高等代数(II)第一次作业情况

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1　补充说明

- 上交作业写清姓名和学号, 多张纸标好页码, 如没有装订须在每页正面写上个人信息;
- 注意叙述的科学性. 两个多项式的最大公因式有且仅有一个, 不应出现"存在/一个……的最大公因式"等类似表述; 在未说明$g|f$时不应出现形如$\frac{f}{g}$的表述.

2　2月21日作业

无.

3　2月24日作业

P8: 4, 5, 6, 7; P13: 3(2), 4, 7, 8

P8: 5. (注意零因子和零的差别)若不然, 设可逆元$r \in R$为左零因子, 那么根据定义存在非零$s \in R$使得$rs = 0$, 于是有

$$s = 1s = r^{-1}rs = r^{-1}0 = 0,$$

这与$s \neq 0$的假设矛盾. 同理r也不可能为右零因子. □

4　2月28日作业

P21: 1(2), 2, 3, 4, 5

P21: 1(2).

$$\begin{aligned}
&(x^4+6x^3-6x^2+6x-7,\ x^3+x^2-7x+5)\\
=&((x+5)(x^3+x^2-7x+5)+(-4x^2+36x-32),\ x^3+x^2-7x+5)\\
=&(-4x^2+36x-32,\ x^3+x^2-7x+5)\\
=&(x^2-9x+8,\ x^3+x^2-7x+5)\\
=&(x^2-9x+8,\ (x+10)(x^2-9x+8)+(75x-75))\\
=&(x^2-9x+8,\ 75x-75)\\
=&(x^2-9x+8,\ x-1)\\
=&((x-8)(x-1),\ x-1)=x-1
\end{aligned}$$

Conversely,

$$\begin{aligned}
&x-1\\
=&\frac{1}{75}(0(x^2-9x+8)+(75x-75))\\
=&\frac{1}{75}(-(x+10)(x^2-9x+8)+(x+10)(x^2-9x+8)+(75x-75))\\
=&\frac{1}{75}\cdot\frac{1}{4}((x+10)(-4x^2+36x-32)+4(x^3+x^2-7x+5))\\
=&\frac{1}{300}((x+10)((x+5)(x^3+x^2-7x+5)+(-4x^2+36x-32))\\
&\qquad+(4-(x+10)(x+5))(x^3+x^2-7x+5))\\
=&\frac{1}{300}((x+10)(x^4+6x^3-6x^2+6x-7)+(-x^2-15x-46)(x^3+x^2-7x+5))\\
=&\frac{x+10}{300}(x^4+6x^3-6x^2+6x-7)+\frac{-x^2-15x-46}{300}(x^3+x^2-7x+5)
\end{aligned}$$

□

题目 4.1. 设f, $g\in\mathbb{F}[x]$, $\lambda\in\mathbb{F}$, 则

$$R(f,(x-\lambda)g)=(-1)^{\deg f}f(\lambda)R(f,g).$$

证明. Let

$$f(x)=\sum_{k=0}^{n}a_kx^k,\quad g(x)=\sum_{l=0}^{m}b_lx^l,$$

then

$$(x-\lambda)g(x)=(x-\lambda)\sum_{l=0}^{m}b_lx^l=\sum_{l=0}^{m}b_lx^{l+1}-\lambda\sum_{l=0}^{m}b_lx^l=b_mx^{m+1}+\sum_{l=1}^{m}(b_{l-1}-\lambda b_l)x^l-\lambda b_0.$$

By definition we have

$$R(f,(x-\lambda)g)=\begin{vmatrix}
a_n & \cdots & \cdots & a_1 & a_0 & & \\
 & \ddots & & & \ddots & \ddots & \\
 & & a_n & \cdots & \cdots & a_1 & a_0\\
b_m & b_{m-1}-\lambda b_m & \cdots & b_0-\lambda b_1 & -\lambda b_0 & & \\
 & \ddots & \ddots & & \ddots & \ddots & \\
 & & b_m & b_{m-1}-\lambda b_m & \cdots & b_0-\lambda b_1 & -\lambda b_0
\end{vmatrix}$$

$$
=\begin{vmatrix}
a_n & a_{n-1}+\lambda a_n & \cdots & a_1+\lambda a_2+\cdots+\lambda^{n-1}a_n & f(\lambda) & \lambda f(\lambda) & \cdots & \lambda^m f(\lambda) \\
 & \ddots & & & \ddots & \ddots & & \vdots \\
 & & a_n & a_{n-1}+\lambda a_n & \cdots & a_1+\lambda a_2+\cdots+\lambda^{n-1}a_n & & f(\lambda) \\
b_m & b_{m-1} & \cdots & b_0 & 0 & & & \\
 & \ddots & \ddots & & \ddots & \ddots & & \\
 & & b_m & b_{m-1} & \cdots & b_0 & & 0
\end{vmatrix}
$$

$$
=\begin{vmatrix}
a_n & a_{n-1} & \cdots & a_1 & a_0 & & \\
 & \ddots & & & \ddots & & \\
 & & a_n & a_{n-1} & \cdots & a_1 & f(\lambda) \\
b_m & b_{m-1} & \cdots & b_0 & 0 & & \\
 & \ddots & \ddots & & \ddots & \ddots & \\
 & & b_m & b_{m-1} & \cdots & b_0 & 0
\end{vmatrix}
$$

$$
=(-1)^{(m+1)+(m+1+n)}f(\lambda)R(f,g)=(-1)^{\deg f}R(f,g)
$$

□

题目 4.2. 设

$$
\begin{aligned}
f &= 2x^3-3x^2+\lambda x+3 \\
g &= x^3+\lambda x+1.
\end{aligned}
$$

λ为何值f和g有非平凡公因子.

证明. 除直接计算结式外, 还可对最小公因式进行变换.

$$
\begin{aligned}
(f,g) &= (2x^3-3x^2+\lambda x+3, x^3+\lambda x+1) \\
&= (x^3-3x^2+2, x^3+\lambda x+1) \\
&= (x-1, x^3+\lambda x+1)(x^2-2x-2, x^3+\lambda x+1) \\
&= (x-1, 1^3+\lambda\cdot 1+1)(x^2-2x-2, 2x^2+(\lambda+2)x+1) \\
&= (x-1, \lambda+2)(x^2-2x-2, (\lambda+6)x+5) \\
&= (x-1, \lambda+2)(x-(1+\sqrt{3}), (\lambda+6)x+5)(x-(1-\sqrt{3}), (\lambda+6)x+5)
\end{aligned}
$$

故f和g有非平凡公因子,即$(f,g)\neq 1$当且仅当$\lambda+2=0$或$(\lambda+6)(1\pm\sqrt{3})+5=0$. 解之可得$\lambda\in\{-2,(-7\pm 5\sqrt{3})/2\}$. □

题目 4.3. 求

$$
R(x^n+x+1, x^2-3x+2).
$$

证明. 利用第一题结论,

$$
\begin{aligned}
R(x^n+x+1, x^2-3x+2) &= R(x^n+x+1, (x-1)(x-2)) \\
&= (-1)^n(1^n+1+1)R(x^n+x+1, x-2) \\
&= (-1)^n(1^n+1+1)(-1)^n(2^n+2+1)R(x^n+x+1, 1) \\
&= 3\cdot(2^n+3)
\end{aligned}
$$

□