High Dimensional Data Analysis With Dependency and Under Limited Memory

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September 12, 2019

Outline

Motivation

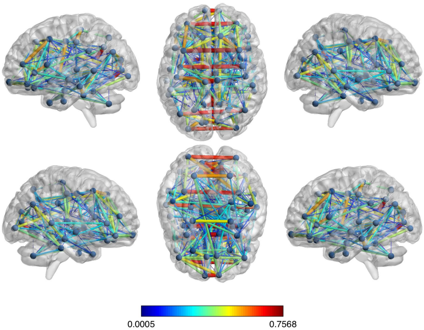


Figure: Interactions Between Regions in Brain

Weakly & strongly stationary time series

Definition (Weak Stationarity)

p-variate time series X is weakly stationary, if $\mathbb{E}X_t = \mathbb{E}X_s$ for any t, s and $\Gamma(\ell) := \mathbb{E}X_t X_{t-\ell}^\top$ only depends on the lag ℓ.

Definition (Strong Stationarity)

p-variate time series X is strongly stationary, if for any sequence $t_1, \cdots, t_n$, $X_{t_1} \cdots X_{t_n}$ has the same distribution of $X_{t_1+\tau} \cdots X_{t_n+\tau}$ for any integer τ,

Gaussian process

Definition (Gaussian Process)

p-variate time series X is Gaussian process if for any sequence $t_1, \cdots, t_n$, $X_{t_1} \cdots X_{t_n}$ are jointly Gaussian distributed.

For Gaussian process, weak stationarity is equivalent to strong stationarity.

Spectral density

Given a weakly stationary p-variate time series X, the spectral density at frequency $\omega \in [-\pi, \pi)$ is defined

$$f(\omega) = \sum_{\ell=-\infty}^{\infty} \Gamma(\ell) e^{-i\omega\ell}$$

where $\Gamma(\ell) = \mathbb{E} X_0 X_{-\ell}^{\top}$. X_t is independent with X_s, $t \neq s$ iff $f_{rs}(\omega) = 0$ for any ω.

Thresholding estimator under weak sparsity- An example

Suppose that we have n observation of p-variate Gaussian distribution as follows.

$$y_i \overset{i.i.d}{\sim} \mathcal{N}\left(\mu, \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \sigma_p^2 \end{bmatrix}\right),$$

$i = 1, \cdots, n$.

An example

The maximum likelihood estimator for μ_j is $\bar{y}_j = \frac{1}{n}\sum_{i=1}^{n} y_{ij}$.
Does not Work Well Under Weak Sparsity

$$\mu \in \left\{ \mu \in \mathbb{R}^p, \sum_{j=1}^{p} |\mu_j|^q \leq c_0(p) \right\}.$$

for some $0 \leq q < 1$ and $c_0(p)$ measures the weak sparsity.

Solution : Thresholding

Suppose $\sigma_i \leq B$, define element-wise thresholding operator

$$T_\lambda(x) = \begin{cases} x & |x| \geq \lambda \\ 0 & \text{else} \end{cases}$$

. Hard thresholding estimator $T_\lambda(\bar{y}_j)$ can be shown asymptotically consistent under weak sparsity where we set

$$\lambda \propto B\sqrt{\frac{\log p}{n}}$$

and assume $\lambda \to 0$.

Two key ingredients for thresholding

Two key ingredients under above example assuming weak sparsity. [Cai & Liu 2011]

- An element-wise concentration inequality :

$$\mathbb{P}(|\bar{y}_j - \mu_j| \geq \eta) \leq 2\exp(-n\eta^2/2\sigma_j^2).$$

- σ_j are uniformly bounded.

Shortcomings for hard thresholding

- σ_j may variate much
- B will appear in the thresholding value making convergence rate slow

Solution: adaptive thresholding

Simply estimate σ_j, say with sample standard deviation:

$$\hat{\sigma}_j = \sqrt{1/(n-1)\sum_{i=1}^{n}(y_{ij} - \bar{y}_j)^2}$$

and replace B: $\lambda_j \propto \hat{\sigma}_j \sqrt{\frac{\log p}{n}}$. Now we can relax constraint in upper bound for σ_j and upper bound will not appear in rate of convergence.

An Similar Example: Covariance Matrix

$$y_i \overset{i.i.d}{\sim} \mathcal{N}(0, \Sigma_{p\times p})$$

Goal: Estimate Σ assuming weak sparsity, $\|\Sigma\|_1 \leq c_0(p)$.
[Bickel & Levina 2008]

A similar example: covariance matrix

Estimate the expectation of a vector of length p^2:

$$[(y_1 y_1^\top)_{rs}, 1 \le r, s \le p].$$

MLE: $\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^n y_i y_i^\top$. But we need to perform thresholding. Remember two ingredients:

- $\mathbb{P}(|\hat{\Sigma}_{rs} - \Sigma_{rs}| \ge \eta) \le c_1 \exp(-c_2 n \eta^2)$
- $\mathbf{var}((y_1 y_1^\top)_{rs}) = \Sigma_{rr}\Sigma_{ss} + \Sigma_{rs}^2 \le 2\max_{r=1}^p \Sigma_{rr}^2$

Thus Bickel & Levina 2008 presents an assumption $\max_{r=1}^p \Sigma_{rr}$ is bounded.

