Gradient dominance of a generalized framework from linear quadratic regulator problem

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June 13, 2020

1 Introduction

Convexification method is widely used in controller design problems. Define a continuous time linear time invariant system

$$\dot{x} = Ax + Bu,\ x(0) = x_0, \tag{1}$$

where x is state and u is input signal, x_0 comes from an initial distribution such that $\mathbf{E}(x_0 x_0^T) \succ 0$, one considers minimizing the LQR loss

$$\min_{u(t)} f(u(t)) := \mathbf{E}_{x_0} \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t)) dt \tag{2}$$

where Q, R are positive definite matrices. It is known that, the input signal that minimizes the loss function $f(u)$ is a state feedback controller

$$u = K^* x = -R^{-1} B P x, \tag{3}$$
$$A^T P + PA + Q - PBR^{-1} BP = 0. \tag{4}$$

Note that once we know the state feedback controller is static, we can write loss $f(u(t))$ as $loss(K)$ which is a function of K instead, and search only in static state feedback controllers.

One approach of finding K^* is to solve the Riccati equations (3,4) to get K^*. It is also of interest of solving (2) by iterative optimization algorithms, which enables us to finish early to get smaller computational complexity, implement in noisy case or when the system parameters A, B are inexactly measured, etc. In that case, one uses a reparametrization approach [1]:

$$loss(K) = h(X, Y) = \mathbf{trace}(QX + Y^T R Y X^{-1}). \tag{5}$$

Let $\mathcal{A}(X) = AX + XA,\ \mathcal{B}(Y) = BY + Y^T B^Y$, they have the relation

$$\mathcal{A}(X) + \mathcal{B}(Y) + \Omega = 0 \tag{6}$$

where $\Omega = \mathbf{E}(x_0 x_0^T)$. One can construct a bijection from X, Y to K, P, and prove that, if we minimize $h(X, Y)$ subject to (6), the optimizer X^*, Y^* will map to the optimal K^*, and this problem is convex, so we can solve it by optimization algorithms.

Recently nonconvex optimization algorithms are widely used in machine learning, so it is also of interest whether we can run gradient algorithm in K space without reparametrization, which means that, we run the gradient flow

$$\dot{K} = -\eta \nabla_K loss(K), \tag{7}$$

can K converge to the optimal controller K^*? [1] answers the question by studying the mapping between the originally and reparametrized spaces, and suggests that gradient flow converges to K^* in linear rate.

This work is a generalization of [1]. We extend the approach to a far more generic sets of problems which covers the original LQR. We will prove that, for the static state feedback controller design problems in the set of problems, gradient dominance always hold. Based on that, we argue that the nonconvex optimization problem in K space can be globally solved by gradient flow.

2 Main result

Theorem 1. *We consider the problems*

$$\min_{K} \quad loss(K), \tag{8a}$$

$$s.t., \quad K \text{ stabilizes} \tag{8b}$$

and

$$\min_{P,L,G} \quad f(L,G,P), \tag{9a}$$

$$s.t., \quad (L,G,P) \in \mathcal{S} \tag{9b}$$

which requires the following:

1. *L, G, P are three matrices in parametrized space. P does not have to exist, i.e., $f(L,G,P) = f(L,G)$ is allowed.*
2. *L, G, P lives in a convex feasible set $\mathcal{S}$.*
3. *Cost function $f(L,G,P)$ (or $f(L,G)$ if P does not exist) is convex.*
4. *There is a bijection between K and L, G such that $\exists P$, $(L,G,P) \in \mathcal{S}$. Specifically, $K = LG^{-1}$ where $G \succeq \lambda_0 I \succ 0$.*
5. *Cost function $loss(K) = \min_{P,(L,G,P)\in\mathcal{S}} f(L,G,P) := f(L,G,\mathscr{P}(L,G))$ when K and L, G form a bijection, and $\mathscr{P}(L,G) \in \operatorname{argmin}_{P,(L,G,P)\in\mathcal{S}} f(L,G,P)$. If P does not exist, $loss(K) = f(L,G)$. Intuitively P is some epigraph of L, G.*

Then if we solve the problem by gradient flow

$$\dot{K} = -t\mathcal{P}_{\mathcal{S}}(\nabla loss(K))/\|\mathcal{P}_{\mathcal{S}}(\nabla f(x))\|_2 \tag{10}$$

then $K(t)$ converges to the global optimizer K^*, and moreover,

1. *if f is linear, the gradient satisfies*

   $$\|\nabla loss(K)\| \geq C(loss(K) - loss(K^*)). \tag{11}$$

   for some constant C.

2. *if f is μ-strongly convex, the gradient satisfies*

   $$\|\nabla loss(K)\| \geq C(\mu(loss(K) - loss(K^*)))^{1/2}. \tag{12}$$

   for some constant C.

