$
$[d]$
$E\|\frac{1}{\sqrt{k}}\mathbf{y}\|_2^2 =$
$\|\mathbf{x}\|_2^2$
$\mathbf{y} =$
$\mathbf{Ax}$
$Ey_r^2 =$
$\|x\|_2^2$

$$Ey_r^2 = E\sum_{s_1=1}^{d}\sum_{s_2=1}^{d}\mathbf{A}(r,s_1)\mathbf{A}(r,s_2)x_{s_1}x_{s_2} = \sum_{s=1}^{d}\mathbf{A}^2(r,s)x_s^2 = \|\mathbf{x}\|_2^2, \tag{1}$$

$E\mathbf{A}(r,s_1)\mathbf{A}(r,s_2) =$
0
$s_1 \neq$
s_2
$E\mathbf{A}^2(r,s) =$
1
??
$\mathbf{B}_1 \in$
$R^{d_1\times k}, \mathbf{B}_2 \in$
$R^{d_2\times k}$
??
$E\mathbf{B}_n^2(r,s) =$
1
$E[\mathbf{B}_n(r_1,s)\mathbf{B}_n(r_2,s)] =$
0
$n =$
$1, 2, s \in$
$[d], r, r_1 \neq$
$r_2 \in$
$[d_n]$
$\mathbf{A} =$
$(\mathbf{B}_1 \odot$
$\mathbf{B}_2)^\top$
$\mathbf{\Omega}$
$1 \leq$
$r_1 \leq$
$d_1, 1 \leq$
$r_2 \leq$
d_2

$$E\mathbf{A}_1^2(k_1,k_2) = E\mathbf{B}_1^2(k_1,1)\mathbf{B}_2^2(k_2,1) = E\mathbf{B}_1^2(k_1,1)E\mathbf{B}_2^2(k_1,1) = 1.(independencebewteen\mathbf{B}_i, i=1,2) \tag{2}$$

$\frac{1}{k}E\|\mathbf{Ax}\|^2 =$
$\|\mathbf{x}\|_2^2$
$E(\|\mathrm{TRP}_T(\mathbf{x})\|_2^2) = \|\mathbf{x}\|_2^2,$

$Ey_1^2 =$
$\|x\|_2^2$
??

$$E\|\mathbf{y}\|_2^4 = \sum_{i=1}^{k} Ey_i^4 + \sum_{i\neq j} Ey_i^2 y_j^2.$$

??
$Ey_i^2 y_j^2 =$
$Ey_i^2 Ey_j^2 =$
$\|\mathbf{x}\|^4$
$E\|\mathbf{y}\|_2^4$
Ey_1^4
y_i
Ω
$\{1, \cdots, \prod_{n=1}^{N} d_n\}$

$$y_1^4 = \left[\sum_{\mathbf{r}\in\Omega} \mathbf{A}(1,\mathbf{r})x_{\mathbf{r}}\right]^4 = \sum_{\mathbf{r}\in\Omega} \mathbf{A}^4(1,\mathbf{r})x_{\mathbf{r}}^4 + 3\sum_{\mathbf{r}_1\neq\mathbf{r}_2\in\Omega} \mathbf{A}^2(1,\mathbf{r}_1)x_{\mathbf{r}_1}^2 \mathbf{A}^2(1,\mathbf{r}_2)x_{\mathbf{r}_2}^2 + 6\sum_{\mathbf{r}_1\neq\mathbf{r}_2\neq\mathbf{r}_3\in\Omega} \mathbf{A}^2(1,\mathbf{r}_1)x_{\mathbf{r}_1}\mathbf{A}(1,\mathbf{r}_2)x_{\mathbf{r}_2}\mathbf{A}(1,\mathbf{r}_3)x_{\mathbf{r}_3}$$

(5)

$$E\mathbf{A}^4(1,\mathbf{r}) = E\mathbf{A}_1^4(1,r_1)\cdots\mathbf{A}_N^4(1,r_N) = \Delta^N.$$

??

$$E\mathbf{A}^2(1,\mathbf{r}_1)\mathbf{A}^2(1,\mathbf{r}_2) = E\mathbf{A}^2(1,\mathbf{r}_1)E\mathbf{A}^2(1,\mathbf{r}_2) = 1.$$

$$E\|\mathrm{TRP}(\mathbf{x})\|^4 = \frac{1}{k^2}\left[k(\Delta^N-3)\|\mathbf{x}\|_4^4 + 3k\|\mathbf{x}\|_2^4 + (k-1)k\|\mathbf{x}\|_2^4\right] = \frac{1}{k}\left[(\Delta^N-3)\|\mathbf{x}\|_4^4 + 2\|\mathbf{x}\|_2^4\right] + \|\mathbf{x}\|_2^4.$$

