# Analysis of Policy Gradient Descent for Control: Global Optimality via Convex Parameterization

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### Abstract

Policy gradient descent, while being a nonconvex optimization method, is recently a popular approach in reinforcement learning for its simplicity. We aim to investigate the policy gradient descent method applied in control problems, and justify its convergence guarantee to global optimums. To do so, we bridge the gap between policy gradient descent and the classical convex parameterization methods in control theory, to show the gradient dominance property of the nonconvex landscape. Such connection between nonconvex and convex landscapes holds for continuous/discrete time LQR, finding optimal distributed controller, minimizing the $\mathcal{L}_2$ gain, system-level parameterization, $\mathcal{H}_2/\mathcal{H}_\infty$ mixed/robust control, etc. To the best of our knowledge, this paper offers the first result unifying the landscape analysis of a broad class of control problems.

## 1  Introduction

Linear quadratic regulator (LQR) is one of the most well studied optimal control problems for decades [1]. Consider the continuous time linear time-invariant dynamical system,

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $A, B$ are constant matrices describing the dynamics. The goal of optimal control is to determine the input series $u(t)$ that minimizes some cost function (that typically depends on the state and input). In the infinite horizon LQR problem, we define constant matrices $Q \in \mathbf{S}_{++}^n, R \in \mathbf{S}_{++}^p$, and minimize the cost as a function of input

$$\text{cost}(u(t)) := \mathbf{E}_{x_0} \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t)) dt. \tag{2}$$

The optimal controller is linear in the state, called static state feedback controller, and can be described as $u(t) = Kx(t)$ for a constant $K \in \mathbb{R}^{p \times n}$ [1].

We can define this cost function with variable $K$,

$$\mathcal{L}(K) := \mathbf{E}_{x_0} \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t)) dt, \text{ s.t. } u(t) = Kx(t). \tag{3}$$

The policy gradient descent method is to minimize $\mathcal{L}(K)$ by running first order method with respect to $K$. In classical control theory literature, this is not commonly used, since the loss function is typically nonconvex in $K$ [2]. Instead, one can introduce another parameterization of the cost to make it convex, and apply convex optimization method with global convergence guarantee. This approach is in sharp contrast to how one would typically minimize a cost function through gradient descent on $K$. In the following discussion, we will first review the convex parameterization methods, and then review the recent progress on the policy gradient methods on LQR and more control problems.

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**Solving LQR – classical way.** LQR is usually solved by the algebraic Riccati equation (ARE) [3, 4]. A large number of works have studied the solution of ARE, including approaches based on iterative algorithms [5], algebraic solution methods [6], and semidefinite programming [7]. Besides LQR, convex parameterizations such as Youla parametrization, $Q$-parameterization, or the more recent System Level Synthesis (SLS) have allowed the reformulation of certain control design problems as semidefinite programs. They are convex optimization methods, which natually can be globally minimized by first order algorithms. On the other hand, one has to know the system parameters to run them, and the paprameterizations are case-by-case.

**LQR with unknown system parameters: model-based and model-free.** There are two major types of algorithms when system parameters are not known. The first type is model-based methods, when we first estimate the system parameters and then a controller is constructed based on the identified system. System identification has a long history, as reviewed in [8]. Recently the paper [9] gave sample complexity bounds for state-observed system. The papers [10–13] describe the joint system identification and optimal control approaches, where the algorithms estimate the system parameter from the intput and output, and apply controllers based on the estimated system. .

The second type of method is model free method, when the controller is directly trained by observing the cost function, without characterizing the dynamics. Here one does not necessarily estimate the system parameters $A, B$. The paper [14] is a review of reinforcement learning area and optimal control, which studies a few fixed point type dynamic programming methods. Q-learning is a typical model free method for reinforcement learning, and it is applied to LQR as in [10, 15, 16].

**Recent works on policy gradient descent.** Policy gradient descent calls for an estimate of the cost and its gradient with respect to controller $K$. The goal is to show that gradient descent with respect to $K$ converges to the optimal controller $K^*$. The policy gradient descent is recently reviewed by [17, 18]. The paper [2] provides a counterexample showing that minimizing the quadratic LQ cost as a function of $K$ is not convex, quasi-convex or star-convex.

There has been recent evidence of the *empirical* success of first order methods in solving nonconvex reinforcement learning problems. The paper [19, Ch. 3] proposes the gradient based method for optimal control and extends to decentralized control. The paper [20] studies feedback control with dynamical controllers, and observes that gradient descent with Youla parameterization is robust within the set of stabilizing controllers while other parameterizations are not. On the *theoretical* side, despite the nonconvexity of $\mathcal{L}(K)$, for certain types of control problems, there are works showing the *gradient dominance* property, which enables first order methods to converge to the global optimum. The paper [2] gives the first result by proving the coercivity and the gradient dominance properties of $\mathcal{L}(K)$ for the discrete time LQR. Based on this, paper [2] shows the linear convergence of gradient based method. Later paper [21] shows a similar result for the continuous time case, papers [22, 23] give a more detailed analysis for both discrete and continuous time LQR. The papers [24, 25] show the convergence for two  types of zero-sum LQ games. The paper [26] studies the convergence of gradient descent on $\mathcal{H}_2$ control with $\mathcal{H}_\infty$ constraint, and shows that gradient descent implicitly makes the controller robust. The paper [27] shows the convergence for finite-horizon distributed control under the quadratic invariance assumption.

Control for nonlinear systems is far more difficult, typically via dynamic programming, solving Bellman equations [28] , or recent deep RL that led to empirical success in control of complex systems . Yet it is still mysterious how deep learning models work in this context, and recent theoretical studies have focused on linear systems in the hope of providing insights into more complex cases.

**Motivation:** Although we reviewed many papers that show convergence of policy gradient descent, they investigate different control problems and the proofs are given case by case. However, we observe that all the results are proven by the gradient dominance property, and all of them are solved by convex parameterization methods in classical control literature. Thus, we ask whether there is a proof that unifies the proofs of the gradient dominance property for different control problems, and bridges nonconvex methods with convex methods in classical control literature.

**Contributions:** In this paper, we make a connection between nonconvex policy gradient descent and known convex parameterization methods with a map between the two parameters. This map maintains the gradient dominance when going from the convex landscape to the nonconvex landscape.

Our result is quite general—we show that continuous-time LQR is a special case that the main theorems apply to, and we generalize the guarantees provided by this method to a range of other control problems. The instances cover optimal control, robust control, mixed design and system level synthesis.   To judge

whether a nonconvex landscape has gradient dominance property, one can directly check if it is covered by the theorems, avoiding a case-by-case analysis. Also, as discussed in [2], theoretical guarantees for first-order methods naturally lead to guarantees for the more practical zeroth-order optimization or sampling-based methods, which do not need access to the gradient of the cost with respect to $K$.

**Roadmap:** The rest of this paper is structured as follows. Sec. 2 reviews the continuous-time LQR problem. Sec. 3 presents our main result on the gradient dominance property for the nonconvex cost. Sec. 4 lists more examples of control problems covered by the main theorem. Although Sec. 4 covered many problems, Sec. 5 further generalizes our main result and Sec. 6 covers examples under the more generic result. Sec. 7 gives a proof sketch with intuitive connections between the nonconvex and convex formulations.

## 2 Review of convex parameterization for continuous time LQR

Convex parameterization (e.g., solving optimal control by linear matrix inequalities (LMI) in [29]) is widely used in optimal control problems, and here we discuss its application for continuous time LQR [21]. We will introduce new variables, construct an equivalent convex optimization problem with new variables, and the pair of variables are proven to be linked by a bijection. In the next section we use the critical properties of the nonconvex and convex problems as an intuition to generalize to a more general form.

Define a continuous time linear time invariant system (1) where $x$ is state and $u$ is input signal, and $x_0$ is the initial state. We assume that $\mathbf{E}(x_0 x_0^\top) = \Sigma \succ 0$. This is a commonly used setup such as in [22, §3.3], [19, Ch. 3].

One can then consider minimizing the linear quadratic (LQ) cost (2) as a function of $u(t)$ where $Q, R$ are positive definite matrices. [1] proves that, the input signal that minimizes the cost function $\mathrm{cost}(u)$ is given by a static state feedback controller, denoted by $u(t) = K^* x(t)$. $K^*$ can be obtained by solving linear equations, called Riccati equations. Once we know that the optimal state feedback controller is static, we can write cost as $\mathcal{L}(K)$ as (3). It is a function of $K$, and we search only static state feedback controllers.

An alternative approach is reparameterizing to obtain a convex formulation, as used in [21]. We will review it here, starting from the Lyapunov equation. Suppose the initial state satisfies $\mathbf{E}(x_0 x_0^\top) = \Sigma \succ 0$, and $\dot{x} = Ax$. Then with a matrix $P \in \mathbf{S}_{++}^{n \times n}$ ($P$ is a positive definite matrix) as the variable, the Lyapunov equation is written as

$$AP + PA^\top + \Sigma = 0.$$

In our setup (1), we use a state feedback controller $u = Kx$, thus we have $\dot{x} = (A + BK)x$. We denote the set of stabilizing controllers as $\mathcal{S}_{K,\mathrm{sta}}$, which is defined as

$$\mathcal{S}_{K,\mathrm{sta}} = \{K : \ \mathrm{Re}(\lambda_i(A + BK)) < 0, \ i = 1, ..., n\}.$$

If a state feedback controller is applied, the cost is only bounded when $K \in \mathcal{S}_{K,\mathrm{sta}}$ and is coercive in $\mathcal{S}_{K,\mathrm{sta}}$ [23]. Replace $A$ by the closed loop system matrix $A + BK$ in the Lyapunov equation, and let $L = KP \in \mathbb{R}^{p \times n}$, we get

$$AP + PA^\top + BL + L^\top B^\top + \Sigma = 0.$$

Let $\mathcal{A}(P) = AP + PA^\top$, $\mathcal{B}(L) = BL + L^\top B^\top$, which are often referred to as Lyapunov maps. Assume $\mathcal{A}$ is invertible, then we have the relation

$$\mathcal{A}(P) + \mathcal{B}(L) + \Sigma = 0. \tag{4}$$

Indeed, once we fix the system and any stabilizing controller $A, B, K$, the matrices $P$ as well as $L = KP$ are uniquely determined. $P$ is the Grammian matrix

$$P = \int_0^\infty e^{t(A+BK)} \, \Sigma \, e^{t(A+BK)^\top} \, dt. \tag{5}$$

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The matrix $P$ is positive definite if $\Sigma \succ 0$. We are interested in the cost function $\mathcal{L}(K)$ when[1] $K \in \mathcal{S}_{K,\mathrm{sta}}$, which corresponds to (2) by inserting $u(t) = Kx(t)$,

$$\mathcal{L}(K) = \begin{cases} \mathbf{Tr}((Q + K^\top RK)P), & K \in \mathcal{S}_{K,\mathrm{sta}}; \\ +\infty, & K \notin \mathcal{S}_{K,\mathrm{sta}}. \end{cases} \tag{6}$$

One can construct a bijection from $P, L$ to $K$, and prove that, if we minimize $f(L, P)$ subject to (4), the optimizer $P^*, L^*$ will map to the optimal $K^*$, and this minimization problem is convex.

**Convex reparameterization for Continuous time LQR:** Suppose the dynamics and costs are (1) and (2), and let $\mathbf{E}(x_0 x_0^\top) = \Sigma \succ 0$. Denote the (static) state feedback controller by $K$, so that $u(t) = Kx(t)$. The optimal control problem then is

$$\min_K \ \mathcal{L}(K), \quad \text{s.t.} \quad K \in \mathcal{S}_{K,\mathrm{sta}} \tag{7}$$

where $\mathcal{L}(K)$ is the cost in (2) with $u = Kx$. This problem can be expressed as the following equivalent convex problem,

$$\min_{L,P,Z} \ f(L, P, Z) := \mathbf{Tr}(QP) + \mathbf{Tr}(ZR) \tag{8a}$$

$$\text{s.t.} \ \mathcal{A}(P) + \mathcal{B}(L) + \Sigma = 0, \ P \succ 0, \tag{8b}$$

$$\begin{bmatrix} Z & L \\ L^\top & P \end{bmatrix} \succeq 0. \tag{8c}$$

The connection between the two problems is distilled in Sec. 3. For all feasible $(L, P, Z)$ triplets in (8), we can take the first two elements $(L, P)$, and they form a bijection with all stabilizing controllers $K$ in (7). The cost function values are equal under the bijection. So we can solve for $L^*, P^*$, and $K^* = L^*(P^*)^{-1}$.

# 3 Main result

Motivated by methods that use gradient descent in the policy space, we ask whether running a gradient-based algorithm and getting $\nabla_K \mathcal{L}(K) = 0$ for some $K$ in fact gives the globally optimum $K^*$. The papers [2,21] show the coercivity and gradient dominance property of $\mathcal{L}(K)$ for discrete- and continuous-time LQR respectively. In this paper, we generalize these results from the special case of continuous-time LQR to a much broader set of control problems, showing the gradient dominance property of the nonconvex costs as functions of policy.

