# Beyond Convexification of LQR: a Generalized Framework

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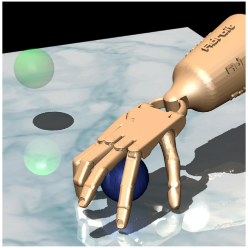
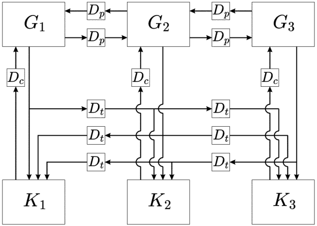
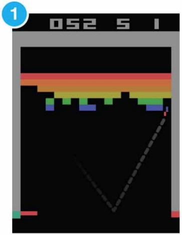
# Optimal control



Robot




Decentralized control




Game

# Linear quadratic regulator – Continuous

- Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0,$$

  $x$ is state, $u$ is input (the controller).

- Loss/Objective:

$$loss(u(t)) := \mathbf{E}_{x_0 \sim \mathcal{N}(0,\Omega)} \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t)) dt$$

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- Optimal controller: state feedback

$$u = K^* x = -R^{-1}BPx,$$
$$AP + PA^\top + Q - PBR^{-1}BP = 0.$$

  Solved by Riccati equations.

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- Convexification

$$\begin{aligned} \min_{Z,L,G} \ & f(L,G,Z) := \mathbf{Tr}(QG + ZR) \\ & \text{s.t., } \mathcal{A}(G) + \mathcal{B}(L) + \Omega = 0,\ G \succ 0, \\ & \begin{bmatrix} Z & L^\top \\ L & G \end{bmatrix} \succeq 0 \end{aligned}$$

And $K^* = L^* G^{*-1}$.

# Policy gradient descent – motivation and related works

**Policy gradient descent:** $K^+ = K - \eta \nabla \mathit{loss}(K)$. Gradient descent on nonconvex objective

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**Policy gradient descent, LQR, discrete:** Fazel et al. 2018; Bu, A. Mesbahi, Fazel, et al. 2019
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**Motivation of our work:**

- Each paper solves one optimal control problem – not easy to generalize, not including optimal $\mathcal{H}_2$ control.
- The proof requires much computation, the intuition is not clear and not linked to convexification.

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- Each paper solves one optimal control problem – not easy to generalize, not including optimal $\mathcal{H}_2$ control.
- The proof requires much computation, the intuition is not clear and not linked to convexification.

If the optimal control problem can be convexified, does policy gradient descent converge to global optimum?

# Distilling the properties of convexification

$$\begin{aligned}\min_{Z,L,G} \quad & f(L,G,Z) := \mathbf{Tr}(QG+ZR) \\ \text{s.t.,} \quad & \mathcal{A}(G)+\mathcal{B}(L)+\Omega=0, \\ & G \succ 0, \\ & \begin{bmatrix} Z & L \\ L^\top & G \end{bmatrix} \succeq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned}\min_{Z,L,G} \quad & f(L,G,Z), \\ \text{s.t.,} \quad & (L,G,Z)\in\mathcal{S}\end{aligned}$$

The following assumptions are required:

1. $\mathcal{S}$ is convex.
2. $f(L,G,Z)$ is convex (special: linear, strongly convex) on $\mathcal{S}$.

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Either of the following holds.

3. 3.1 Let $G$ be invertible, $K = LG^{-1}$ defines a bijection $K \leftrightarrow (L, G)$. For any such bijection $K \leftrightarrow (L, G)$, $\exists Z$, such that $(L, G, Z) \in \mathcal{S}$.
   3.2 $loss(K) = \min_Z\ f(L, G, Z)$ subject to $(L, G, Z) \in \mathcal{S}$.
4. We can express $loss(K)$ as:

$$\begin{aligned} loss(K) = \min_{L,G,Z}\ & f(L,G,Z)\\ \text{s.t., } & (L,G,Z) \in \mathcal{S},\ K = LG^{-1}. \end{aligned}$$

More generally, $K = LG^{-1}$ can be replaced by a general map $K = F(L, G)$ with well-defined first order derivative (handles structured LQR [Furieri, Zheng, and Kamgarpour 2020]).

# Theorem

$$\begin{aligned} \min_{Z,L,G} \quad & f(L, G, Z), \\ \text{s.t.,} \quad & (L, G, Z) \in \mathcal{S} \end{aligned}$$

**Theorem (simplified)**

Under assumptions 1,2,3 or 1,2,4, $\nabla loss(K) = 0 \iff K = K^*$. More specifically,

1. If $f$ is linear, $\|\nabla loss(K)\| \gtrsim loss(K) - loss(K^*)$.
2. If $f$ is $\mu$ strongly convex, $\|\nabla loss(K)\| \gtrsim (\mu(loss(K) - loss(K^*)))^{1/2}$.

# Minimizing $\mathcal{L}_2$ gain

- Dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), \;\; y = Cx(t) + Du(t)$$

$x$ is state, $u$ is input, $w$ is a perturbation. We hope to find the optimal state feedback controller $u(t) = K^* x(t)$ that minimizes the following loss.