(6)

$$\mathrm{Var}(\|\mathrm{TRP}(\mathbf{x})\|_2^2) = E\|\mathrm{TRP}(\mathbf{x})\|_2^4 - (E\|\mathrm{TRP}(\mathbf{x})\|_2^2)^2 = \frac{1}{k}\left[(\Delta^N-3)\|\mathbf{x}\|_4^4 + 2\|\mathbf{x}\|_2^4\right].$$

??
$E\|\mathrm{TRP}_T(\mathbf{x})\|_2^2 =$
$\|\mathbf{x}\|_2^2$
$E\|\mathrm{TRP}_T(\mathbf{x})\|_2^4$

$$\|\mathrm{TRP}_T(\mathbf{x})\|_2^4 = \frac{1}{T^2}\left[\sum_{t=1}^{T}\|\mathrm{TRP}^{(t)}(\mathbf{x})\|_2^2 + \sum_{t_1\neq t_2}\langle \mathrm{TRP}^{(t_1)}(\mathbf{x}), \mathrm{TRP}^{(t_2)}(\mathbf{x})\rangle\right]^2 = \frac{1}{T^2}\Big[\sum_{t=1}^{T}\|\mathrm{TRP}^{(t)}(\mathbf{x})\|_2^4 + \sum_{t_1\neq t_2}\|\mathrm{TRP}^{(t_1)}(\mathbf{x})\|_2^2\|\mathrm{T}$$

(7)

$E(rest) =$
0
$\mathbf{y}$

$$E\|\mathrm{TRP}^{(t_1)}(\mathbf{x})\|_2^2\|\mathrm{TRP}^{(t_2)}(\mathbf{x})\|_2^2 = \|\mathbf{x}\|_2^4,$$

(8)

$$E\langle \mathrm{TRP}^{(t_1)}(\mathbf{x}), \mathrm{TRP}^{(t_2)}(\mathbf{x})\rangle^2 = \frac{1}{k^2}E\left[\sum_{i=1}^{k} y_i^{(t_1)}y_i^{(t_2)}\right]^2 = \frac{1}{k}E[y_1^{(t_1)}y_1^{(t_2)}]^2 = \frac{1}{k}\|x\|_2^4.$$

(9)

$$\mathrm{Var}(\|\mathrm{TRP}_T(\mathbf{x})\|_2^2) = E\|\mathrm{TRP}_T(\mathbf{x})\|_2^4 - (E\|\mathrm{TRP}_T(\mathbf{x})\|_2^2)^2 = \frac{1}{T^2}\Big[\frac{T}{k}\left[(\Delta^N-3)\|\mathbf{x}\|_4^4 + 2\|\mathbf{x}\|_2^4\right] + T(T-1)\|\mathbf{x}\|_2^4 + T\|\mathbf{x}\|_2^4 + 2T(T$$

(10)

$\mathbf{u} =$
$\mathbf{Ax}$
$\mathbf{v} =$
$\mathbf{Ay}$

$$(u_1v_1)^2 = [\sum_{r\in\Omega}\mathbf{A}(1,\mathbf{r})x_{\mathbf{r}}]^2[\sum_{r\in\Omega}\mathbf{A}(1,\mathbf{r})y_{\mathbf{r}}]^2 = \sum_{\mathbf{r}}\mathbf{A}(1,\mathbf{r})^4x_{\mathbf{r}}^2y_{\mathbf{r}}^2 + \sum_{\mathbf{r}_1\neq\mathbf{r}_2}\mathbf{A}(1,\mathbf{r}_1)^2\mathbf{A}(1,\mathbf{r}_2)^2x_{\mathbf{r}_1}^2y_{\mathbf{r}_2}^2 + 2\sum_{\mathbf{r}_1\neq\mathbf{r}_2}\mathbf{A}(1,\mathbf{r}_1)^2\mathbf{A}(1,\mathbf{r}_2)^2x_{\mathbf{r}_1}x$$

(11)

$$E(\langle\mathrm{TRP}(\mathbf{x})\mathrm{TRP},(\mathbf{y})\rangle)^2 = \frac{1}{k}[(\Delta^N-3)\sum_{\mathbf{r}}x_{\mathbf{r}_1}^2y_{\mathbf{r}_2}^2 + \|\mathbf{x}\|_2^2\|\mathbf{y}\|_2^2 + 2\langle\mathbf{x},\mathbf{y}\rangle^2]$$

(12)

$$E\mathbf{A}^4(1,\mathbf{r}) = E\mathbf{A}_1^4(1,r_1)\cdots\mathbf{A}_N^4(1,r_N) = \Delta^N.$$