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Pen & Paper Exercise 4

Social Networks

1 Power-law Distributions

Suppose you were studying educational institutions in the region and want to build models to answer the following questions:

(i) As a function of k, what fraction of classes at RWTH Aachen have k students enrolled?

(ii) As a function of k, what fraction of 3rd-grade elementary school classrooms in North Rhine-Westphalia have k pupils?

Which one of these functions would you expect to more closely follow a power-law distribution as a function of k? Give a brief explanation for your answer.

2 Preferential Attachment

Assume we are given a connected network of $N_0 = 15$ nodes and $M_0 = 40$ edges, in which node 1 has 10 neighbors, i.e. $k_1 = 10$, while node 15 has only one neighbor, i.e. $k_{15} = 1$. Now we want to grow this network by adding consecutively adding nodes with $m = 1$ link according to the preferential attachment model by Barabasi and Albert as presented in the lecture. What is the probability that after adding two nodes...

(i) ...we have $k_1 = 12$?

(ii) ...we have $k_1 = 10$?

(iii) ...we have $k_{15} = 1$?

(iv) ...we have $k_{15} = 3$?

3 The Kronecker Graph Model

A more recently proposed graph model is built on creating huge graphs out of small base graphs by utilizing the Kronecker matrix product.
For two matrices $A \in \mathbb{R}^{m \times n}, B \in B \in \mathbb{R}^{p \times r}$, their Kronecker product C is defined as

$$C := A \otimes B := \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{pmatrix} \in \mathbb{R}^{mp \times nr}$$

The main idea behind the model is to take the adjacency matrix A of a small given initiator graph G, and then define the Kronecker power graph G^k as the graph whose adjacency matrix is the k-th power of A with respect to the Kronecker product. Thus, for G^2, the adjacency matrix would be $A \otimes A$, for G^3 the adjacency matrix would be $A \otimes A \otimes A$, and so on.

Prove the following two statements (you may assume that G is undirected):

(i) If a graph G is disconnected, then all of its Kronecker powers G^k are disconnected as well.

(ii) If G is connected with diameter d, and all of its nodes have a self-loop, then all Kronecker power graphs G^k also have diameter d.