We present our main result in Theorem 1. We consider a pair of problems satisfying Assumptions 1, 3. In Sec. 4 we catalog a number of examples showing the generality of this result.

We begin by considering an abstract description of the pair of problems (7) and (8). These problem descriptions cover LQR as discussed in the last section, as well as more problems discussed in Sec. 4. Consider the problems

$$\min_K \ \mathcal{L}(K), \quad \text{s.t.} \ K \in \mathcal{S}_K, \tag{9}$$

and

$$\min_{L,P,Z} \ f(L, P, Z), \quad \text{s.t.} \quad (L, P, Z) \in \mathcal{S}, \tag{10}$$

where the sets $\mathcal{S}_K, \mathcal{S}$ capture the control constraints. They are defined differently for each specific example in Sec. 4. For example, for continuous time LQR, $\mathcal{S}_K$ is the set of all stabilizing controllers (7) and $\mathcal{S}$ is the intersection of (8b) & (8c). In infinite horizon problems, we usually need a stabilizing $K$ so that $\mathcal{S}_K$ is equal to or a subset of the set of stabilizing controllers. We allow special cases when (10) depends only on $L, P$,

$$\min_{L,P} \ f(L, P), \quad \text{s.t.} \quad (L, P) \in \mathcal{S}. \tag{11}$$

We distill three properties of the two problems (9) and (10) that will be critical for Theorem 1, and allow us to cover more problems as discussed in Sec. 4.

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[1] If $K$ is not a stabilizing controller, we define $\mathcal{L}(K) = +\infty$.

**Assumption 1.** *The feasible set $\mathcal{S}$ is convex in $(L, P, Z)$. The cost function $f(L, P, Z)$ is convex, bounded, and differentiable, over an open domain that contains the set $\mathcal{S}$*

Assumption 1 implies the second problem is convex. Next, we extract the property of the connection between (7) and (8), and give an abstract description of the assumptions for (9) and (10).

**Assumption 2.** *Let $P$ be invertible whenever $(L, P, Z) \in \mathcal{S}$.* [2] *Assume we can express $\mathcal{L}(K)$ as follows,*

$$\mathcal{L}(K) = \min_{L,P,Z} \ f(L, P, Z)$$

$$s.t. \ (L, P, Z) \in \mathcal{S}, \ LP^{-1} = K.$$

Denote $\nabla \mathcal{L}(K)[V] := \mathbf{Tr}(V^\top \nabla \mathcal{L}(K))$ as the directional derivative of $\mathcal{L}(K)$ is the direction $V$. With the assumptions above, we will present the main theorem.

**Theorem 1.** *Suppose assumptions 1,2 hold, and consider the two problems (9) and (10). Let $K^*$ denote the global minimizer of $\mathcal{L}(K)$ in $S_K$. Then there exist constants $C_1, C_2 > 0$ independent of the suboptimality $\mathcal{L}(K) - \mathcal{L}(K^*)$, and a direction $V$, with $\|V\|_F = 1$, in the descent cone of $\mathcal{S}_K$ at $K$ such that,*

*1. if $f$ is convex, the gradient of $\mathcal{L}$ satisfies[3]*

$$\nabla \mathcal{L}(K)[V] \leq -C_1(\mathcal{L}(K) - \mathcal{L}(K^*)). \tag{12}$$

*2. if $f$ is $\mu$-strongly convex, the gradient satisfies*

$$\nabla \mathcal{L}(K)[V] \leq -C_2(\mu(\mathcal{L}(K) - \mathcal{L}(K^*)))^{1/2}. \tag{13}$$

*The constants $C_1, C_2$ are discussed below and in Appendix B.*

**Remark 1.** *The constants in the above theorem are case by case. They typically depend on the norm of system parameters and the radius of the feasible domain[4]. We study continuous time LQR as an example. Let the sublevel set be where $\mathcal{L}(K) \leq a$, and define*

$$\nu = 4a \left( \sigma_{\max}(A)\lambda_{\min}^{-1/2}(Q) + \sigma_{\max}(B)\lambda_{\min}^{-1/2}(R) \right)^2,$$

$$C_{1,1} = 2a\nu\lambda_{\min}^{-1}(\Sigma)\lambda_{\min}^{-1/2}(Q)\lambda_{\min}^{-1/2}(R),$$

$$C_{1,2} = 2a^2\nu^2\lambda_{\min}^{-2}(\Sigma)\lambda_{\min}^{-3/2}(Q)\lambda_{\min}^{-1/2}(R),$$

*. Then $C_1 = (\max\{C_{1,1}, C_{1,2}\})^{-1}$. The paper [21] gives another convex formulation with strong convexity and we can get $C_2$ for that form,*

$$C_2 = \max\{\nu\lambda_{\min}^{-1}(\Sigma), \ a\nu^2\lambda_{\min}^{-2}(\Sigma)\lambda_{\min}^{-1/2}(Q)\lambda_{\min}^{-1/2}(R)\}.$$

*The details are in Appendix B.*

Our lower bound of the gradient, $\|\nabla\mathcal{L}(K)\|_F \gtrsim (\mathcal{L}(K) - \mathcal{L}(K^*))^\alpha$, is known as Lojasiewicz inequality [30]. When $\alpha = 1/2$, it is also called the *gradient dominance* property. If Lojasiewicz inequality holds for all $K$, all local minimums of the objective function are global minimum, and an iterative method in which the norm of the gradient decreases to zero will have to converge to global minimum. Nonconvex functions that satisfy Lojasiewicz inequality are easily optimized, compared to those with spurious local minimums. In practice, gradient dominance often holds in a neighborhood of a local minimum, and gradient dominance is typically used as a tool for local convergence analysis (it is rare that it holds for $\mathcal{L}(K)$ globally, however that is true for example problems in this paper).

Assumption 2 is a weak assumption. Assumption 3 is a stronger one covered by Assumption 2 that, we assume that there is a bijection between $K$ and $(L, P)$. This is true for many control problems including continuous time LQR. Theorem 1 also holds with Assumptions 1,2. We emphasize the special case for Assumptions 1,3. In Sec. 7, we will use the bijection to map back and forth the points and gradients in the nonconvex and convex problem, and then illustrate the critical proof step.

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[2]The invertibility of $P$ holds for all instances in Sec. 4 , and guarantees a well defined map between $L, P$ and $K$.

[3]We always consider the directional derivative of a feasible direction within descent cone.

[4]Although the radius can infinity, when we run gradient descent with respect to $cL(K)$, the cost is usually bounded by the initial value $\mathcal{L}(K_0)$ so the iterates are in a sublevel set. The radius of a finite sublevel set usually finite.

**Assumption 3.** *1. (Bijection between two feasible sets) Let $P$ be invertible, and let $K = LP^{-1}$ define a bijection[5] $K \leftrightarrow (L, P)$. For any such bijection $K \leftrightarrow (L, P)$, $\exists Z$, such that $(L, P, Z) \in \mathcal{S}$.*

*2. (Equivalence of functions) Choose a controller $K \in \mathcal{S}_K$ with corresponding $(L, P) \in \mathcal{S}$. Then $\mathcal{L}(K) = \min_Z f(L, P, Z)$ subject to $(L, P, Z) \in \mathcal{S}$.*

Thm. 1 suggests that, when the original nonconvex optimization problem can be mapped to a convex optimization problem that satisfies Assumptions 1,2 or 1,3, all stationary points of the nonconvex objective are global minimums. So if we can evaluate the gradient of nonconvex objective and run gradient descent algorithm, the iterates converge to the optimal controller.

# 4 Control problems covered by main theorem

In order to reach the conclusion, Theorem 1 requires an optimal control problem (9), its convex form (10) and a few assumptions. This is an abstract and general description that does not need the exact continuous time LQR formulation in Sec. 2. We can easily check that the continuous time LQR satisfies Assumptions 1,3, thus we can directly apply Theorem 1 to argue that the continuous time LQR cost $\mathcal{L}(K)$ satisfies (34).

Below, we will propose more examples, showing that Theorem 1 covers a wide range of optimal control problems. This shows the **generality** of Theorem 1. If one encounters optimal control problems from the following set, and moreover, if one encounters a new problem, we hope that one can check if the problem satisfies the assumptions in Theorem 1. If so, one can directly claim that the original cost function has no spurious local minimum and can be optimized by policy gradient method, which can be applied in model free setup.

## 4.1 Discrete time LQR

We consider a discrete time linear system

$$x(t + 1) = Ax(t) + Bu(t), \ x(0) = x_0,$$

The goal is to find a state feedback controller $K$ such that the cost function

$$\mathcal{L}(K) = \mathbf{E}_{x_0} \sum_{i=0}^{\infty} x(t)^\top Q x(t) + u(t)^\top R u(t), \ u = Kx$$

is minimized. In other words, we will solve

$$\min_K \ \mathcal{L}(K), \ \text{s.t. } K \text{ stabilizes.}$$

Similar to the continuous time system, one can choose the same parameterization $P, L, Z$ and another PSD matrix $G \in \mathbb{R}^{n \times n} \succeq 0$ and solve the following problem

$$\min_{L,P,Z,G} \ f(L, P, Z, G) := \mathbf{Tr}(QP) + \mathbf{Tr}(ZR), \tag{14a}$$

$$\text{s.t. } P \succ 0, \ G - P + \Sigma = 0, \tag{14b}$$

$$\begin{bmatrix} Z & L \\ L^\top & P \end{bmatrix} \succeq 0, \ \begin{bmatrix} G & AP + BL \\ (AP + BL)^\top & P \end{bmatrix} \succeq 0. \tag{14c}$$

The goal is to argue that $\mathcal{L}(K)$ and (14) has the connection such that Theorem 1 applies, so that the stationary point of $\mathcal{L}(K)$ has to be the global optimum.

**Lemma 1.** *The LQR problem* $\min \mathcal{L}(K)$ *with stabilizing $K$, and problem* (14), *satisfy Assumption 1, 2.*

**Proof.** (14) is a convex optimization problem. Now we prove Assumption 2, i.e., we prove that $L(K)$ equals the minimum of the problem (14) with an extra constraint $K = LP^{-1}$.

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[5]Note that generally $K = LP^{-1}$ cannot guarantee a bijection. However bijection is possible with the extra constraint $(L, P) \in \mathcal{S}$.

- We first minimize over $Z$, the minimizer is $Z = LP^{-1}L^\top$. Now replace $L$ by $KP$ and the cost becomes $\mathbf{Tr}((Q + K^\top RK)P)$.

- Eliminate $G$ by

$$G - P + \Sigma = 0, \quad \begin{bmatrix} G & AP + BL \\ (AP + BL)^\top & P \end{bmatrix} \succeq 0.$$

Using Schur complement, it is equivalent to

$$(AP + BL)P^{-1}(AP + BL)^\top - P + \Sigma \preceq 0.$$

Plug in $L = KP$, we have

$$(A + BK)P(A + BK)^\top - P + \Sigma \preceq 0.$$

The cost does not involve $G$ so it does not change.

- Now, we need to prove that $\mathcal{L}(K)$ is equal to

$$\min_P \ \mathbf{Tr}((Q + K^\top RK)P),$$
$$\text{s.t. } (A + BK)P(A + BK)^\top - P + \Sigma \preceq 0. \tag{15}$$

The last constraint can be written as

$$(A + BK)P(A + BK)^\top - P + \Theta = 0, \ \Theta \succeq \Sigma.$$

- Denote the solution to $(A + BK)P(A + BK)^\top - P + \Theta = 0$ as $P(\Theta)$. $P(\Theta)$ for all $\Theta \succeq \Sigma$ covers the feasible points of (15). $P(\Theta)$ is expressed as:

$$P(\Theta) = \sum_{t=0}^{\infty} (A + BK)^t \Theta ((A + BK)^\top)^t.$$

So $P(\Theta) \succeq P(\Sigma)$, $\forall \Theta \succeq \Sigma$. Since $Q$ and $K^\top RK$ are positive semidefinite, $\mathbf{Tr}((Q + K^\top RK)P)$ achieves the minimum at $P = P(\Sigma)$.

- At the end, $P(\Sigma)$ is the Grammian $\mathbf{E} \sum_{t=0}^{\infty} x(t)x(t)^\top$ when $\mathbf{E}x(0)x(0)^\top = \Sigma$. We studied the connection between continous time Grammian (5) and the cost (6), a similar result holds for discrete time LQR:

$$\mathbf{Tr}((Q + K^\top RK)P(\Sigma)) = \mathcal{L}(K).$$

We build the connection between minimizing $\mathcal{L}(K)$, and the convex optimization (14). We argued this pair of problems satisfies the assumptions of Theorem 1. Theorem 1 suggests that $\mathcal{L}(K)$ is gradient dominant, so we can approach $K^*$ by gradient descent on $K$. This is essentially the conclusion of [2, 22]. Note that the proof of discrete time LQR [2, 22] and continuous time LQR [21, 23] cannot trivially extend to each other.