A similar example: covariance matrix

hard thresholding: $\lambda_{rs} \propto (\max_{r=1}^{p} \Sigma_{rr})\sqrt{\frac{\log p}{n}}$

adaptive thresholding: $\lambda_{rs} \propto \sqrt{\widehat{\mathbf{var}(y_1 y_1^\top})_{rs}}\sqrt{\frac{\log p}{n}}$ where

$$\widehat{\mathbf{var}(y_1 y_1^\top})_{rs} = \frac{1}{n-1}\sum_{i=1}^{n}\left[(y_i y_i^\top)_{rs} - \frac{1}{n}\sum_{i=1}^{n}(y_i y_i^\top)_{rs}\right]^2$$

Discrete Fourier coefficient

Suppose $\mathcal{X}_{n\times p} = [X_1 : \ldots : X_n]^\top$ is a data matrix from, discrete Fourier coefficient is defined as $d(\omega) = \mathcal{X}^\top (C(\omega) - iS(\omega))$, where

$$\begin{aligned} C(\omega) &= \frac{1}{\sqrt{n}}(1, \cos\omega, \ldots, \cos(n-1)\omega)^\top, \\ S(\omega) &= \frac{1}{\sqrt{n}}(1, \sin\omega, \ldots, \sin(n-1)\omega)^\top. \end{aligned} \tag{1}$$

$\omega_k = 2\pi k/n,\ k \in F_n$, the set of Fourier frequencies. To be precise, F_n denotes the set $\left\{-[\frac{n-1}{2}], \ldots, [\frac{n}{2}]\right\}$ where $[x]$ is the integer part of x. F_n contains exactly the same frequencies used to calculate discrete Fourier transformation.

Asymptotic distribution of discrete Fourier coefficient

Assumption

$\sum_{\ell=-\infty}^{\infty} \|\Gamma(\ell)\| < \infty.$

Lemma

Suppose $\mathcal{X}_{n\times p} = [X_1 : \ldots : X_n]^\top$ is a data matrix from a strongly stationary Gaussian time series X_t, and assumption 1.1 is satisfied, we have for all $j \in F_n$ with $\omega_j \neq 0$ or π,

$$\text{vec}(d_n(\omega_j)) = \begin{bmatrix} \mathbf{Re}(d_n(\omega_j)) \\ \mathbf{Im}(d_n(\omega_j)) \end{bmatrix} = \begin{bmatrix} \mathcal{X}^\top c_j \\ \mathcal{X}^\top s_j \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{2}\begin{bmatrix} \mathbf{Re}(f(\omega_j) & -\mathbf{Im}(f(\omega_j)) \\ \mathbf{Im}(f(\omega_j)) & \mathbf{Re}(f(\omega_j)) \end{bmatrix}\right. \tag{2}$$

For $k \notin \{j, -j\}$, $\begin{bmatrix} \mathcal{X}^\top c_j \\ \mathcal{X}^\top s_j \end{bmatrix}$ is asymptotically independent of $\begin{bmatrix} \mathcal{X}^\top c_k \\ \mathcal{X}^\top s_k \end{bmatrix}$.

Here the convergence is convergence in distribution.

Smoothing periodogram

periodogram

$$I(\omega_j) = d_n(\omega_j) d_n^\dagger(\omega_j)$$

Let $\mathcal{B}_j^m$ be the set containing all indices nearest to j excluding $0, [n/2]$ and all possible pairs $\{j, -j\}$. Assuming $m < n/2$. Key Ingredient: within $\mathcal{B}_j$, $d_n(\omega_j)$ behaves like i.i.d. , so smoothing periodogram is like MLE:

$$\hat{f}(\omega_j) = \frac{1}{m} \sum_{k \in \mathcal{B}_j} I(\omega_k).$$

Comparison to covariance matrix

$$\Sigma = \mathbb{E} y_1 y_1^\top$$

$$\mathbf{var}((y_1 y_1^\top)_{rs}) = \Sigma_{rr}\Sigma_{ss} + \Sigma_{rs}^2$$

$$\leq 2\Sigma_{rr}\Sigma_{ss} \leq 2 \max_{r=1}^{p} \Sigma_{rr}^2$$

$$\text{MLE} : \frac{1}{n} y_i y_i^\top$$

$$\mathbf{var}((y_1 y_1^\top)_{rs}) \sim \Sigma_{rr}\Sigma_{ss}$$

Let $d_\infty(\omega_j)$ be r.v. whose distribution is same with limiting distribution of $d_n(\omega_j)$

$$f(\omega_j) = \mathbb{E} d_\infty(\omega_j) d_\infty^\dagger(\omega_j)$$

$$\mathbf{var}(|d_\infty(\omega_j) d_\infty^\dagger(\omega_j)|)$$

$$\leq \mathbb{E}|d_{\infty,r}(\omega_j)|^2 |d_{\infty,s}(\omega_j)|^2$$

$$\leq \sup_\omega \max_{r=1}^{p} f_{rr}(\omega)$$

$$\text{'MLE'} : \frac{1}{m} \sum_{k \in \mathcal{B}_j} I(\omega_k) \tag{3}$$

Hard thresholding for spectral density

$$T_\lambda(\hat{f}_{rs}(\omega_j)) = \begin{cases} \hat{f}_{rs}(\omega_j) & \text{if } |\hat{f}_{rs}(\omega_j)| \geq \lambda \\ 0 & \text{if } |\hat{f}_{rs}(\omega_j)| < \lambda, \end{cases} \tag{4}$$

