A virtual element method for 2D contact analysis
VEMによる二次元接触問題解析

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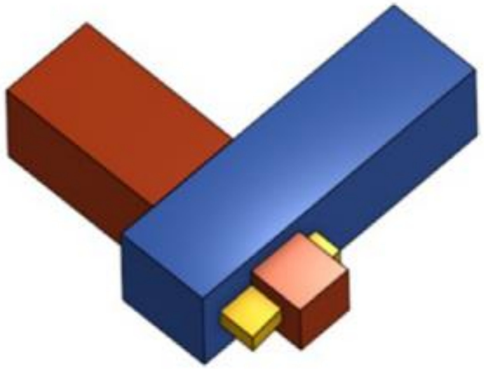
Background

Contact analysis (接触解析):組合せ構造の設計では、複数の部材間に生じる接触現象の評価が重要になり、固体力学における重要な問題の一つである

様々な適用例 (Many applications):



tenon joint (ほぞ継ぎ)

Source: L. Aharoni et al. Topology optimization of rigid interlocking assemblies, 2021

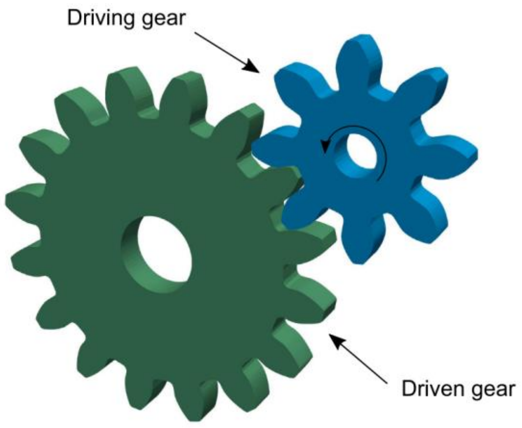
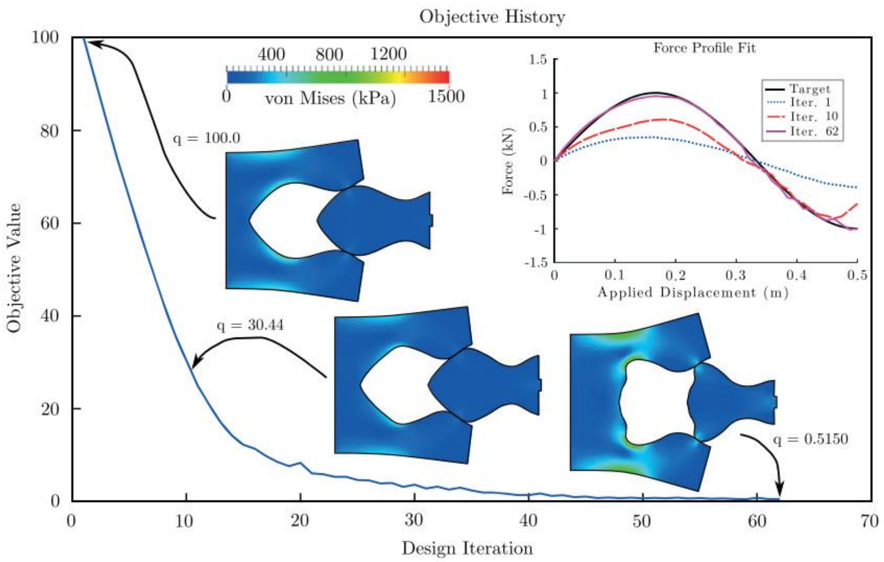


Fig. 29. Model of the gear contact.

Gear contact (歯車)

Source: W. Xing et al. A node-to-node scheme for three-dimensional contact problems using the scaled boundary finite element method

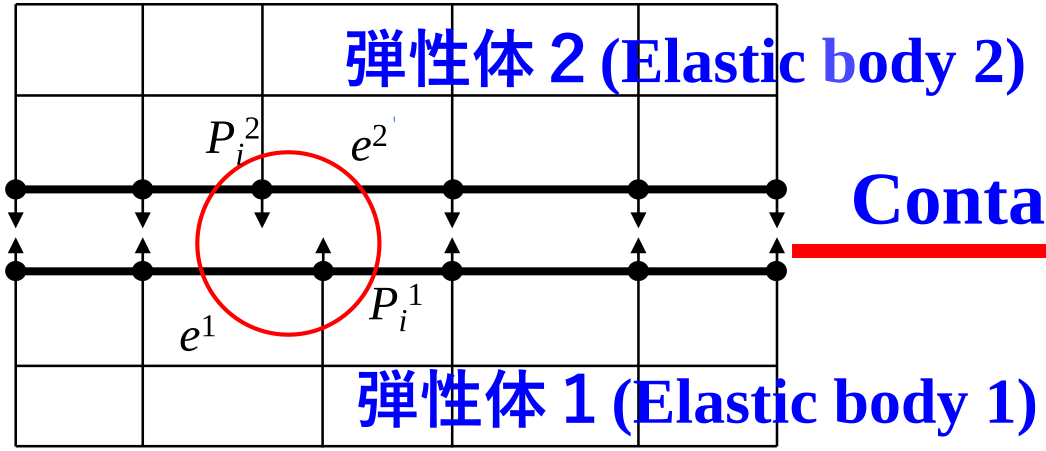


topology optimization (トポロジー最適化)

Source: M. Lawry, K. Maute. Level set shape and topology optimization of finite strain bilateral contact problems

Challenge

Non-matching mesh(不整合メッシュ):