## 4.2 LQR with Markov jump linear system

We generalize the discrete time linear system to multiple linear systems with transitions in this part. Suppose there are $N$ linear systems, the $i$-th one being

$$x(t + 1) = A_i x(t) + B_i u(t).$$

Now we study the Markov jump linear system [31]. At each time $t$, the dynamics linking $x(t + 1)$ and the past state and input $x(t), u(t)$ is given by

$$x(t + 1) = A_{w(t)} x(t) + B_{w(t)} u(t), \ w(t) \in [N] := \{1, ..., N\}.$$

So at any time $t$, the system is chosen from the $N$ candidate linear systems. The transition of the linear systems, or the transition of $w(t)$, follows a probabilistic model

$$\mathbf{Pr}(w(t+1) = j | w(t) = i) = \rho_{ij} \in [0, 1], \ \forall t \geq 0.$$

Suppose $\mathbf{Pr}(w(0) = i) = p_i$. For the $i$-th system, we will use a state feedback controller $K_i$. Let $K = [K_1, ..., K_N]$. Define the cost as

$$\mathcal{L}(K) = \mathbf{E}_{w,x_0} \sum_{t=0}^{\infty} x(t)^\top Q x(t) + u(t)^\top R u(t), \ u(t) = K_{w(t)} x(t), \ \mathbf{Pr}(w(0) = i) = p_i.$$

**Grammian matrix.** Let $P_i$ be the Grammian matrix

$$P_i = \mathbf{E}(\sum_{t=0}^{\infty} x(t) x(t)^\top \mid w(0) = i), \ \text{when } \mathbf{E}(x(0)x(0)^\top) = \Sigma.$$

$P_i$ is time invariant. In other words,

$$P_i = \mathbf{E}(\sum_{t=\tau}^{\infty} x(t) x(t)^\top \mid w(\tau) = i), \ \text{when } \mathbf{E}(x(\tau)x(\tau)^\top) = \Sigma,$$

and for any $\Gamma \in \mathbb{R}^{n \times n}$,

$$\Gamma P_i \Gamma^\top = \mathbf{E}(\sum_{t=0}^{\infty} x(t) x(t)^\top \mid w(0) = i), \ \text{when } \mathbf{E}(x(0)x(0)^\top) = \Gamma \Sigma \Gamma^\top.$$

If $w(0) = i$, then the covariance of $x(1)$ is $(A_i + B_i K_i) \Sigma (A_i + B_i K_i)^\top$.

$$(A_i + B_i K_i) \left( \sum_{j=1}^{N} P_j \cdot \mathbf{Pr}(w(1) = j | w(0) = i) \right) (A_i + B_i K_i) = \mathbf{E}(\sum_{t=1}^{\infty} x(t) x(t)^\top \mid w(0) = i),$$

$$\text{when } \mathbf{E}(x(1)x(1)^\top) = (A_i + B_i K_i) \Sigma (A_i + B_i K_i)^\top.$$

So

$$(A_i + B_i K_i)(\sum_{j=1}^{N} \rho_{ij} P_j)(A_i + B_i K_i) = \mathbf{E}(\sum_{t=1}^{\infty} x(t) x(t)^\top \mid w(0) = i),$$

$$\text{when } \mathbf{E}(x(0)x(0)^\top) = \Sigma,$$

and we can build the following recursion:

$$\mathbf{E}(\sum_{t=1}^{\infty} x(t) x(t)^\top \mid w(0) = i)$$

$$= \mathbf{E}(\sum_{t=0}^{\infty} x(t) x(t)^\top \mid w(0) = i) - \mathbf{E}(x(0)x(0)^\top)$$

$$= P_i - \Sigma.$$

As a result $(A_i + B_i K_i)(\sum_{j=1}^{N} \rho_{ij} P_j)(A_i + B_i K_i) = P_i - \Sigma$.

**Convex formulation.** We propose the following convex formulation: Denote $P_0, P_1, ..., P_N \in \mathbb{R}^{n \times n} \succ 0$, $L_1, ..., L_N \in \mathbb{R}^{p \times n}$, $Z_0, Z_1, ..., Z_N \in \mathbb{R}^{p \times p} \succeq 0$. Then

$$\min \ \mathbf{Tr}(Q P_0) + \mathbf{Tr}(Z_0 R), \tag{16a}$$

$$\text{s.t. } P_0 = \sum_{i=1}^{N} p_i P_i, \ Z_0 = \sum_{i=1}^{N} p_i Z_i, \ P_i \succ 0, \ P^i = \sum_{j=1}^{N} \rho_{ij} P_j, \tag{16b}$$

$$\begin{bmatrix} Z_i & L_i \\ L_i^\top & P_i \end{bmatrix} \succeq 0, \ \begin{bmatrix} P_i - \Sigma & A_i P^i + B_i L_i \\ (A_i P^i + B_i L_i)^\top & P^i \end{bmatrix} \succeq 0, \ \forall i \in [N]. \tag{16c}$$

The variables are $P^i, P_i, L_i, Z_i$ for all $i \in [N]$ and $P_0, Z_0$. With this formulation, $P_i$ and $P^i$ associate with

$$\mathbf{E}(\sum_{t=0}^{\infty} x(t)x(t)^\top \mid w(0) = i), \ \mathbf{E}(\sum_{t=1}^{\infty} x(t)x(t)^\top \mid w(0) = i)$$

respectively. The mapping between the controller $K_i$ and the new variables are $K_i = L_i(P^i)^{-1}$.

We prove the $\mathcal{L}(K)$ and the convex formulation satisfy Assumptions 1, 2 in Appendix C.1, so that all local minimums of $\mathcal{L}(K)$ are global minimum.

## 4.3 Minimizing $\mathcal{L}_2$ gain

We quote from [29] the problem of minimizing the $\mathcal{L}_2$ gain with static state feedback controller $K$. As discussed in [29, §6.3.2], this problem has an associated convex optimization problem and we can show it satisfies Assumptions 1,2.

We consider minimizing the $\mathcal{L}_2$ gain of a closed loop system. The continuous time linear dynamical system is

$$\dot{x} = Ax + Bu + B_w w, \ y = Cx + Du.$$

For any signal $z$, denote

$$\|z\|_2 := \left(\int_0^\infty \|z(t)\|_2^2 dt\right)^{1/2}$$

Suppose we use a state feedback controller $u = Kx$, and aim to find the optimal controller $K^*$ that minimizes the $\mathcal{L}_2$ gain. We minimize the squared $\mathcal{L}_2$ gain as

$$\min_K \ \mathcal{L}(K) := (\sup_{\|w\|_2=1} \|y\|_2)^2.$$

This problem can be further reformulated as [29, Sec 7.5.1]

$$\min_{L,P,\gamma} \ f(L, P, \gamma) := \gamma, \ \text{s.t.}$$
$$\begin{bmatrix} AP + PA^\top + BL + L^\top B^\top + B_w B_w^\top & (CP + DL)^\top \\ CP + DL & -\gamma I \end{bmatrix} \preceq 0. \tag{17}$$

The minimum $\mathcal{L}_2$ gain is $\sqrt{\gamma^*}$ and $K^* = L^* P^{*-1}$. We will show in the Appendix C.2 that the parameters $K$ and $(L, P, \gamma)$, with cost $\mathcal{L}(K)$ and $f(L, P, \gamma)$, satisfy Assumptions 1,2. Thus we can claim that all stationary points of $\mathcal{L}(K)$ are global minimum.

[29, §6.3.2] suggests that $\mathcal{L}_2$ gain is also the $\mathcal{H}_\infty$ norm of transfer function.

## 4.4 Dissipativity

We use the dynamical system

$$\dot{x} = Ax + Bu + B_w w, \ y = Cx + Du + D_w w \tag{18}$$

The notion of dissipativity can be found in [29, §6.3.3, §7.5.2]. Our goal is to maximize the dissipativity, which is defined and formulated as with a convex parameterization [29, Sec. 6.3.3, 7.5.2].

The dissipativity is defined as all $\eta > 0$ (we usually take the maximum one) that satisfy

$$\int_0^T w^\top y - \eta w^\top w \, dt \geq 0, \ \forall T > 0.$$

Same as the last example, we use a state feedback controller $K$, and the goal is to find $K^*$ that maximizes the dissipativity $\eta$. Same as before, let $K$ be factorized as $LP^{-1}$. We can maximize the dissipativity $\eta$ as a function of $K$. From [29, §7.5.2], we maximize $\eta$ subject to the dissipativity constraint (19),

$$\max_{\eta, L, P} \eta,$$

$$\text{s.t.} \begin{bmatrix} AP + PA^\top + BL + L^\top B^\top & B_w - PC^\top - (DL)^\top \\ B_w^\top - CP - DL & 2\eta I - (D + D^\top) \end{bmatrix} \preceq 0. \tag{19}$$

With similar reasoning as in Sec. 4.4, the assumptions for Theorem 1 holds. Thus we can claim that all stationary points of $\mathcal{L}(K)$ are global minimum.

## 4.5 System level synthesis (SLS) for finite horizon time varying discrete LQR

This problem and its convex form are introduced in [32]. It satisfies Assumption 1,3. We consider the following linear dynamical system

$$x(t+1) = A(t)x(t) + B(t)u(t) + w(t) \tag{20}$$

over a finite horizon $0, \ldots T$. Let the state be $x$ and the input be $u$. Define

$$X = \begin{bmatrix} x(0) \\ \ldots \\ x(T) \end{bmatrix}, \ U = \begin{bmatrix} u(0) \\ \ldots \\ u(T) \end{bmatrix},$$

$$W = \begin{bmatrix} x(0) \\ w(0) \\ \ldots \\ w(T-1) \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \ldots & & & & \\ 0 & 0 & \ldots & I & 0 \end{bmatrix},$$

$$\mathcal{A} = \text{diag}(A(0), ..., A(T-1), 0),$$
$$\mathcal{B} = \text{diag}(B(0), ..., B(T-1), 0).$$

Now we consider the time varying controller $K$ that links state and input as

$$u(t) = \sum_{i=0}^{t} K(t, t-i)x(i), \tag{21}$$

and let

$$\mathcal{K} = \begin{bmatrix} K(0,0) & 0 & \ldots & 0 \\ K(1,1) & K(1,0) & \ldots & 0 \\ \ldots & & & \\ K(T,T) & K(T,T-1) & \ldots & K(T,0) \end{bmatrix}.$$

We will minimize some cost function with the constraint. For example, in the discrete time LQR regime, let the input be (21) and define (More examples of nonquadratic cost in [32, Sec 2.2])

$$\mathcal{L}(\mathcal{K}) = \sum_{t=0}^{T} x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t), \tag{22}$$

here $Q(t), R(t) \succeq 0$. We will minimize $\mathcal{L}(\mathcal{K})$ where $\mathcal{K}$ is the variable.

Parameterization: The dynamics (20) can be written as

$$X = Z\mathcal{A}X + Z\mathcal{B}U + W = Z(\mathcal{A} + \mathcal{B}\mathcal{K})X + W$$

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We define the mapping from $W$ to $X, U$ by

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} W.$$

where $\Phi_X, \Phi_U$ are block lower triangular. There is a constraint on $\Phi_X, \Phi_U$:

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I. \tag{23}$$

It is proven in [32, Thm 2.1] that $\mathcal{K} = \Phi_U \Phi_X^{-1}$. $\mathcal{K}$ and $\Phi_X, \Phi_U$ is a bijection given $\Phi_X, \Phi_U$ satisfying (23).

Let $\mathcal{Q} = \mathrm{diag}(Q(0), ..., Q(T))$, $\mathcal{R} = \mathrm{diag}(R(0), ..., R(T))$, the LQR cost with $x(0) \sim \mathcal{N}(0, \Sigma)$ and no noise is

$$f(\Phi_X, \Phi_U) = \left\| \mathrm{diag}(\mathcal{Q}^{1/2}, \mathcal{R}^{1/2}) \begin{bmatrix} \Phi_X(:, 0) \\ \Phi_U(:, 0) \end{bmatrix} \Sigma^{1/2} \right\|_F^2 ;$$

the LQR cost with $x(0), w(t)$ being i.i.d from $\mathcal{N}(0, \Sigma)$ is

$$f(\Phi_X, \Phi_U) = \left\| \mathrm{diag}(\mathcal{Q}^{1/2}, \mathcal{R}^{1/2}) \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} \Sigma^{1/2} \right\|_F^2 .$$

If we solve

$$\min_{\mathcal{K}} \ \mathcal{L}(\mathcal{K}), \ \mathcal{K} \text{ is lower left triangular}$$

with the above two costs of $w(t)$, both can be minimized with constraint (23):

$$\min_{\Phi_X, \Phi_U} \ f(\Phi_X, \Phi_U), \ \text{s.t.} \ \begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I,$$

$$\Phi_X, \Phi_U \text{ are lower left triangular.}$$

This problem is convex and satisfy Assumption 1. The theorem [32, Thm 2.1] suggests the relation between $\mathcal{L}$ and $f$ satisfying Assumption 3 for Theorem 1. With Theorem 1, we can argue that all stationary points of $\mathcal{L}(\mathcal{K})$ are global minimum.

# 5 A more general description of Assumption 2

In this section, we will give a more general theorem, based on replacing the map $K = LP^{-1}$ by arbitrary function $\Phi$ defined below. This allows the theorem to cover more examples in Sec. 6.