As discussed before, λ should be proportional to an upper bound $\sup_\omega \max_{rr}$, Sun, Li, Kuceyeski & Basu 2018 choose a loose bound $\mathcal{M}(f_X) = \sup_\omega \|f(\omega)\|$.

Technical Challenge

Quantify the error using i.i.d. approximation

- Gap between finite sample to limiting distribution
- Dependency in finite sample data

Theory for hard thresholding

Quantify the error using i.i.d. approximation

- Gap between finite sample to limiting distribution
- Dependency in finite sample data

Bias in Expectation: $2\left[\frac{m+1/2\pi}{n}\Omega_n(f)+\frac{1}{2\pi}L_n(f)\right]$

Theory for hard thresholding estimator

$$|||f|||_q = \sup_{\omega \in [-\pi,\pi]} \|f(\omega)\|_q.$$

Theorem

Assume $X_t, t = 1, \cdots, n$, are n consecutive observations from a stable Gaussian time series there exist universal constants $c_1, c_2 > 0$ such that choosing a threshold

$$\lambda = 2R|||f|||\sqrt{\frac{\log p}{m}} + 2\left[\frac{m + 1/2\pi}{n}\Omega_n(f) + \frac{1}{2\pi}L_n(f)\right], \quad (5)$$

estimation error of thresholded averaged periodogram satisfies

$$\mathbb{P}\left(\left\| T_\lambda(\hat{f}(\omega_j)) - f(\omega_j)\right\| \geq 7|||f|||_q^q \lambda^{(1-q)}\right) \leq c_1 \exp\left[-(c_2 R^2 - 2)\log p\right].$$

Adaptive thresholding

We propose to perform adaptive thresholding for real and imaginary part respectively.

$$\begin{aligned} &\mathbf{var}(\mathbf{Re}(d_\infty(\omega_j)d_\infty^\dagger(\omega_j))_{rs}) \\ &= \frac{1}{2}\left[f_{rr}(\omega_j)f_{ss}(\omega_j) + \mathbf{Re}(f_{rs}(\omega_j))^2 - \mathbf{Im}(f_{rs}(\omega_j))^2\right] \end{aligned} \tag{6}$$

Not at the same order of $f_{rr}(\omega_j)f_{ss}(\omega_j)$, which is key for building theory in [Cai & Liu 2011].

Modified periodogram

$$I(\omega_j) = d(\omega_j)d^\dagger(\omega_j),$$

$$H(\omega_j) = 2\mathbf{Re}(d(\omega_j))\mathbf{Re}(d(\omega_j))^\top + 2\mathbf{Im}(d(\omega_j))\mathbf{Re}(d(\omega_j))^\top i$$

Same expectation: $\mathbb{E}I(\omega) = \mathbb{E}H(\omega_j)$.

Adaptive thresholding estimator for real part

- 'MLE':

$$\frac{1}{m}\sum_{k\in\mathcal{B}_j}\mathbf{Re}(H(\omega_k)) = \frac{2}{m}\sum_{k\in\mathcal{B}_j}\mathbf{Re}(d(\omega_k))\mathbf{Re}(d(\omega_k))^\top$$

- Variance

$$\mathbf{var}(\mathbf{Re}(H_{\infty,rs}(\omega_j))) = \left[f_{rr}(\omega_j)f_{ss}(\omega_j) + \mathbf{Re}(f_{rs}(\omega_j))^2\right]$$

Same order of $f_{rr}(\omega_j)f_{ss}(\omega_j)$

Adaptive thresholding estimator

- Variance estimator:

$$\hat{\theta}_{j,rs}^{(r)} = \frac{1}{m-1} \sum_{q \in \mathcal{B}_j} \left[\mathbf{Re}(H_{rs}(\omega_q)) - \frac{1}{m} \sum_{k \in \mathcal{B}_j} \mathbf{Re}(H_{rs}(\omega_k)) \right]^2$$

- Adaptive thresholding estimator

$$\lambda_{rs}^{(r)} = \sqrt{\hat{\theta}_{j,rs}^{(r)}} \lambda^{(r)}$$

$$\lambda^{(r)} = \sqrt{\hat{\theta}_{j,rs}^{(r)}} \sqrt{\frac{\log p}{m}} + \text{Bias}$$

Theoretical guarantees

- Sparse class:

$$\mathcal{U}^a(q, c_0(p), \omega) = \left\{ f(\omega) : \max_{r=1}^{p} \sum_{s=1}^{p} (f_{rr}(\omega) f_{ss}(\omega))^{(1-q)/2} |f_{rs}(\omega)|^q \leq c_0(p) \right\}$$

