Analysis of Policy Gradient Descent for Control: Global Optimality via Convex Parameterization

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Abstract

Policy gradient descent is a popular approach in reinforcement learning due to its simplicity. We aim to investigate the optimality and convergence properties of this method when applied in control problems. To do so, we make a connection between policy gradient descent and classical convex parameterization techniques in control theory, to show the *gradient dominance* property for the nonconvex cost function. Such a connection between nonconvex and convex landscapes holds for continuous/discrete time LQR, distributed optimal control, minimizing the $\mathcal{L}_2$ gain, and $\mathcal{H}_2/\mathcal{H}_\infty$ mixed/robust control, among others. To the best of our knowledge, this paper offers the first result unifying the landscape analysis of a broad class of control problems.

1 Introduction

This paper proposes a framework that builds the mapping between a few control problems with their associated convex parameterized form. With the mapping, we show that all stationary points of the cost functions, as functions of the policy, are global minima despite their nonconvexity. The fact allows first order optimization methods (i.e., policy gradient method) to converge the globally optimal controller. We give a comprehensive theory covering many control problems, including continuous/discrete time LQR, distributed optimal control, minimizing the $\mathcal{L}_2$ gain, and $\mathcal{H}_2/\mathcal{H}_\infty$ mixed/robust control that unifies the conclusion of each specific work.

We start from introducing linear quadratic regulator (LQR), which is one of the most well studied optimal control problems for decades [1]. Consider the continuous time linear time-invariant dynamical system,

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and A, B are constant matrices describing the dynamics. The goal of optimal control is to determine the input series $u(t)$ that minimizes some cost function (that typically depends on the state and input). In the infinite horizon LQR problem, we define constant matrices $Q \in \mathbf{S}^n_{++}, R \in \mathbf{S}^p_{++}$, and minimize the cost as a function of input

$$\mathrm{cost}(u(t)) := \mathbf{E}_{x_0} \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t)) \mathrm{d}t. \tag{2}$$

The optimal controller is linear in the state, called static state feedback controller, and can be described as $u(t) = Kx(t)$ for a constant $K \in \mathbb{R}^{p \times n}$ [1].

We can define this cost function with variable K,

$$\mathcal{L}(K) := \mathbf{E}_{x_0} \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t)) \mathrm{d}t, \text{ s.t. } u(t) = Kx(t). \tag{3}$$

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Policy gradient method. The policy gradient descent method is to minimize $\mathcal{L}(K)$ by running first order method with respect to K, for example,

$$K \leftarrow K - \eta \nabla \mathcal{L}(K).$$

It is shown that, the loss function is typically nonconvex in K, e.g., discrete time LQR [2]. However, gradient descent for nonconvex optimization is widely used in machine learning, or control tasks with the context of reinforcement learning.

Convex parameterization. In classical control theory literature, due to the nonconvex nature, policy gradient method is not commonly used. Instead, one can introduce another parameterization of the cost to make it convex, and apply convex optimization method with global convergence guarantee. This approach is in sharp contrast to how one would typically minimize a cost function through gradient descent on K. In the following discussion, we will first review the convex parameterization methods, and then review the recent progress on the policy gradient methods on LQR and more control problems.

Generality of this paper. Below, we will review the techniques of solving LQR, since most of the recent works on policy gradient descent show that, the linear quadratic loss or its close variants are gradient dominant. However, our paper proposes a generic analysis of the gradient dominance landscape, than can be applied to continuous/discrete time LQR, distributed optimal control, minimizing the $\mathcal{L}_2$ gain, system-level parameterization, and $\mathcal{H}_2/\mathcal{H}_\infty$ mixed/robust control, etc.

Solving LQR – classical way. LQR is usually solved by the algebraic Riccati equation (ARE) [3,4]. A large number of works have studied the solution of ARE, including approaches based on iterative algorithms [5], algebraic solution methods [6], and semidefinite programming [7]. Besides LQR, convex parameterizations such as Youla parametrization, Q-parameterization, or the more recent System Level Synthesis (SLS) have allowed the reformulation of certain control design problems as semidefinite programs. They are convex optimization methods, which natually can be globally minimized by first order algorithms. On the other hand, one has to know the system parameters to run them, and the papparameterizations are case-by-case.

LQR with unknown system parameters: model-based and model-free. There are two major types of algorithms when system parameters are not known. The first type is model-based methods, when we first estimate the system parameters and then a controller is constructed based on the identified system. System identification has a long history, as reviewed in [8]. Recently the paper [9] gave sample complexity bounds for state-observed system. The papers [10–13] describe the joint system identification and optimal control approaches, where the algorithms estimate the system parameter from the intput and output, and apply controllers based on the estimated system. .

The second type of method is model free method, when the controller is directly trained by observing the cost function, without characterizing the dynamics. Here one does not necessarily estimate the system parameters A, B. The paper [14] is a review of reinforcement learning area and optimal control, which studies a few fixed point type dynamic programming methods. Q-learning is a typical model free method for reinforcement learning, and it is applied to LQR as in [10, 15, 16].

Recent works on policy gradient descent. Policy gradient descent calls for an estimate of the cost and its gradient with respect to controller K. The goal is to show that gradient descent with respect to K converges to the optimal controller (we can call it K^*). The policy gradient descent is recently reviewed by [17, 18]. The paper [2] provides a counterexample showing that minimizing the quadratic LQ cost as a function of K is not convex, quasi-convex or star-convex.

There has been recent evidence of the *empirical* success of first order methods in solving nonconvex reinforcement learning problems. The paper [19, Ch. 3] proposes the gradient based method for optimal control and extends to decentralized control. The paper [20] studies feedback control with dynamical controllers, and observes that gradient descent with Youla parameterization is robust within the set of stabilizing controllers while other parameterizations are not. On the *theoretical* side, despite the nonconvexity of $\mathcal{L}(K)$, for certain types of control problems, there are works showing the *gradient dominance* property, which enables first order methods to converge to the global optimum. The paper [2] gives the first result by proving the coercivity and the gradient dominance properties of $\mathcal{L}(K)$ for the discrete time LQR. Based on this, paper [2] shows the linear convergence of gradient based method. Later paper [21] shows a similar result for the continuous time case, papers [22, 23] give a more detailed analysis for both discrete and continuous time LQR. The papers [24, 25] show the convergence for two types of zero-sum LQ games. The paper [26] studies the convergence of gradient descent on $\mathcal{H}_2$ control with $\mathcal{H}_\infty$ constraint, and shows that gradient descent implicitly makes the controller robust. The paper [27] shows the convergence for finite-horizon distributed control under the quadratic invariance assumption.

Control for nonlinear systems is far more difficult, typically via dynamic programming, solving Bellman equations [28] , or recent deep RL that led to empirical success in control of complex systems . Yet it is still mysterious how deep learning models work in this context, and recent theoretical studies have focused on linear systems in the hope of providing insights into more complex cases.

Motivation: Although we reviewed many papers that show convergence of policy gradient descent, they investigate different control problems and the proofs are given case by case. However, we observe that all the results are proven by the gradient dominance property, and all of them are solved by convex parameterization methods in classical control literature. Thus, we ask whether there is a proof that unifies the proofs of the gradient dominance property for different control problems, and bridges nonconvex methods with convex methods in classical control literature.

Contributions: In this paper, we make a connection between nonconvex policy gradient descent and known convex parameterization methods with a map between the two parameters. This map maintains the gradient dominance when going from the convex landscape to the nonconvex landscape.

Our result is quite general—we show that continuous-time LQR is a special case that the main theorems apply to, and we generalize the guarantees provided by this method to a range of other control problems. The instances cover optimal control, robust control, mixed design and system level synthesis. To judge whether a nonconvex landscape has gradient dominance property, one can directly check if it is covered by the theorems, avoiding a case-by-case analysis. Also, as discussed in [2], theoretical guarantees for first-order methods naturally lead to guarantees for the more practical zeroth-order optimization or sampling-based methods, which do not need access to the gradient of the cost with respect to K.

Roadmap: The rest of this paper is structured as follows. Sec. 2 reviews the continuous-time LQR problem. Sec. 3 presents our main result on the gradient dominance property for the nonconvex cost. Sec. 4 lists more examples of control problems covered by the main theorem. Although Sec. 4 covered many problems, Sec. 5 further generalizes our main result and Sec. 6 covers examples under the more generic result. Sec. 7 gives a proof sketch with intuitive connections between the nonconvex and convex formulations.

2 Review of convex parameterization for continuous-time LQR

Convex parameterization (e.g., solving optimal control by linear matrix inequalities (LMI) in [29]) is widely used in optimal control problems, and here we discuss its application for continuous time LQR [21]. We will introduce new variables, construct an equivalent convex optimization problem with new variables, and the pair of variables are proven to be linked by a bijection. In the next section we use the critical properties of the nonconvex and convex problems as an intuition to generalize to a more general form.

Define a continuous time linear time invariant system (1) where x is state and u is input signal, and x_0 is the initial state. We assume that $\Sigma := \mathbf{E}(x_0 x_0^\top) \succ 0$. This is a commonly used setup such as in the theoretical study [22, §3.3], and the practical work [19, Ch. 3]. With $\Sigma \succ 0$, the optimal controller is not state-dependent; when Σ is low rank, then a controller K that gives finite LQR cost does not stabilize the system for all initial state $x_0 \in \mathrm{null}(\Sigma)$.

One can then consider minimizing the linear quadratic (LQ) cost (2) as a function of $u(t)$ where Q, R are positive definite matrices. The paper [1] proves that, the input signal that minimizes the cost function $\mathrm{cost}(u)$ is given by a static state feedback controller, denoted by $u(t) = K^* x(t)$. K^* can be obtained by solving linear equations, called Riccati equations. Once we know that the optimal state feedback controller is static, we can write cost as $\mathcal{L}(K)$ as (3). It is a function of K, and we search only static state feedback controllers.

An alternative approach is reparameterization, to obtain a convex optimization problem, as used in [21]. We will review it here, starting from the Lyapunov equation. Suppose the initial state satisfies $\mathbf{E}(x_0 x_0^\top) = \Sigma \succ 0$, and $\dot{x}(t) = Ax(t)$. Then with a matrix $P \in \mathbf{S}_{++}^{n \times n}$ (P is a positive definite matrix) as the variable, the Lyapunov equation is written as

$$AP + PA^\top + \Sigma = 0.$$

In our setup (1), we use a state feedback controller $u = Kx$, thus we have $\dot{x} = (A + BK)x$. We denote the set of stabilizing controllers as $\mathcal{S}_{K,\text{sta}}$, which is defined as

$$\mathcal{S}_{K,\text{sta}} = \{K : \ \text{Re}(\lambda_i(A + BK)) < 0, \ i = 1, ..., n\}.$$

If a state feedback controller is applied, the cost is only bounded when $K \in \mathcal{S}_{K,\text{sta}}$ and is coercive in $\mathcal{S}_{K,\text{sta}}$ [23]. Replace A by the closed loop system matrix $A + BK$ in the Lyapunov equation, and let $L = KP \in \mathbb{R}^{p\times n}$, we get

$$AP + PA^\top + BL + L^\top B^\top + \Sigma = 0.$$

Let $\mathcal{A}(P) = AP + PA^\top$, $\mathcal{B}(L) = BL + L^\top B^\top$, which are referred to as Lyapunov maps. Assume $\mathcal{A}$ is invertible, then we have the relation

$$\mathcal{A}(P) + \mathcal{B}(L) + \Sigma = 0. \tag{4}$$

Indeed, once we fix the system and any stabilizing controller A, B, K, the matrices P as well as $L = KP$ are uniquely determined. P is the Grammian matrix

$$P = \int_0^\infty e^{t(A+BK)} \ \Sigma \ e^{t(A+BK)^\top} \ \text{d}t. \tag{5}$$

The matrix P is positive definite if $\Sigma \succ 0$. We are interested in the cost function $\mathcal{L}(K)$ when[1] $K \in \mathcal{S}_{K,\text{sta}}$, which corresponds to (2) by inserting $u(t) = Kx(t)$,

$$\mathcal{L}(K) = \begin{cases} \mathbf{Tr}((Q + K^\top RK)P), & K \in \mathcal{S}_{K,\text{sta}}; \\ +\infty, & K \notin \mathcal{S}_{K,\text{sta}}. \end{cases} \tag{6}$$

One can construct a bijection from P, L to K, and prove that, if we minimize $f(L, P)$ subject to (4), the optimizer P^*, L^* will map to the optimal K^*, and this minimization problem is convex.