We chose $K = LP^{-1}$ because this is frequently used for the convex parameterization of the optimal control problem. For example, with the continuous time LQR problem motivated in Sec. 2, the mapping between $K$ and $L, P$ is (almost) the only widely used convex parameterization method. If we choose another change of variable, the resulting objective function is usually not convex in the new variables.

On the other hand, although the mapping $K = LP^{-1}$ is studied, we can generalize Thm. 1 with arbitrary mappings if the reformulated problem is convex – of course, the new mappings have to satisfy a few assumptions to preserve gradient dominance.

Here we will propose the following assumptions which replace the mapping $K = LP^{-1}$ by an abstract mapping $\Phi$.

Suppose we consider the problems

$$\min_{K} \ \mathcal{L}(K), \quad \text{s.t. } K \in \mathcal{S}_K, \tag{24}$$

and

$$\min_{P} \ f(P), \quad \text{s.t. } P \in \mathcal{S}. \tag{25}$$

The matrix $P$ can be a concatenation of many variables, just as a short expression. For example, $P$ represents $(P, L, Z)$ of continuous LQR. We will study the original optimization problem (24), and map it to a convex optimization problem (25) where the mapping between $K$ and the variables of the other problem $P$ is abstractly denoted by $K = \Phi(P)$ in (26).

**Assumption 4.** *The feasible set $\mathcal{S}$ is convex in $P$. The cost function $f(P)$ is convex, finite and differentiable in $P \in \mathcal{S}$. $\mathcal{L}(K)$ is Lipschitz in $K$.*

**Assumption 5.** *Assume we can express $\mathcal{L}(K)$ as:*

$$\mathcal{L}(K) = \min_P \; f(P), \; s.t. \; P \in \mathcal{S}, \; K = \Phi(P). \tag{26}$$

*And we assume the first order Taylor expansion of the mapping $\Phi$ can be written as*

$$\Phi(P + \mathrm{d}P) = \Phi(P) + \Psi(P)[\mathrm{d}P] + o(\mathrm{d}P).$$

*for any $P \in \mathcal{S}$ and any perturbation $\mathrm{d}P$ such that $\mathrm{d}P$ is in the descent cone of $\mathcal{S}$ at $P$.*

Thm. 1 holds with the above Assumptions 4, 5, which means the cost $\mathcal{L}(K)$ is gradient dominant in $K$ given the above assumptions. It generalizes beyond the specific mapping $\Phi(P, L) = LP^{-1}$ to a more general definition, and we propose some instances of convex formulations with different $\Phi$ in the subsections. We propose the following theorem and the proof is in Sec. A.

**Theorem 2.** *Denote $\Delta K = \Psi(P)[P^* - P]$. Let $\nabla\mathcal{L}(K)[\Delta K]$ be the directional derivative of $\mathcal{L}(K)$ in direction $\Delta K$. Then with Assumptions 4, 5 we have*

$$\nabla\mathcal{L}(K)[\Delta K] \le \mathcal{L}(K^*) - \mathcal{L}(K).$$

If $K$ is not optimal, then the right hand side is strictly less than 0, which means the directional derivative of $\mathcal{L}$ is not 0. Therefore $\nabla\mathcal{L}(K) = 0$ only at the global minimums.

# 6 Control problems with generalized map

This section will cover examples where the parameterization is based on the general map $\Phi$.

## 6.1 Distributed finite horizon LQR

[27] We consider the time varying linear system

$$x(t + 1) = A(t)x(t) + B(t)u(t) + w(t),$$
$$y(t) = C(t)x(t) + v(t).$$

This is in finite time horizon $t = 0, ..., T$. The state evolution is same as the setup in our SLS example (Sec. 4.5), and we can use the same notation $X, U, W, Z, \mathcal{A}, \mathcal{B}$. We further define

$$Y = \begin{bmatrix} y(0) \\ ... \\ y(T) \end{bmatrix}, \; V = \begin{bmatrix} v(0) \\ ... \\ v(T) \end{bmatrix}, \; \mathcal{C} = \mathrm{diag}(C(0), ..., C(T-1), 0).$$

Now we will consider the control policy

$$u(t) = \sum_{i=0}^{t} K(t, t-i)y(i).$$

The search space of policy is same as SLS, and we define $\mathcal{K}$ matrix in the same way. The paper [27] studies the problem under the context of distributed control. One searches for the controller $K \in \mathcal{S}_K$ where $\mathcal{S}_K$ a subset of controllers. In distributed control, there is a graph model for controllers such that the $i$-th controller

might not be able to access the state $j$ for $(i, j)$ in a set of indices $\mathcal{S}_{\mathrm{idx}}$. In this case, $K_{i,j} = 0$ is an extra constraint for the control problem. Therefore, if one searches for the optimal controller in $\mathcal{S}_K$, we can define the subspace

$$\mathcal{S}_K := \{ K \mid K_{i,j} = 0, \ \forall (i,j) \in \mathcal{S}_{\mathrm{idx}} \}.$$

The extra constraint is not always easily handled, but [27, Sec. 3] proposes an extra assumption, called quadratic invariance (QI), and introduces the equivalent convex optimization.

QI means that, for all $K \in \mathcal{S}_K$, $KCP_{12}K \in \mathcal{S}_K$. The cost function is same as (22):

$$\mathcal{L}(\mathcal{K}) = \sum_{t=0}^{T} x(t)^\top Q(t) x(t) + u(t)^\top R(t) u(t).$$

It is proven that, define

$$P_{12} = (I - Z\mathcal{A})^{-1} Z\mathcal{B}, \ \Phi(\mathcal{G}) = (I + \mathcal{G}\mathcal{C}P_{12})^{-1}\mathcal{G}.$$

Then we can get a new variable $\mathcal{G}$ and a function $\Phi$. With $\mathcal{K} = \Phi(\mathcal{G})$, the cost can be proven to be convex in $\mathcal{G}$. $\mathcal{G}$ is in the same subspace as $\mathcal{K}$ determined by $\mathcal{S}_K$. Indeed, the mapping satisfies Assumptions 4, 5, and the exact formulation of the two optimization problems are described in [27, Append. A, Lem. 5]. Define $\mathcal{Q} = \mathrm{diag}(Q(0), ..., Q(T))$, $\mathcal{R} = \mathrm{diag}(R(0), ..., R(T))$. Let $w(t)$ be Gaussian random vectors with stationary covariance, $w(t_1) \perp w(t_2)$, $\forall t_1 \neq t_2$. $\Sigma_w = I_T \otimes \mathrm{Cov}(w)$, $\Sigma_x = \mathrm{diag}(\mathbf{E}(x_0 x_0^\top), 0, ..., 0)$. The convex cost function takes the form

$$\left\| \mathcal{Q}^{1/2} \mathcal{C} (I + P_{12}\mathcal{G}\mathcal{C}) P_{11} \begin{bmatrix} \Sigma_w^{1/2} & \Sigma_x^{1/2} \end{bmatrix} \right\|_F^2 + \left\| \mathcal{R}^{1/2} \mathcal{G} \mathcal{C} P_{11} \begin{bmatrix} \Sigma_w^{1/2} & \Sigma_x^{1/2} \end{bmatrix} \right\|_F^2.$$

Thus we can also claim via Thm. 1 that, such distributed LQ regulator problem with $\mathcal{K}$ as variable has no spurious local minimum.

## 6.2  Multi-objective and mixed controller design

In this part, we study a few synthesis problems with dynamical controllers, where the objectives are transfer functions of the closed form system. We study the dynamical system with state, disturbance, input, output, and controller's input $x, w, u, z, y$ with the following dynamics

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B \\ C_z & D & E \\ C & F & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}.$$

The dynamical controller follows

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}$$

and we will design the controller $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$ to minimize certain cost objectives in the next few subsections. [33, eq(4.2.15)] studies the parameterization of the problem, by introducing a list of variables

$$v = [X, Y, K, L, M, N]$$

and invertible matrices $U, V$ such that $UV^\top = I - XY$. The Lyapunov matrix $P$ of the closed loop system and $A_c, B_c, C_c, D_c$ are the unique solution of

$$\begin{bmatrix} K & L \\ M & N \end{bmatrix} = \begin{bmatrix} U & XB \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ CY & I \end{bmatrix} + \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix}. \tag{27}$$

The change of variable enables us to make some control problems as convex optimization, listed below. For simplicity of notation, let

$$\mathscr{X} = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}, \ \mathscr{A} = \begin{bmatrix} AY + BM & A + BNC \\ K & AX + LC \end{bmatrix}, \tag{28}$$

$$\mathscr{B} = \begin{bmatrix} B_w + BNF \\ XB_w + LF \end{bmatrix}, \ \mathscr{C} = \begin{bmatrix} C_zY + EM & C_z + ENC \end{bmatrix}, \ \mathscr{D} = D + ENF. \tag{29}$$

**Remark 2.** *The mapping in (27) can be written as*

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \Phi(v)$$

*where* $\Phi$ *plays the role in (26). We propose a few control problems with convex forms in the next few subsections. The variables of nonconvex objective functions are* $A_c, B_c, C_c, D_c$, *the new objective functions with respect to* $v = [X, Y, K, L, M, N]$ *are convex, and the two forms satisfy Assumptions 4, 5. Thus the cost functions with respect to matrix* $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$ *are gradient dominant.*

In the following subsections, we refer to the result of [33] that, the optimal $\mathcal{H}_\infty$ design, $\mathcal{H}_2$ design and the multi-objective design that is a combination of the previous two, can be made convex optimization problems with our proposed way. Thus, the objectives with respect to controller $K$ are all gradient dominant.

### 6.2.1 $\mathcal{H}_\infty$ design

[33, Sec. 4.2.3] The goal in this part is to minimize the $\mathcal{H}_\infty$ norm of the closed loop system's transfer function by designing the optimal controller. Let the transfer function of the closed form system be $\mathscr{T}$. The problem with its raw form is to minimize the $\|\mathscr{T}\|_{\mathcal{H}_\infty}$ over $A_c, B_c, C_c, D_c$, and we will propose the convex formulation – the change of variable trick such that the argument we minimize over becomes $v$. The problem takes the form:

$$\min \ \gamma,$$

$$\text{s.t.} \quad \mathscr{X} \succeq 0, \ \begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B} & \mathscr{C}^\top \\ \mathscr{B}^\top & -\gamma I & \mathscr{D}^\top \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{bmatrix} \preceq 0.$$

If we fix all other parameters and optimize over $\gamma$, then $\gamma^*(v)$ is the $\mathcal{H}_\infty$ value of the closed loop system with the mapping from $v$ to controller by (27). If we minimize over $\gamma$ and $v$, then we can get optimal $\mathcal{H}_\infty$ design.

### 6.2.2 $\mathcal{H}_2$ design

[33, Sec. 4.2.5] This part is similar to $\mathcal{H}_\infty$ design. Suppose the goal is to minimize $\|\mathscr{T}\|_{\mathcal{H}_2}$, one can alternatively solve

$$\min \ \gamma,$$

$$\text{s.t.} \quad \begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B} \\ \mathscr{B}^\top & -\gamma I \end{bmatrix} \preceq 0, \ \mathscr{D} = 0, \ \begin{bmatrix} \mathscr{X} & \mathscr{C}^\top \\ \mathscr{C} & Z \end{bmatrix} \succeq 0, \ \mathbf{Tr}(Z) \leq \gamma.$$

If we fix all other parameters and optimize over $\gamma, Z$, then $\gamma^*(v)$ is the $\mathcal{H}_2$ value of the closed loop system with the mapping from $v$ to controller by (27). If we minimize over $\gamma, Z$ and $v$, then we can get optimal $\mathcal{H}_2$ design.

### 6.2.3 Multi-objective synthesis

[33, Sec. 4.3] Let the system be

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & B \\ C_1 & D_1 & D_{12} & E_1 \\ C_2 & D_{21} & D_2 & E_2 \\ C & F_1 & F_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ u \end{bmatrix} \tag{30}$$

Now we study the mixed design for $\mathcal{H}_\infty$ design from $z_1$ to $w_1$ and $\mathcal{H}_2$ design from $z_2$ to $w_2$. We keep the mapping (27) and the change of parameter (28), but replace (29) by

$$\mathscr{B}_i = \begin{bmatrix} B_i + BNF_i \\ XB_i + LF_i \end{bmatrix}, \ \mathscr{C}_i = \begin{bmatrix} C_iY + E_iM & C_i + E_iNC \end{bmatrix}, \ \mathscr{D}_i = D_i + E_iNF_i.$$

for $i = 1, 2$. Suppose we are given a positive number $\lambda$ and hope to study $\|\mathscr{T}_1\|_{\mathcal{H}_\infty} + \lambda\|\mathscr{T}_2\|_{\mathcal{H}_2}$ where $\mathscr{T}_i$ is the transfer function of the $i$-th system ($z_1$ to $w_1$, $z_2$ to $w_2$), then we can write

$$\min \ \gamma_1 + \lambda\gamma_2, \tag{31}$$

$$\text{s.t.} \quad \begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B}_1 & \mathscr{C}_1^\top \\ \mathscr{B}_1^\top & -\gamma_1 I & \mathscr{D}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 & -\gamma_1 I \end{bmatrix} \preceq 0, \tag{32}$$

$$\begin{bmatrix} \mathscr{A}^\top + \mathscr{A} & \mathscr{B}_2 \\ \mathscr{B}_2^\top & -\gamma_2 I \end{bmatrix} \preceq 0, \ \mathscr{D}_2 = 0, \ \begin{bmatrix} \mathscr{X} & \mathscr{C}_2^\top \\ \mathscr{C}_2 & Z \end{bmatrix} \succeq 0, \ \mathbf{Tr}(Z) \leq \gamma_2. \tag{33}$$

If we fix all other parameters and optimize over $\gamma_1, \gamma_2, Z$, then the function value is the mixed $\mathcal{H}_\infty/\mathcal{H}_2$ value of the closed loop system with the mapping from $v$ to controller by (27). If we minimize over $\gamma_1, \gamma_2, Z$ and $v$, then we can get the optimal mixed $\mathcal{H}_\infty/\mathcal{H}_2$ design.

