Beyond Convexification of LQR: a Generalized Framework

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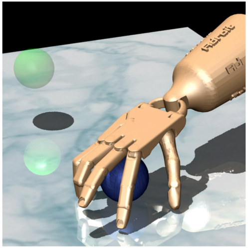
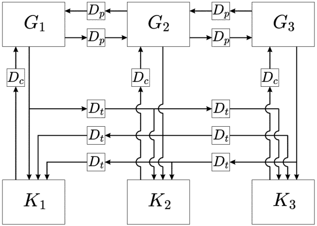
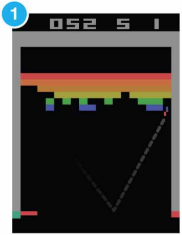
Optimal control



Robot



Decentralized control



Game

Linear quadratic regulator – Continuous

- Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0,$$

  x is state, u is input (the controller).

- Loss/Objective:

$$loss(u(t)) := \mathbf{E}_{x_0 \sim \mathcal{N}(0,\Omega)} \int_0^\infty (x(t)^\top Qx(t) + u(t)^\top Ru(t))dt$$

  State tends to 0 while the energy of input is small.

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- Optimal controller: state feedback

$$u = K^* x = -R^{-1}BPx,$$
$$AP + PA^\top + Q - PBR^{-1}BP = 0.$$

Solved by Riccati equations.

Linear quadratic regulator – Continuous

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- Optimal control

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- Convexification

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\min_{Z,L,G} \;& f(L, G, Z) := \mathbf{Tr}(QG + ZR) \\
\text{s.t., } & \mathcal{A}(G) + \mathcal{B}(L) + \Omega = 0, \; G \succ 0, \\
& \begin{bmatrix} Z & L^\top \\ L & G \end{bmatrix} \succeq 0
\end{aligned}
$$

And $K^* = L^* G^{*-1}$.

Policy gradient descent – motivation and related works

Policy gradient descent: $K^+ = K - \eta \nabla \mathit{loss}(K)$. Gradient descent on nonconvex objective

- In today's machine learning, people run gradient descent on nonconvex functions.
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Policy gradient descent, LQR, continuous: Mohammadi et al. 2019; Bu, A. Mesbahi, and M. Mesbahi 2020
Policy gradient descent, LQR, discrete: Fazel et al. 2018; Bu, A. Mesbahi, Fazel, et al. 2019
Policy gradient descent, robust LQR, continuous: Zhang, Hu, and Basar 2020
Policy gradient descent, structured finite horizon LQR, discrete: Furieri, Zheng, and Kamgarpour 2020

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Motivation of our work:

- Each paper solves one optimal control problem – not easy to generalize, not including optimal $\mathcal{H}_2$ control.
- The proof requires much computation, the intuition is not clear and not linked to convexification.

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- The proof requires much computation, the intuition is not clear and not linked to convexification.

If the optimal control problem can be convexified, does policy gradient descent converge to global optimum?

Distilling the properties of convexification

$$\begin{aligned} \min_{Z,L,G} \quad & f(L,G,Z) := \mathbf{Tr}(QG + ZR) \\ \text{s.t.,} \quad & \mathcal{A}(G) + \mathcal{B}(L) + \Omega = 0, \\ & G \succ 0, \\ & \begin{bmatrix} Z & L \\ L^\top & G \end{bmatrix} \succeq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min_{Z,L,G} \quad & f(L,G,Z), \\ \text{s.t.,} \quad & (L,G,Z) \in \mathcal{S} \end{aligned}$$

The following assumptions are required:

1. $\mathcal{S}$ is convex.
2. $f(L,G,Z)$ is convex (special: linear, strongly convex) on $\mathcal{S}$.

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Either of the following holds.

3. 3.1 Let G be invertible, $K = LG^{-1}$ defines a bijection $K \leftrightarrow (L, G)$. For any such bijection $K \leftrightarrow (L, G)$, $\exists Z$, such that $(L, G, Z) \in \mathcal{S}$.
   3.2 $loss(K) = \min_Z \; f(L, G, Z)$ subject to $(L, G, Z) \in \mathcal{S}$.
4. We can express $loss(K)$ as:

$$\begin{aligned} loss(K) = \min_{L,G,Z} \;& f(L,G,Z) \\ \text{s.t.}, \;& (L,G,Z) \in \mathcal{S}, \; K = LG^{-1}. \end{aligned}$$

More generally, $K = LG^{-1}$ can be replaced by a general map $K = F(L, G)$ with well-defined first order derivative (handles structured LQR [Furieri, Zheng, and Kamgarpour 2020]).

Theorem

$$\begin{aligned} \min_{Z,L,G} \quad & f(L, G, Z), \\ \text{s.t.,} \quad & (L, G, Z) \in \mathcal{S} \end{aligned}$$

Theorem (simplified)

Under assumptions 1,2,3 or 1,2,4, $\nabla loss(K) = 0 \;\Leftrightarrow\; K = K^*$. More specifically,

1. If f is linear, $\|\nabla loss(K)\| \gtrsim loss(K) - loss(K^*)$.
2. If f is μ strongly convex, $\|\nabla loss(K)\| \gtrsim (\mu(loss(K) - loss(K^*)))^{1/2}$.

Minimizing $\mathcal{L}_2$ gain

- Dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), \; y = Cx(t) + Du(t)$$

  x is state, u is input, w is a perturbation. We hope to find the optimal state feedback controller $u(t) = K^* x(t)$ that minimizes the following loss.