Convex reparameterization for Continuous time LQR: Suppose the dynamics and costs are (1) and (2), and let $\mathbf{E}(x_0 x_0^\top) = \Sigma \succ 0$. Denote the (static) state feedback controller by K, so that $u(t) = Kx(t)$. The optimal control problem then is

$$\min_K \ \mathcal{L}(K), \quad \text{s.t.} \quad K \in \mathcal{S}_{K,\text{sta}} \tag{7}$$

where $\mathcal{L}(K)$ is the cost in (2) with $u = Kx$. This problem can be expressed as the following equivalent convex problem,

$$\min_{L,P,Z} \ f(L, P, Z) := \mathbf{Tr}(QP) + \mathbf{Tr}(ZR) \tag{8a}$$

$$\text{s.t. } \mathcal{A}(P) + \mathcal{B}(L) + \Sigma = 0, \ P \succ 0, \tag{8b}$$

$$\begin{bmatrix} Z & L \\ L^\top & P \end{bmatrix} \succeq 0. \tag{8c}$$

The connection between the two problems is distilled in Sec. 3. For all feasible (L, P, Z) triplets in (8), we can take the first two elements (L, P), and they form a bijection with all stabilizing controllers K in (7). The cost function values are equal under the bijection. So we can solve for L^*, P^*, and $K^* = L^*(P^*)^{-1}$.

3 Main result

Motivated by methods that use gradient descent in the policy space, we ask whether running a gradient-based algorithm and getting $\nabla_K \mathcal{L}(K) = 0$ for some K in fact gives the globally optimum K^*. The papers [2,21] show the coercivity and gradient dominance property of $\mathcal{L}(K)$ for discrete- and continuous-time LQR respectively. In this paper, we generalize these results from the special case of continuous-time LQR to a much broader set of control problems, showing the gradient dominance property of the nonconvex costs as functions of policy.

[1]If K is not a stabilizing controller, we define $\mathcal{L}(K) = +\infty$.

We present our main result in Theorem 1. We consider a pair of problems satisfying Assumptions 1, 3. In Sec. 4 we catalog a number of examples showing the generality of this result.

We begin by considering an abstract description of the pair of problems (7) and (8). These problem descriptions cover LQR as discussed in the last section, as well as more problems discussed in Sec. 4. Consider the problems

$$\min_{K} \quad \mathcal{L}(K), \quad \text{s.t. } K \in \mathcal{S}_K, \tag{9}$$

and

$$\min_{L,P,Z} \quad f(L,P,Z), \quad \text{s.t.} \quad (L,P,Z) \in \mathcal{S}, \tag{10}$$

where the sets $\mathcal{S}_K, \mathcal{S}$ capture the control constraints. They are defined differently for each specific example in Sec. 4. For example, for continuous time LQR, $\mathcal{S}_K$ is the set of all stabilizing controllers (7) and $\mathcal{S}$ is the intersection of (8b) & (8c). In infinite horizon problems, we usually need a stabilizing K so that $\mathcal{S}_K$ is equal to or a subset of the set of stabilizing controllers. We allow special cases when (10) depends only on L, P,

$$\min_{L,P} \quad f(L,P), \quad \text{s.t.} \quad (L,P) \in \mathcal{S}. \tag{11}$$

We distill three properties of the two problems (9) and (10) that will be critical for Theorem 1, and allow us to cover more problems as discussed in Sec. 4.

Assumption 1. *The feasible set $\mathcal{S}$ is convex in (L,P,Z). The cost function $f(L,P,Z)$ is convex, bounded, and differentiable, over an open domain that contains the set $\mathcal{S}$*

Assumption 1 implies the second problem is convex. Next, we extract the property of the connection between (7) and (8), and give an abstract description of the assumptions for (9) and (10).

Assumption 2. *Let P be invertible*[2] *whenever $(L,P,Z) \in \mathcal{S}$. Assume we can express $\mathcal{L}(K)$ as follows,*

$$\begin{aligned}\mathcal{L}(K) = \min_{L,P,Z} \ & f(L,P,Z) \\ s.t. \ & (L,P,Z) \in \mathcal{S}, \ LP^{-1} = K.\end{aligned}$$

Denote $\nabla\mathcal{L}(K)[V] := \mathbf{Tr}(V^\top \nabla\mathcal{L}(K))$ as the directional derivative of $\mathcal{L}(K)$ is the direction V. With the assumptions above, we will present the main theorem.

Theorem 1. *Suppose assumptions 1,2 hold, and consider the two problems (9) and (10). Let K^* denote the global minimizer of $\mathcal{L}(K)$ in S_K. Then there exist constants $C_1, C_2 > 0$ independent of the suboptimality $\mathcal{L}(K) - \mathcal{L}(K^*)$, and a direction V, with $\|V\|_F = 1$, in the descent cone of $\mathcal{S}_K$ at K such that,*

1. *if f is convex, the gradient of $\mathcal{L}$ satisfies*[3]

$$\nabla\mathcal{L}(K)[V] \le -C_1(\mathcal{L}(K) - \mathcal{L}(K^*)). \tag{12}$$

2. *if f is μ-strongly convex, the gradient of $\mathcal{L}$ satisfies*

$$\nabla\mathcal{L}(K)[V] \le -C_2(\mu(\mathcal{L}(K) - \mathcal{L}(K^*)))^{1/2}. \tag{13}$$

The constants C_1, C_2 are discussed below and in Appendix B.

[2]The invertibility of P holds for all instances in Sec. 4.

[3]We always consider the directional derivative of a feasible direction within descent cone.

Remark 1. *The constants in the above theorem can be computed or bounded in a case by case manner. They typically depend on the norm of system parameters and the radius of the feasible domain*[4] *We study continuous time LQR as an example. Let the sublevel set be where $\mathcal{L}(K) \le a$, and define*

$$\nu = 4a\left(\sigma_{\max}(A)\lambda_{\min}^{1/2}(Q) + \sigma_{\max}(B)\lambda_{\min}^{-1/2}(R)\right)^2,$$

$$C_{1,1} = \min\left\{\frac{1}{2a\nu}\lambda_{\min}(\Sigma)\lambda_{\min}^{1/2}(Q)\lambda_{\min}^{1/2}(R),\ \frac{1}{2a^2\nu^2}\lambda_{\min}^2(\Sigma)\lambda_{\min}^{3/2}(Q)\lambda_{\min}^{1/2}(R)\right\}.$$

The paper [21] gives another convex formulation with strong convexity and we can get C_2 for that form,

$$C_2 = \min\{\frac{1}{\nu}\lambda_{\min}(\Sigma),\ \frac{1}{a\nu^2}\lambda_{\min}^2(\Sigma)\lambda_{\min}^{1/2}(Q)\lambda_{\min}^{1/2}(R)\}.$$

See Appendix B for more details.

The lower bound $\|\nabla\mathcal{L}(K)\|_F \gtrsim (\mathcal{L}(K) - \mathcal{L}(K^*))^\alpha$ on the norm of the gradient is known as Lojasiewicz inequality [30]. When $\alpha = 1/2$, it is also called the *gradient dominance* property. If Lojasiewicz inequality holds for all K, all stationary points of the objective function are global minima, and an iterative method in which the norm of the gradient decreases to zero will have to converge to a global minimum. Nonconvex functions that satisfy Lojasiewicz inequality are easily optimized, compared to those with spurious local minima. In practice, gradient dominance often holds in a neighborhood of a local minimum, and gradient dominance is typically used as a tool for local convergence analysis (it is rare that gradient dominance holds for $\mathcal{L}(K)$ globally, but it holds for example problems in this paper).

Assumption 2 is a weak assumption. Assumption 3 is a stronger one covered by Assumption 2 where we assume that there is a bijection between K and (L, P). This is true for many control problems including continuous time LQR. Theorem 1 also holds with Assumptions 1,3. We emphasize the special case with the bijection for illustration. In Sec. 7, we will illustrate the critical proof steps. We use the fact that the convex function $f(L, P)$ is gradient dominant, and apply the bijection between K and (L, P) to calculate $\nabla\mathcal{L}(K)$.

Assumption 3.
1. *(Bijection between two feasible sets) Let P be invertible, and let $K = LP^{-1}$ define a bijection*[5] *$K \leftrightarrow (L, P)$. For any such bijection $K \leftrightarrow (L, P)$, there exists an auxiliary variable Z such that $(L, P, Z) \in \mathcal{S}$.*
2. *(Equivalence of functions) Choose a controller $K \in \mathcal{S}_K$ with corresponding $(L, P) \in \mathcal{S}$. Then $\mathcal{L}(K) = \min_Z f(L, P, Z)$ subject to $(L, P, Z) \in \mathcal{S}$.*

Thm. 1 suggests that when the original nonconvex optimization problem can be mapped to a convex optimization problem that satisfies Assumptions 1,2 or 1,3, all stationary points of the nonconvex objective are global minima. So if we can evaluate the gradient of nonconvex objective and run gradient descent algorithm, the iterates converge to the optimal controller.

4 Control problems covered by main theorem

Theorem 1 requires an optimal control problem (9), its convex form (10) that satisfies a few assumptions. This is an abstract and general description that does not need the exact continuous time LQR formulation in Sec. 2. Sec. 2 implies that the continuous time LQR satisfies Assumptions 1,3, thus we can directly apply Theorem 1 to argue that the continuous time LQR cost $\mathcal{L}(K)$ satisfies (12).

Below, we will propose more examples, showing that Theorem 1 covers a wide range of optimal control problems. This shows the **generality** of Theorem 1. If one encounters optimal control problems from the following set, and moreover, if one encounters a new problem, we hope that one can check if the problem satisfies the assumptions for Theorem 1. If so, one can directly claim that the stationary points of the original cost function all global minima and can be optimized by policy gradient method.

[4] Although the set can be unbounded, when we run gradient descent with respect to $\mathcal{L}(K)$, the cost is typically bounded by the initial value $\mathcal{L}(K_0)$ so the iterates are in a sublevel set, therefore boundedness of this sublevel set suffices for our purpose.

[5] Note that generally $K = LP^{-1}$ cannot guarantee a bijection. However bijection is possible with the extra constraint $(L, P) \in \mathcal{S}$.

4.1 Discrete time LQR

We will show that minimizing the LQ cost as a function of the state feedback controller K, and the convex form, satisfy the assumptions for Thm. 1. So that all stationary points of the LQ cost as a function of K are global minima, same as the result in [2].

We consider a discrete time linear system

$$x(t+1) = Ax(t) + Bu(t),\ x(0) = x_0,$$

The goal is to find a state feedback controller K such that the cost function

$$\mathcal{L}(K) = \mathbf{E}_{x_0} \sum_{i=0}^{\infty} x(t)^\top Q x(t) + u(t)^\top R u(t),\ u = Kx$$

is minimized. In other words, we will solve

$$\min_K \ \mathcal{L}(K),\ \text{s.t. } K \text{ stabilizes.} \tag{14}$$

Here we assume that $\mathbf{E}x_0 x_0^\top = \Sigma$. Similar to the continuous time system, one can choose the same parameterization P, L, Z and another PSD matrix $G \in \mathbb{R}^{n\times n} \succeq 0$ and solve the following problem

$$\min_{L,P,Z,G} \ f(L,P,Z,G) := \mathbf{Tr}(QP) + \mathbf{Tr}(ZR), \tag{15a}$$

$$\text{s.t. } P \succ 0,\ G - P + \Sigma = 0, \tag{15b}$$

$$\begin{bmatrix} Z & L \\ L^\top & P \end{bmatrix} \succeq 0,\ \begin{bmatrix} G & AP+BL \\ (AP+BL)^\top & P \end{bmatrix} \succeq 0. \tag{15c}$$

The goal is to argue that $\mathcal{L}(K)$ and (15) has the connection such that Theorem 1 applies, so that the stationary point of $\mathcal{L}(K)$ has to be the global optimum.

Lemma 1. *The LQR problems* (14) *and* (15) *satisfy Assumptions 1, 2.*

Proof. (15) is a convex optimization problem. Now we prove Assumption 2, i.e., we prove that $L(K)$ equals the minimum of the problem (15) with an extra constraint $K = LP^{-1}$.