### 6.2.4   Robust state feedback control

[33, Sec. 8.1.2] We study the robust state feedback control problem, where the robustness corresponds to a system with uncertain parameters, denoted by $\Delta$ below. We apply the system model (30). "State feedback" means that $C = I$ and $F_1, F_2 = 0$. Let $N_\Delta$ be a positive integer. The connection between $w_1$ and $z_1$ is an uncertain channel

$$w_1(t) = \Delta(t)z_1(t)$$

for any

$$\Delta(t) \in \Delta_c := \text{conv}\{0, \Delta_1, ..., \Delta_{N_\Delta}\}.$$

The goal is to minimize a certain norm of the transfer function from $z_2$ to $w_2$, which can be found in the previous part. We consider minimizing the norm under an extra constraint when the closed loop system achieves robust stability with $z_1$ to $w_1$ ($z_1$ with finite norm) and robust quadratic performance with $z_2$ to $w_2$ via a matrix $P_p$. The robust quadratic performance is defined as: there exists a matrix $P_p$ such that

$$P_p = \begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^\top & \tilde{R}_p \end{bmatrix}, \ P_p^{-1} = \begin{bmatrix} Q_p & S_p \\ S_p^\top & R_p \end{bmatrix}$$

such that $\tilde{R}_p \succ 0, Q_p \prec 0$, and

$$\int_0^\infty \begin{bmatrix} w_2(t) \\ z_2(t) \end{bmatrix}^\top P_p \begin{bmatrix} w_2(t) \\ z_2(t) \end{bmatrix} dt \leq \epsilon\|w_2\|_{\mathcal{H}_2}^2$$

for some $\epsilon > 0$.

[33, Sec. 8.1.2] proposes a convex formulation of the extra constraint. Define new variables $Q, S, R$ in addition to $v = [X, Y, K, L, M, N]$, and let $\mathscr{M}$ replace

$$\mathscr{M} \leftarrow \begin{bmatrix} -(AY + BM)^\top & -(C_1Y + E_1M)^\top & -(C_2Y + E_2M)^\top \\ I & 0 & 0 \\ -B_1^\top & -D_1^\top & -D_{21}^\top \\ 0 & I & 0 \\ -B_2^\top & -D_{12}^\top & -D_2^\top \\ 0 & 0 & I \end{bmatrix}.$$

The constraints, which is proven to be convex [33, Sec. 8.1.2] can be written as

$$R \succ 0, \ Q \prec 0, \ \begin{bmatrix} I \\ -\Delta_j \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} I \\ -\Delta_j \end{bmatrix} \prec 0, \ \forall j \in [N_\Delta]$$

$$Y \succ 0, \ \mathscr{M}^\top \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^\top & R & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^\top & R_p \end{bmatrix} \mathscr{M} \succ 0.$$

For example, if we aim to minimize the $\mathcal{H}_2$ norm of the transfer function from $z_2$ to $w_2$, then we can minimize $\gamma_2$ subject to (33) and the constraints above. The main theorem of this paper suggests that, with the convex formulation, if we apply policy gradient descent with respect to $\mathcal{H}_2$ norm of the transfer function from $z_2$ to $w_2$ with robust stability ($z_1$ with finite norm) and robust quadratic performance constraints, then policy gradient descent converges to globally optimal controller.

### 6.2.5 Discrete time system

[33, Sec. 4.6] Suppose we study the discrete time system, and we define the system in a similar way of defining the continuous time system:

$$\begin{bmatrix} x(t+1) \\ z_1(t) \\ z_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & B \\ C_1 & D_1 & D_{12} & E_1 \\ C_2 & D_{21} & D_2 & E_2 \\ C & F_1 & F_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_1(t) \\ w_2(t) \\ u(t) \end{bmatrix}, \ \begin{bmatrix} x_c(t+1) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}.$$

Now we study the mixed design for $\mathcal{H}_\infty$ design from $z_1$ to $w_1$ and $\mathcal{H}_2$ design from $z_2$ to $w_2$. Suppose we are given a positive number $\lambda$ and hope to study $\|\mathscr{T}_1\|_{\mathcal{H}_\infty} + \lambda\|\mathscr{T}_2\|_{\mathcal{H}_2}$ where $\mathscr{T}_i$ is the transfer function of the $i$-th system ($z_1$ to $w_1$, $z_2$ to $w_2$), then we can write

$$\min \ \gamma_1 + \lambda\gamma_2,$$

$$\text{s.t.} \ \begin{bmatrix} \mathscr{X} & 0 & \mathscr{A}^\top & \mathscr{C}_1^\top \\ 0 & \gamma_1 I & \mathscr{B}_1^\top & \mathscr{D}_1 \\ \mathscr{A} & \mathscr{B}_1 & \mathscr{X} & 0 \\ \mathscr{C}_1 & \mathscr{D}_1 & 0 & \gamma_1 I \end{bmatrix} \succ 0, \ \mathbf{Tr}(Z) \le \gamma_2,$$

$$\begin{bmatrix} \mathscr{X} & \mathscr{A} & \mathscr{B}_2 \\ \mathscr{A}^\top & \mathscr{X} & 0 \\ \mathscr{B}_2^\top & 0 & \gamma_2 I \end{bmatrix} \succ 0, \ \begin{bmatrix} \mathscr{X} & 0 & \mathscr{C}_2 \\ 0 & \mathscr{X} & \mathscr{D}_2 \\ \mathscr{C}_2 & \mathscr{D}_2 & Z \end{bmatrix} \succ 0.$$

If we fix all other parameters and optimize over $\gamma_1, \gamma_2, Z$, then the function value is the mixed $\mathcal{H}_\infty/\mathcal{H}_2$ value of the closed loop system with the mapping from $v$. If we minimize over $\gamma_1, \gamma_2, Z$ and $v$, then we can get the optimal mixed $\mathcal{H}_\infty/\mathcal{H}_2$ design.

## 7 Proof Sketch

We put the full proof of Theorem 1 in Appendix A, and give a sketch of the proof in this section. We illustrate the idea in Figure 1, which, on the high level, maps the original space of controller $K$ where the cost is nonconvex, and the parameterized space with $L, P, Z$ where the cost is convex.

For simplicity, we sketch the proof using Assumptions 1,3. For any point $K$, we can find a point $(L, P, Z)$ in the parameterized space. If it is not the optimizer, we can find the line segment linking $(L, P, Z)$ and the optimizer $(L^*, P^*, Z^*)$. Note that the optimization problem is convex in this space so that $\langle \nabla f(L, P, Z), (L^*, P^*, Z^*) - (L, P, Z) \rangle$ is upper bounded by $f(L^*, P^*, Z^*) - f(L, P, Z)$. Then with the help of our assumptions, we can map the directional derivative back to the original $K$ space, and show that the directional derivative in $\mathcal{L}(K)$ is not 0.
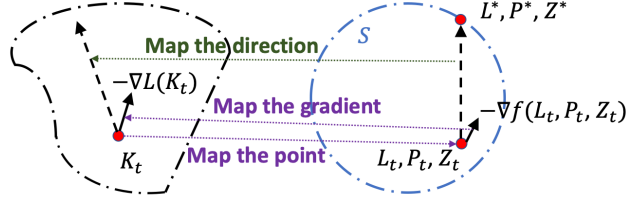
Figure 1: Mapping between nonconvex and convex landscapes. Suppose we run gradient descent at iteration $t$, for any controller $K$, we can map it to $L, P, Z$ in the other parameterized space. and then we map the direction $(L^*, P^*, Z^*) - (L, P, Z)$ and the gradient $\nabla f(L, P, Z)$ back to the original $K$ space. Since in $(L, P, Z)$ space the objective function is convex, then $\langle \nabla f(L, P, Z), (L^*, P^*, Z^*) - (L, P, Z) \rangle < 0$. We prove that similar correlation holds for the nonconvex objective.

Before concluding, we remark that the assumptions in Theorem 1 come from an optimization theory perspective, and we do not dive into the control theoretic interpretations of the constants and assumptions. Our approach has the benefit that it unifies the analysis of many control problems in a single abstract result. We leave it to future work to refine the analysis to obtain the best case-specific convergence rates, and to provide an interpretation of the associated constants in terms of control theoretic notions.

# References

[1] R. E. Kalman *et al.*, "Contributions to the theory of optimal control," *Bol. soc. mat. mexicana*, vol. 5, no. 2, pp. 102–119, 1960.

[2] M. Fazel, R. Ge, S. M. Kakade, and M. Mesbahi, "Global convergence of policy gradient methods for linearized control problems," *arXiv preprint arXiv:1801.05039*, 2018.

[3] R. F. Stengel, *Optimal control and estimation.* Courier Corporation, 1994.

[4] G. E. Dullerud and F. Paganini, *A course in robust control theory: a convex approach.* Springer Science & Business Media, 2013, vol. 36.

[5] G. Hewer, "An iterative technique for the computation of the steady state gains for the discrete optimal regulator," *IEEE Transactions on Automatic Control*, vol. 16, no. 4, pp. 382–384, 1971.

[6] P. Lancaster and L. Rodman, *Algebraic riccati equations.* Clarendon press, 1995.

[7] V. Balakrishnan and L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 30–41, 2003.

[8] L. Ljung, "System identification: theory for the user," *PTR Prentice Hall, Upper Saddle River, NJ*, pp. 1–14, 1999.

[9] M. Simchowitz, H. Mania, S. Tu, M. I. Jordan, and B. Recht, "Learning without mixing: Towards a sharp analysis of linear system identification," in *Conference On Learning Theory*, 2018, pp. 439–473.

[10] S. J. Bradtke, B. E. Ydstie, and A. G. Barto, "Adaptive linear quadratic control using policy iteration," in *Proceedings of 1994 American Control Conference-ACC'94*, vol. 3. IEEE, 1994, pp. 3475–3479.

[11] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu, "On the sample complexity of the linear quadratic regulator," *arXiv preprint arXiv:1710.01688*, 2017.

[12] S. Dean, S. Tu, N. Matni, and B. Recht, "Safely learning to control the constrained linear quadratic regulator," in *2019 American Control Conference (ACC).* IEEE, 2019, pp. 5582–5588.

[13] H. Mania, S. Tu, and B. Recht, "Certainty equivalent control of LQR is efficient," *arXiv preprint arXiv:1902.07826*, 2019.

[14] F. L. Lewis and D. Vrabie, "Reinforcement learning and adaptive dynamic programming for feedback control," *IEEE circuits and systems magazine*, vol. 9, no. 3, pp. 32–50, 2009.

[15] J. Y. Lee, J. B. Park, and Y. H. Choi, "Integral q-learning and explorized policy iteration for adaptive optimal control of continuous-time linear systems," *Automatica*, vol. 48, no. 11, pp. 2850–2859, 2012.

[16] D. Lee and J. Hu, "Primal-dual q-learning framework for LQR design," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3756–3763, 2018.

[17] S. M. Kakade, "A natural policy gradient," in *Advances in neural information processing systems*, 2002, pp. 1531–1538.

[18] A. Rajeswaran, K. Lowrey, E. V. Todorov, and S. M. Kakade, "Towards generalization and simplicity in continuous control," in *Advances in Neural Information Processing Systems*, 2017, pp. 6550–6561.

[19] K. Mårtensson, "Gradient methods for large-scale and distributed linear quadratic control," Ph.D. dissertation, Lund University, 2012.

[20] J. W. Roberts, I. R. Manchester, and R. Tedrake, "Feedback controller parameterizations for reinforcement learning," in *2011 IEEE Symposium on Adaptive Dynamic Programming and Reinforcement Learning (ADPRL)*. IEEE, 2011, pp. 310–317.

[21] H. Mohammadi, A. Zare, M. Soltanolkotabi, and M. R. Jovanović, "Global exponential convergence of gradient methods over the nonconvex landscape of the linear quadratic regulator," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 7474–7479.

[22] J. Bu, A. Mesbahi, M. Fazel, and M. Mesbahi, "LQR through the lens of first order methods: Discrete-time case," *arXiv preprint arXiv:1907.08921*, 2019.