-

$$\lambda = Rc_0(p)\sqrt{\frac{\log p}{m}} + 2B_f/\phi_0, B_f \text{ is some bias} \qquad (7)$$

where $B_f = \frac{m}{n}\Omega_n(f_X) + \frac{1}{2\pi}\left(\frac{\Omega_n(f_X)}{n} + L_n(f_X)\right) + \frac{\Omega_n}{2\pi n}$, assuming $f(\omega_j) \in \mathcal{U}^a(q, c_0(p), \omega)$, the estimation error of adaptive thresholding average modified periodogram satisfies

$$\mathbb{P}\left(\left\| T_\lambda(\hat{f}(\omega_j)) - f(\omega_j) \right\| \geq 7\lambda^{(1-q)/2}\right) \leq c_1 \exp\left[-(c_2 R^2 - 2)\log p\right].$$

Outline

Motivation

We listed three scenarios for Motivation Borrowed from Professor Udell's Recent Talk

Big data, small laptop

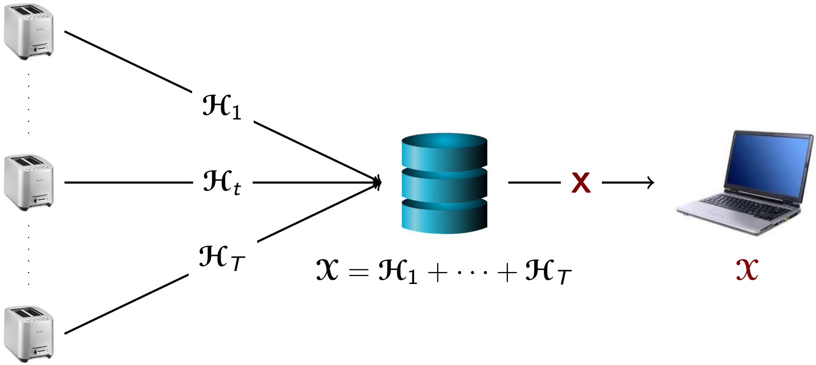
$\xrightarrow{\mathcal{H}_T}$

$\xrightarrow{\mathcal{X}}$

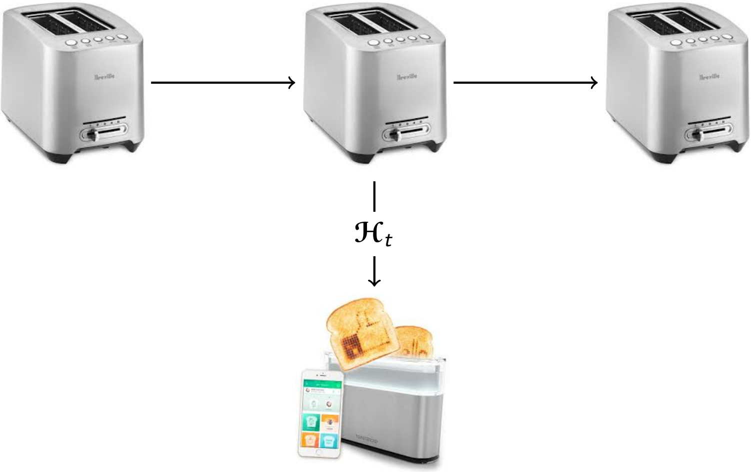
$$\mathcal{X} = \mathcal{H}_1 + \cdots + \mathcal{H}_T$$

$$\mathcal{X}$$

Distributed data



Streaming data



$$\mathcal{X}^{(t)} = \mathcal{H}_1 + \cdots + \mathcal{H}_t$$

Notation

tensor to compress:

- tensor $\mathcal{X} \in \mathbf{R}^{I_1 \times \cdots \times I_N}$ with N modes
- sometimes assume $I_1 = \cdots = I_N = I$ for simplicity

indexing:

- $[N] = 1, \ldots, N$
- $I_{(-n)} = I_1 \times \cdots \times I_{n-1} \times I_{n+1} \times I_N$

tensor operations:

- mode n product: for $\mathcal{A} \in \mathbf{R}^{k \times I_n}$,
  $\mathcal{X} \times_n \mathbf{A} \in \mathbf{R}^{I_1 \times \cdots \times I_{n-1} \times k \times I_{n+1} \times \cdots \times I_N}$
- unfolding $\mathbf{X}^{(n)} \in \mathbf{R}^{I_n \times I_{(-n)}}$ stacks mode-n fibers of $\mathcal{X}$ as columns of matrix

Review of our tool: linear sketch

A linear random projection can be represented as a random matrix $\mathbf{\Omega} \in \mathbf{R}^{d \times k}$, operating on a vector $\mathbf{x} \in \mathbf{R}^{d}$ or a matrix $\mathbf{X} \in \mathbf{R}^{m \times d}$ to reduce the dimension:

$$\begin{aligned} \mathbf{x} \in \mathbf{R}^{n} &\rightarrow \mathbf{\Omega}^{\top}\mathbf{x} \in \mathbf{R}^{k} \\ \mathbf{X} \in \mathbf{R}^{m \times d} &\rightarrow \mathbf{X}\mathbf{\Omega} \in \mathbf{R}^{m \times k}. \end{aligned} \tag{8}$$