- We first minimize over Z, the minimizer is $Z = LP^{-1}L^\top$. Now we plug $Z = LP^{-1}L^\top$ into cost, replace L by KP and the cost becomes $\mathbf{Tr}((Q + K^\top RK)P)$.
- Eliminate G by

  $$G - P + \Sigma = 0,\ \begin{bmatrix} G & AP+BL \\ (AP+BL)^\top & P \end{bmatrix} \succeq 0.$$

  Using Schur complement, it is equivalent to

  $$(AP+BL)P^{-1}(AP+BL)^\top - P + \Sigma \preceq 0.$$

  Plug in $L = KP$, we have

  $$(A+BK)P(A+BK)^\top - P + \Sigma \preceq 0.$$

  The cost does not involve G so it does not change.
- Now, we need to prove that $\mathcal{L}(K)$ is equal to

  $$\begin{aligned} &\min_P \ \mathbf{Tr}((Q + K^\top RK)P), \\ &\text{s.t. } (A+BK)P(A+BK)^\top - P + \Sigma \preceq 0. \end{aligned} \tag{16}$$

  The last constraint can be written as

  $$(A+BK)P(A+BK)^\top - P + \Theta = 0,\ \Theta \succeq \Sigma.$$

- Denote the solution to $(A+BK)P(A+BK)^\top - P + \Theta = 0$ as $P(\Theta)$. $P(\Theta)$ for all $\Theta \succeq \Sigma$ covers the feasible points of (16). $P(\Theta)$ is expressed as:

$$P(\Theta) = \sum_{t=0}^{\infty}(A+BK)^t\Theta((A+BK)^\top)^t.$$

So $P(\Theta) \succeq P(\Sigma)$, for all $\Theta \succeq \Sigma$. Since Q and $K^\top RK$ are positive semidefinite, the cost $\mathbf{Tr}((Q+K^\top RK)P)$ achieves the minimum at $P = P(\Sigma)$.

- At the end, $P(\Sigma)$ is the Grammian $\mathbf{E}\sum_{t=0}^{\infty} x(t)x(t)^\top$ when $\mathbf{E}x(0)x(0)^\top = \Sigma$. We studied the connection between continous time Grammian (5) and the cost (6), and a similar result holds for discrete time LQR:

$$\mathbf{Tr}((Q+K^\top RK)P(\Sigma)) = \mathcal{L}(K).$$

□

We build the connection between minimizing $\mathcal{L}(K)$, and the convex optimization (15). We argued this pair of problems satisfies the assumptions of Theorem 1. Theorem 1 suggests that $\mathcal{L}(K)$ is gradient dominant, so we can approach K^* by gradient descent on K. This is essentially the conclusion of [2,22]. Note that the proof of discrete time LQR [2,22] and continuous time LQR [21,23] cannot trivially extend to each other, but our result can cover both continuous and discrete time cases.

4.2 LQR with Markov jump linear system

We generalize the discrete time linear system to multiple linear systems with transitions, called Markov jump linear system in this part. We show that, the LQR with Markov jump linear system can be covered by the conclusion of Thm. 1. It means all stationary points of the linear quadratic cost as a function of policy/controllers are global minima.

Markov jump linear system. Suppose there are N linear systems, the i-th one being

$$x(t+1) = A_i x(t) + B_i u(t).$$

Now we study the Markov jump linear system [31]. At each time t, the dynamics linking $x(t+1)$ and the past state and input $x(t), u(t)$ is given by

$$x(t+1) = A_{w(t)}x(t) + B_{w(t)}u(t),\ w(t) \in [N] := \{1, ..., N\}.$$

At time t, a system $w(t)$ from number 1 to N is randomly chosen by some probabilistic model. The transition of the linear systems, or the transition of $w(t)$, follows the following probabilistic model

$$\mathbf{Pr}(w(t+1) = j | w(t) = i) = \rho_{ij} \in [0,1],\ \forall t \geq 0.$$

Suppose $\mathbf{Pr}(w(0) = i) = p_i$. For the i-th system, we will use a state feedback controller K_i. Let $K = [K_1, ..., K_N]$. Define the cost as

$$\mathcal{L}(K) = \mathbf{E}_{w,x_0}\sum_{t=0}^{\infty} x(t)^\top Qx(t) + u(t)^\top Ru(t),\ \text{s.t. } u(t) = K_{w(t)}x(t),\ \mathbf{Pr}(w(0) = i) = p_i.$$

The nonconvex problem we target to solve is

$$\min_K\ \mathcal{L}(K), \quad \text{s.t. } \mathcal{L}(K) \text{ is finite.} \tag{17}$$

Convex formulation. We propose the following convex formulation. Denote $X_0, X_1, ..., X_N \in \mathbb{R}^{n\times n} \succ 0$, $L_1, ..., L_N \in \mathbb{R}^{p\times n}$, $Z_0, Z_1, ..., Z_N \in \mathbb{R}^{p\times p} \succeq 0$, $U_{ji} \in \mathbb{R}^{n\times n} \succ 0$ for $i, j \in [N]$. The following problem is convex:

$$
\begin{aligned}
&\min \ \mathbf{Tr}(QX_0) + \mathbf{Tr}(Z_0 R),\\
&\text{s.t. } X_0 = \sum_{i=1}^N X_i, \ Z_0 = \sum_{i=1}^N Z_i, \ \begin{bmatrix} Z_i & L_i \\ L_i^\top & X_i \end{bmatrix} \succeq 0,\\
&X_i - p_i \Sigma = \sum_{j=1}^N U_{ji}, \ \begin{bmatrix} \rho_{ji}^{-1} U_{ji} & A_j X_j + B_j L_j \\ (A_j X_j + B_j L_j)^\top & X_j \end{bmatrix} \succeq 0, \ \forall i, j \in [N].
\end{aligned}
$$

The mapping between the controller K_i and the new variables are $K_i = L_i (X_i)^{-1}$.

We prove that (17) and the convex formulation satisfy Assumptions 1, 2 in Appendix C.1, so that we apply Theorem 1 to claim that all stationary points of $\mathcal{L}(K)$ are global minima.

4.3 Minimizing $\mathcal{L}_2$ gain

We quote from [29] the problem of minimizing the $\mathcal{L}_2$ gain with static state feedback controller K and the convex formulation, and show we can apply Thm. 1 to argue that all stationary points of $\mathcal{L}_2$ gain as a function of K are global minima. The $\mathcal{L}_2$ gain is also the $\mathcal{H}_\infty$ norm of transfer function [29, §6.3.2]. This problem has an associated convex optimization problem and we can show that they satisfy Assumptions 1,2.

We consider minimizing the $\mathcal{L}_2$ gain of a closed loop system. The continuous time linear dynamical system is

$$
\dot{x} = Ax + Bu + B_w w, \ y = Cx + Du.
$$

For any signal z, denote

$$
\|z\|_2 := \left(\int_0^\infty \|z(t)\|_2^2 \mathrm{d}t \right)^{1/2}
$$

Suppose we use a state feedback controller $u = Kx$, and aim to find the optimal controller K^* that minimizes the $\mathcal{L}_2$ gain. We minimize the squared $\mathcal{L}_2$ gain as

$$
\min_K \ \mathcal{L}(K) := (\sup_{\|w\|_2 = 1} \|y\|_2)^2.
$$

This problem can be further reformulated as the formulation in [29, Sec 7.5.1]

$$
\begin{aligned}
&\min_{L,P,\gamma} \ f(L,P,\gamma) := \gamma, \ \text{s.t.}\\
&\begin{bmatrix} AP + PA^\top + BL + L^\top B^\top + B_w B_w^\top & (CP + DL)^\top \\ CP + DL & -\gamma I \end{bmatrix} \preceq 0.
\end{aligned} \tag{19}
$$

The minimum $\mathcal{L}_2$ gain is $\sqrt{\gamma^*}$ and $K^* = L^* {P^*}^{-1}$. We will show in the Appendix C.2 that the above nonconvex and convex problems satisfy Assumptions 1,2. Thus we can claim that all stationary points of $\mathcal{L}(K)$ are global minimum.

4.4 Dissipativity

We quote from [29] the problem of maximizing the dissipativity with static state feedback controller K and the convex formulation, and apply Thm. 1 to show that all stationary points of the dissipativity as a function of K are global minima.

We study the dynamical system

$$
\dot{x} = Ax + Bu + B_w w, \ y = Cx + Du + D_w w \tag{20}
$$

The notion of dissipativity can be found in [29, §6.3.3, §7.5.2]. Our goal is to maximize the dissipativity, which is defined and formulated as with a convex parameterization [29, Sec. 6.3.3, 7.5.2].

The dissipativity is defined as all $\eta > 0$ (we usually take the maximum one) that satisfy

$$\int_0^T w^\top y - \eta w^\top w \mathrm{d}t \geq 0, \ \forall T > 0.$$

Same as the last example, we use a state feedback controller K, and the goal is to find K^* that maximizes the dissipativity η. Same as before, let K be factorized as LP^{-1}. We can maximize the dissipativity η as a function of K. From the formulation in [29, §7.5.2], we maximize η subject to the dissipativity constraint (21),

$$\begin{aligned} &\max_{\eta, L, P} \ \eta, \\ &\text{s.t.} \begin{bmatrix} AP + PA^\top + BL + L^\top B^\top & B_w - PC^\top - (DL)^\top \\ B_w^\top - CP - DL & 2\eta I - (D + D^\top) \end{bmatrix} \preceq 0. \end{aligned} \tag{21}$$

With similar reasoning as in Sec. 4.4, the assumptions for Theorem 1 holds. Thus we can claim that all stationary points of $\mathcal{L}(K)$ are global minima.

4.5 System level synthesis (SLS) for finite horizon time varying discrete time LQR

In this part, we switch to the discrete time system in finite horizon. We study the finite horizon time varying LQR problem, and its solution using SLS, and show that it satisfies Assumptions 1,3. Hence we can apply Thm. 1 to conclude that all stationary points of the nonconvex objective functions are global minima.

This problem and its convex form are introduced in [32]. We consider the following linear dynamical system

$$x(t+1) = A(t)x(t) + B(t)u(t) + w(t) \tag{22}$$

over a finite horizon $0, \ldots T$. Let the state be x and the input be u. Define

$$\begin{aligned} X &= \begin{bmatrix} x(0) \\ \ldots \\ x(T) \end{bmatrix}, \ U = \begin{bmatrix} u(0) \\ \ldots \\ u(T) \end{bmatrix}, \\ W &= \begin{bmatrix} x(0) \\ w(0) \\ \ldots \\ w(T-1) \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \ldots \\ 0 & 0 & \ldots & I & 0 \end{bmatrix}, \\ \mathcal{A} &= \mathrm{diag}(A(0), \ldots, A(T-1), 0), \\ \mathcal{B} &= \mathrm{diag}(B(0), \ldots, B(T-1), 0). \end{aligned}$$

Now we consider the time varying controller K that links state and input as

$$u(t) = \sum_{i=0}^{t} K(t, t-i)x(i), \tag{23}$$

and let

$$\mathcal{K} = \begin{bmatrix} K(0,0) & 0 & \ldots & 0 \\ K(1,1) & K(1,0) & \ldots & 0 \\ \ldots \\ K(T,T) & K(T, T-1) & \ldots & K(T,0) \end{bmatrix}.$$

We will minimize some cost function with the constraint. For example, in the discrete time LQR regime (more examples of nonquadratic cost in [32, Sec 2.2]), let the input be (23) and define

$$\mathcal{L}(\mathcal{K}) = \sum_{t=0}^{T} x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t), \tag{24}$$

here $Q(t), R(t) \succeq 0$. We will minimize $\mathcal{L}(\mathcal{K})$ where $\mathcal{K}$ is the variable.

Parameterization: The dynamics (22) can be written as

$$X = Z\mathcal{A}X + Z\mathcal{B}U + W = Z(\mathcal{A} + \mathcal{B}\mathcal{K})X + W$$

We define the mapping from W to X, U by

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} W.$$

where Φ_X, Φ_U are block lower triangular. There is a constraint on Φ_X, Φ_U:

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I. \tag{25}$$

It is proven in [32, Thm 2.1] that $\mathcal{K} = \Phi_U \Phi_X^{-1}$. $\mathcal{K}$ and Φ_X, Φ_U is a bijection given Φ_X, Φ_U satisfying (25).

