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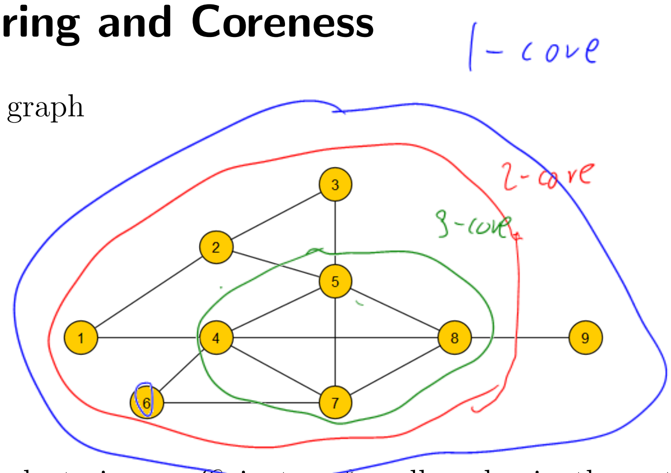
Pen & Paper Exercise 2

Social Networks

1 Local Clustering and Coreness

Consider the following graph



there cannot be a 4-core, because in the 3-core subgraph, no node has degree 4

a) Compute the local clustering coefficient c_i for all nodes in the network. What is the average local clustering?

b) Compute the coreness of each node in the network!

c) Calculate the closeness centralities of nodes 1 and 5!

Corenesses:

1	2	3	4	5	6	7	8	9
2	2	2	3	3	2	3	3	1

a) Clustering coeff of node i:

$$C_i = \frac{2e_i}{k_i(k_i-1)}$$

e_i = # links between neighbors of i

k_i = degree of node i

$$c_1 = \frac{2\cdot 0}{2\cdot 1} = 0$$

$$c_2 = \frac{\cancel{2}\cdot 1}{3\cdot \cancel{2}} = \frac{1}{3}$$

$$c_3 = \frac{2\cdot 1}{2\cdot 1} = 1$$

$$c_4 = \frac{2\cdot \cancel{4}}{5\cdot \cancel{4}} = \frac{2}{5}$$

$$c_5 = \frac{2}{5}$$

$$c_6 = 1$$

$$c_7 = \frac{2}{3}$$

$$c_8 = \frac{1}{2}$$

$$c_9 = 0$$

avg local clustering

$$C_{avg} = \frac{1}{9}\sum_{i=1}^{9} c_i$$

$$= \frac{43}{90}$$

c) Closeness Centralities

Let $d_i = \frac{1}{N-1} \sum_{j \neq i} d_{ij}$ with d_{ij} is distance between nodes i and j

↳ avg distance

Then closeness centrality of node i is defined as

$$C_i = \frac{1}{d_i}$$

$$d_1 = \frac{1}{8} \cdot (1+2+1+2+2+2+2+3) = \frac{15}{8}$$

$$\hookrightarrow C_1 = \frac{1}{d_1} = \frac{8}{15}$$

$$d_5 = \frac{1}{8}(2+1+1+2+1+1+2) = \frac{11}{8}$$

$$\hookrightarrow C_5 = \frac{8}{11}$$

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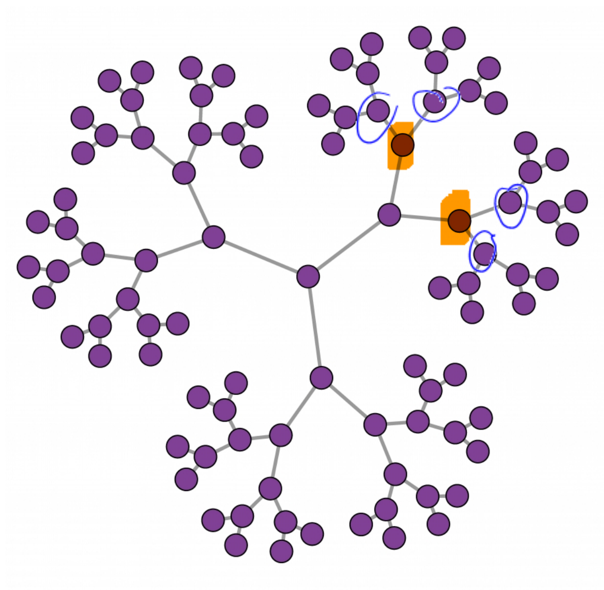
2 Small Worlds

A Cayley tree is a symmetric regular tree in which each node is connected to the same number of k others, until we get out to the leaves. A Cayley tree with $k = 3$ looks like this:



Show that the number of nodes which are reachable in exactly d steps from the central node is $k(k-1)^{d-1}$. Further, find an expression for the diameter of the network in terms of k and the number of nodes n. Does the Cayley tree display the small-world effect? Explain your answer.

let $r(d)$ be the number of nodes reachable from the center in d steps. We want to show that $r(d) = k(k-1)^{d-1}$

↳ we prove this claim via mathematical induction

Base Case: $d=1$

- the central node has k neighbors, so $r(1) = k = k\underbrace{(k-1)^{\overbrace{1-1}^{0}}}_{1}$ ✓

Induction step $d \to d+1$:

- by induction hypothesis, we have that $r(d) = k(k-1)^{d-1}$.

now every node that is reachable in d steps, has $k-1$ neighbors reachable in the next step.