Properties preserved after projection

Lemma (Arriaga & Vempala 2006)

Let $\mathbf{x} \in \mathbf{R}^d$, assume that the entries in $\mathbf{\Omega} \in \mathbf{R}^{d\times k}$ are sampled independently from $\mathcal{N}(0,1)$. Then

$$\textbf{Prob}\left((1-\epsilon)\|\mathbf{x}\|^2 \le \left\|\frac{1}{\sqrt{k}}\mathbf{\Omega}^\top \mathbf{x}\right\| \le (1+\epsilon)\|\mathbf{x}\|^2\right) \le 1{-}2e^{-(\epsilon^2-\epsilon^3)k/4}. \tag{9}$$

Lemma (Halko, Martinsson & Tropp 2011)

Let $\mathbf{X} \in \mathbf{R}^{m\times d}$, assume that the entries in $\mathbf{\Omega} \in \mathbf{R}^{d\times(k+p)}$ are sampled independently from $\mathcal{N}(0,1)$. Then let $\mathbf{Q}$ be the orthonormal matrix from QR factorization $\mathbf{X\Omega} = \mathbf{QR}$, then

$$\|\mathbf{X} - \mathbf{QQ}^\top\mathbf{X}\|_F \le \left(1+\frac{k}{p-1}\right)^{1/2}\left(\sum_{j>k}\sigma_j^2\right)^{1/2}. \tag{10}$$

Tucker factorization

rank $\mathbf{r} = (r_1, \ldots, r_N)$ **Tucker factorization** of $\mathcal{X} \in \mathbf{R}^{I_1 \times \cdots \times I_N}$:

$$\mathcal{X} \;=\; \mathcal{G} \times_1 \mathbf{U}_1 \cdots \times_N \mathbf{U}_N =: [\![\mathcal{G}; \mathbf{U}_1, \ldots, \mathbf{U}_N]\!]$$

where

- $\mathcal{G} \in \mathbf{R}^{r_1 \times \cdots \times r_N}$ is the **core matrix**
- $\mathbf{U}_n \in \mathbf{R}^{I_n \times r_n}$ is the **factor matrix** for each mode $n \in [N]$

(sometimes assume $r_1 = \cdots = r_N = r$ for simplicity)

Tucker is useful for compression: when N is small,

- Tucker stores $O(rNI)$ numbers for rank r^3 approximation
- CP stores $O(rNI)$ numbers for rank r approximation

The sketch

approximate factor matrices and core:

- **Factor sketch** ($\mathbf{k}$). For each $n \in [N]$,
  fix random DRM $\mathbf{\Omega}_n \in \mathbb{R}^{I_{(-n)} \times k_n}$ and compute the sketch

$$\mathbf{V}_n = \mathbf{X}^{(n)} \mathbf{\Omega}_n \quad \in \mathbf{R}^{I_n \times k_n}.$$

- **Core sketch** ($\mathbf{s}$). For each $n \in [N]$,
  fix random DRM $\mathbf{\Phi}_n \in \mathbf{R}^{I_n \times s_n}$. Compute the sketch

$$\mathcal{H} = \mathcal{X} \times_1 \mathbf{\Phi}_1^\top \cdots \times_N \mathbf{\Phi}_N^\top \quad \in \mathbf{R}^{s_1 \times \cdots \times s_N}.$$

- *Rule of thumb.* Pick $\mathbf{k}$ as big as you can afford, pick $\mathbf{s} = 2\mathbf{k}$.
- define $(\mathcal{H}, \mathbf{V}_1, \ldots, \mathbf{V}_N) = \textsc{Sketch}\left(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n \in [N]}\right)$

Low memory DRMs

factor sketch DRMs are big! Same size of the tensor

- $I_{(-n)} \times k_n$ for each $n \in [N]$
- Solution: Generate random matrix $\mathbf{A}_n \in \mathbf{R}^{I_n \times k}$ [Sun, Guo, Luo, Tropp & Udell 2019]

$$\mathbf{\Omega} := (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} A_{11}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & \cdots & A_{mn}\mathbf{B} \end{bmatrix}.$$

We let $\mathbf{X} \odot \mathbf{Y}$ denotes the *Khatri-Rao product*, $\mathbf{A} \in \mathbb{R}^{I \times K}, \mathbf{B} \in \mathbb{R}^{J \times K}$, i.e. the "matching column-wise" Kronecker product. The resulting matrix of size $(IJ) \times K$ is given by:

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{A}_{(1,.)} \otimes \mathbf{B}_{(1,.)}, \ldots, \mathbf{A}_{(K,.)} \otimes \mathbf{B}_{(K,.)}]. \qquad (11)$$

Two pass algorithm

Algorithm Two Pass Sketch and Low Rank Recovery

Given: tensor $\mathcal{X}$, DRMs $\{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n\in[N]}$ with parameters $\mathbf{k}$ and $\mathbf{s} \geq \mathbf{k}$