Let $\mathcal{Q} = \text{diag}(Q(0), ..., Q(T))$, $\mathcal{R} = \text{diag}(R(0), ..., R(T))$, the LQR cost with $x(0) \sim \mathcal{N}(0, \Sigma)$ and no noise is

$$f(\Phi_X, \Phi_U) = \left\| \text{diag}(\mathcal{Q}^{1/2}, \mathcal{R}^{1/2}) \begin{bmatrix} \Phi_X(:,0) \\ \Phi_U(:,0) \end{bmatrix} \Sigma^{1/2} \right\|_F^2;$$

the LQR cost with $x(0), w(t)$ being i.i.d from $\mathcal{N}(0, \Sigma)$ is

$$f(\Phi_X, \Phi_U) = \left\| \text{diag}(\mathcal{Q}^{1/2}, \mathcal{R}^{1/2}) \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} \Sigma^{1/2} \right\|_F^2.$$

If we solve

$$\min_{\mathcal{K}} \ \mathcal{L}(\mathcal{K}), \ \mathcal{K} \text{ is lower left triangular}$$

with the above two costs of $w(t)$, both can be minimized with constraint (25):

$$\begin{aligned} \min_{\Phi_X, \Phi_U} \ f(\Phi_X, \Phi_U), \ \text{s.t.} \ &\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I, \\ &\Phi_X, \Phi_U \text{ are lower left triangular.} \end{aligned}$$

This problem is convex and satisfy Assumption 1. The theorem [32, Thm 2.1] suggests the relation between $\mathcal{L}$ and f satisfying Assumption 3 for Theorem 1. With Theorem 1, we can argue that all stationary points of $\mathcal{L}(\mathcal{K})$ are global minimum.

5 A more general description of Assumption 2

In this section, we will give a more general theorem, based on replacing the map $K = LP^{-1}$ by arbitrary function Φ defined below. This allows the theorem to cover more examples in Sec. 6.

We chose $K = LP^{-1}$ because this is frequently used for the convex parameterization of the optimal control problem. For example, with the continuous time LQR problem motivated in Sec. 2, the mapping between K and L, P is (almost) the only widely used convex parameterization method. If we choose another change of variable, the resulting objective function is usually not convex in the new variables.

On the other hand, although the mapping $K = LP^{-1}$ is studied, we can generalize Thm. 1 with arbitrary mappings if the reformulated problem is convex – of course, the new mappings have to satisfy a few assumptions to preserve gradient dominance.

Here we will propose the following assumptions which replace the mapping $K = LP^{-1}$ by an abstract mapping Φ.

Suppose we consider the problems

$$\min_{K} \quad \mathcal{L}(K), \quad \text{s.t. } K \in \mathcal{S}_K, \tag{26}$$

and

$$\min_{P} \quad f(P), \quad \text{s.t.} \quad P \in \mathcal{S}. \tag{27}$$

The matrix P can be a concatenation of many variables, just as a short expression. For example, P represents (P, L, Z) of continuous LQR. We will study the original optimization problem (26), and map it to a convex optimization problem (27) where the mapping between K and the variables of the other problem P is abstractly denoted by $K = \Phi(P)$ in (28).

Assumption 4. *The feasible set $\mathcal{S}$ is convex in P. The cost function $f(P)$ is convex, finite and differentiable in $P \in \mathcal{S}$. $\mathcal{L}(K)$ is Lipschitz in K.*

Assumption 5. *Assume we can express $\mathcal{L}(K)$ as:*

$$\mathcal{L}(K) = \min_{P} \; f(P), \; \textit{s.t.} \; P \in \mathcal{S}, \; K = \Phi(P). \tag{28}$$

And we assume the first order Taylor expansion of the mapping Φ can be written as

$$\Phi(P + \mathrm{d}P) = \Phi(P) + \Psi(P)[\mathrm{d}P] + o(\mathrm{d}P).$$

for any $P \in \mathcal{S}$ and any perturbation $\mathrm{d}P$ such that $\mathrm{d}P$ is in the descent cone of $\mathcal{S}$ at P.

Thm. 1 holds with the above Assumptions 4, 5, which means the cost $\mathcal{L}(K)$ is gradient dominant in K given the above assumptions. It generalizes beyond the specific mapping $\Phi(P, L) = LP^{-1}$ to a more general definition, and we propose some instances of convex formulations with different Φ in the subsections. We propose the following theorem and the proof is in Sec. A.

Theorem 2. *Denote $\Delta K = \Psi(P)[P^* - P]$. Let $\nabla\mathcal{L}(K)[\Delta K]$ be the directional derivative of $\mathcal{L}(K)$ in direction ΔK. Then with Assumptions 4, 5 we have*

$$\nabla\mathcal{L}(K)[\Delta K] \leq \mathcal{L}(K^*) - \mathcal{L}(K).$$

If K is not optimal, then the right hand side is strictly less than 0, which means the directional derivative of $\mathcal{L}$ is not 0. Therefore $\nabla\mathcal{L}(K) = 0$ only at the global minima.

6 Control problems with generalized map

This section will cover examples where the parameterization is based on the general map Φ.

6.1 Distributed finite horizon LQR

The paper [19, Ch. 3] is an *empirical* study (i.e., proposing an algorithm without a proof of convergence) of the gradient descent method for distributed control synthesis.

For such a problem, the controller is distributed with a graph structure. If controller i has no access to state j, then $K_{ij} = 0$. Thus there is an extra subspace constraint regarding the graph structure of K, and [19, Ch. 3] applies projected gradient descent on (2) with respect to K. It allows a fixed or random of initial state as in (2). Generally it is NP-hard to find a global optimum with the subspace constraint, so the paper only proposes an algorithm without a proof.

With an extra condition called quadratic invariance, the problem is not NP-hard. We review the solutions in [27] with the connection to our framework.

We consider the time varying linear system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + w(t), \\ y(t) &= C(t)x(t) + v(t). \end{aligned}$$

This is in finite time horizon $t = 0, ..., T$. The state evolution is same as the setup in our SLS example (Sec. 4.5), and we can use the same notation $X, U, W, Z, \mathcal{A}, \mathcal{B}$. We further define

$$Y = \begin{bmatrix} y(0) \\ ... \\ y(T) \end{bmatrix}, \ V = \begin{bmatrix} v(0) \\ ... \\ v(T) \end{bmatrix}, \ \mathcal{C} = \operatorname{diag}(C(0), ..., C(T)).$$

Now we will consider the control policy

$$u(t) = \sum_{i=0}^{t} K(t, t-i) y(i).$$

The search space of policy is same as SLS, and we define $\mathcal{K}$ matrix in the same way. The paper [27] studies the problem under the context of distributed control. One searches for the controller $K \in \mathcal{S}_K$ where $\mathcal{S}_K$ a subset of controllers. In distributed control, there is a graph model for controllers such that the i-th controller might not be able to access the state j for (i, j) in a set of indices $\mathcal{S}_{\mathrm{idx}}$. In this case, $K_{i,j} = 0$ is an extra constraint for the control problem. Therefore, if one searches for the optimal controller in $\mathcal{S}_K$, we can define the subspace

$$\mathcal{S}_K := \{K \mid K_{i,j} = 0, \ \forall (i, j) \in \mathcal{S}_{\mathrm{idx}}\}.$$

The extra constraint is not always easily handled, but [27, Sec. 3] proposes an extra assumption, called quadratic invariance (QI), and introduces the equivalent convex optimization.

Remember we defined

$$\mathcal{K} = \begin{bmatrix} K(0,0) & 0 & ... & 0 \\ K(1,1) & K(1,0) & ... & 0 \\ ... \\ K(T,T) & K(T,T-1) & ... & K(T,0) \end{bmatrix}, \ \mathcal{C} = \operatorname{diag}(C(0), ..., C(T)).$$

And define

$$P_{11} = (I - Z\mathcal{A})^{-1}, \ P_{12} = (I - Z\mathcal{A})^{-1} Z\mathcal{B}.$$

QI means that, for all $\mathcal{K} \in \mathcal{S}_K$, $\mathcal{K}\mathcal{C}P_{12}\mathcal{K} \in \mathcal{S}_K$.

The cost function is:

$$\mathcal{L}(\mathcal{K}) = \sum_{t=0}^{T} y(t)^\top Q(t) y(t) + u(t)^\top R(t) u(t).$$

Define

$$\Phi(\mathcal{G}) = (I + \mathcal{G}\mathcal{C}P_{12})^{-1}\mathcal{G}.$$

Then we can get a new variable $\mathcal{G}$ and a function Φ. With $\mathcal{K} = \Phi(\mathcal{G})$, the cost can be proven to be convex in $\mathcal{G}$. The variable $\mathcal{G}$ is in the same subspace as $\mathcal{K}$ determined by $\mathcal{S}_K$. Indeed, the mapping satisfies Assumptions 4, 5, and the exact formulation of the two optimization problems are described in [27, Append. A, Lem. 5]. Define $\mathcal{Q} = \operatorname{diag}(Q(0), ..., Q(T))$, $\mathcal{R} = \operatorname{diag}(R(0), ..., R(T))$. Let $w(t)$ be Gaussian random vectors with stationary covariance, $w(t_1) \perp w(t_2)$, $\forall t_1 \neq t_2$. $\Sigma_w = I_T \otimes \operatorname{Cov}(w)$, $\Sigma_x = \operatorname{diag}(\mathbf{E}(x_0 x_0^\top), 0, ..., 0)$. The convex cost function takes the form

$$f(\mathcal{G}) = \left\| \mathcal{Q}^{1/2}\mathcal{C}(I + P_{12}\mathcal{G}\mathcal{C})P_{11} \begin{bmatrix} \Sigma_w^{1/2} & \Sigma_x^{1/2} \end{bmatrix} \right\|_F^2 + \left\| \mathcal{R}^{1/2}\mathcal{G}\mathcal{C}P_{11} \begin{bmatrix} \Sigma_w^{1/2} & \Sigma_x^{1/2} \end{bmatrix} \right\|_F^2.$$

In summary, we have a pair of problems: 1) minimize $\mathcal{L}(\mathcal{K})$ over $\mathcal{K}$ and 2) minimize $f(\mathcal{G})$ over $\mathcal{G}$. They are related under the Assumptions 1, 3 of Thm. 1. Thus we can claim via Thm. 1 that, such distributed LQ regulator problem with $\mathcal{K}$ as variable has no spurious local minimum.

6.2 Multi-objective and mixed controller design

In this part, we study a few synthesis problems with dynamical controllers, where the objectives are about (e.g., norms of) transfer functions of the closed form system. We study the dynamical system with state, disturbance, input, output, and controller's input x, w, u, z, y with the following dynamics

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B \\ C_z & D & E \\ C & F & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}.$$

The dynamical controller follows

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}.$$

We will denote the transfer function of the closed loop system as $\mathscr{T}$, and the control problems below are typically related to $\mathscr{T}$.

In the next few subsections, we will present a few control problems:

1. The **variables** are the controller parameters $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$.
2. The **objective functions** are $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm of $\mathscr{T}$ and the weighted sum of norms.
3. The book [33, eq(4.2.15)] defines the parameterization of the problem, by introducing the **variables that typically make the objective functions convex**:
$$v = [X, Y, K, L, M, N].$$
4. **Mapping of the variables.** Define invertible matrices U, V such that $UV^\top = I - XY$. The matrices A_c, B_c, C_c, D_c are the unique solution of
$$\begin{bmatrix} K & L \\ M & N \end{bmatrix} = \begin{bmatrix} U & XB \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ CY & I \end{bmatrix} + \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

The change of variable enables us to make some control problems as convex optimization, listed below. For simplicity of notation, let

$$\mathscr{X} = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}, \ \mathscr{A} = \begin{bmatrix} AY + BM & A + BNC \\ K & AX + LC \end{bmatrix}, \tag{30}$$

$$\mathscr{B} = \begin{bmatrix} B_w + BNF \\ XB_w + LF \end{bmatrix}, \ \mathscr{C} = \begin{bmatrix} C_zY + EM & C_z + ENC \end{bmatrix}, \ \mathscr{D} = D + ENF. \tag{31}$$

Remark 2. *The mapping in* (29) *can be written as*

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \Phi(v)$$

where Φ plays the role in (28)*. We propose a few control problems with convex forms in the next few subsections. The variables of nonconvex objective functions are A_c, B_c, C_c, D_c, the new objective functions with respect to $v = [X, Y, K, L, M, N]$ are convex, and the two forms satisfy Assumptions 4, 5. Thus the cost functions with respect to matrix $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$ are gradient dominant.*

In the following subsections, we refer to the result of [33] that, the optimal $\mathcal{H}_\infty$ design, $\mathcal{H}_2$ design and the multi-objective and robust designs, can be made convex optimization problems with the proposed way. Thus, the objectives with respect to controller K are all gradient dominant.

