Linear system identification with redundancy in output

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June 6, 2020

In this work we discuss the overfitting linear dynamics model in [1, Sec 3]. This paper has the background in reinforcement learning in video games, and the output can be seen as the frames shown on the screen. Since the frames are high dimension, this paper discuss the overfitting issue in this setting. In the latter LQR section, the regime is simplified to an LTI system whose state is lower dimensional than the output. Let $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^m$ be the state, input and output of the system, and $m > 2n$. The model in [1, Sec 3] is

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t + \xi_t. \end{aligned} \tag{1}$$

where ξ_t is the noise of the system, which we assume to be iid Gaussian with covariance $\sigma_\xi I$. Let $C = [W_c^T, W_\theta^T]^T$ where $W_c \in \mathbb{R}^{n \times n}$, and W_c, W_θ be semi-orthogonal, and we recognize W_c as the meaningful part for the system and W_θ is a redundant part. It's proved that the optimal controller for LQR is not unique due tothe $W_\theta x$ part, and it's named as overfitting.

However, this overfitting naturally happens since the second part contains as important information of the state as the first part, so the two parts are not truly different. Though we assume that the first part is more meaningful, actually it's not.

This drives us think that what "overfitting" really means. In the following section, we adopt a different model with two parts, however, the second part contains strictly less information that the first, but it's not random noise that's independent to, or has zero correlation with the first part which contains full information of the state. We consider the system identification regime, where we do not know the parameters of the dynamics, and we hope to identify it[1]. The following is the problem formulation.

Consider the linear dynamics

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t + \xi_t + Ne_t. \end{aligned} \tag{2}$$

Ne_t is an environment that may carry or not carry the information of x. However, we assume that $C^T N = 0$, $D^T N = 0$, which means the role of $(x, u)_t$ and e_t are isolated. In other words, we have that

$$C^\dagger y_t = x_t + C^\dagger D u_t + C^\dagger \xi_t$$

which recovers states x_t from output y_t which is only contaminated by ξ_t. Now we feed in input and observe output, and hope to identify the system parameters. This is cast as a denoising problem where the Ne_t term is seen as a noise term. This model is different from the usual identification works [2, 3, 4] that assume only Gaussian noise (ξ term).

Since not all the information of y is needed, we want to fit a mapping from input u to $\bar{C}y$ where $\bar{C}N$ is close to 0.

We recover the system with impulse length n. We consider the problem

$$\begin{aligned} &\min_{P \in \mathbb{S}^n, K \in \mathbb{R}^{m \times (np)}} \sum_{t=1}^{T} \|y_t - Py_t\|_F^2 \\ &\text{subject to} \quad \sum_{t=1}^{T} \|Py_t - \sum_{j=1}^{n} K_j u_{t-j}\|_F^2 \leq \delta^2. \end{aligned} \tag{3}$$

Here $K_j \in \mathbb{R}^{m \times p}$ is the jth block of K.

Denote $U_t = [u_t^T, ..., u_{t-n+1}^T]^T$, then we have $y_t \sim GU_t + \xi_t + Ne_t + CA^{n-1}u_{t-n}$ where G is an impulse response matrix

$$G = [D, CB, CAB, ..., CA^{n-2}B]$$

[1]With a good identification of the linear system, we can design the optimal controller easily.

and $G^TN = 0$. Denote

$$\begin{aligned} U_t &= [u_t^T, ..., u_{t-n+1}^T]^T, \\ U &= [U_n, ..., U_T], \\ Y &= [y_n, ..., y_T], \\ \Xi &= [\xi_n, ..., \xi_T] \\ E &= [e_n, ..., e_T], \\ X_- &= [x_0, ..., x_{T-n}]. \end{aligned} \tag{4}$$

then

$$Y = GU + \Xi + NE + CA^{n-1}X_-. \tag{5}$$

Let V be the matrix consisting of a basis of orthogonal complement of column space of N, so that VV^T is the projection matrix onto the subspace, then $VV^TC = C$, $VV^TD = D$, $VV^TN = 0$. We will show that P^* is close to VV^T.

Theorem 1 *Let the system (2) satisfy $C^TN = 0$, $D^TN = 0$. VV^T is the projection matrix onto the orthogonal complement of N, then $VV^TC = C$, $VV^TD = D$, $VV^TN = 0$. Denote the concatenation of time series as (4). We solve the problem (3) where δ^2 is the optimal value of regressing VV^TY on U proven in [3]. Assume the smallest singular value of $[U^T,\ E^T]^T$ is bigger than $\kappa\sqrt{T}$, $\|\Xi + CA^{n-1}X_-\|_2 \le \sigma_T$. Then we have*

$$\|P^* - VV^TP^*VV^T\|_F \le \frac{2\delta + \sigma_T\|P^* - VV^T\|_F}{\sigma_{\min}(N)\kappa\sqrt{T}}, \tag{6}$$

$$\|VV^TP^*VV^TY - VV^TY\|_F \le \frac{2(2\delta + 2\sigma_T\|P^* - VV^T\|_F)}{\sigma_{\min}(N)\kappa\sqrt{T}}\|Y\|_2. \tag{7}$$