- Loss/Objective:

$$loss(K) := \sup_{\|w\|_2 = 1} \|y\|_2.$$

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- Convexification

$$\min_{L,G,\gamma} \; f(L, G, \gamma) := \gamma$$

$$\text{s.t.,} \; \begin{bmatrix} AG + GA^\top + BL + L^\top B^\top + B_w B_w^\top & (CG + DL)^\top \\ CG + DL & -\gamma^2 I \end{bmatrix} \preceq 0,$$

$$\gamma \geq 0.$$

And $K^* = L^* {G^*}^{-1}$.

# Maximizing dissipativity

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Dissipativity $\eta$:

$$\int_0^T w^\top y - \eta w^\top w dt \geq 0,\ \forall T > 0.$$

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$$\max_{L,G,\eta} \; f(L,G,\eta) := \eta$$
$$\text{s.t.,} \; \begin{bmatrix} AG + GA^\top + BL + L^\top B^\top & B_w - GC^\top \\ B_w^\top - CG & 2\eta I - (D + D^\top) \end{bmatrix} \preceq 0.$$

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# System level synthesis (SLS)

(Anderson et al. 2019) Dynamics $x(t+1) = A(t)x(t) + B(t)u(t) + w(t)$.

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$$X = \begin{bmatrix} x(0) \\ ... \\ x(T) \end{bmatrix}, \ U = \begin{bmatrix} u(0) \\ ... \\ u(T) \end{bmatrix}, \ W = \begin{bmatrix} x(0) \\ w(0) \\ ... \\ w(T-1) \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & ... & 0 & 0 \\ I & 0 & ... & 0 & 0 \\ 0 & I & ... & 0 & 0 \\ ... & & & & \\ 0 & 0 & ... & I & 0 \end{bmatrix}, \ \mathcal{K} = \begin{bmatrix} K(0,0) & 0 & ... & 0 \\ K(1,1) & K(1,0) & ... & 0 \\ ... & & & \\ K(T,T) & K(T,T-1) & ... & K(T,0) \end{bmatrix},$$

$$\mathcal{A} = \mathrm{diag}(A(0), ..., A(T-1), 0), \ \mathcal{B} = \mathrm{diag}(B(0), ..., B(T-1), 0)$$

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$$\mathcal{A} = \mathrm{diag}(A(0), \dots, A(T-1), 0), \; \mathcal{B} = \mathrm{diag}(B(0), \dots, B(T-1), 0)$$

Constraint:

$$X = Z\mathcal{A}X + Z\mathcal{B}U + W = Z(\mathcal{A} + \mathcal{B}\mathcal{K})X + W$$

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We define the mapping from $W$ to $X, U$ by

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} W.$$

where $\Phi_X, \Phi_U$ are block lower triangular, $\mathcal{K} = \Phi_U \Phi_X^{-1}$. There is a constraint on $\Phi_X, \Phi_U$:

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I.$$

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The LQR loss $\mathcal{L}(\mathcal{K}) := \sum_{t=0}^{T} x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t)$
with $x(0), w(t)$ being i.i.d from $\mathcal{N}(0, \Sigma)$ is

$$f(\Phi_X, \Phi_U) = \left\| \mathrm{diag}(\mathcal{Q}^{1/2}, \mathcal{R}^{1/2}) \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} \Sigma^{1/2} \right\|_F^2.$$

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**Convex:** Minimize $f(\Phi_X, \Phi_U)$ subject to $\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I$.

Recover $\mathcal{K}$ by $\mathcal{K} = \Phi_U \Phi_X^{-1}$.

# Distilling the properties of convexification

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# Proof sketch

$$\min_{K} \quad loss(K), \qquad \text{s.t.,} \quad K \text{ stabilizes}$$

$$\Rightarrow$$

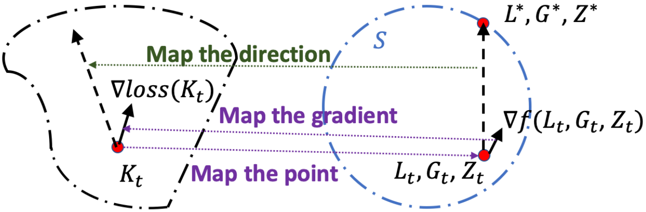
$$\min_{Z,L,G} \quad f(L,G,Z), \qquad \text{s.t.,} \quad (L,G,Z) \in \mathcal{S}$$

$$\Downarrow$$

$$\|\mathcal{P}_{\mathcal{S}} \nabla f(L,G,Z)\| \gtrsim \Phi(f(L,G,Z) - f(L^*,G^*,Z^*))$$

$$\Leftarrow$$

$$\|\nabla loss(K)\| \gtrsim \Phi(loss(K) - loss(K^*))$$



# Conclusion

- For a family of optimal control problems that can be convexified, policy gradient descent converges to the global optimum despite the nonconvexity.
- We propose a concise proof that bridges the nonconvex landscape with the convexified problems.

**Future:** Understanding the discrete optimal control problem. Beat the convergence rate with related papers on specific problems.

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