6.2.1 $\mathcal{H}_\infty$ design

([33, Sec. 4.2.3]) The goal in this part is to minimize the $\mathcal{H}_\infty$ norm of the closed loop system's transfer function by designing the optimal controller. Let the transfer function of the closed form system be $\mathscr{T}$. The problem with its raw form is to minimize the $\|\mathscr{T}\|_{\mathcal{H}_\infty}$ over A_c, B_c, C_c, D_c, and we will propose the convex formulation – the change of variable trick such that the argument we minimize over becomes v. The problem takes the form:

$$
\begin{aligned}
&\min \ \gamma, \\
&\text{s.t.} \quad \mathscr{X} \succeq 0, \ \begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B} & \mathscr{C}^\top \\ \mathscr{B}^\top & -\gamma I & \mathscr{D}^\top \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{bmatrix} \preceq 0.
\end{aligned}
$$

If we fix all other parameters and optimize over γ, then γ^* (that depends on v) is the $\mathcal{H}_\infty$ value of the closed loop system with the mapping from v to controller by (29). If we minimize over γ and v, then we can get optimal $\mathcal{H}_\infty$ design.

6.2.2 $\mathcal{H}_2$ design

([33, Sec. 4.2.5]) This part is similar to $\mathcal{H}_\infty$ design. Suppose the goal is to minimize $\|\mathscr{T}\|_{\mathcal{H}_2}$, one can alternatively solve

$$
\begin{aligned}
&\min \ \gamma, \\
&\text{s.t.} \quad \begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B} \\ \mathscr{B}^\top & -\gamma I \end{bmatrix} \preceq 0, \ \mathscr{D} = 0, \ \begin{bmatrix} \mathscr{X} & \mathscr{C}^\top \\ \mathscr{C} & Z \end{bmatrix} \succeq 0, \ \mathbf{Tr}(Z) \leq \gamma.
\end{aligned}
$$

If we fix all other parameters and optimize over γ, Z, then γ^* (that depends on v) is the $\mathcal{H}_2$ value of the closed loop system with the mapping from v to controller by (29). If we minimize over γ, Z and v, then we can get optimal $\mathcal{H}_2$ design.

6.2.3 Multi-objective synthesis

([33, Sec. 4.3]) Let the system be

$$
\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & B \\ C_1 & D_1 & D_{12} & E_1 \\ C_2 & D_{21} & D_2 & E_2 \\ C & F_1 & F_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ u \end{bmatrix} \tag{32}
$$

Now we study the mixed design for $\mathcal{H}_\infty$ design from z_1 to w_1 and $\mathcal{H}_2$ design from z_2 to w_2. We keep the mapping (29) and the change of parameter (30), but replace (31) by

$$
\mathscr{B}_i = \begin{bmatrix} B_i + BNF_i \\ XB_i + LF_i \end{bmatrix}, \ \mathscr{C}_i = \begin{bmatrix} C_iY + E_iM & C_i + E_iNC \end{bmatrix}, \ \mathscr{D}_i = D_i + E_iNF_i.
$$

for $i = 1, 2$. Suppose we are given a positive number λ and hope to study $\|\mathscr{T}_1\|_{\mathcal{H}_\infty} + \lambda\|\mathscr{T}_2\|_{\mathcal{H}_2}$ where $\mathscr{T}_i$ is the transfer function of the i-th system (z_1 to w_1, z_2 to w_2), then we can write

$$
\min \ \gamma_1 + \lambda\gamma_2, \tag{33}
$$

$$
\text{s.t.} \quad \begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B}_1 & \mathscr{C}_1^\top \\ \mathscr{B}_1^\top & -\gamma_1 I & \mathscr{D}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 & -\gamma_1 I \end{bmatrix} \preceq 0, \tag{34}
$$

$$
\begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B}_2 \\ \mathscr{B}_2^\top & -\gamma_2 I \end{bmatrix} \preceq 0, \ \mathscr{D}_2 = 0, \ \begin{bmatrix} \mathscr{X} & \mathscr{C}_2^\top \\ \mathscr{C}_2 & Z \end{bmatrix} \succeq 0, \ \mathbf{Tr}(Z) \leq \gamma_2. \tag{35}
$$

If we fix all other parameters and optimize over γ_1, γ_2, Z, then the function value is the mixed $\mathcal{H}_\infty/\mathcal{H}_2$ value of the closed loop system with the mapping from v to controller by (29). If we minimize over γ_1, γ_2, Z and v, then we can get the optimal mixed $\mathcal{H}_\infty/\mathcal{H}_2$ design.

6.2.4 Robust state feedback control

([33, Sec. 8.1.2]) We study the robust state feedback control problem, where the robustness corresponds to a system with uncertain parameters, denoted by Δ below. We apply the system model (32). "State feedback" means that $C = I$ and $F_1, F_2 = 0$. Let N_Δ be a positive integer. The connection between w_1 and z_1 is an uncertain channel

$$w_1(t) = \Delta(t) z_1(t)$$

for any

$$\Delta(t) \in \Delta_c := \mathrm{conv}\{0, \Delta_1, ..., \Delta_{N_\Delta}\}.$$

The goal is to minimize a certain norm of the transfer function from z_2 to w_2, which can be $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norms in the previous part. We consider minimizing the norm under an extra constraint when the closed loop system achieves robust stability with z_1 to w_1 (z_1 with finite norm) and robust quadratic performance with z_2 to w_2 via a matrix P_p. The robust quadratic performance is defined as: there exists a matrix P_p such that

$$P_p = \begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^\top & \tilde{R}_p \end{bmatrix}, \; P_p^{-1} = \begin{bmatrix} Q_p & S_p \\ S_p^\top & R_p \end{bmatrix}$$

such that $\tilde{R}_p \succ 0, Q_p \prec 0$, and

$$\int_0^\infty \begin{bmatrix} w_2(t) \\ z_2(t) \end{bmatrix}^\top P_p \begin{bmatrix} w_2(t) \\ z_2(t) \end{bmatrix} \mathrm{d}t \le \epsilon \|w_2\|_{\mathcal{H}_2}^2$$

for some $\epsilon > 0$.

Define new variables Q, S, R in addition to $v = [X, Y, K, L, M, N]$, and let $\mathscr{M}$ replace

$$\mathscr{M} \leftarrow \begin{bmatrix} -(AY + BM)^\top & -(C_1 Y + E_1 M)^\top & -(C_2 Y + E_2 M)^\top \\ I & 0 & 0 \\ -B_1^\top & -D_1^\top & -D_{21}^\top \\ 0 & I & 0 \\ -B_2^\top & -D_{12}^\top & -D_2^\top \\ 0 & 0 & I \end{bmatrix}.$$

The constraints, which is proven to be convex [33, Sec. 8.1.2] can be written as

$$R \succ 0, \; Q \prec 0, \; \begin{bmatrix} I \\ -\Delta_j \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} I \\ -\Delta_j \end{bmatrix} \prec 0, \; \forall j \in [N_\Delta]$$

$$Y \succ 0, \; \mathscr{M}^\top \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^\top & R & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^\top & R_p \end{bmatrix} \mathscr{M} \succ 0.$$

For example, if we aim to minimize the $\mathcal{H}_2$ norm of the transfer function from z_2 to w_2, then we can minimize γ_2 subject to (35) and the constraints above. The main theorem of this paper suggests that, with the convex formulation, if we apply policy gradient descent with respect to $\mathcal{H}_2$ norm of the transfer function from z_2 to w_2 with robust stability (z_1 with finite norm) and robust quadratic performance constraints, then policy gradient descent converges to globally optimal controller.

6.2.5 Discrete time system

([33, Sec. 4.6]) Suppose we study the discrete time system, and we define the system in a similar way of defining the continuous time system:

$$\begin{bmatrix} x(t+1) \\ z_1(t) \\ z_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & B \\ C_1 & D_1 & D_{12} & E_1 \\ C_2 & D_{21} & D_2 & E_2 \\ C & F_1 & F_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_1(t) \\ w_2(t) \\ u(t) \end{bmatrix}, \quad \begin{bmatrix} x_c(t+1) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}.$$

Now we study the mixed design for $\mathcal{H}_\infty$ design from z_1 to w_1 and $\mathcal{H}_2$ design from z_2 to w_2. Suppose we are given a positive number λ and hope to study $\|\mathscr{T}_1\|_{\mathcal{H}_\infty} + \lambda\|\mathscr{T}_2\|_{\mathcal{H}_2}$ where $\mathscr{T}_i$ is the transfer function of the i-th system (z_1 to w_1, z_2 to w_2), then we can write

$$\begin{aligned} &\min \;\; \gamma_1 + \lambda\gamma_2, \\ &\text{s.t.} \;\; \begin{bmatrix} \mathscr{X} & 0 & \mathscr{A}^\top & \mathscr{C}_1^\top \\ 0 & \gamma_1 I & \mathscr{B}_1^\top & \mathscr{D}_1 \\ \mathscr{A} & \mathscr{B}_1 & \mathscr{X} & 0 \\ \mathscr{C}_1 & \mathscr{D}_1 & 0 & \gamma_1 I \end{bmatrix} \succ 0, \;\; \mathbf{Tr}(Z) \le \gamma_2, \\ &\begin{bmatrix} \mathscr{X} & \mathscr{A} & \mathscr{B}_2 \\ \mathscr{A}^\top & \mathscr{X} & 0 \\ \mathscr{B}_2^\top & 0 & \gamma_2 I \end{bmatrix} \succ 0, \;\; \begin{bmatrix} \mathscr{X} & 0 & \mathscr{C}_2 \\ 0 & \mathscr{X} & \mathscr{D}_2 \\ \mathscr{C}_2 & \mathscr{D}_2 & Z \end{bmatrix} \succ 0. \end{aligned}$$

If we fix all other parameters and optimize over γ_1, γ_2, Z, then the function value is the mixed $\mathcal{H}_\infty/\mathcal{H}_2$ value of the closed loop system with the mapping from v. If we minimize over γ_1, γ_2, Z and v, then we can get the optimal mixed $\mathcal{H}_\infty/\mathcal{H}_2$ design.

7 Proof Sketch

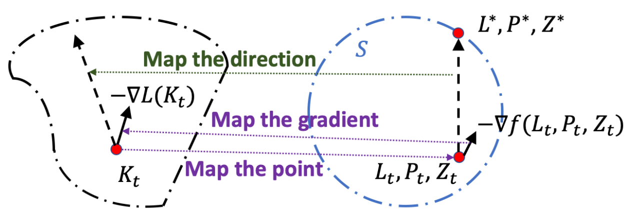

Figure 1: Mapping between nonconvex and convex landscapes. Suppose we run gradient descent at iteration t, for any controller K, we can map it to L, P, Z in the other parameterized space. and then we map the direction $(L^*, P^*, Z^*) - (L, P, Z)$ and the gradient $\nabla f(L, P, Z)$ back to the original K space. Since in (L, P, Z) space the objective function is convex, then $\langle \nabla f(L, P, Z), (L^*, P^*, Z^*) - (L, P, Z)\rangle < 0$. We prove that similar correlation holds for the nonconvex objective.

We put the full proof of Theorem 1 in Appendix A, and give a sketch of the proof in this section. We illustrate the idea in Figure 1, which, on the high level, maps the original space of controller K where the cost is nonconvex, and the parameterized space with L, P, Z where the cost is convex.