Remark 1 *If, we further assume that $\sigma_1(Y)/\sigma_m(Y) = \kappa_Y < \infty$, then (7) becomes*

$$\|VV^TP^*VV^T - VV^T\|_F \le \frac{2\kappa_Y(\delta + 2\sigma_T\|P^* - VV^T\|_F)}{\sigma_{\min}(N)\kappa\sqrt{T}}. \tag{8}$$

Combining with (6), if $\frac{\sigma_{\min}(N)\kappa\sqrt{T}}{1+\kappa_Y} > \sigma_T$, we have

$$\|P^* - VV^T\|_F \le 2\delta(\frac{\sigma_{\min}(N)\kappa\sqrt{T}}{1 + \kappa_Y} - \sigma_T)^{-1}. \tag{9}$$

Let δ be big enough so that $P = VV^T, K = G$ is feasible.

We assume $\rho(A)^n$ is small and Ξ is bounded. VV^T is feasible means that

$$\delta \geq \|VV^TY - GU\|_F \tag{10}$$
$$= \|VV^T(GU + \Xi + NE + CA^{n-1}X_-) - GU\|_F \tag{11}$$
$$= \|VV^T(\Xi + CA^{n-1}X_-)\|_F. \tag{12}$$

Let $P^* - VV^T = M,\ K^* = G + L$, we have

$$\delta \geq \|VV^T(\Xi + CA^{n-1}X_-) + M(GU + \Xi + NE + CA^{n-1}X_-) - LU\|_F \tag{13}$$
$$\geq \|M(GU + \Xi + NE + CA^{n-1}X_-) - LU\|_F - \|VV^T(\Xi + CA^{n-1}X_-)\|_F. \tag{14}$$

So that

$$\|(MG - L)U + MNE\|_F \leq 2\delta + \|M(\Xi + CA^{n-1}X_-)\|_F.$$

If, we further assume that the smallest singular value of $[U^T,\ E^T]^T$ is bigger than $\kappa\sqrt{T}$, then

$$\kappa\sqrt{T}(\|MG - L\|_F^2 + \|MN\|_F^2)^{1/2} \leq 2\delta + \|\Xi + CA^{n-1}X_-\|_2\|M\|_F \tag{15}$$
$$:= 2\delta + \sigma_T\|M\|_F \tag{16}$$

So that

$$\|MG - L\|_F,\ \|MN\|_F \leq \frac{2\delta + \sigma_T\|M\|_F}{\kappa\sqrt{T}}. \tag{17}$$

We can also write

$$\|MVV^TG - L\|_F,\ \|M(I - VV^T)N\|_F \leq \frac{2\delta + \sigma_T\|M\|_F}{\kappa\sqrt{T}}. \tag{18}$$

The second inequality says

$$\|(I - VV^T)M\|_F = \|M(I - VV^T)\|_F \leq \frac{2\delta + \sigma_T\|M\|_F}{\sigma_{\min}(N)\kappa\sqrt{T}}.$$

Now, let $M = VV^TMVV^T + M_2$ where $M_2 = (I - VV^T)M + VV^TM(I - VV^T)$, $\|M_2\|_F \leq \frac{2\delta + 2\sigma_T\|M\|_F}{\sigma_{\min}(N)\kappa\sqrt{T}}$. The original objective is

$$\begin{aligned}
\|Y - PY\|_F^2 &= \|NE + (I - VV^T)(\Xi + CA^{n-1}X_-) - M(GU + \Xi + NE + CA^{n-1}X_-)\|_F^2 \\
&= \|NE + (I - VV^T)(\Xi + CA^{n-1}X_-) - (VV^T + I - VV^T)MY\|_F^2 \\
&= \|NE + (I - VV^T)(\Xi + CA^{n-1}X_-) - (I - VV^T)MY\|_F^2 + \|VV^TMY\|_F^2 \\
&= \|NE + (I - VV^T)(\Xi + CA^{n-1}X_-) - (I - VV^T)MY\|_F^2 \\
&\quad + \|VV^TMVV^TY + VV^TM(I - VV^T)Y\|_F^2.
\end{aligned}$$

The first term is independent of VV^TMVV^T. So we have

$$\begin{aligned}
\|VV^TMVV^TY\|_F &\leq 2\|VV^TM(I - VV^T)Y\|_F \\
&\leq 2\|M_2\|_F\|Y\|_2 \\
&\leq \frac{2(2\delta + 2\sigma_T\|M\|_F)}{\sigma_{\min}(N)\kappa\sqrt{T}}\|Y\|_2
\end{aligned}$$

References

[1] X. Song, Y. Jiang, Y. Du, and B. Neyshabur, "Observational overfitting in reinforcement learning," *arXiv preprint arXiv:1912.02975*, 2019.

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