[23] J. Bu, A. Mesbahi, and M. Mesbahi, "Policy gradient-based algorithms for continuous-time linear quadratic control," *arXiv preprint arXiv:2006.09178*, 2020.

[24] J. Bu, L. J. Ratliff, and M. Mesbahi, "Global convergence of policy gradient for sequential zero-sum linear quadratic dynamic games," *arXiv preprint arXiv:1911.04672*, 2019.

[25] K. Zhang, Z. Yang, and T. Basar, "Policy optimization provably converges to nash equilibria in zero-sum linear quadratic games," *Advances in Neural Information Processing Systems*, vol. 32, pp. 11 602–11 614, 2019.

[26] K. Zhang, B. Hu, and T. Basar, "Policy optimization for $\mathcal{H}_2$ linear control with $\mathcal{H}_\infty$ robustness guarantee: Implicit regularization and global convergence," in *Learning for Dynamics and Control*, 2020, pp. 179–190.

[27] L. Furieri, Y. Zheng, and M. Kamgarpour, "Learning the globally optimal distributed lq regulator," in *Learning for Dynamics and Control*. PMLR, 2020, pp. 287–297.

[28] R. Bellman, "Dynamic programming," *Science*, vol. 153, no. 3731, pp. 34–37, 1966.

[29] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. SIAM, 1994.

[30] S. Lojasiewicz, "A topological property of real analytic subsets," *Coll. du CNRS, Les équations aux dérivées partielles*, vol. 117, pp. 87–89, 1963.

[31] J. P. Jansch-Porto, B. Hu, and G. E. Dullerud, "Convergence guarantees of policy optimization methods for markovian jump linear systems," in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 2882–2887.

[32] J. Anderson, J. C. Doyle, S. H. Low, and N. Matni, "System level synthesis," *Annual Reviews in Control*, vol. 47, pp. 364–393, 2019.

[33] C. Scherer and S. Weiland, "Linear matrix inequalities in control," *Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands*, vol. 3, no. 2, 2000.

[34] A. Zare, H. Mohammadi, N. K. Dhingra, T. T. Georgiou, and M. R. Jovanović, "Proximal algorithms for large-scale statistical modeling and sensor/actuator selection," *IEEE Transactions on Automatic Control*, vol. 65, no. 8, pp. 3441–3456, 2019.

# A   Proof of the main theorems

**Theorem 1.** *Suppose assumptions 1,2 hold, and consider the two problems* (9) *and* (10). *Let $K^*$ denote the global minimizer of $\mathcal{L}(K)$ in $S_K$. Then there exist constants $C_1, C_2 > 0$ independent of the suboptimality $\mathcal{L}(K) - \mathcal{L}(K^*)$, and a direction $V$, with $\|V\|_F = 1$, in the descent cone of $\mathcal{S}_K$ at $K$ such that,*

1. *if $f$ is convex, the gradient of $\mathcal{L}$ satisfies*[6]

$$\nabla\mathcal{L}(K)[V] \leq -C_1(\mathcal{L}(K) - \mathcal{L}(K^*)). \tag{34}$$

2. *if $f$ is $\mu$-strongly convex, the gradient satisfies*

$$\nabla\mathcal{L}(K)[V] \leq -C_2(\mu(\mathcal{L}(K) - \mathcal{L}(K^*)))^{1/2}. \tag{35}$$

*where (the constants can be bounded with simple constraints bounding norms of $L, P$ or $K$)*

$$C_1 = (\max\{\max_L \|L - L^*\|_F \sigma_{\min}^{-1}(P), \max_P \|P - P^*\|_F \sigma_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1},$$
$$C_2 = 2(\max\{\sigma_{\min}^{-1}(P), \sigma_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1}.$$

*Proof.* Denote $\mathcal{P}_{\mathcal{S}}(\nabla f(x))$ as the projection of $\nabla f(x)$ onto the descent cone of $\mathcal{S}$ at $x$. First, for any convex function $f(x)$, let the minimum be $x^*$, and $x - x^* = \Delta$. Let $\nabla f(x) = g$. For any non-stationary point, $f(x) \leq f(x^*) + g^\top \Delta$. Since $\mathcal{S}$ is a convex set, $-\Delta$ belongs to the descent cone of $\mathcal{S}$ at $x$, so the direction $\frac{\Delta}{\|\Delta\|}$ is feasible, $f(x) - f(x - t\frac{\Delta}{\|\Delta\|}) > tg^\top \frac{\Delta}{\|\Delta\|}$, $t \to 0$, so the norm of projected gradient $\|\mathcal{P}_{\mathcal{S}}(\nabla f(x))\| \geq g^\top \frac{\Delta}{\|\Delta\|} = \frac{f(x) - f(x^*)}{\|x - x^*\|}$.

Let $K^*$ be the optimal $K$ and $(Z^*, L^*, P^*)$ be the optimal point in the parameterized space. We have $\mathcal{L}(K^*) = f(Z^*, L^*, P^*)$. Denote $V$ as any matrix in $K$ space, $\mathcal{P}_V$ is projection of a vector onto direction $V$. In other words,

$$\mathcal{P}_V(M) = \frac{\mathbf{Tr}(V^\top M)}{\|V\|_F^2}V,$$

then we denote $\nabla\mathcal{L}(K)[\cdot]$ as a linear operator such that $\nabla\mathcal{L}(K)[V] = \mathbf{Tr}(V^\top \nabla\mathcal{L}(K))$ (a more general discussion about Taylor expansion is in Sec. 5),

$$\nabla\mathcal{L}(K)^\top \nabla\mathcal{L}(K) \geq (\mathcal{P}_V \nabla\mathcal{L}(K))^\top \mathcal{P}_V \nabla\mathcal{L}(K) = \left(\frac{\nabla\mathcal{L}(K)[V]}{\|V\|_F}\right)^2. \tag{36}$$

We denote $\mathscr{Z}(L, P) \in \text{argmin}_Z f(L, P, Z)$ subject to $(L, P, Z) \in \mathcal{S}$ (if there are multiple minimizers we pick any one). With either Assumption 3 or 2, we can define the mapping from $K$ to $(L, P, Z)$ respectively in one of the following ways:

1. (Assumption 3) let $K$ map to $(L, P)$ and $Z = \mathscr{Z}(L, P)$.

2. (Assumption 2) let

$$(L, P, Z) = \text{argmin}_{L', P', Z'} \ f(L', P', Z')$$
$$\text{s.t. } (L', P', Z') \in \mathcal{S}, \ P' \succ 0, \ L'P'^{-1} = K.$$

---

[6]We always consider the directional derivative of a feasible direction within descent cone.

Note $f$ is convex, so

$$
\begin{aligned}
&\nabla f(L, P, Z)[(L, P, Z) - (L^*, P^*, Z^*)] \\
&\geq f(L, P, Z) - f(L^*, P^*, Z^*) \\
&= f(\mathscr{Z}(L, P), L, P) - f(\mathscr{Z}(L^*, P^*), L^*, P^*) \\
&= \mathcal{L}(K) - \mathcal{L}(K^*).
\end{aligned}
\tag{37}
$$

Now we consider the directional derivative in $K$ space. By definition,

$$
\nabla \mathcal{L}(K)[V] = \lim_{t \to 0^+} (\mathcal{L}(K + tV) - \mathcal{L}(K))/t.
$$

Let $\Delta L = L^* - L$, $\Delta P = P^* - P$, and $V = \Delta L P^{-1} - L P^{-1} \Delta P P^{-1}$. Then

$$
\begin{aligned}
\nabla \mathcal{L}(K)[V] &= \lim_{t \to 0^+} (\mathcal{L}(K + tV) - \mathcal{L}(K))/t \\
&= \lim_{t \to 0^+} (\mathcal{L}(L P^{-1} + t(\Delta L P^{-1} - L P^{-1} \Delta P P^{-1})) - \mathcal{L}(L P^{-1}))/t \\
&= \lim_{t \to 0^+} (\mathcal{L}((L + t\Delta L)(P + t\Delta P)^{-1}) - \mathcal{L}(L P^{-1}))/t.
\end{aligned}
$$

The last line uses $(P + t\Delta P)^{-1} = P^{-1} - t P^{-1} \Delta P P^{-1} + o(t)$. Denote $\Delta(L, P, Z) = (L^*, P^*, Z^*) - (L, P, Z)$, $\Delta(L, P, Z)$ is in the descent cone of $\mathcal{S}$ at $(L, P, Z)$ due to the convexity of $\mathcal{S}$. With Assumption 3, we continue with

$$
\begin{aligned}
\nabla \mathcal{L}(K)[V] &= \lim_{t \to 0^+} (f(L + t\Delta L, P + t\Delta P, \mathscr{Z}(L + t\Delta L, P + t\Delta P)) - f(L, P, \mathscr{Z}(L, P)))/t \\
&\leq \lim_{t \to 0^+} (f(L + t\Delta L, P + t\Delta P, Z + t\Delta Z) - f(L, P, \mathscr{Z}(L, P)))/t \\
&= \nabla f(L, P, Z)[\Delta(L, P, Z)].
\end{aligned}
$$

With Assumption 2, we continue with

$$
\begin{aligned}
\nabla \mathcal{L}(K)[V] = \lim_{t \to 0^+} \min_{L', P', Z'} \ & f(L', P', Z') - f(L, P, Z) \\
\text{s.t. } & (L', P', Z') \in \mathcal{S}, \ P' \succ 0, \\
& L' P'^{-1} = (L + t\Delta L)(P + t\Delta P)^{-1}.
\end{aligned}
$$

$(L + t\Delta L, P + t\Delta P, Z + t\Delta Z)$ is a feasible point of the optimization problem, thus is less than or equal to the minimum, and then

$$
\begin{aligned}
\nabla \mathcal{L}(K)[V] &\leq \lim_{t \to 0^+} (f(L + t\Delta L, P + t\Delta P, Z + t\Delta Z) - f(L, P, \mathscr{Z}(L, P)))/t \\
&= \nabla f(L, P, Z)[\Delta(L, P, Z)].
\end{aligned}
$$

So the final inequality holds either by Assumption 3 or 2. So

$$
\nabla \mathcal{L}(K)[V] \leq \nabla f(L, P, Z)[\Delta(L, P, Z)] < 0.
$$

Using (36) and (37), we have

$$
\|\nabla \mathcal{L}(K)\|_F^2 \geq \frac{1}{\|V\|_F^2} (\mathcal{L}(K) - \mathcal{L}(K^*))^2.
\tag{38}
$$

If $f(L, P, Z)$ is $\mu$ strongly convex, then we can restrict $f$ in the line segment $(L, P, Z) - (L^*, P^*, Z^*)$ and

get and then

$$
\begin{aligned}
(\nabla \mathcal{L}(K)[V])^2 &\geq \frac{1}{\|V\|_F^2} (\nabla f(L, P, Z)[\Delta(L, P, Z)])^2 \\
&\geq \frac{\mu \|\Delta(L, P, Z)\|_F^2}{\|V\|_F^2} \cdot (f(L, P, Z) - f(L^*, P^*, Z^*)) \\
&= \frac{\mu(\|L^* - L\|^2 + \|P^* - P\|^2 + \|Z^* - Z\|^2)}{\|(L^* - L)P^{-1} - LP^{-1}(P^* - P)P^{-1}\|_F^2} \cdot (f(L, P, Z) - f(L^*, P^*, Z^*)) \\
&\geq \frac{\mu(\|L^* - L\|^2 + \|P^* - P\|^2)}{\|(L^* - L)P^{-1} - LP^{-1}(P^* - P)P^{-1}\|_F^2} \cdot (f(L, P, Z) - f(L^*, P^*, Z^*)) \\
&\geq \frac{4\mu(f(L, P, Z) - f(L^*, P^*, Z^*))}{(\max\{\sigma_{\min}^{-1}(P), \sigma_{\min}^{-2}(P)\sigma_{\max}(L)\})^2}.
\end{aligned}
$$