1. *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \dots, \mathbf{V}_N) = \textsc{Sketch}\left(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n\in[N]}\right)$
2. *Recover factor matrices.* For $n \in [N]$,
$$(\mathbf{Q}_n, \sim) \leftarrow \mathrm{QR}(\mathbf{V}_\mathrm{n})$$
3. *Recover core.*
$$\mathcal{W} \leftarrow \mathcal{X} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N$$

Return: Tucker approximation $\tilde{\mathcal{X}} = [\![\mathcal{W}; \mathbf{Q}_1, \dots, \mathbf{Q}_N]\!]$ with rank $\leq \mathbf{k}$

accesses $\mathcal{X}$ twice: 1) to sketch 2) to recover core

Intuition: one pass core recovery

- we want to know $\mathcal{W}$:
  compression of $\mathcal{X}$ using factor range approximations $\mathbf{Q}_n$
- we observe $\mathcal{H}$:
  compression of $\mathcal{X}$ using random projections $\mathbf{\Phi}_n$

how to approximate $\mathcal{W}$?

$$
\begin{aligned}
\mathcal{X} &\approx \mathcal{X} \times_1 \mathbf{Q}_1\mathbf{Q}_1^\top \times \cdots \times_N \mathbf{Q}_N\mathbf{Q}_N^\top \\
&= \left(\mathcal{X} \times_1 \mathbf{Q}_1^\top \times_N \cdots \times \mathbf{Q}_N^\top\right) \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \\
&= \mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N \\
\underbrace{\mathcal{X} \times_1 \mathbf{\Phi}_1^\top \cdots \times_N \mathbf{\Phi}_N^\top}_{\mathcal{H}} &\approx \mathcal{W} \times_1 \mathbf{\Phi}_1^\top\mathbf{Q}_1 \times \cdots \times_N \mathbf{\Phi}_N^\top\mathbf{Q}_N
\end{aligned}
$$

we can solve for $\mathcal{W}$: $\mathbf{s} > \mathbf{k}$, so each $\mathbf{\Phi}_n^\top\mathbf{Q}_n$ has a left inverse (whp):

$$\mathcal{W} \approx \mathcal{H} \times_1 (\mathbf{\Phi}_1^\top\mathbf{Q}_1)^\dagger \times \cdots \times_N (\mathbf{\Phi}_N^\top\mathbf{Q}_N)^\dagger$$

One pass algorithm

Algorithm One Pass Sketch and Low Rank Recovery

Given: tensor $\mathcal{X}$, rank $\mathbf{r} = (r_1, \ldots, r_N)$, DRMs $\{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n \in [N]}$

- *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \ldots, \mathbf{V}_N) = \textsc{Sketch}\left(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n \in [N]}\right)$
- *Recover factor matrices.* For $n \in [N]$,

$$(\mathbf{Q}_n, \sim) \leftarrow \mathrm{QR}(\mathbf{V}_\mathrm{n})$$

- *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{H} \times_1 (\mathbf{\Phi}_1^\top \mathbf{Q}_1)^\dagger \times \cdots \times_N (\mathbf{\Phi}_N^\top \mathbf{Q}_N)^\dagger$$

Return: Tucker approximation $\hat{\mathcal{X}} = [\![\mathcal{W}; \mathbf{Q}_1, \ldots, \mathbf{Q}_N]\!]$

accesses $\mathcal{X}$ only once, to sketch

Source: [Sun et al. 2019]

Fixed rank approximation

to truncate reconstruction to rank $\mathbf{r}$, truncate core:

Lemma

For a tensor $\mathcal{W} \in \mathbb{R}^{k_1 \times \cdots \times k_N}$, orthogonal matrices $\mathbf{Q}_n \in \mathbb{R}^{k_n \times r_n}$,

$$[\![\mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N]\!]_{\mathbf{r}} = [\![\mathcal{W}]\!]_{\mathbf{r}} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N,$$

where $[\![\cdot]\!]$ denotes the best rank $\mathbf{r}$ Tucker approximation.

$\implies$ compute fixed rank approximation using, e.g., HOOI on (small) core approximation $\mathcal{W}$

Tail Energy

For each unfolding $\mathbf{X}^{(n)}$, define its ρ*th tail energy* as

$$(\tau_\rho^{(n)})^2 := \sum_{k>\rho}^{\min(I_n, I_{(-n)})} \sigma_k^2(\mathbf{X}^{(n)}),$$

where $\sigma_k(\mathbf{X}^{(n)})$ is the kth largest singular value of $\mathbf{X}^{(n)}$.