↳ $r(d+1) = r(d) \cdot (k-1) \overset{\text{Ind. Hyp.}}{=} k(k-1)^{d-1} \cdot (k-1) = k(k-1)^{(d+1)-1}$

→ the claim follows by the principle of mathematical induction

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- Express number of nodes N in terms of d and k, consider the diameter is $2 \cdot d$

$$N = \underbrace{1}_{\text{central node}} + \underbrace{\sum_{l=1}^{d} r(l)}_{\text{\# nodes nodes reachable in } l=1,\dots,d \text{ steps}} = 1 + \sum_{l=1}^{d} k(k-1)^{l-1}$$

$$= 1 + k \sum_{l=1}^{d} (k-1)^{l-1}$$

$$\overset{\text{reindex}}{=} 1 + k \underbrace{\sum_{l=0}^{d-1} (k-1)^{l}}_{\text{geometric sum}}$$

$$= 1 + k \cdot \frac{(k-1)^{(d-1)+1} - 1}{(k-1) - 1}$$

$$\hookrightarrow N = 1 + k \cdot \frac{(k-1)^{d} - 1}{k-2} \quad (*)$$

geometric sum

$$\sum_{l=0}^{n} a^l = \frac{a^{n+1} - 1}{a - 1}$$

$a = k-1$
$n = d-1$

↳ we want to express diameter in terms of N and k
↳ express d in terms of N, k

$(*)$ is equivalent to $d = \frac{\log(N(k-2)+2) - \log(k)}{\log(k-1)}$

↳ so in essence we have that $d \in O(\log(N))$
↳ since diameter is $2d$, diameter is in $O(\log(N))$ as well
↳ the network displays the small world property

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3 Betweenness Centrality in a Ring Graph

Consider a network of $2n$ nodes arranged in a ring. Each node has two neighbors. In such a network all nodes have the same betweenness centrality. Derive the formula for this value as a function of n.

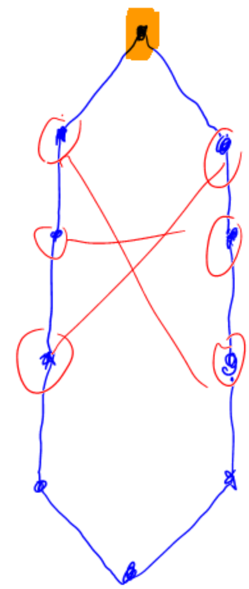
Betweenness centrality of node v:

$$b(v) = \sum_{u \neq v \neq w} \frac{\sigma_{uw}(v)}{\sigma_{uw}}$$

$d=2$
↳ 1 pair

$d=3$
↳ 2 pairs

$d=4$
↳ 3 pairs



$n=5$
$2n=10$ nodes

→ always $d-1$ node pairs !

σ_{uw} = # number of shortest paths between u and w

$\sigma_{uw}(v)$ = —— " —— which go throug v

- Case 1: nodes which are opposite in the graph
  ↳ have maximum possible distance n
  ↳ each of these opposite node pairs has 2 shortest paths, of which only 1 passes through node v → these contribute $\frac{1}{2}$ to $b(v)$
  ↳ there are $n-1$ such pairs in total
- Case 2: node pairs with distance d between 2 and n → $2 \leq d \leq n-1$
  ↳ How many node pairs with distance d are there which pass through v? $d-1$!
  ↳ each of these pairs has exactly one shortest path, which goes through v, so for these pairs u,w we have $\frac{\sigma_{uw}(v)}{\sigma_{uw}} = 1$

↳ If we sum up the contributions of all node pairs of cases 1+2, we get

$$b(v) = \underbrace{\frac{1}{2}\cdot(n-1)}_{\text{case 1}} + \underbrace{\sum_{d=2}^{n-1}(d-1)\cdot 1}_{\text{case 2}} = \frac{n-1}{2} + \sum_{d=1}^{n-2} d = \frac{n-1}{2} + \frac{(n-1)(n-2)}{2} = \frac{(n-1)^2}{2}$$

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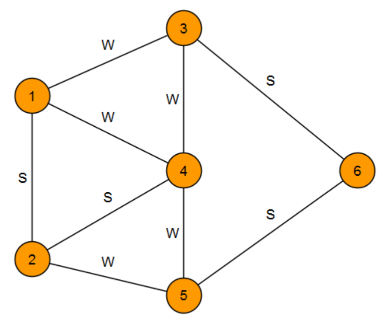
4 Weak Ties and Strong Triadic Closure

Consider the following labeled graph, in which the are labeled as weak (W) or strong (S):



Do all nodes in this graph fulfill the strong triadic closure property? Provide an explanation to your answer!

Triadic Closure: is given iff for any 3 nodes A, B, C, from A having strong ties with B and C it follows that there is an edge between nodes B and C

↳ not given, because 6 has strong ties to 3+5, which are not connected ↯