Contact

Node-to-Node (点-点接触)
Cannot be applied

Cannot pass the patch test
接触圧力の分位が不均一

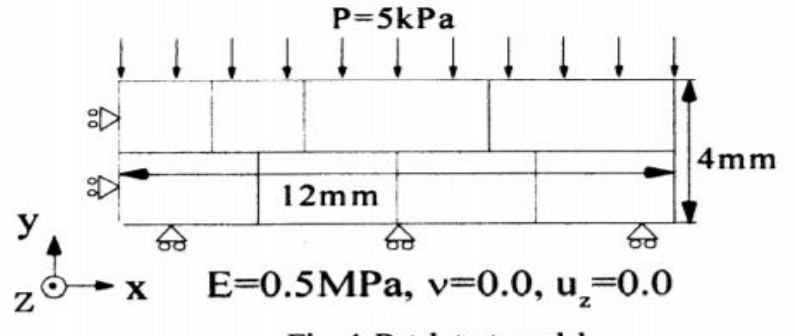


Fig. 4 Patch test model

接触問題PATCH TESTをパスする有限要素解析アルゴリズムの開発　陳 献, 久田 俊明　2006

Other alternatives:

Question

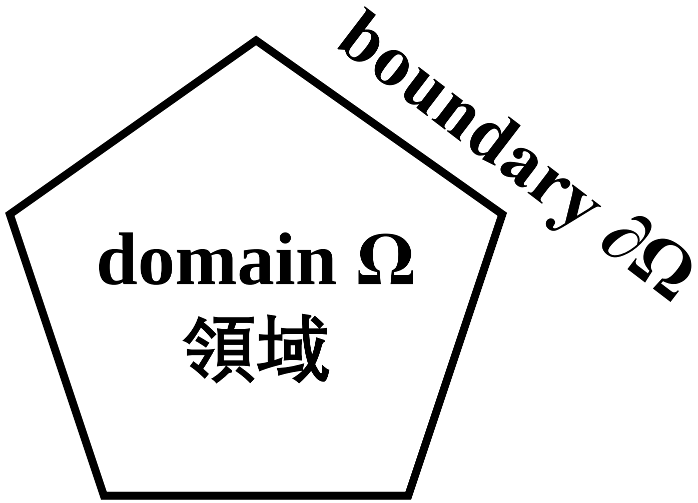
Node-to-segment (点-面接触)
segment-to-segment (面-面接触)

Virtual element method (VEM)

VEM is a generalization of FEM (Beirão da Veiga et al., 2013)

- **Able to deal with arbitrary polygonal meshes (多角形メッシュ)**
- **No domain integral is needed (境界上の積分計算のみ)**

Suppose we have a pentagon element, the variational formulation (弱形式) of the boundary value problem (境界値問題) is: to find displacement *u* such that



pentagon element

$$\boxed{a(\boldsymbol{u},\boldsymbol{v})} = \boxed{f(\boldsymbol{v})}, \ \forall \boldsymbol{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$$ **in which**

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, d\Omega,$$ **bilinear form (双線型形式);**

$$f(\boldsymbol{v}) = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{t}\, d\partial\Omega,$$ **linear form (線型汎関数);**

***v*: test function (試験関数);** $H_0^1(\Omega)$ **Sobolev space (空間);**

Construction of $a\,(\boldsymbol{u},\boldsymbol{v})$

$$\boxed{a\left(\boldsymbol{u},\boldsymbol{v}\right)} = f\left(\boldsymbol{v}\right),\ \forall \boldsymbol{v} \in H_0^1\left(\Omega\right) \times H_0^1\left(\Omega\right)$$

$f(v)$ is the same as in FEM

Stress tensor

$$\boldsymbol{\sigma}\left(\boldsymbol{u}\right) = \mathbb{C}\boldsymbol{\varepsilon}\left(\boldsymbol{u}\right)$$

$\mathbb{C}$: elasticity tensor (弾性テンソル)

$$a\left(\boldsymbol{u},\boldsymbol{v}\right) = \int_\Omega \boxed{\boldsymbol{\sigma}\left(\boldsymbol{u}\right)} : \boxed{\boldsymbol{\varepsilon}\left(\boldsymbol{v}\right)}\, d\Omega$$

$$\boldsymbol{\varepsilon}\left(\boldsymbol{v}\right) = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\right)$$

∇ : gradient operator (勾配)

Strain tensor

First we define a local virtual element space (関数空間) $V \times V$

$$V \times V = \mathrm{span}\left\{\begin{bmatrix} \boxed{\varphi_1} & 0 & \varphi_2 & 0 & \cdots & \varphi_N & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & \cdots & 0 & \varphi_N \end{bmatrix}\right\} = \mathrm{span}\left\{\boxed{\boldsymbol{\varphi}_i}\right\}_{i=1}^{2N}$$

$(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_{2N}$ 張る線形空間)

constant-valued function (関数)

Vector-valued function (ベクトル値関数)

and we define $\boldsymbol{\varphi}_i$ as the *i*th vector-valued shape function (ベクトル形状関数)

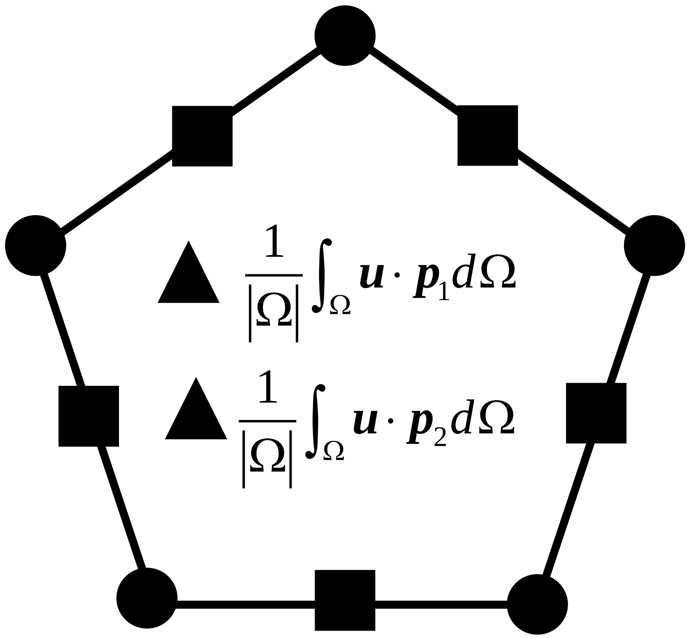
Degree of freedom (DOF)