$\square$

**Theorem 2.** *Denote $\Delta K = \Psi(P)[P^* - P]$. Let $\nabla \mathcal{L}(K)[\Delta K]$ be the directional derivative of $\mathcal{L}(K)$ in direction $\Delta K$. Then with Assumptions 4, 5 we have*

$$
\nabla \mathcal{L}(K)[\Delta K] \leq \mathcal{L}(K^*) - \mathcal{L}(K).
$$

*Proof.* Suppose $f(P)$ is convex in $P$, and the optimizer of (25) is $P^*$. Denote

$$
P = \operatorname{argmin}_{P'} f(P'), \text{ s.t. } P' \in \mathcal{S}, \ K = \Phi(P'),
$$

and

$$
\Delta P = P^* - P, \ \Delta K = \Psi(P)[\Delta P].
$$

We take the directional derivative and get (explanation of key steps below the last line)

$$
\begin{aligned}
\nabla \mathcal{L}(K)[\Delta K] &= \lim_{t \to 0^+} \frac{\mathcal{L}(K + t\Delta K) - \mathcal{L}(K)}{t} \\
&= \lim_{t \to 0^+} \frac{\mathcal{L}(K + t\Psi(P)[\Delta P]) - f(P)}{t} \quad (39) \\
&= \lim_{t \to 0^+} \frac{\mathcal{L}(\Phi(P) + t\Psi(P)[\Delta P]) - f(P)}{t} \quad (40) \\
&= \lim_{t \to 0^+} \frac{\mathcal{L}(\Phi(P + t\Delta P) - o(t)) - f(P)}{t} \quad (41) \\
&= \lim_{t \to 0^+} \frac{\mathcal{L}(\Phi(P + t\Delta P)) - f(P)}{t} \\
&= \lim_{t \to 0^+} \frac{\min_{P' \in \mathcal{S}, \ \Phi(P+t\Delta P) = \Phi(P')} f(P') - f(P)}{t} \quad (42) \\
&= \lim_{t \to 0^+} \frac{\min_{P' \in \mathcal{S}, \ \Phi(P+t\Delta P) = \Phi(P')} f(P') - f(P + t\Delta P) + f(P + t\Delta P) - f(P)}{t} \\
&= \lim_{t \to 0^+} \frac{\min_{P' \in \mathcal{S}, \ \Phi(P+t\Delta P) = \Phi(P')} f(P') - f(P + t\Delta P)}{t} + \nabla f(P)[\Delta P]. \quad (43)
\end{aligned}
$$

(39) and (40) replace $\Delta K$ and $K$ with expressions in $P$ and $\Delta P$. (41) applies the Taylor expansion of $\Phi$:

$$
\Phi(P + t\Delta P) - (\Phi(P) + t\Psi(P)[\Delta P]) = o(t).
$$

(42) applies Assumption 5, and we plug in $K = \Phi(P + t\Delta P)$. (43) applied the definition of directional derivative

$$
\nabla f(P)[\Delta P] = \lim_{t \to 0^+} \frac{f(P + t\Delta P) - f(P)}{t}.
$$

21

Now we bound the first term of (43). Note that, since $P + t\Delta P$ for $t > 0$ and $t \to 0^+$ belongs to the line segment from $P$ to $P^*$. Since $\mathcal{S}$ is a convex set, we know that the line segment between to feasible points $P^*$ and $P$ is in $\mathcal{S}$. then

$$P + t\Delta P \in \{P' \mid P' \in \mathcal{S}, \ \Phi(P + t\Delta P) = \Phi(P')\},$$

so that $f(P + t\Delta P)$ is no less than the minimum of the optimization problem (26),

$$\lim_{t \to 0^+} \frac{\min_{P' \in \mathcal{S}, \ \Phi(P+t\Delta P) = \Phi(P')} f(P') - f(P + t\Delta P)}{t} \leq 0.$$

$\nabla f(P)[\Delta P]$ is the directional derivative of $f(P)$ in the direction of $P^* - P$, for a convex function $f$, if $P$ is not an optimizer, $\nabla f(P)[\Delta P]$ is upper bounded by $f(P^*) - f(P) = \mathcal{L}(K^*) - \mathcal{L}(K) < 0$. $\qquad \square$

# B   Constants for continuous time LQR

Thm. 1 asks for two constants $C_1, C_2$. They are bounded differently for different examples. As an instance, we will calculate the constants for continuous time LQR, quoted from [21] Appendix B. First $P \succ 0$, so we replace singular value by eigenvalue with $P$

$$C_1 = (\max\{\max_L \|L - L^*\|_F \lambda_{\min}^{-1}(P), \max_P \|P - P^*\|_F \lambda_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1},$$

$$C_2 = 2(\max\{\lambda_{\min}^{-1}(P), \lambda_{\min}^{-2}(P)\sigma_{\max}(L)\})^{-1}.$$

We need upper bounds for $P, L$ and a lower bound for $\lambda_{\min}(P)$ to guarantee $C_1, C_2$ finite. We will show the bounds within the sublevel set that $\{K : \mathcal{L}(K) \leq a\}$. Since we can randomly initialize a feasible $K_0$ and run (projected) gradient descent method with respect to $K$, if $\mathcal{L}(K)$ is gradient dominant, it is reasonable to assume that during all iterations of the optimization algorithm, the function value is always upper bounded by $\mathcal{L}(K_0)$, or some values related to $\mathcal{L}(K_0)$. So our derivation with a finite sublevel set is reasonable.

Suppose the matrices $Q, R \succ 0$, and we consider the sublevel set when $\mathcal{L}(K) \leq a$. The sublevel set gives $\mathbf{Tr}(QP) + \mathbf{Tr}(LP^{-1}L^\top R) \leq a$, so

$$\begin{aligned}
\lambda_{\min}(R)\lambda_{\max}^{-1}(P)\|L\|_F^2 &\leq \lambda_{\min}(R)\|LP^{-1/2}\|_F^2 \\
&\leq \mathbf{Tr}(LP^{-1}L^\top R) \\
&\leq \mathbf{Tr}(QP) + \mathbf{Tr}(LP^{-1}L^\top R) \leq a.
\end{aligned}$$

So $\|L\|_F \leq a(\lambda_{\max}(P)\lambda_{\min}^{-1}(R))^{1/2}$, and we know from [21, eq(34)] $\mathbf{Tr}(P) \leq a\lambda_{\min}^{-1}(Q)$. So we can bound $P, L$

$$\mathbf{Tr}(P) \leq a\lambda_{\min}^{-1}(Q),$$

$$\|L\|_F \leq a(\lambda_{\min}(Q)\lambda_{\min}(R))^{-1/2}.$$

Define

$$\nu = 4a \left(\sigma_{\max}(A)\lambda_{\min}^{-1/2}(Q) + \sigma_{\max}(B)\lambda_{\min}^{-1/2}(R)\right)^2.$$

[34, eq(38,40)] suggests $\lambda_{\min}(P) \geq \lambda_{\min}(\Sigma)/\nu$. In summary, we upper bounded $L$, and upper and lower bounded $P$ in the sublevel set $\mathcal{L}(K) \leq a$, and those bounds are also true for $L^*, P^*$. We can complete the calculation by inserting the bounds into $C_1$.

$$C_1 = (\max\{2a\nu\lambda_{\min}^{-1}(\Sigma)\lambda_{\min}^{-1/2}(Q)\lambda_{\min}^{-1/2}(R), 2a^2\nu^2\lambda_{\min}^{-3/2}(Q)\lambda_{\min}^{-1/2}(R)\lambda_{\min}^{-2}(\Sigma)\})^{-1}.$$

$C_2$ is calculated similarly with upper bound on $P, L, P^{-1}$.

$$C_2 = \max\{\nu\lambda_{\min}^{-1}(\Sigma), \ a\nu^2\lambda_{\min}^{-2}(\Sigma)\lambda_{\min}^{-1/2}(Q)\lambda_{\min}^{-1/2}(R)\}.$$

## B.1 Strongly convex parameter of continuous time LQR

In our previous convex formulation of continuous time LQR (8), we translate the objective function as a linear function in the new variables $L, P, Z$. The problem (8) can be slightly reformulated as

$$\min_{L,P} \ f(L,P) := \mathbf{Tr}(QP) + \mathbf{Tr}(LP^{-1}L^\top R), \tag{44a}$$

$$\text{s.t. } \mathcal{A}(P) + \mathcal{B}(L) + \Sigma = 0, \ P \succ 0. \tag{44b}$$

Compared with (8), (44) does not contain the variable $Z$. Below, we will prove that the new objective function $f(L, P)$, restricted within the feasible set, is a strongly convex function, which is not the case for the linear objective (8). In Thm. 1, there is another result with strongly convex $f$ and the gradient dominance parameter depends on the strongly convex parameter $\mu$. We also calculate $\mu$ of $f(L, P)$ below.

**Lemma 2.** *Define a sublevel set of of $f$ at level $a$, consisting of all $L, P$ such that $f(L, P) \leq a$. Define*

$$\nu = \frac{\lambda_{\min}(\Sigma)}{4}\left(\|A\|\lambda_{\min}^{-1/2}(Q) + \|B\|\lambda_{\min}^{-1/2}(R)\right), \ \eta = \left(\nu^{1/2}\|\mathcal{B}\|\lambda_{\min}(\Sigma)\lambda_{\min}(Q)\lambda_{\min}^{1/2}(R)\right)^{-1},$$

$$\mu_0 = \frac{2\lambda_{\min}(Q)\lambda_{\min}(R)}{a(1+a^2\eta)^2}.$$

*The strongly convex parameter $\mu$ of the function $f(L, P)$ restricted within the feasible set in (44) satisfies*

$$\mu \geq (\|\mathcal{A}^{-1} \circ \mathcal{B}\| + 1)^{-1}\mu_0.$$

*Proof.* Denote $\mathcal{A}^{-1}$ as the inverse of $\mathcal{A}$, a linear operator such that $\mathcal{A}^{-1}(\mathcal{A}(P)) = P$. [21, Proposition 1] concludes that the following function $h(\cdot)$ is $\mu_0$ strongly convex.

$$h(L) = f(L, -\mathcal{A}^{-1}(\mathcal{B}(L) + \Sigma)).$$

Define a perturbation direction $(\tilde{L}, \tilde{P})$. Any feasible perturbation at the point $L, P$ will satisfy $\mathcal{A}(\tilde{P}) + \mathcal{B}(\tilde{L}) = 0$, so $\tilde{P} = -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))$.

Let the strongly convex parameter of $f$ in the feasible directions be $\mu$, we will show its connection with $\mu_0$. Let $L$ be perturbed by $\tilde{L}$.

$$\nabla^2 h(L)[\tilde{L}, \tilde{L}] = \nabla^2 f(L, P)[(\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))), (\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L})))], \tag{45}$$

Due to the strong convexity of $h$,

$$\nabla^2 h(L)[\tilde{L}, \tilde{L}] \geq \frac{\mu_0\|\tilde{L}\|_F^2}{2}.$$

We perturb $f$ at $(L, P)$ in direction $(\tilde{L}, \tilde{P}) = (\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L})))$. The strongly convex parameter of $f$ in feasible directions is defined as the positive number $mu$ such that

$$\nabla^2 f(L, P)[(\tilde{L}, \tilde{P}), (\tilde{L}, \tilde{P})] \geq \frac{\mu(\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2)}{2}$$

for all $(\tilde{L}, \tilde{P})$ such that $\tilde{P} = -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))$. The directional Hessian is

$$\nabla^2 f(L, P)[(\tilde{L}, \tilde{P}), (\tilde{L}, \tilde{P})] = \nabla^2 f(L, P)[(\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))), (\tilde{L}, -\mathcal{A}^{-1}(\mathcal{B}(\tilde{L})))]. \tag{46}$$

(46) equals (45). So that

$$\nabla^2 f(L, P)[(\tilde{L}, \tilde{P}), (\tilde{L}, \tilde{P})] \geq \frac{\mu_0\|\tilde{L}\|_F^2}{2}$$

$$= \frac{\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2}{2} \cdot \frac{\mu_0\|\tilde{L}\|_F^2}{\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2}$$

$$= \frac{\|\tilde{P}\|_F^2 + \|\tilde{L}\|_F^2}{2} \cdot \frac{\mu_0\|\tilde{L}\|_F^2}{\|\mathcal{A}^{-1}(\mathcal{B}(\tilde{L}))\|_F^2 + \|\tilde{L}\|_F^2}.$$

So

$$\mu \geq (\|\mathcal{A}^{-1} \circ \mathcal{B}\| + 1)^{-1}\mu_0.$$

$\square$

# C  Checking the assumptions for examples

## C.1  Markov jump linear system

**Example 1.** *(Assumptions 1,2) We study the system*

$$x(t+1) = A_{w(t)}x(t) + B_{w(t)}u(t), \ w(t) \in [N].$$

*The transition model is*

$$\mathbf{Pr}(w(t+1) = j|w(t) = i) = \rho_{ij} \in [0,1], \ \forall t \geq 0.$$

*Let $K = [K_1, ..., K_N]$. Define the cost as*

$$\mathcal{L}(K) = \mathbf{E}_{w,x_0} \sum_{t=0}^{\infty} x(t)^\top Q x(t) + u(t)^\top R u(t), \ u(t) = K_{w(t)}x(t), \ \mathbf{Pr}(w(0) = i) = p_i.$$

*Let the convex formulation be*

$$\min \ \mathbf{Tr}(QP_0) + \mathbf{Tr}(Z_0 R),$$

$$s.t. \ P_0 = \sum_{i=1}^{N} p_i P_i, \ Z_0 = \sum_{i=1}^{N} p_i Z_i, \ P_i \succ 0, \ P^i = \sum_{j=1}^{N} \rho_{ij} P_j,$$

$$\begin{bmatrix} Z_i & L_i \\ L_i^\top & P_i \end{bmatrix} \succeq 0, \ \begin{bmatrix} P_i - \Sigma & A_i P^i + B_i L_i \\ (A_i P^i + B_i L_i)^\top & P^i \end{bmatrix} \succeq 0, \ \forall i \in [N].$$

*Then the pair of problems satisfy Assumptions 1,2.*

*Proof.* If we fix the other variables and minimize over $Z_i$, then $Z_i = L_i P_i L_i^\top$. Later we will fix all $Z_i = L_i P_i L_i^\top$.