- Loss/Objective:

$$loss(K) := \sup_{\|w\|_2 = 1} \|y\|_2.$$

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- Convexification

$$\min_{L, G, \gamma} \ f(L, G, \gamma) := \gamma$$

$$\text{s.t.,} \begin{bmatrix} AG + GA^\top + BL + L^\top B^\top + B_w B_w^\top & (CG + DL)^\top \\ CG + DL & -\gamma^2 I \end{bmatrix} \preceq 0,$$

$$\gamma \geq 0.$$

And $K^* = L^* {G^*}^{-1}$.

Maximizing dissipativity

- Dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t),\ \ y = Cx(t) + Du_t + D_w w(t)$$

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Dissipativity η:

$$\int_0^T w^\top y - \eta w^\top w dt \geq 0,\ \forall T > 0.$$

Maximizing dissipativity

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$$\int_0^T w^\top y - \eta w^\top w dt \geq 0,\ \forall T > 0.$$

- Convexification

$$\max_{L,G,\eta}\ f(L, G, \eta) := \eta$$
$$\text{s.t.,}\ \begin{bmatrix} AG + GA^\top + BL + L^\top B^\top & B_w - GC^\top - (DL)^\top \\ B_w^\top - CG - DL & 2\eta I - (D_w + D_w^\top) \end{bmatrix} \preceq 0.$$

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System level synthesis (SLS)

(Anderson et al. 2019) Dynamics $x(t+1) = A(t)x(t) + B(t)u(t) + w(t)$.

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$$X = \begin{bmatrix} x(0) \\ ... \\ x(T) \end{bmatrix},\ U = \begin{bmatrix} u(0) \\ ... \\ u(T) \end{bmatrix},\ W = \begin{bmatrix} x(0) \\ w(0) \\ ... \\ w(T-1) \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & ... & 0 & 0 \\ I & 0 & ... & 0 & 0 \\ 0 & I & ... & 0 & 0 \\ ... & & & & \\ 0 & 0 & ... & I & 0 \end{bmatrix},\ \mathcal{K} = \begin{bmatrix} K(0,0) & 0 & ... & 0 \\ K(1,1) & K(1,0) & ... & 0 \\ ... & & & \\ K(T,T) & K(T,T-1) & ... & K(T,0) \end{bmatrix},$$

$$\mathcal{A} = \mathrm{diag}(A(0), ..., A(T-1), 0),\ \mathcal{B} = \mathrm{diag}(B(0), ..., B(T-1), 0)$$

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$$\mathcal{A} = \mathrm{diag}(A(0), ..., A(T-1), 0),\ \mathcal{B} = \mathrm{diag}(B(0), ..., B(T-1), 0)$$

Constraint:

$$X = Z\mathcal{A}X + Z\mathcal{B}U + W = Z(\mathcal{A} + \mathcal{B}\mathcal{K})X + W$$

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We define the mapping from W to X, U by

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} W.$$

where Φ_X, Φ_U are block lower triangular, $\mathcal{K} = \Phi_U \Phi_X^{-1}$. There is a constraint on Φ_X, Φ_U:

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I.$$

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The LQR loss $\mathcal{L}(\mathcal{K}) := \sum_{t=0}^{T} x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t)$
with $x(0), w(t)$ being i.i.d from $\mathcal{N}(0, \Sigma)$ is

$$f(\Phi_X, \Phi_U) = \left\| \mathrm{diag}(\mathcal{Q}^{1/2}, \mathcal{R}^{1/2}) \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} \Sigma^{1/2} \right\|_F^2 .$$

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Convex: Minimize $f(\Phi_X, \Phi_U)$ subject to $\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \Phi_X \\ \Phi_U \end{bmatrix} = I$.

Recover $\mathcal{K}$ by $\mathcal{K} = \Phi_U \Phi_X^{-1}$.

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Proof sketch

$$\min_{K} \quad loss(K), \qquad \text{s.t.,} \quad K \text{ stabilizes}$$

$$\Rightarrow$$

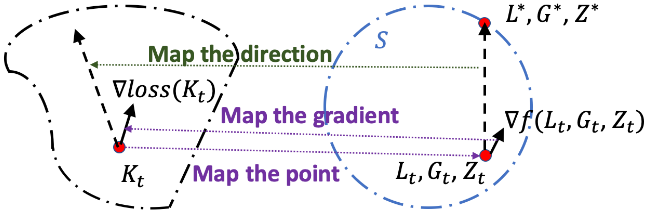
$$\min_{Z,L,G} \quad f(L,G,Z), \qquad \text{s.t.,} \quad (L,G,Z) \in \mathcal{S}$$

$$\Downarrow$$

$$\|\mathcal{P}_{\mathcal{S}} \nabla f(L,G,Z)\| \gtrsim \Phi(f(L,G,Z) - f(L^*,G^*,Z^*))$$

$$\Leftarrow$$

$$\|\nabla loss(K)\| \gtrsim \Phi(loss(K) - loss(K^*))$$



Conclusion

- For a family of optimal control problems that can be convexified, policy gradient descent converges to the global optimum despite the nonconvexity.
- We propose a concise proof that bridges the nonconvex landscape with the convexified problems.

Future: Understanding the discrete optimal control problem. Beat the convergence rate with related papers on specific problems.

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