Functions *u* and *v* are approximated by Galerkin method (Galerkin 近似)

$$\boldsymbol{u}=\begin{bmatrix}u_x\\u_y\end{bmatrix}\approx\sum_{i=1}^{2N}\boxed{\mathrm{dof}_i}(\boldsymbol{u})\boldsymbol{\varphi}_i;\ \boldsymbol{v}=\begin{bmatrix}v_x\\v_y\end{bmatrix}\approx\sum_{i=1}^{2N}\boxed{\mathrm{dof}_i}(\boldsymbol{v})\boldsymbol{\varphi}_i$$

$\mathrm{dof}_i(\cdot)$: ith degree of freedom (**自由度**)

Take second-order VEM (二次VEM) as example, the DOF is defined as



- ● **Values of *u* at the 5 vertices (頂点変位);**
- ■ **Values of *u* at the 5 midpoints (中間節点の変位);**
- ▲ **Two integrals of *u* with** $\boldsymbol{p}_1=[1,0]^T$ **and** $\boldsymbol{p}_2=[0,1]^T$

$$\frac{1}{\boxed{|\Omega|}}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{p}_1 d\Omega;\ \frac{1}{|\Omega|}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{p}_2 d\Omega$$

$|\Omega|$ **Domain area(面積)**

Entry of stiffness matrix

Then similar to FEM, $a\ (\boldsymbol{u},\boldsymbol{v})$ is approximated by

To be solved (未知数)

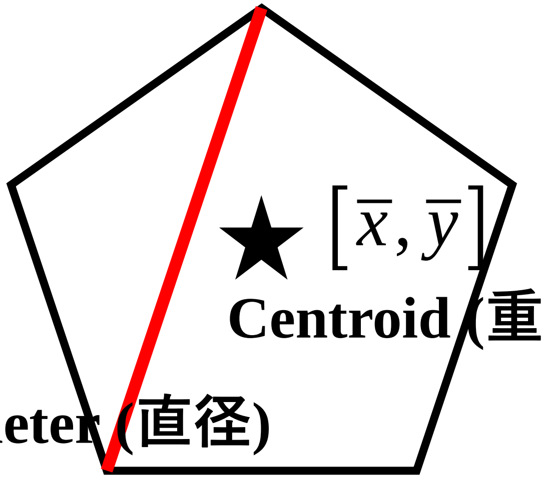
$$a(\boldsymbol{u},\boldsymbol{v}) \approx a\left(\sum_{i=1}^{2N}\mathrm{dof}_i(\boldsymbol{u})\boldsymbol{\varphi}_i, \sum_{i=1}^{2N}\mathrm{dof}_i(\boldsymbol{v})\boldsymbol{\varphi}_i\right) = \sum_{i=1}^{2N}\sum_{j=1}^{2N}\boxed{\mathrm{dof}_i(\boldsymbol{u})}\,\mathrm{dof}_i(\boldsymbol{v})\,\boxed{a(\boldsymbol{\varphi}_i,\boldsymbol{\varphi}_j)}$$

Entry of stiffness matrix (剛性行列)

Define the second-order polynomial space (二次多項式関数のなす空間) $P\times P$

12 basis functions(基底関数)

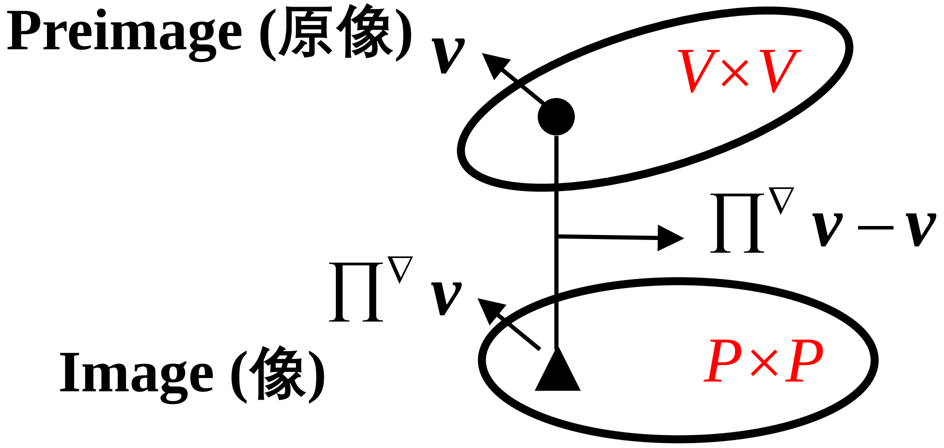
$$P\times P = \mathrm{span}\left\{\begin{bmatrix} 1 & 0 & \eta & \xi & 0 & \eta & \xi^2 & 0 & \eta^2 & 0 & \xi\eta & 0 \\ 0 & 1 & -\xi & 0 & \eta & \xi & 0 & \xi^2 & 0 & \eta^2 & 0 & \xi\eta \\ \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 & \boldsymbol{p}_5 & \boldsymbol{p}_6 & \boldsymbol{p}_7 & \boldsymbol{p}_8 & \boldsymbol{p}_9 & \boldsymbol{p}_{10} & \boldsymbol{p}_{11} & \boldsymbol{p}_{12} \end{bmatrix}\right\}$$



