Prof. Dr. Markus Strohmaier
Dr. Fabian Flöck
Ivan Smirnov, Ph.D.
Tobias Schumacher

Pen & Paper Exercise 1

Social Networks

1 Basic Graph Properties

Consider the graph G represented by the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where the corresponding set of nodes is $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, with node i corresponding to the i-th row/column of A for all $i \in V$.

a) Is G a simple graph? Explain your answer by using arguments exclusively relating to A.

b) Is G connected? If yes, what is its diameter? If not, how many connected components does G consist of?

c) Does a path from node 1 to node 5 exist? If yes, how many shortest paths between these nodes exist? What is its/their length?

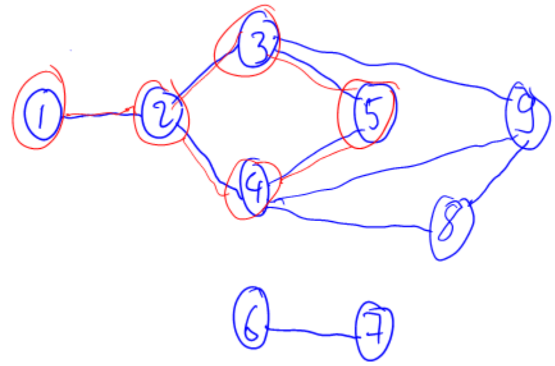
d) Compute the average degree $\langle k \rangle$ of the nodes in G.

e) Compute the density of graph G. Is G sparse? Explain your answer.

Prof. Dr. Markus Strohmaier
Dr. Fabian Flöck
Ivan Smirnov, Ph.D.
Tobias Schumacher

a) yes, it is simple, as there are no self-loops, no multiple/weighted edges, and it is undirected (A symmetric)

b) not connected



c) yes, there exist two path from 1 to 5, each have length 3.

d) $\langle k \rangle = \frac{2L}{N} = \frac{20}{9}$

$L = 10$ # links in the graph

$N = 9$

e) $d = \frac{L}{\underbrace{\left(\frac{N(N-1)}{2}\right)}_{\binom{N}{2} = \text{\# possible links}}} = \frac{2L}{N(N-1)} = \frac{2 \cdot 10}{9 \cdot 8} = \frac{5}{18}$

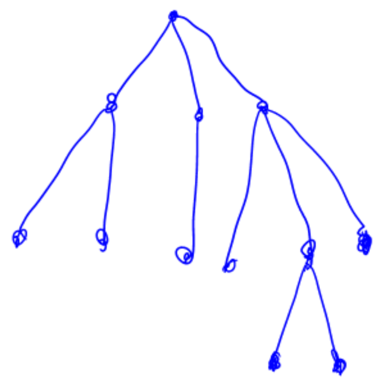
- Yes, it is sparse, because there is only 1 more edge than there are nodes.

Prof. Dr. Markus Strohmaier
Dr. Fabian Flöck
Ivan Smirnov, Ph.D.
Tobias Schumacher

2 Tree Graphs

Consider a tree graph consisting of N nodes. How many edges does it have? Explain/Prove your claim!

- it has N-1 edges
- except for the root, every node has one unique link to its parent, and these links constitute all links in the network



Prof. Dr. Markus Strohmaier
Dr. Fabian Flöck
Ivan Smirnov, Ph.D.
Tobias Schumacher

3 Bipartite Networks

Consider a bipartite network with N_1 and N_2 nodes in the two sets.

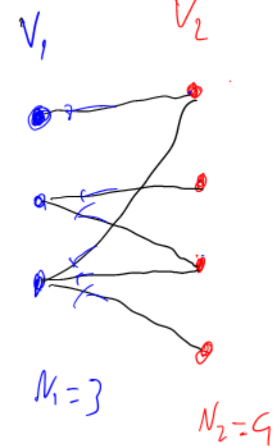
a) What is the maximum number of links L^{bp}_{max} the network can have?

b) How many links cannot occur compared to a non-bipartite network of size $N = N_1 + N_2$?

c) If $N_1 \ll N_2$, what can you say about the network density?

d) Find an expression connecting N_1, N_2 and the average degree for the two sets in the bipartite network, $\langle k_1 \rangle$ and $\langle k_2 \rangle$.

$\langle k_1 \rangle = \frac{L}{N_1}$



a) $L^{bp}_{max} = N_1 \cdot N_2$

↳ There N_1 nodes in V_1, each of them can connect to max. N_2 nodes in V_2 (and vice versa).

b) $L_{max} = \frac{N(N-1)}{2} = \frac{(N_1+N_2)(N_1+N_2-1)}{2}$

$= \frac{1}{2}[N_1^2 + N_1N_2 - N_1 + N_1N_2 + N_2^2 - N_2]$

$= \frac{1}{2}[N_1^2 + 2N_1N_2 - N_1 - N_2 + N_2^2]$

$L_{max} - L^{bp}_{max} = \frac{1}{2}[N_1^2 + 2\cancel{N_1N_2} - N_1 - N_2 + N_2^2] - \cancel{N_1N_2}$

$= \frac{1}{2}N_1^2 + \frac{1}{2}\cdot 2 \cdot N_1N_2 - \frac{1}{2}N_1 - \frac{1}{2}N_2 + \frac{1}{2}N_2^2 - N_1N_2$

$= \frac{1}{2}[N_1^2 - N_1 - N_2 + N_2^2]$

c) $N_1 << N_2$

$N = N_1 + N_2$

↳ $N \sim N_2$

$d = \frac{2L}{N(N-1)} \sim \frac{2N}{N(N-1)} \sim \frac{1}{N}$

$L^{bp}_{max} = N_1 \cdot N_2 \sim N_2 \sim N$ → the maximum # of links is roughly the amount of nodes

↳ $L \sim N$ approximately

→ the network gets very sparse

d) $\langle k_1 \rangle = \frac{L}{N_1}$ $\langle k_2 \rangle = \frac{L}{N_2}$ $\Rightarrow N_1 \langle k_1 \rangle = N_2 \langle k_2 \rangle \Leftrightarrow \langle k_1 \rangle = \frac{N_2}{N_1} \langle k_2 \rangle$

$\Leftrightarrow L = N_1 \langle k_1 \rangle$ $L = N_2 \langle k_2 \rangle$

Prof. Dr. Markus Strohmaier
Dr. Fabian Flöck
Ivan Smirnov, Ph.D.
Tobias Schumacher

4 Gatekeepers

For any given graph $G = (V, E)$, we say that a node v is a *gatekeeper* if for some other two nodes u and w, every path from u to w passes through v, with v not being equal to u or w. In addition, we say that a node v is a *local* gatekeeper if there are two neighbors of v, say u and w, that are not connected by an edge.

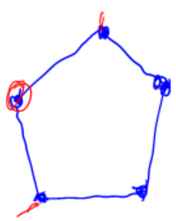
a) Give an example of a graph in which more than half of all nodes are gatekeepers.

b) Give an example of a graph in which there are no gatekeepers, but in which every node is a local gatekeeper.

a)

every inner node of a line graph is a gatekeeper which blocks the outer nodes

b)



every circle graph with $N > 3$ nodes fulfills this property, since no node's neighbors are connected, but there is no "global" gatekeeper, as there is always one path on the other graph of the circle.