We fix the stabilizing controllers $K$, in other words we fix the closed loop systems. Denote $\mathfrak{A}_i(P)$ : $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} = (A_i + BK_i)P(A_i + BK_i)^\top$. $\mathfrak{A}_i$'s are positive definite operators and the operator norms of $\mathfrak{A}_i$ are all less than 1. Denote

$$\mathfrak{A} : \mathbb{R}^{Nn\times n} \to \mathbb{R}^{Nn\times n} = \mathrm{diag}(\mathfrak{A}_1, ..., \mathfrak{A}_N).$$

Denote $\Omega \in \mathbb{R}^{Nn\times Nn}$ as a block matrix, each block being $\mathbb{R}^{n\times n}$, and the $i,j$-th block is $\rho_{ij}I_n$.

Now we study the following constraints:

$$P^i = \sum_{j=1}^{N} \rho_{ij} P_j, \ \begin{bmatrix} P_i - \Sigma & A_i P^i + B_i L_i \\ (A_i P^i + B_i L_i)^\top & P^i \end{bmatrix} \succeq 0.$$

denote $P = \mathrm{diag}(P_1, ..., P_N)$, and $\Sigma^{[N]} =\in \mathbb{R}^{Nn\times Nn} = \mathrm{diag}(\Sigma, ..., \Sigma)$, then we have

$$\mathfrak{A}(\Omega P) - P + \Sigma^{[N]} \preceq 0. \tag{48}$$

Fix any $\Theta \succeq \Sigma$ with $x(0)x(0)^\top = \Theta$. Let

$$P_i(\Theta) = \mathbf{E}(\sum_{t=0}^{\infty} x(t)x(t)^\top \mid w(0) = i), \ \text{when } \mathbf{E}(x(0)x(0)^\top) = \Theta,$$

$$= \mathbf{E}(\sum_{t=\tau}^{\infty} x(t)x(t)^\top \mid w(\tau) = i), \ \text{when } \mathbf{E}(x(\tau)x(\tau)^\top) = \Sigma.$$

Define $P(\Theta) = \mathrm{diag}(P_1(\Theta), ..., P_N(\Theta))$.

The Grammian $P(\Theta), P(\Sigma)$ satisfy

$$\mathfrak{A}(\Omega P(\Sigma)) - P(\Sigma) + \Sigma^{[N]} = 0,$$
$$\mathfrak{A}(\Omega P(\Theta)) - P(\Theta) + \Theta^{[N]} = 0,$$
$$\Theta \succeq \Sigma.$$

Compare with (48). The set of all feasible Grammians is the set of $P(\Theta)$ with $\Theta \succeq \Sigma$.

Denote

$$P_i(\Theta, s) = \mathbf{E}(x(s)x(s)^\top \mid w(0) = i), \text{ when } \mathbf{E}(x(0)x(0)^\top) = \Theta.$$

We will show that $P_i(\Theta, s) \succeq P_i(\Sigma, s)$ for all $s, i$. First, the case when $s = 0$ is trivial.

For any $i$, suppose $P_i(\Theta, s') \succeq P_i(\Sigma, s')$ for all $s' \leq s$. Then we prove $P_i(\Theta, s+1) \succeq P_i(\Sigma, s+1)$. The probability $\mathbf{Pr}(w(s) = j)$ does not depend on the initial condition $\Theta$.

$$P_i(\Theta, s+1) = \sum_{j=1}^{N} \left( \mathbf{Pr}(w(s) = j) \cdot (A_j + B_j K_j)P_i(\Theta, s)(A_j + B_j K_j)^\top \right).$$

Using the following fact

$$P_i(\Theta, s) \succeq P_i(\Sigma, s) \succeq 0$$
$$\Rightarrow (A_j + B_j K_j)P_i(\Theta, s)(A_j + B_j K_j)^\top \succeq (A_j + B_j K_j)P_i(\Sigma, s)(A_j + B_j K_j)^\top \succeq 0,$$

we get $P_i(\Theta, s+1) \succeq P_i(\Sigma, s+1)$. So we proved $P_i(\Theta, s) \succeq P_i(\Sigma, s)$ for all $s$.

$$P(\Theta) = \text{diag}(P_1(\Theta), ..., P_N(\Theta))$$
$$= \text{diag}(\sum_{s=0}^{\infty} P_1(\Theta, s), ..., \sum_{s=0}^{\infty} P_N(\Theta, s))$$

so that $P(\Theta) \succeq P(\Sigma)$.

Let $P_0(\Theta) = \sum_{i=1}^{N} p_i P_i(\Theta)$. The objective function is $\mathbf{Tr}((Q + K^\top R K)P_0(\Theta))$. If we fix $K$, the objective function is linear in $P_0(\Theta)$. We call it $f(P_0(\Theta))$. And, if $P_0(\Theta) \succeq P_0(\Sigma)$, then $f(P_0(\Theta)) \geq f(P_0(\Sigma))$, so the minimum is $f(P_0(\Sigma)) = \mathcal{L}(K)$.

$\square$

If we fix $K$ and minimize (16) with extra constraints $K_i = L_i(P^i)^{-1}$, then the optimal value is $\mathcal{L}(K)$. This verifies Assumption 2, and overall, we conclude that $\mathcal{L}(K)$ is gradient dominant, satisfying our main theorem.

## C.2   Minimizing $\mathcal{L}_2$ gain

In this section, we will show that the minimizing $\mathcal{L}_2$ gain example in Sec. 4.3 satisfy Assumptions 1,2. Thus Theorem 1 applies to this problem, so all stationary points of $\mathcal{L}_2$ gain as a cost function of the controller $K$ are global minimum.

**Example 2.** *(Assumptions 1,2) We consider minimizing the $\mathcal{L}_2$ gain of a closed loop system. The input output system is*

$$\dot{x} = Ax + Bu + B_w w, \ y = Cx + Du \tag{49}$$

*and we use the state feedback controller $u = Kx$, and let*

$$\mathcal{L}(K) := (\sup_{\|w\|_2=1} \|y\|_2)^2.$$

*If we minimize the function $\mathcal{L}(K)$, this problem can be reformulated as*

$$\min_{L,P,\gamma} \ f(L, P, \gamma) := \gamma$$
$$s.t. \ \begin{bmatrix} AP + PA^\top + BL + L^\top B^\top + B_w B_w^\top & (CP + DL)^\top \\ CP + DL & -\gamma I \end{bmatrix} \preceq 0.$$

*And $K^* = L^* P^{*-1}$. This pair of problems satisfy Assumptions 1,2.*

*Proof.* We will check Assumption 2, which means checking

$$\mathcal{L}(K) = \min_{L,P,\gamma} \gamma \tag{50a}$$

$$\text{s.t.} \begin{bmatrix} AP + PA^\top + BL + L^\top B^\top + B_w B_w^\top & (CP + DL)^\top \\ CP + DL & -\gamma I \end{bmatrix} \preceq 0, \ LP^{-1} = K. \tag{50b}$$

Note that, the intermediate step [29, Sec 7.5.1] is

$$\mathcal{L}(K) = \min_{P,\gamma} \gamma, \quad \text{s.t.} \tag{51a}$$

$$\begin{bmatrix} (A + BK)P + P(A + BK)^\top + B_w B_w^\top & P^\top (C + DK)^\top \\ (C + DK)P & -\gamma I \end{bmatrix} \preceq 0. \tag{51b}$$

Denote the optimizer of (50) by $\hat{L}, \hat{P}, \hat{\gamma}$, and the optimizer of (51) by $\check{P}, \check{\gamma}$.

Note $\hat{\gamma} \leq \check{\gamma}$. If (50) is not true, $\hat{\gamma} < \check{\gamma}$, we can replace $\check{P}, \check{\gamma}$ with $\hat{P}, \hat{\gamma}$ in (51) and it's still feasible. Thus the optimality condition of $\check{P}, \check{\gamma}$ in (51) is violated, which contradicts the assumption that (50) is not true. Then we claim that (50) is true. □

# D   System Level Synthesis with Infinite Horizon

In this work, we studied the landscape of the optimal control problem where the variables are matrices (which are finite dimensional), and SLS is an example. Generally, SLS also works with the infinite horizon problem. In this regime, the variables are *transfer functions* and they are infinite dimensional. In practice, when the problem is made convex, one can parameterize the transfer function (say as finite impulse response) and minimize the cost with respect to the finite dimensional parameters. However, Thm. 1 does not apply to the infinite dimensional optimization problems, and it is not obvious that the parameterization satisfies the assumptions for our main theorem. We review the infinite horizon SLS here. A future direction is to judge whether the gradient dominance holds in the space of transfer function or its parameterized form, and how to analyze it using SLS.

**Example 3.** *(System level synthesis with infinite horizon) [32] Suppose one has a discrete time dynamical system with*

$$x(t+1) = Ax(t) + Bu(t) + w(t).$$

*One can apply a dynamic controller $K(z)$. The goal is to find the optimial controller which minimizes the LQR cost where $u(z) = K(z)x(z)$*

$$\mathcal{L}(K) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} x(t)^\top Q x(t) + u(t)^\top R u(t).$$

*Suppose $x_0, w_t$ are i.i.d. from $\mathcal{N}(0,\Sigma)$. The SLS defines two transfer functions $\Phi_X(z), \Phi_U(z)$, and solve the following convex optimization problem*

$$\min_{\Phi_X(z),\Phi_U(z)} \left\| \begin{bmatrix} Q^{1/2}\Phi_X(z) \\ R^{1/2}\Phi_U(z) \end{bmatrix} \Sigma^{1/2} \right\|_{\mathcal{H}_2},$$

$$\text{s.t.} \ \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \Phi_X(z) \\ \Phi_U(z) \end{bmatrix} = I,$$

$$\Phi_X(z), \Phi_U(z) \in \frac{1}{z}\mathcal{RH}_\infty.$$

*Let the optimizer be $\Phi_U^*(z), \Phi_X^*(z)$. The optimal controller is $K^*(z) = \Phi_U^*(z)(\Phi_X^*(z))^{-1}$.*

# E   Conditions of convexifiable nonconvex cost

We consider the pair of problems in Theorem 1, and ask the question: what property of the nonconvex cost function $\mathcal{L}(K)$ allows us to reformulate the problem (9) as a *convex* optimization problem (10)? In this section we propose the following lemma.

**Lemma 3.** *Suppose Assumptions 1, 3 hold, and $\mathcal{L}(LP^{-1})$ as a function of $L, P$ is differentiable. We define the notation $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2]$ as in (52). If $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2] > 0$ for all $(L, P) \in \mathcal{S}$ and all $(\Gamma_1, \Gamma_2)$ such that $\mathcal{A}(\Gamma_2) + \mathcal{B}(\Gamma_1) = 0$, then we can define a convex function $f(L, P)$ so that Assumption 1 holds. We can apply Theorem 1 so that all stationary points of $\mathcal{L}(K)$ are global minimum.*

  **Proof.** Suppose we observe the simple version (11). We know from Assumption 3 that, $f(L, P) = \mathcal{L}(K) = \mathcal{L}(LP^{-1})$ is convex in $L, P$. We take the Hessian and ask for

$$\nabla \begin{bmatrix} \nabla\mathcal{L}(LP^{-1})P^{-1} \\ -P^{-1}L^{\top}\nabla\mathcal{L}(LP^{-1})P^{-1} \end{bmatrix} \succ 0.$$

Note that this is a tensor and it is positive definite. For simplicity, we analyze the directional Hessian as the following. We expand the left hand side of the inequality above and define $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2]$ as

$$
\begin{aligned}
&\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2] \\
&:= \nabla^2\mathcal{L}(LP^{-1})[\Gamma_1 G^{-2}, \Gamma_1] - 2\nabla^2\mathcal{L}(LP^{-1})[\Gamma_1, LP^{-3}\Gamma_2] \\
&\quad - 2\langle\Gamma_1, \nabla\mathcal{L}(LP^{-1})P^{-1}\Gamma_2 P^{-1}\rangle + 2\langle\Gamma_2, LP^{-1}\Gamma_2 P^{-1}\nabla\mathcal{L}(LP^{-1})P^{-1}\rangle \\
&\quad + \nabla^2\mathcal{L}(LP^{-1})[LP^{-2}\Gamma_2, LP^{-2}\Gamma_2].
\end{aligned}
\tag{52}
$$

This is the directional Hessian of $\mathcal{L}$ with respect to $(L, P)$ in direction $(\Gamma_1, \Gamma_2)$. Thus, if $\nabla^2_{L,P}\mathcal{L}(LP^{-1})[\Gamma_1, \Gamma_2] > 0$ for all $(L, P) \in \mathcal{S}$ and all $(\Gamma_1, \Gamma_2)$ such that $\mathcal{A}(\Gamma_2) + \mathcal{B}(\Gamma_1) = 0$ (which is a condition on nonconvex cost $\mathcal{L}$), we know that $f(L, P)$ is convex in $L, P$ and the convex formulation can be made.