Guarantees for two pass

Theorem ([Sun, Guo, Tropp & Udell 2018])

Sketch the tensor $\mathcal{X}$ using a Tucker sketch with parameters $\mathbf{k}$ using DRMs with i.i.d. Gaussian $\mathcal{N}(0,1)$ entries. Then the approximation $\hat{\mathcal{X}}_2$ computed with the two pass method satisfies

$$\mathbb{E}\|\mathcal{X} - \hat{\mathcal{X}}_2\|_F^2 \leq \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^{N} \left(1 + \frac{\rho_n}{k_n - \rho_n - 1}\right) (\tau_{\rho_n}^{(n)})^2.$$

Guarantees for one pass

Theorem ([Sun et al. 2018])

Sketch $\mathcal{X}$ with Gaussian DRMs of parameters $\mathbf{k}$, $\mathbf{s} \geq 2\mathbf{k} + 1$. Form a rank $\mathbf{r}$ Tucker approximation $\hat{\mathcal{X}}$ using the one pass algorithm. Then

$$\mathbb{E}\|\mathcal{X} - \hat{\mathcal{X}}\|_F^2 \leq (1 + \Delta) \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^{N} \left(1 + \frac{\rho_n}{k_n - \rho_n - 1}\right) (\tau_{\rho_n}^{(n)})^2$$

where $\Delta = \max_{n=1}^{N} k_n/(s_n - k_n - 1)$

Comparison to other methods in pseudo optimality

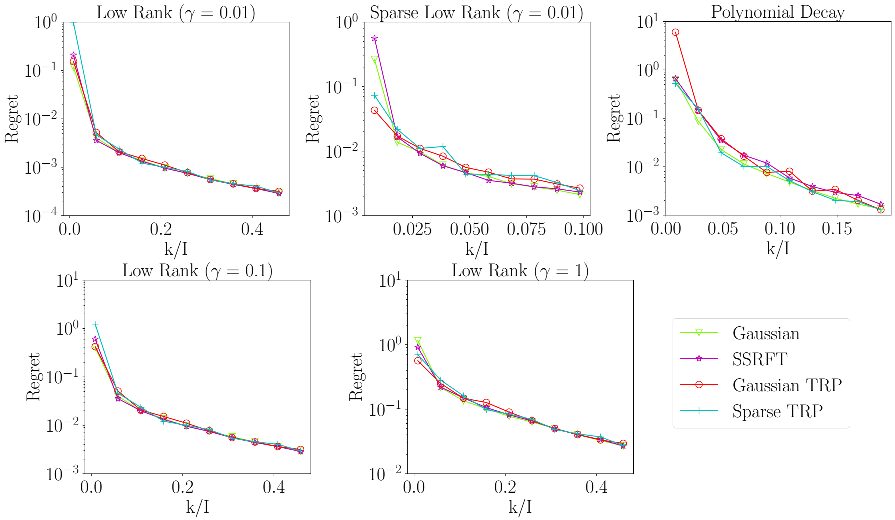
- HOSVD and ST-HOSVD is pseudo optimal with factor N:

$$\|\mathcal{X} - [\![\mathcal{X}]\!]_{\mathbf{ST-r}}\|_F \leq \sqrt{\sum_{n=1}^{N} (\tau_{r_n}^{(n)})^2} \leq \sqrt{N}\|\mathcal{X} - [\![\mathcal{X}]\!]_{\mathbf{r}}\|_F, \tag{12}$$

- Set $\mathbf{k} = 2\mathbf{r} + 1$ and $\mathbf{s} = 2\mathbf{k} + 1$, and use truncated QR factorization to get $\mathbf{Q} \in \mathbf{R}^{I_n \times r_n}$ from factor sketch.

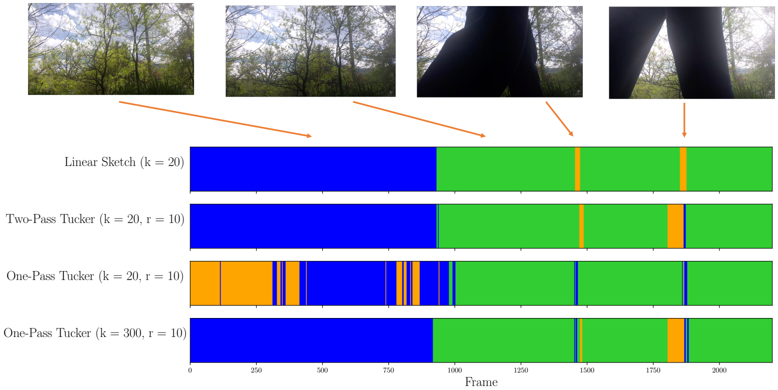
$$\|\mathcal{X} - \hat{\mathcal{X}}_2|_F \leq \sqrt{\sum_{n=1}^{N} (\tau_{r_n}^{(n)})^2} \leq \sqrt{2N}\|\mathcal{X} - [\![\mathcal{X}]\!]_{\mathbf{r}}\|_F. \tag{13}$$

Different DRMs perform similarly



Comments: Synthetic data, $I = 600$ and $\mathbf{r} = (5, 5, 5)$. $k/I = .4 \implies 20\times$ compression.

Video scene classification



Comments: Video data $2200 \times 1080 \times 1980$. Classify scenes using k-means on: 1) linear sketch along the time dimension $k = 20$ (Row 1); 2) The Tucker factor along the time dimension, computed via our two pass (Row 2) and one pass (Row 3) sketching algorithm $(r, k, s) = (10, 20, 41)$. 3) The Tucker factor along the time dimension, computed via our one pass (Row 4) sketching algorithm $(r, k, s) = (10, 300, 601)$.