Scaled coordinates: $\xi = \dfrac{x-\bar{x}}{h},\quad \eta = \dfrac{y-\bar{y}}{h}$

Entry of stiffness matrix

And define a projector Π^{∇}: which maps(写像) $V\times V \to P\times P$ (Beirão da Veiga et al., 2013) and satisfies



$$a\left(\boldsymbol{p}_{\alpha}, \prod^{\nabla} \boldsymbol{v} - \boldsymbol{v}\right) = 0, \quad \text{for } \forall \boldsymbol{p}_{\alpha} \in P\times P \text{ and } \boldsymbol{v} \in V\times V$$

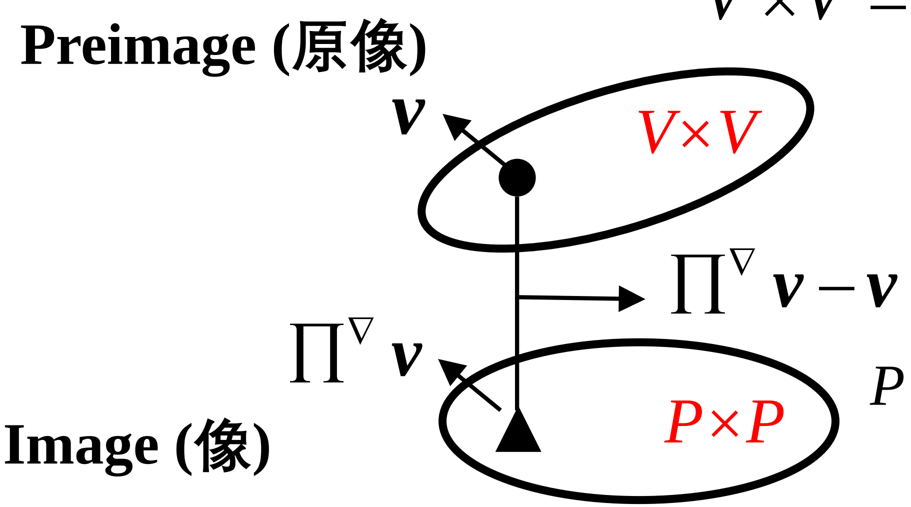
Then the entry $a\left(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j\right)$ **can be reformulated as (to avoid $\boldsymbol{\varphi}_i$)**

$$a\left(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j\right) = a\left(\prod^{\nabla} \boldsymbol{\varphi}_i + \left(\boldsymbol{\varphi}_i - \prod^{\nabla} \boldsymbol{\varphi}_i\right), \prod^{\nabla} \boldsymbol{\varphi}_j + \left(\boldsymbol{\varphi}_j - \prod^{\nabla} \boldsymbol{\varphi}_j\right)\right) =$$

$$\boxed{\underbrace{a\left(\prod^{\nabla} \boldsymbol{\varphi}_i, \prod^{\nabla} \boldsymbol{\varphi}_j\right)}_{\text{consistency}}} + \boxed{\underbrace{a\left(\boldsymbol{\varphi}_i - \prod^{\nabla} \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j - \prod^{\nabla} \boldsymbol{\varphi}_j\right)}_{\text{stability}}}$$

Derivation of Π^{∇}

Recall that:



$$V\times V=\operatorname{span}\left\{\begin{bmatrix}\varphi_1 & 0 & \varphi_2 & 0 & \cdots & \varphi_N & 0\\ 0 & \varphi_1 & 0 & \varphi_2 & \cdots & 0 & \varphi_N\end{bmatrix}\right\}$$

$$P\times P=\operatorname{span}\left\{\begin{array}{c}\begin{bmatrix}1 & 0 & \eta & \xi & 0 & \eta & \xi^2 & 0 & \eta^2 & 0 & \xi\eta & 0\\ 0 & 1 & -\xi & 0 & \eta & \xi & 0 & \xi^2 & 0 & \eta^2 & 0 & \xi\eta\end{bmatrix}\\ \begin{array}{cccccccccccc}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 & \boldsymbol{p}_5 & \boldsymbol{p}_6 & \boldsymbol{p}_7 & \boldsymbol{p}_8 & \boldsymbol{p}_9 & \boldsymbol{p}_{10} & \boldsymbol{p}_{11} & \boldsymbol{p}_{12}\end{array}\end{array}\right\}$$

So the projector: $\Pi^{\nabla}: V\times V\rightarrow P\times P$ is a 2N-by-12 matrix $\boldsymbol{\Pi}$

$$\boldsymbol{\Pi}=\begin{bmatrix}\boxed{\begin{matrix}S_{1,1}\\ \vdots\\ S_{12,1}\end{matrix}} & \begin{matrix}\cdots\\ \ddots\\ \cdots\end{matrix} & \begin{matrix}S_{1,2N}\\ \vdots\\ S_{12,2N}\end{matrix}\end{bmatrix}$$

For each column: $\prod^{\nabla}\boldsymbol{\varphi}_i=\sum_{\alpha=1}^{12}\boxed{S_{i,\alpha}}\boldsymbol{p}_{\alpha}$ **Linear combination (線型結合)**

Details can be found in (Beirão da Veiga et al., 2013)

Calculation of $a\,(\varphi_i\,,\,\varphi_j)$

$$a\left(\varphi_i,\varphi_j\right)=\boxed{\underbrace{a\left(\Pi^{\nabla}\varphi_i,\Pi^{\nabla}\varphi_j\right)}_{\text{consistency}}}+\underbrace{a\left(\varphi_i-\Pi^{\nabla}\varphi_i,\varphi_j-\Pi^{\nabla}\varphi_j\right)}_{\text{stability}}$$

(Beirão da Veiga et al., 2013)