For simplicity, we sketch the proof using Assumptions 1,3. For any point K, we can find a point (L, P, Z) in the parameterized space. If it is not the optimizer, we can find the line segment linking (L, P, Z) and the optimizer (L^*, P^*, Z^*). Note that the optimization problem is convex in this space so that $\langle \nabla f(L, P, Z), (L^*, P^*, Z^*) - (L, P, Z)\rangle$ is upper bounded by $f(L^*, P^*, Z^*) - f(L, P, Z)$. Then with the help of our assumptions, we can map the directional derivative back to the original K space, and show that the directional derivative in $\mathcal{L}(K)$ is not 0.

Before concluding, we remark that the assumptions in Theorem 1 come from an optimization theory perspective, and we do not dive into the control theoretic interpretations of the constants and assumptions.

Our approach has the benefit that it unifies the analysis of many control problems in a single abstract result, showing that all stationary points of the objective functions are global minima, so that one can apply policy gradient method to find the globally optimal policy. The future work is to refine the analysis to obtain the best case-specific convergence rates, and to provide an interpretation of the associated constants in terms of control theoretic notions.

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A Proof of the main theorems

Theorem 1. *Suppose assumptions 1,2 hold, and consider the two problems* (9) *and* (10)*. Let* K^* *denote the global minimizer of* $\mathcal{L}(K)$ *in* S_K*. Then there exist constants* $C_1, C_2 > 0$ *independent of the suboptimality* $\mathcal{L}(K) - \mathcal{L}(K^*)$*, and a direction* V*, with* $\|V\|_F = 1$*, in the descent cone of* $\mathcal{S}_K$ *at* K *such that,*

1. *if* f *is convex, the gradient of* $\mathcal{L}$ *satisfies*[6]

$$\nabla\mathcal{L}(K)[V] \le -C_1(\mathcal{L}(K) - \mathcal{L}(K^*)). \tag{36}$$

2. *if* f *is* μ*-strongly convex, the gradient satisfies*

$$\nabla\mathcal{L}(K)[V] \le -C_2(\mu(\mathcal{L}(K) - \mathcal{L}(K^*)))^{1/2}. \tag{37}$$

where (the constants can be bounded with simple constraints bounding norms of L, P *or* K*)*

$$\begin{aligned}
C_1 &= (\max\{\max_L \|L - L^*\|_F \sigma_{\min}^{-1}(P), \max_P \|P - P^*\|_F \sigma_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1}, \\
C_2 &= (\max\{\sigma_{\min}^{-1}(P), \sigma_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1}.
\end{aligned}$$

Proof. Denote $\mathcal{P}_{\mathcal{S}}(\nabla f(x))$ as the projection of $\nabla f(x)$ onto the descent cone of $\mathcal{S}$ at x. First, for any convex function $f(x)$, let the minimum be x^*, and $x - x^* = \Delta$. Let $\nabla f(x) = g$. For any non-stationary point, $f(x) \le f(x^*) + g^\top \Delta$. Since $\mathcal{S}$ is a convex set, $-\Delta$ belongs to the descent cone of $\mathcal{S}$ at x, so the direction $\frac{\Delta}{\|\Delta\|}$ is feasible, $f(x) - f(x - t\frac{\Delta}{\|\Delta\|}) > tg^\top \frac{\Delta}{\|\Delta\|}$, $t \to 0$, so the norm of projected gradient $\|\mathcal{P}_{\mathcal{S}}(\nabla f(x))\| \ge g^\top \frac{\Delta}{\|\Delta\|} = \frac{f(x) - f(x^*)}{\|x - x^*\|}$.

Let K^* be the optimal K and (Z^*, L^*, P^*) be the optimal point in the parameterized space. We have $\mathcal{L}(K^*) = f(Z^*, L^*, P^*)$. Denote V as any matrix in K space, $\mathcal{P}_V$ is projection of a vector onto direction V. In other words,

$$\mathcal{P}_V(M) = \frac{\mathbf{Tr}(V^\top M)}{\|V\|_F^2} V,$$

then we denote $\nabla\mathcal{L}(K)[\cdot]$ as a linear operator such that $\nabla\mathcal{L}(K)[V] = \mathbf{Tr}(V^\top \nabla\mathcal{L}(K))$ (a more general discussion about Taylor expansion is in Sec. 5),

$$\nabla\mathcal{L}(K)^\top \nabla\mathcal{L}(K) \ge (\mathcal{P}_V \nabla\mathcal{L}(K))^\top \mathcal{P}_V \nabla\mathcal{L}(K) = \left(\frac{\nabla\mathcal{L}(K)[V]}{\|V\|_F}\right)^2. \tag{38}$$

We denote $\mathscr{Z}(L, P) \in \operatorname{argmin}_Z f(L, P, Z)$ subject to $(L, P, Z) \in \mathcal{S}$ (if there are multiple minimizers we pick any one). With either Assumption 3 or 2, we can define the mapping from K to (L, P, Z) respectively in one of the following ways:

1. (Assumption 3) let K map to (L, P) and $Z = \mathscr{Z}(L, P)$.
2. (Assumption 2) let

$$\begin{aligned}
&(L, P, Z) = \operatorname{argmin}_{L', P', Z'} \; f(L', P', Z') \\
&\text{s.t. } (L', P', Z') \in \mathcal{S}, \; P' \succ 0, \; L'P'^{-1} = K.
\end{aligned}$$

Note f is convex, so

$$\begin{aligned}
&\nabla f(L, P, Z)[(L, P, Z) - (L^*, P^*, Z^*)] \\
&\ge f(L, P, Z) - f(L^*, P^*, Z^*) \\
&= f(\mathscr{Z}(L, P), L, P) - f(\mathscr{Z}(L^*, P^*), L^*, P^*) \\
&= \mathcal{L}(K) - \mathcal{L}(K^*).
\end{aligned} \tag{39}$$

[6] We always consider the directional derivative of a feasible direction within descent cone.

Now we consider the directional derivative in K space. By definition,

$$\nabla \mathcal{L}(K)[V] = \lim_{t\to 0^+} (\mathcal{L}(K+tV) - \mathcal{L}(K))/t.$$

Let $\Delta L = L^* - L$, $\Delta P = P^* - P$, and $V = \Delta L P^{-1} - LP^{-1}\Delta P P^{-1}$. Then

$$\begin{aligned}\nabla \mathcal{L}(K)[V] &= \lim_{t\to 0^+} (\mathcal{L}(K+tV) - \mathcal{L}(K))/t \\ &= \lim_{t\to 0^+} (\mathcal{L}(LP^{-1} + t(\Delta L P^{-1} - LP^{-1}\Delta P P^{-1})) - \mathcal{L}(LP^{-1}))/t \\ &= \lim_{t\to 0^+} (\mathcal{L}((L+t\Delta L)(P+t\Delta P)^{-1}) - \mathcal{L}(LP^{-1}))/t.\end{aligned}$$

The last line uses $(P+t\Delta P)^{-1} = P^{-1} - tP^{-1}\Delta P P^{-1} + o(t)$. Denote $\Delta(L,P,Z) = (L^*,P^*,Z^*) - (L,P,Z)$, $\Delta(L,P,Z)$ is in the descent cone of $\mathcal{S}$ at (L,P,Z) due to the convexity of $\mathcal{S}$. With Assumption 3, we continue with

$$\begin{aligned}\nabla \mathcal{L}(K)[V] &= \lim_{t\to 0^+} (f(L+t\Delta L, P+t\Delta P, \mathscr{Z}(L+t\Delta L, P+t\Delta P)) - f(L,P,\mathscr{Z}(L,P)))/t \\ &\le \lim_{t\to 0^+} (f(L+t\Delta L, P+t\Delta P, Z+t\Delta Z) - f(L,P,\mathscr{Z}(L,P)))/t \\ &= \nabla f(L,P,Z)[\Delta(L,P,Z)].\end{aligned}$$

With Assumption 2, we continue with

$$\begin{aligned}\nabla \mathcal{L}(K)[V] = \lim_{t\to 0^+} \min_{L',P',Z'} \ & f(L',P',Z') - f(L,P,Z) \\ \text{s.t. } & (L',P',Z') \in \mathcal{S}, \ P' \succ 0, \\ & L'P'^{-1} = (L+t\Delta L)(P+t\Delta P)^{-1}.\end{aligned}$$

$(L+t\Delta L, P+t\Delta P, Z+t\Delta Z)$ is a feasible point of the optimization problem, thus is less than or equal to the minimum, and then

$$\begin{aligned}\nabla \mathcal{L}(K)[V] &\le \lim_{t\to 0^+} (f(L+t\Delta L, P+t\Delta P, Z+t\Delta Z) - f(L,P,\mathscr{Z}(L,P)))/t \\ &= \nabla f(L,P,Z)[\Delta(L,P,Z)].\end{aligned}$$

So the final inequality holds either by Assumption 3 or 2. So

$$\nabla \mathcal{L}(K)[V] \le \nabla f(L,P,Z)[\Delta(L,P,Z)] < 0.$$

Using (38) and (39), we have

$$\|\nabla \mathcal{L}(K)\|_F^2 \ge \frac{1}{\|V\|_F^2} (\mathcal{L}(K) - \mathcal{L}(K^*))^2. \tag{40}$$

If $f(L,P,Z)$ is μ strongly convex, then we can restrict f in the line segment $(L,P,Z) - (L^*,P^*,Z^*)$ and get and then

$$\begin{aligned}(\frac{\nabla \mathcal{L}(K)[V]}{\|V\|_F})^2 &\ge \frac{1}{\|V\|_F^2} (\nabla f(L,P,Z)[\Delta(L,P,Z)])^2 \\ &\ge \frac{\mu \|\Delta(L,P,Z)\|_F^2}{\|V\|_F^2} \cdot (f(L,P,Z) - f(L^*,P^*,Z^*)) \\ &= \frac{\mu(\|L^*-L\|^2 + \|P^*-P\|^2 + \|Z^*-Z\|^2)}{\|(L^*-L)P^{-1} - LP^{-1}(P^*-P)P^{-1}\|_F^2} \cdot (f(L,P,Z) - f(L^*,P^*,Z^*)) \\ &\ge \frac{\mu(\|L^*-L\|^2 + \|P^*-P\|^2)}{\|(L^*-L)P^{-1} - LP^{-1}(P^*-P)P^{-1}\|_F^2} \cdot (f(L,P,Z) - f(L^*,P^*,Z^*)) \\ &\ge \frac{\mu(f(L,P,Z) - f(L^*,P^*,Z^*))}{(\max\{\sigma_{\min}^{-1}(P), \sigma_{\min}^{-2}(P)\sigma_{\max}(L)\})^2}.\end{aligned}$$

□

Theorem 2. *Denote $\Delta K = \Psi(P)[P^* - P]$. Let $\nabla \mathcal{L}(K)[\Delta K]$ be the directional derivative of $\mathcal{L}(K)$ in direction ΔK. Then with Assumptions 4, 5 we have*

$$\nabla \mathcal{L}(K)[\Delta K] \leq \mathcal{L}(K^*) - \mathcal{L}(K).$$

Proof. Suppose $f(P)$ is convex in P, and the optimizer of (27) is P^*. Denote

$$P = \mathrm{argmin}_{P'}\ f(P'),\ \text{s.t. } P' \in \mathcal{S},\ K = \Phi(P'),$$

and

$$\Delta P = P^* - P,\ \Delta K = \Psi(P)[\Delta P].$$

We take the directional derivative and get (explanation of key steps below the last line)

$$\begin{aligned}
\nabla \mathcal{L}(K)[\Delta K] &= \lim_{t\to 0^+} \frac{\mathcal{L}(K + t\Delta K) - \mathcal{L}(K)}{t} \\
&= \lim_{t\to 0^+} \frac{\mathcal{L}(K + t\Psi(P)[\Delta P]) - f(P)}{t} && (41)\\
&= \lim_{t\to 0^+} \frac{\mathcal{L}(\Phi(P) + t\Psi(P)[\Delta P]) - f(P)}{t} && (42)\\
&= \lim_{t\to 0^+} \frac{\mathcal{L}(\Phi(P + t\Delta P) - o(t)) - f(P)}{t} && (43)\\
&= \lim_{t\to 0^+} \frac{\mathcal{L}(\Phi(P + t\Delta P)) - f(P)}{t} \\
&= \lim_{t\to 0^+} \frac{\min_{P'\in\mathcal{S},\ \Phi(P+t\Delta P) = \Phi(P')}\ f(P') - f(P)}{t} && (44)\\
&= \lim_{t\to 0^+} \frac{\min_{P'\in\mathcal{S},\ \Phi(P+t\Delta P) = \Phi(P')}\ f(P') - f(P+t\Delta P) + f(P+t\Delta P) - f(P)}{t} \\
&= \lim_{t\to 0^+} \frac{\min_{P'\in\mathcal{S},\ \Phi(P+t\Delta P) = \Phi(P')}\ f(P') - f(P+t\Delta P)}{t} + \nabla f(P)[\Delta P]. && (45)
\end{aligned}$$