Property of tensor random projection

Preserve Pair-wise Distance: Fix $\mathbf{x} \in \mathbf{R}^{\prod_{n=1}^{N} d_n}$. Generate random matrix

$$\boldsymbol{\Omega} = (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$$

Theorem

Fix $\mathbf{x} \in \mathbb{R}^{\prod_{n=1}^{N} d_n}$. Form a TRP and TRP_T of order N with range k composed of independent matrices with independent columns whose entries are mean zero, variance one, fourth moment Δ, and within each column every pair of elements has covariance zero. Then

$$\begin{aligned} \mathbb{E}\left\|\frac{1}{\sqrt{k}}\boldsymbol{\Omega}^\top \mathbf{x}\right\|^2 &= \|\mathbf{x}\|^2 \\ \textbf{var}(\|\frac{1}{\sqrt{k}}\boldsymbol{\Omega}^\top \mathbf{x}\|^2) &= \frac{1}{k}(\Delta^N - 3)\|\mathbf{x}\|_4^4 + \frac{2}{k}\|\mathbf{x}\|_2^4 \end{aligned} \tag{14}$$

Property of tensor random projection

Preserve Column Space: Fix $\mathbf{x} \in \mathbf{R}^{\prod_{n=1}^{N} d_n}$. Generate random matrix

$$\boldsymbol{\Omega} = (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$$

Theorem

Let $\mathbf{X}_n \in \mathbb{R}^{m_i \times d_n}$ be a series of matrix and $\boldsymbol{\Omega}_n \in \mathbb{R}^{d_n \times (k+p_n)}$ with eact element sampled from standard Gaussian distribution, let $\tau_n(k) = \sum_{j>k} \sigma_j^2((x))$ be the tail energy for $\mathbf{X}_i$. Let $\mathbf{Q} \in \mathbb{R}^{d \times k}$ be the orthonormal matrix from QR factorization:

$$\mathbf{Q}, - = \mathrm{QR}[(\mathbf{X}_1 \otimes \cdots \otimes \mathbf{X}_\mathrm{N})(\boldsymbol{\Omega}_1 \odot \cdots \odot \boldsymbol{\Omega}_\mathrm{N})]$$

we have

$$\begin{aligned} &\|(\mathbf{X}_1 \otimes \cdots \otimes \mathbf{X}_N) - \mathbf{Q}\mathbf{Q}^\top(\mathbf{X}_1 \otimes \cdots \otimes \mathbf{X}_N)\|_F^2 \\ &\leq \prod_{i=1}^{N} \left(1 + \frac{k}{p_n - 1}\right) \tau_n^2(k). \end{aligned} \tag{15}$$

Outline

A small example to connect two researches

Suppose we have a p-variate time series $\{\mathcal{X}_t\}$ which reveal its column one by one.

- Before we know the exactly number of time points, we cannot calculate Fourier coefficients (depends in sample size)
- the data matrix $\mathcal{X}$ is too big to store

Sketching $\mathcal{X}$

Using sketching algorithm [Tropp, Yurtsever, Udell & Cevher 2017] to get an approximation of $\mathcal{X}$: $\hat{\mathcal{X}}$. $\hat{d}_j = \hat{\mathcal{X}}^\top e_j$. Simple observation: $\|e_j\| = 1$, which does averaging and averaging will not make error bound worse.

$$\|\hat{d}_j - d_j\| \leq \|\hat{\mathcal{X}} - \mathcal{X}\|.$$

Outline

Multivariate Spectral Density Estimation Under Weak Sparsity

Low Rank Tucker Approximation of a Tensor from Streaming Data

A Small Example to Connect Two Researches

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References

Arriaga, R. I. & Vempala, S. (2006). An algorithmic theory of learning: Robust concepts and random projection. *Machine Learning*, *63*(2), 161–182.

Bickel, P. J. & Levina, E. (2008). Covariance regularization by thresholding. *The Annals of Statistics*, 2577–2604.

Cai, T. & Liu, W. (2011). Adaptive thresholding for sparse covariance matrix estimation. *Journal of the American Statistical Association*, *106*(494), 672–684.

Halko, N., Martinsson, P.-G., & Tropp, J. A. (2011). Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM review*, *53*(2), 217–288.

Sun, Y., Guo, Y., Luo, C., Tropp, J. A., & Udell, M. (2019). Low rank tucker approximation of a tensor from streaming data. *In preparation*.

Sun, Y., Guo, Y., Tropp, J. A., & Udell, M. (2018). Tensor random projection for low memory dimension reduction. In *NeurIPS Workshop on Relational Representation Learning*.

Sun, Y., Li, Y., Kuceyeski, A., & Basu, S. (2018). Large spectral density matrix estimation by thresholding. *arXiv preprint arXiv:1812.00532*.

Tropp, J. A., Yurtsever, A., Udell, M., & Cevher, V. (2017). Practical sketching algorithms for low-rank matrix approximation. *SIAM Journal on Matrix Analysis and Applications*, *38*(4), 1454–1485.