Consistency part:

$$a\left(\Pi^{\nabla}\varphi_i,\Pi^{\nabla}\varphi_j\right)=\int_{\Omega}\boldsymbol{\sigma}\left(\Pi^{\nabla}\varphi_i\right):\boldsymbol{\varepsilon}\left(\Pi^{\nabla}\varphi_j\right)d\Omega=\int_{\Omega}\boldsymbol{\sigma}\left(\sum_{\alpha=1}^{12}S_{i,\alpha}\boldsymbol{p}_{\alpha}\right):\boldsymbol{\varepsilon}\left(\sum_{\beta=1}^{12}S_{j,\beta}\boldsymbol{p}_{\beta}\right)d\Omega$$

$$==\sum_{\alpha=1}^{12}\sum_{\beta=1}^{12}S_{i,\alpha}S_{j,\beta}\int_{\Omega}\boldsymbol{\sigma}\left(\boldsymbol{p}_{\alpha}\right):\boldsymbol{\varepsilon}\left(\boldsymbol{p}_{\beta}\right)d\Omega=\sum_{\alpha=1}^{12}\sum_{\beta=1}^{12}S_{i,\alpha}S_{j,\beta}a\left(\boldsymbol{p}_{\alpha},\boldsymbol{p}_{\beta}\right)==\sum_{\alpha=1}^{12}\sum_{\beta=1}^{12}\boldsymbol{\Pi}_{i\alpha}\boldsymbol{\Pi}_{\beta j}\boldsymbol{G}_{\alpha\beta}=\boxed{\left(\boldsymbol{\Pi}^{T}\boldsymbol{G}\boldsymbol{\Pi}\right)_{ij}}$$

and 12-by-12 matrix G is calculated as

$$\boldsymbol{G}=\begin{bmatrix} a\left(\boldsymbol{p}_1,\boldsymbol{p}_1\right) & \cdots & a\left(\boldsymbol{p}_1,\boldsymbol{p}_{12}\right)\\ \vdots & \ddots & \vdots\\ a\left(\boldsymbol{p}_{12},\boldsymbol{p}_1\right) & \cdots & a\left(\boldsymbol{p}_{12},\boldsymbol{p}_{12}\right)\end{bmatrix}$$

Divergence theorem (発散定理)

$$a\left(\boldsymbol{p}_{\alpha},\boldsymbol{p}_{\beta}\right)=\int_{\Omega^K}\boldsymbol{\sigma}\left(\boldsymbol{p}_{\alpha}\right):\boldsymbol{\varepsilon}\left(\boldsymbol{p}_{\beta}\right)d\Omega$$

$$=\int_{\partial\Omega}\mathbb{C}\boldsymbol{\varepsilon}\left(\boldsymbol{p}_{\alpha}\right)\mathrm{n}_{\partial\Omega}\cdot\boldsymbol{p}_{\beta}d\Gamma-\int_{\Omega}\boldsymbol{p}_{\beta}\cdot\left[\nabla\cdot\left(\mathbb{C}\boldsymbol{\varepsilon}\left(\boldsymbol{p}_{\alpha}\right)\right)\right]d\Omega,\ \ \alpha,\beta=1,2,\cdots,12$$

Calculation of $a\,(\varphi_i\,,\,\varphi_j)$

$$a\left(\varphi_i,\varphi_j\right)=\underbrace{a\left(\prod^{\nabla}\varphi_i,\prod^{\nabla}\varphi_j\right)}_{\text{consistency}}+\boxed{\underbrace{a\left(\varphi_i-\prod^{\nabla}\varphi_i,\varphi_j-\prod^{\nabla}\varphi_j\right)}_{\text{stability}}}$$

(Beirão da Veiga et al., 2013)

Stability part: First define a 2N-by-12 matrix D with entry

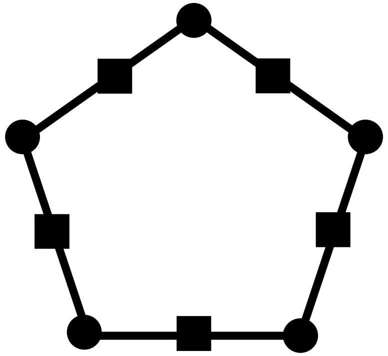
$$\boldsymbol{D}_{i\alpha}=\mathrm{dof}_i\left(\boldsymbol{p}_\alpha\right)\quad\text{for } i=1,2,\cdots 2N \text{ and } \alpha=1,2,\cdots 12$$

and $a\left(\varphi_i-\prod^{\nabla}\varphi_i,\varphi_j-\prod^{\nabla}\varphi_j\right)$ **is approximated by**

$$a\left(\varphi_i-\prod^{\nabla}\varphi_i,\varphi_j-\prod^{\nabla}\varphi_j\right)\approx\gamma\tau*\left(\boldsymbol{I}-\boldsymbol{D\Pi}\right)_i^T\left(\boldsymbol{I}-\boldsymbol{D\Pi}\right)_j$$

***I*: identity matrix (単位行列)**

γ, τ*: user-defined parameters (ユーザー定義パラメータ)