(41) and (42) replace ΔK and K with expressions in P and ΔP. (43) applies the Taylor expansion of Φ:

$$\Phi(P + t\Delta P) - (\Phi(P) + t\Psi(P)[\Delta P]) = o(t).$$

(44) applies Assumption 5, and we plug in $K = \Phi(P + t\Delta P)$. (45) applied the definition of directional derivative

$$\nabla f(P)[\Delta P] = \lim_{t\to 0^+} \frac{f(P + t\Delta P) - f(P)}{t}.$$

Now we bound the first term of (45). Note that, since $P + t\Delta P$ for $t > 0$ and $t \to 0^+$ belongs to the line segment from P to P^*. Since $\mathcal{S}$ is a convex set, we know that the line segment between to feasible points P^* and P is in $\mathcal{S}$. then

$$P + t\Delta P \in \{P' \mid P' \in \mathcal{S},\ \Phi(P + t\Delta P) = \Phi(P')\},$$

so that $f(P + t\Delta P)$ is no less than the minimum of the optimization problem (28),

$$\lim_{t\to 0^+} \frac{\min_{P'\in\mathcal{S},\ \Phi(P+t\Delta P) = \Phi(P')}\ f(P') - f(P+t\Delta P)}{t} \leq 0.$$

$\nabla f(P)[\Delta P]$ is the directional derivative of $f(P)$ in the direction of $P^* - P$, for a convex function f, if P is not an optimizer, $\nabla f(P)[\Delta P]$ is upper bounded by $f(P^*) - f(P) = \mathcal{L}(K^*) - \mathcal{L}(K) < 0$. $\square$

B Constants for continuous time LQR

Thm. 1 asks for two constants C_1, C_2. They are bounded differently for different examples. As an instance, we will calculate the constants for continuous time LQR, quoted from [21] Appendix B. First $P \succ 0$, so we replace singular value by eigenvalue with P

$$\begin{aligned}
C_1 &= (\max\{\max_L \|L - L^*\|_F \lambda_{\min}^{-1}(P), \max_P \|P - P^*\|_F \lambda_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1}, \\
C_2 &= (\max\{\lambda_{\min}^{-1}(P), \lambda_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1}.
\end{aligned}$$

We need upper bounds for P, L and a lower bound for $\lambda_{\min}(P)$ to guarantee C_1, C_2 finite. We will show the bounds within the sublevel set that $\{K : \mathcal{L}(K) \leq a\}$. Since we can randomly initialize a feasible K_0 and run (projected) gradient descent method with respect to K, if $\mathcal{L}(K)$ is gradient dominant, it is reasonable to assume that during all iterations of the optimization algorithm, the function value is always upper bounded by $\mathcal{L}(K_0)$, or some values related to $\mathcal{L}(K_0)$. So our derivation with a finite sublevel set is reasonable.

Suppose the matrices $Q, R \succ 0$, and we consider the sublevel set when $\mathcal{L}(K) \leq a$. The sublevel set gives $\mathbf{Tr}(QP) + \mathbf{Tr}(LP^{-1}L^\top R) \leq a$, so

$$\begin{aligned}
\lambda_{\min}(R)\lambda_{\max}^{-1}(P)\|L\|_F^2 &\leq \lambda_{\min}(R)\|LP^{-1/2}\|_F^2 \\
&\leq \mathbf{Tr}(LP^{-1}L^\top R) \\
&\leq \mathbf{Tr}(QP) + \mathbf{Tr}(LP^{-1}L^\top R) \leq a.
\end{aligned}$$

So $\|L\|_F \leq a(\lambda_{\max}(P)\lambda_{\min}^{-1}(R))^{1/2}$, and we know from [21, eq(34)] $\mathbf{Tr}(P) \leq a\lambda_{\min}^{-1}(Q)$. So we can bound P, L

$$\begin{aligned}
\mathbf{Tr}(P) &\leq a\lambda_{\min}^{-1}(Q), \\
\|L\|_F &\leq a(\lambda_{\min}(Q)\lambda_{\min}(R))^{-1/2}.
\end{aligned}$$

Define

$$\nu = 4a\left(\sigma_{\max}(A)\lambda_{\min}^{-1/2}(Q) + \sigma_{\max}(B)\lambda_{\min}^{-1/2}(R)\right)^2.$$

[34, eq(38,40)] suggests $\lambda_{\min}(P) \geq \lambda_{\min}(\Sigma)/\nu$. In summary, we upper bounded L, and upper and lower bounded P in the sublevel set $\mathcal{L}(K) \leq a$, and those bounds are also true for L^*, P^*. We can complete the calculation by inserting the bounds into C_1.

$$C_1 = (\max\{2a\nu\lambda_{\min}^{-1}(\Sigma)\lambda_{\min}^{-1/2}(Q)\lambda_{\min}^{-1/2}(R), 2a^2\nu^2\lambda_{\min}^{-3/2}(Q)\lambda_{\min}^{-1/2}(R)\lambda_{\min}^{-2}(\Sigma)\})^{-1}.$$

C_2 is calculated similarly with upper bound on P, L, P^{-1}.

$$C_2 = \min\{\nu^{-1}\lambda_{\min}(\Sigma),\ a^{-1}\nu^{-2}\lambda_{\min}^2(\Sigma)\lambda_{\min}^{1/2}(Q)\lambda_{\min}^{1/2}(R)\}.$$

B.1 Strongly convex parameter of continuous time LQR

In our previous convex formulation of continuous time LQR (8), we translate the objective function as a linear function in the new variables L, P, Z. The problem (8) can be slightly reformulated as

$$\min_{L,P}\ f(L,P) := \mathbf{Tr}(QP) + \mathbf{Tr}(LP^{-1}L^\top R), \tag{46a}$$

$$\text{s.t. } \mathcal{A}(P) + \mathcal{B}(L) + \Sigma = 0,\ P \succ 0. \tag{46b}$$

Compared with (8), (46) does not contain the variable Z. Below, we will prove that the new objective function $f(L,P)$, restricted within the feasible set, is a strongly convex function, which is not the case for the linear objective (8). In Thm. 1, there is another result with strongly convex f and the gradient domminance parameter depends on the strongly convex parameter μ. We also calculate μ of $f(L,P)$ below.

Lemma 2. *Define a sublevel set of of f at level a, consisting of all L, P such that $f(L,P) \leq a$. Define*

$$\nu = \frac{\lambda_{\min}(\Sigma)}{4}\left(\|A\|\lambda_{\min}^{-1/2}(Q) + \|B\|\lambda_{\min}^{-1/2}(R)\right), \ \eta = \left(\nu^{1/2}\|\mathcal{B}\|\lambda_{\min}(\Sigma)\lambda_{\min}(Q)\lambda_{\min}^{1/2}(R)\right)^{-1},$$
$$\mu_0 = \frac{2\lambda_{\min}(Q)\lambda_{\min}(R)}{a(1+a^2\eta)^2}.$$

The strongly convex parameter μ of the function $f(L,P)$ restricted within the feasible set in (46) *satisfies*

$$\mu \geq (\|\mathcal{A}^{-1} \circ \mathcal{B}\| + 1)^{-1}\mu_0.$$

Proof. Denote $\mathcal{A}^{-1}$ as the inverse of $\mathcal{A}$, a linear operator such that $\mathcal{A}^{-1}(\mathcal{A}(P)) = P$. [21, Proposition 1] concludes that the following function $h(\cdot)$ is μ_0 strongly convex.

$$h(L) = f(L, -\mathcal{A}^{-1}(\mathcal{B}(L) + \Sigma)).$$

Define a perturbation direction $(\tilde{L}, \tilde{P})$. Any feasible perturbation at the point L, P will satisfy $\mathcal{A}(\tilde{P}) + \mathcal{B}(\tilde{L}) = 0$, so $\tilde{P} = -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))$.

Let the strongly convex parameter of f in the feasible directions be μ, we will show its connection with μ_0.

Let L be perturbed by $\tilde{L}$.

$$\nabla^2 h(L)[\tilde{L}, \tilde{L}] = \nabla^2 f(L,P)[(\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))), (\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L})))], \tag{47}$$

Due to the strong convexity of h,

$$\nabla^2 h(L)[\tilde{L}, \tilde{L}] \geq \frac{\mu_0\|\tilde{L}\|_F^2}{2}.$$

We perturb f at (L,P) in direction $(\tilde{L}, \tilde{P}) = (\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L})))$. The strongly convex parameter of f in feasible directions is defined as the positive number mu such that

$$\nabla^2 f(L,P)[(\tilde{L}, \tilde{P}), (\tilde{L}, \tilde{P})] \geq \frac{\mu(\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2)}{2}$$

for all $(\tilde{L}, \tilde{P})$ such that $\tilde{P} = -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))$. The directional Hessian is

$$\nabla^2 f(L,P)[(\tilde{L}, \tilde{P}), (\tilde{L}, \tilde{P})] = \nabla^2 f(L,P)[(\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))), (\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L})))]. \tag{48}$$

(48) equals (47). So that

$$\begin{aligned}
&\nabla^2 f(L,P)[(\tilde{L}, \tilde{P}), (\tilde{L}, \tilde{P})] \geq \frac{\mu_0\|\tilde{L}\|_F^2}{2} \\
&= \frac{\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2}{2} \cdot \frac{\mu_0\|\tilde{L}\|_F^2}{\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2} \\
&= \frac{\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2}{2} \cdot \frac{\mu_0\|\tilde{L}\|_F^2}{\|\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))\|_F^2 + \|\tilde{L}\|_F^2}.
\end{aligned}$$

So

$$\mu \geq (\|\mathcal{A}^{-1} \circ \mathcal{B}\| + 1)^{-1}\mu_0.$$

∎

C Checking the assumptions for examples

C.1 Markov jump linear system

Example 1. *(Assumptions 1,2) We study the system*

$$x(t+1) = A_{w(t)}x(t) + B_{w(t)}u(t),\ w(t) \in [N].$$

The transition model is

$$\mathbf{Pr}(w(t+1) = j|w(t) = i) = \rho_{ij} \in [0,1],\ \forall t \geq 0.$$

Let $K = [K_1, ..., K_N]$. Define the cost as

$$\mathcal{L}(K) = \mathbf{E}_{w,x_0} \sum_{t=0}^{\infty} x(t)^\top Qx(t) + u(t)^\top Ru(t),\ u(t) = K_{w(t)}x(t),\ \mathbf{Pr}(w(0) = i) = p_i.$$

Let the convex formulation be

$$\min\ \mathbf{Tr}(QX_0) + \mathbf{Tr}(Z_0 R), \tag{49a}$$

$$s.t.\ X_0 = \sum_{i=1}^{N} X_i,\ Z_0 = \sum_{i=1}^{N} Z_i,\ \begin{bmatrix} Z_i & L_i \\ L_i^\top & X_i \end{bmatrix} \succeq 0, \tag{49b}$$

$$X_i - p_i\Sigma = \sum_{j=1}^{N} U_{ji},\ \begin{bmatrix} \rho_{ji}^{-1}U_{ji} & A_jX_j + B_jL_j \\ (A_jX_j + B_jL_j)^\top & X_j \end{bmatrix} \succeq 0,\ \forall i,j \in [N]. \tag{49c}$$

Then the pair of problems satisfy Assumptions 1,2.