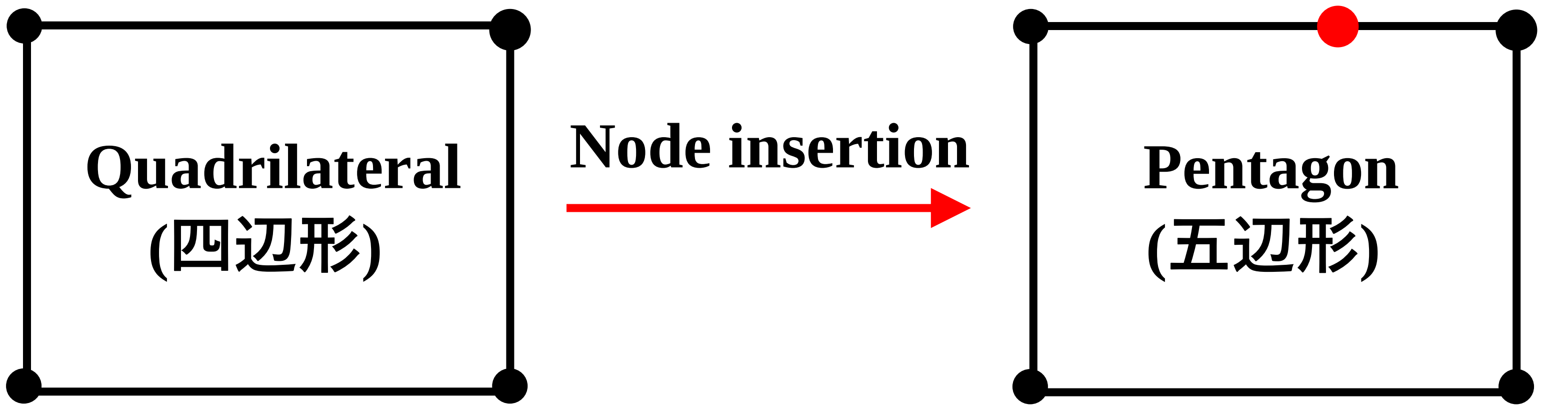
After obtaining $a\,(\varphi_i\,,\,\varphi_j)$, the element stiffness matrix (要素剛性行列)

$$\boxed{k_{ij}=a\left(\varphi_i,\varphi_j\right)}\xrightarrow{\textbf{Assembly}}KU=F$$

Equilibrium equation (平衡方程式)

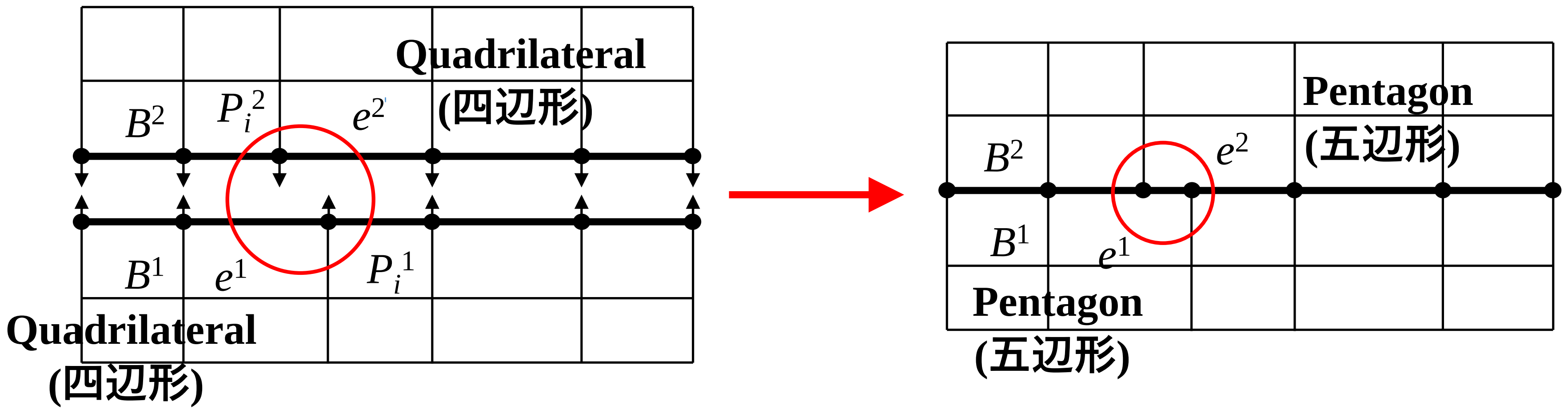
Node insertion algorithm

Merit of VEM: Allow hanging node on boundary (境界の分割可能)



Number of Shape functions (形状関数) increase for the red node only for this element

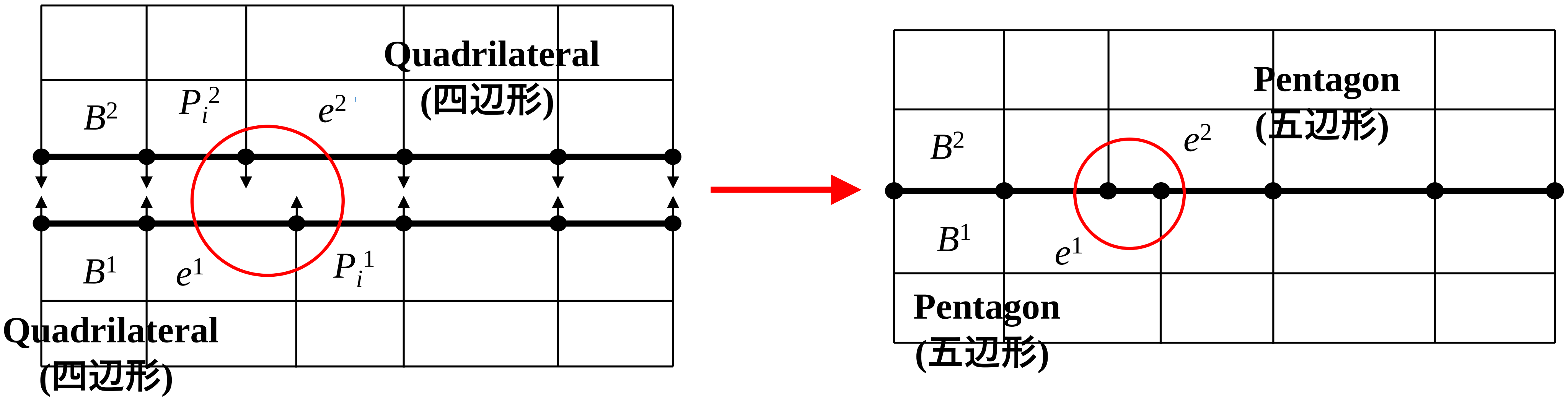
For non-matching contact interface (不整合メッシュ)