Proof. We start from the Grammian matrices below. Let $Y_i(t) = \mathbf{E}(x(t)x(t)^\top \mathbf{1}_{w(t)=i})$, and $X_i = \sum_{t=0}^{\infty} Y_i(t)$. Then [31] suggests

$$Y_i(t+1) = \sum_{j=1}^{N} \rho_{ji}(A_j + B_jK_j)Y_j(t)(A_j + B_jK_j)^\top.$$

Then we can sum over the equation over time t,

$$\begin{aligned}
&\sum_{j=1}^{N} \rho_{ji}(A_j + B_jK_j)(\sum_{t=0}^{\infty} Y_j(t))(A_j + B_jK_j)^\top \\
&= \sum_{t=0}^{\infty}\sum_{j=1}^{N} \rho_{ji}(A_j + B_jK_j)Y_j(t)(A_j + B_jK_j)^\top \\
&= \sum_{t=0}^{\infty} Y_i(t+1) = \sum_{t=1}^{\infty} Y_i(t) \\
&= \sum_{t=0}^{\infty} Y_i(t) - Y_i(0)
\end{aligned}$$

So that

$$\sum_{j=1}^{N} \rho_{ji}(A_j + B_jK_j)X_j(A_j + B_jK_j)^\top = X_i - Y_i(0).$$

Let $L_i = K_iX_i$. We will relax the recursion as

$$\sum_{j=1}^{N} \rho_{ji}(A_jX_j + B_jL_j)X_j^{-1}(A_jX_j + B_jL_j)^\top \preceq X_i - Y_i(0). \tag{50}$$

In our setting $\mathbf{E}(x(0)x(0)^\top) = \Sigma$ so that $Y_i(0) = p_i\Sigma$.

Next, we will show that, if we solve the problem (49) with the extra constraints $K_i = L_iX_i^{-1}$, then the function value is equal to the LQ cost of the system with controllers K_i's.

First, if we minimize over Z_i's, then we have $Z_i = L_iX_iL_i^\top$. Moreover, the constraints (49c) are equivalent to the relaxation (50). Suppose the equal signs are achieved in (50), then X_i's will be the Grammian of the system $\sum_{t=0}^\infty \mathbf{E}(x(t)x(t)^\top \mathbf{1}_{w(t)=i})$ and hence the function value is equal to the LQ cost [35, §4.4.2, Prop. 4.8]. Now, it remains to show that, if not all (50) (with enumerating different j's) achieve equal signs, then the function value will only increase and not be optimal.

We define N matrices $W_1, ..., W_N$, such that $W_i \succeq Y_i(0) = p_i\Sigma$, and

$$\sum_{j=1}^N \rho_{ji}(A_j + B_jK_j)X_j(A_j + B_jK_j)^\top = X_i - W_i.$$

This corresponds to the Markov jump system

$$\tilde{x}(t+1) = A_{w(t)}\tilde{x}(t) + B_{w(t)}u(t),\ \ w(t) \in [N].$$

with the same parameters, transition probability matrix, controllers and a different initial condition

$$\mathbf{E}(\tilde{x}(t)\tilde{x}(t)^\top \mathbf{1}_{w(t)=i}) = W_i \succeq p_i\Sigma = \mathbf{E}(x(t)x(t)^\top \mathbf{1}_{w(t)=i}). \tag{51}$$

Let $\tilde{Y}_i(t) = \mathbf{E}(\tilde{x}(t)\tilde{x}(t)^\top \mathbf{1}_{w(t)=i})$ (so that $\tilde{Y}_i(0) = W_i$), and let $\tilde{X}_i = \sum_{t=0}^\infty \tilde{Y}_i(t)$. We will show that $\tilde{Y}_i(t) \succeq Y_i(t)$ for all $i = 1, ..., N$ and all $t \geq 0$.

We use induction over t. When $t = 0$, we assumed in (51) that $\tilde{Y}_i(0) \succeq Y_i(0)$ hold for all $i \in [N]$. And we have the recursions

$$\tilde{Y}_i(t+1) = \sum_{j=1}^N \rho_{ji}(A_j + B_jK_j)\tilde{Y}_j(t)(A_j + B_jK_j)^\top,$$
$$Y_i(t+1) = \sum_{j=1}^N \rho_{ji}(A_j + B_jK_j)Y_j(t)(A_j + B_jK_j)^\top.$$

If $\tilde{Y}_i(t) \succeq Y_i(t)$ for a certain $t \geq 0$ and for all $i \in [N]$, then the recursion implies that $\tilde{Y}_i(t+1) \succeq Y_i(t+1)$ for all $i \in [N]$. We sum over t and get $\tilde{X}_i \succeq X_i$, so that the objective function with $\tilde{X}_i$'s is larger than with X_i's unless $\tilde{X}_i = X_i$ for all $i \in [N]$.

As a result, the optimization problem (49) with the extra constraints $K_i = L_iX_i^{-1}$ achieves minimum when $Z_i = L_iX_iL_i^\top$ and (50) achieves equality for all $i \in [N]$. This means all X_i's are the Grammians $\sum_{t=0}^\infty \mathbf{E}(x(t)x(t)^\top \mathbf{1}_{w(t)=i})$ of the system, so that the objective function value is equal to LQ cost.

□

C.2 Minimizing $\mathcal{L}_2$ gain

In this section, we will show that the minimizing $\mathcal{L}_2$ gain example in Sec. 4.3 satisfy Assumptions 1,2. Thus Theorem 1 applies to this problem, so all stationary points of $\mathcal{L}_2$ gain as a cost function of the controller K are global minimum.

Example 2. *(Assumptions 1,2) We consider minimizing the $\mathcal{L}_2$ gain of a closed loop system. The input output system is*

$$\dot{x} = Ax + Bu + B_ww,\ \ y = Cx + Du \tag{52}$$

and we use the state feedback controller $u = Kx$, and let

$$\mathcal{L}(K) := (\sup_{\|w\|_2=1} \|y\|_2)^2.$$

If we minimize the function $\mathcal{L}(K)$, this problem can be reformulated as

$$\begin{aligned}
&\min_{L,P,\gamma} \ f(L,P,\gamma) := \gamma \\
&s.t. \begin{bmatrix} AP + PA^\top + BL + L^\top B^\top + B_w B_w^\top & (CP+DL)^\top \\ CP+DL & -\gamma I \end{bmatrix} \preceq 0.
\end{aligned}$$

And $K^* = L^* {P^*}^{-1}$. This pair of problems satisfy Assumptions 1,2.

Proof. We will check Assumption 2, which means checking

$$\mathcal{L}(K) = \min_{L,P,\gamma} \gamma \tag{53a}$$

$$\text{s.t.} \begin{bmatrix} AP + PA^\top + BL + L^\top B^\top + B_w B_w^\top & (CP+DL)^\top \\ CP+DL & -\gamma I \end{bmatrix} \preceq 0, \ LP^{-1} = K. \tag{53b}$$

Note that, the intermediate step [29, Sec 7.5.1] is

$$\mathcal{L}(K) = \min_{P,\gamma} \gamma, \quad \text{s.t.} \tag{54a}$$

$$\begin{bmatrix} (A+BK)P + P(A+BK)^\top + B_w B_w^\top & P^\top (C+DK)^\top \\ (C+DK)P & -\gamma I \end{bmatrix} \preceq 0. \tag{54b}$$

Denote the optimizer of (53) by $\hat{L}, \hat{P}, \hat{\gamma}$, and the optimizer of (54) by $\check{P}, \check{\gamma}$.

Note $\hat{\gamma} \le \check{\gamma}$. If (53) is not true, $\hat{\gamma} < \check{\gamma}$, we can replace $\check{P}, \check{\gamma}$ with $\hat{P}, \hat{\gamma}$ in (54) and it's still feasible. Thus the optimality condition of $\check{P}, \check{\gamma}$ in (54) is violated, which contradicts the assumption that (53) is not true. Then we claim that (53) is true. □

D System Level Synthesis with Infinite Horizon

In this work, we studied the landscape of the optimal control problem where the variables are matrices (which are finite dimensional), and SLS is an example. Generally, SLS also works with the infinite horizon problem. In this regime, the variables are *transfer functions* and they are infinite dimensional. In practice, when the problem is made convex, one can parameterize the transfer function (say as finite impulse response) and minimize the cost with respect to the finite dimensional parameters. However, Thm. 1 does not apply to the infinite dimensional optimization problems, and it is not obvious that the parameterization satisfies the assumptions for our main theorem. We review the infinite horizon SLS here. A future direction is to judge whether the gradient dominance holds in the space of transfer function or its parameterized form, and how to analyze it using SLS.

Example 3. *(System level synthesis with infinite horizon) [32] Suppose one has a discrete time dynamical system with*

$$x(t+1) = Ax(t) + Bu(t) + w(t).$$

One can apply a dynamic controller $K(z)$. The goal is to find the optimial controller which minimizes the LQR cost where $u(z) = K(z)x(z)$

$$\mathcal{L}(K) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} x(t)^\top Q x(t) + u(t)^\top R u(t).$$

Suppose x_0, w_t are i.i.d. from $\mathcal{N}(0,\Sigma)$. The SLS defines two transfer functions $\Phi_X(z), \Phi_U(z)$, and solve the following convex optimization problem

$$\begin{aligned}
&\min_{\Phi_X(z),\Phi_U(z)} \left\| \begin{bmatrix} Q^{1/2}\Phi_X(z) \\ R^{1/2}\Phi_U(z) \end{bmatrix} \Sigma^{1/2} \right\|_{\mathcal{H}_2}, \\
&s.t. \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \Phi_X(z) \\ \Phi_U(z) \end{bmatrix} = I, \\
&\Phi_X(z), \Phi_U(z) \in \frac{1}{z}\mathcal{RH}_\infty.
\end{aligned}$$

Let the optimizer be $\Phi_U^*(z), \Phi_X^*(z)$. The optimal controller is $K^*(z) = \Phi_U^*(z)(\Phi_X^*(z))^{-1}$.

E Conditions of convexifiable nonconvex cost

We consider the pair of problems in Theorem 1, and ask the question: what property of the nonconvex cost function $\mathcal{L}(K)$ allows us to reformulate the problem (9) as a *convex* optimization problem (10)? In this section we propose the following lemma.

Lemma 3. *Suppose Assumptions 1, 3 hold, and $\mathcal{L}(LP^{-1})$ as a function of L, P is differentiable. We define the notation $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2]$ as in (55). If $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2] > 0$ for all $(L, P) \in \mathcal{S}$ and all (Γ_1, Γ_2) such that $\mathcal{A}(\Gamma_2) + \mathcal{B}(\Gamma_1) = 0$, then we can define a convex function $f(L, P)$ so that Assumption 1 holds. We can apply Theorem 1 so that all stationary points of $\mathcal{L}(K)$ are global minimum.*

Proof. Suppose we observe the simple version (11). We know from Assumption 3 that, $f(L, P) = \mathcal{L}(K) = \mathcal{L}(LP^{-1})$ is convex in L, P. We take the Hessian and ask for

$$\nabla \begin{bmatrix} \nabla\mathcal{L}(LP^{-1})P^{-1} \\ -P^{-1}L^\top\nabla\mathcal{L}(LP^{-1})P^{-1} \end{bmatrix} \succ 0.$$

Note that this is a tensor and it is positive definite. For simplicity, we analyze the directional Hessian as the following. We expand the left hand side of the inequality above and define $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2]$ as

$$\begin{aligned} &\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2] \\ :=\ &\nabla^2\mathcal{L}(LP^{-1})[\Gamma_1 G^{-2}, \Gamma_1] - 2\nabla^2\mathcal{L}(LP^{-1})[\Gamma_1, LP^{-3}\Gamma_2] \\ &- 2\langle\Gamma_1, \nabla\mathcal{L}(LP^{-1})P^{-1}\Gamma_2P^{-1}\rangle + 2\langle\Gamma_2, LP^{-1}\Gamma_2P^{-1}\nabla\mathcal{L}(LP^{-1})P^{-1}\rangle \\ &+ \nabla^2\mathcal{L}(LP^{-1})[LP^{-2}\Gamma_2, LP^{-2}\Gamma_2]. \end{aligned} \tag{55}$$

This is the directional Hessian of $\mathcal{L}$ with respect to (L, P) in direction (Γ_1, Γ_2). Thus, if $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2] > 0$ for all $(L, P) \in \mathcal{S}$ and all (Γ_1, Γ_2) such that $\mathcal{A}(\Gamma_2) + \mathcal{B}(\Gamma_1) = 0$ (which is a condition on nonconvex cost $\mathcal{L}$), we know that $f(L, P)$ is convex in L, P and the convex formulation can be made.