Matching mesh
(整合メッシュ)

Contact scheme

Node-to-Node contact scheme (点対点接触)



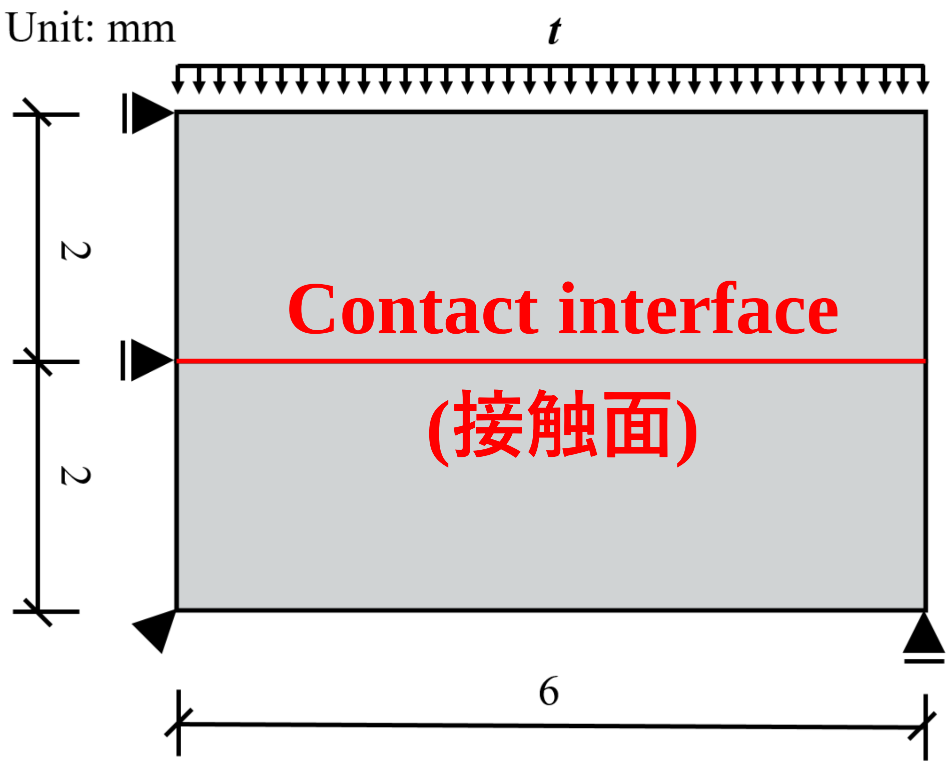
Normal contact: Lagrange multiplier method

(法線方向：ラグランジェ乗数法)

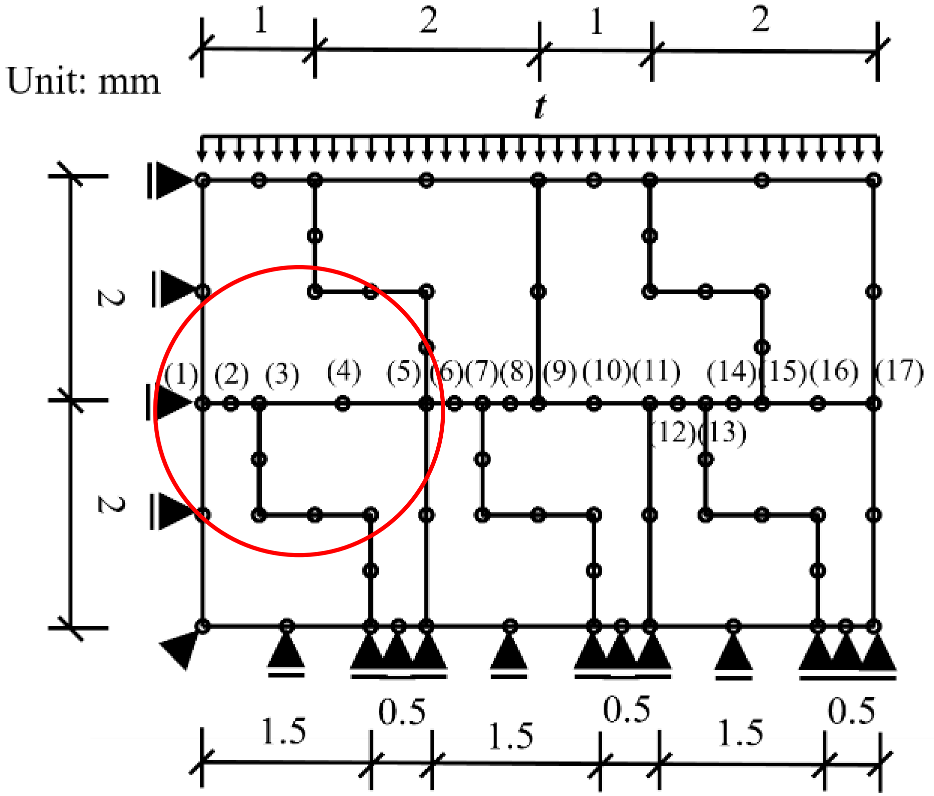
Tangential contact: Penalty method, stick condition
(接線方向：ペナルティ法、固着状態)

Patch test

Young's modulus: 7000 MPa; Poisson ratio: 0.3

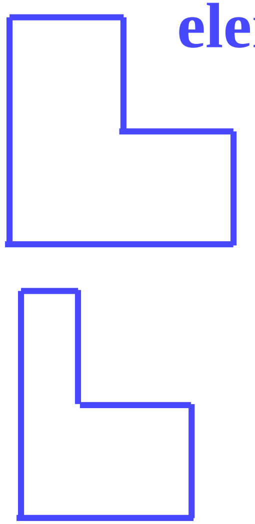


Geometry model
(接触する２弾性体)



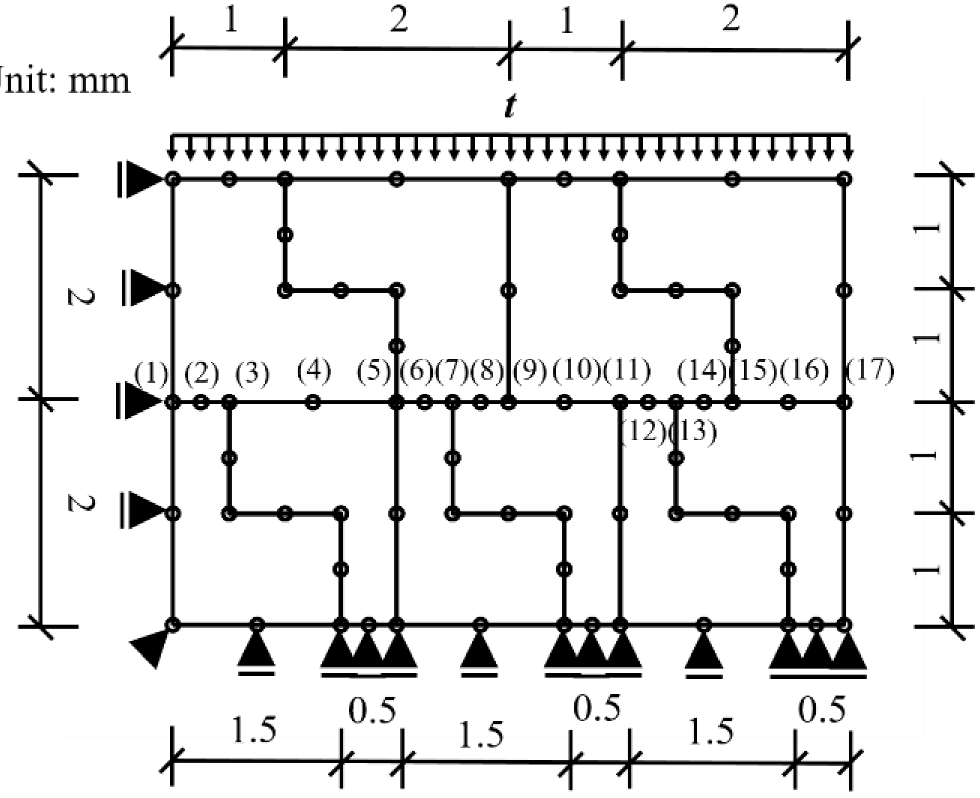
Discretization
(要素分割)

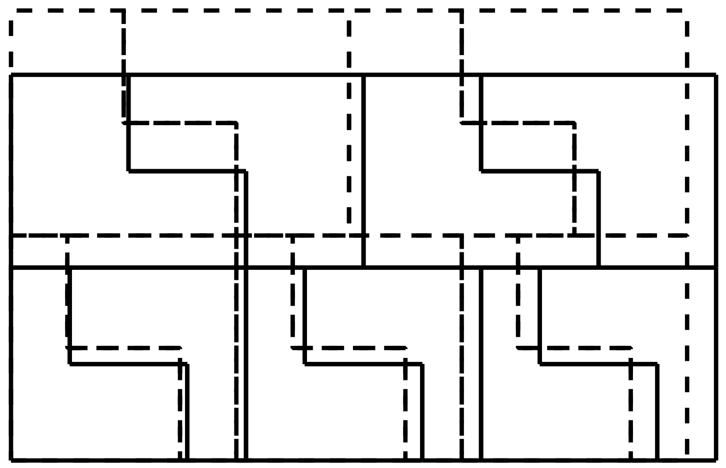
L-shaped element



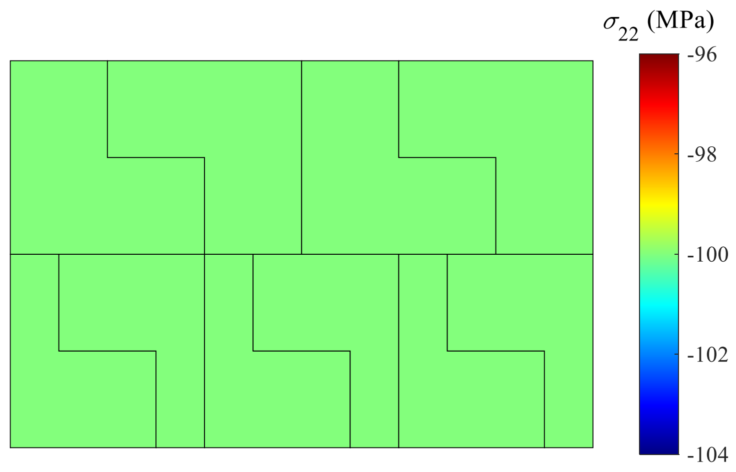
Patch test

Deformation and stress results





Axial Deformation
(一様な変形)



Uniform stress
(一様な接触圧力)

Conclusion and future work

- **The proposed method has the following conclusions:**
  - **By introducing second-order VEM, non-matching mesh can be transformed to matching mesh (不整合メッシュ → 整合メッシュ)**
  - **Then Node-to-Node contact scheme (点-点接触) can be easily applied by either Lagrange multiplier method or penalty method**
  - **Patch test is passed by the proposed method**

- **Future work:**
  - **3-dimensional application (ソリッド要素);**
  - **Large deformation/nonlinear analysis (大変形/非線形解析);**

Thanks for your kind attention

ご清聴ありがとうございました