Remark 1. *For continuous LQR, the cost function is* $f(L,G,P) = \mathrm{Tr}(QG + PR)$*.* $\mathcal{S}$ *is intersection of* $\mathcal{A}(G) + \mathcal{B}(L) + \Omega = 0$*,* $G \succ 0$ *and* $[P, L^T; L, G] \succeq 0$*.* $K = LG^{-1}$*.*

Lemma 1. *A linear function on a convex set* $\mathcal{S}$ *is gradient dominant.*

Proof. Say the function is $f(x)$, the minimum is x^*, and $x - x^* = \Delta$. Let $\nabla f(x) = g$. For any non-stationary point, $f(x) = f(x^*) + g^T\Delta$. Since $\mathcal{S}$ is a convex set, $-\Delta$ belongs to the horizon of $\mathcal{S}$ at x, so there is a direction $\frac{\Delta}{\|\Delta\|}$ such that $f(x) - f(x - t\frac{\Delta}{\|\Delta\|}) > tg^T\frac{\Delta}{\|\Delta\|}$, $t \to 0$, so the norm of projected gradient $\|\mathcal{P}_{\mathcal{S}}(\nabla f(x))\| \geq g^T\frac{\Delta}{\|\Delta\|} = \frac{f(x)-f(x^*)}{\|x-x^*\|}$.

Proof. Denote u as any matrix in K space, $\mathcal{P}_u$ is projection of a vector onto direction u, then

$$\nabla loss(K)^T \nabla loss(K) \geq (\mathcal{P}_u \nabla loss(K))^T \mathcal{P}_u \nabla loss(K) = \left(\frac{\nabla loss(K)[u]}{\|u\|_F}\right)^2. \tag{13}$$

At current iteration K_t, let it map to (L_t, G_t) and $P_t = \mathscr{P}(L_t, G_t)$. Note f is convex, so

$$\begin{aligned} &\nabla f(P_t, L_t, G_t)[(P_t, L_t, G_t) - (P^*, L^*, G^*)] \\ &\geq f(P_t, L_t, G_t) - f(P^*, L^*, G^*) \\ &= f(\mathscr{P}(L_t, G_t), L_t, G_t) - f(\mathscr{P}(L^*, G^*), L^*, G^*) \\ &= loss(K_t) - loss(K^*). \end{aligned} \tag{14}$$

Now we consider the directional derivative in K space. By definition,

$$\nabla loss(K)[u] = \lim_{t\to 0^+} (loss(K + tu) - loss(K))/t.$$

Let $\Delta L = L^* - L_t$, $\Delta G = G^* - G_t$, and $u = \Delta L G_t^{-1} - L_t G_t^{-1} \Delta G G_t^{-1}$. Then

$$\begin{aligned} \nabla loss(K)[u] &= \lim_{t\to 0^+} (loss(K + tu) - loss(K))/t \\ &= \lim_{t\to 0^+} (loss(L_t G_t^{-1} + t(\Delta L G_t^{-1} - L_t G_t^{-1} \Delta G G_t^{-1})) - loss(L_t G_t^{-1}))/t \\ &= \lim_{t\to 0^+} (loss((L_t + t\Delta L)(G_t + t\Delta G)^{-1}) - loss(L_t G_t^{-1}))/t \\ &= \lim_{t\to 0^+} (f(L_t + t\Delta L, G_t + t\Delta G, \mathscr{P}(L_t + t\Delta L, G_t + t\Delta G)) - f(L_t, G_t, \mathscr{P}(L_t, G_t)))/t \\ &\leq \lim_{t\to 0^+} (f(L_t + t\Delta L, G_t + t\Delta G, P_t + t\Delta P) - f(L_t, G_t, \mathscr{P}(L_t, G_t)))/t \\ &= \nabla f(P_t, L_t, G_t)[(P^*, L^*, G^*) - (P_t, L_t, G_t)] \end{aligned}$$

So

$$\nabla loss(K)[-u] \geq \nabla f(P_t, L_t, G_t)[(P_t, L_t, G_t) - (P^*, L^*, G^*)] > 0.$$

Using (13) and (14), we have

$$\nabla loss(K_t)^T \nabla loss(K_t) \geq \frac{1}{\|u\|_F^2}(loss(K_t) - loss(K^*))^2$$

If $f(P, L, G)$ is μ strongly convex, then we have

$$\|\mathcal{P}_{(P_t,L_t,G_t)-(P^*,L^*,G^*)} \nabla f(P_t, L_t, G_t)\| \geq \mu^{1/2}(f(P_t, L_t, G_t) - f(P^*, L^*, G^*))^{1/2}$$

then we have that

$$
\begin{aligned}
&\nabla f(P_t, L_t, G_t)[(P_t, L_t, G_t) - (P^*, L^*, G^*)] \\
&= \|\mathcal{P}_{(P_t, L_t, G_t)-(P^*, L^*, G^*)} \nabla f(P_t, L_t, G_t)\| \cdot \|(P_t, L_t, G_t) - (P^*, L^*, G^*)\| \\
&\geq \mu^{1/2} (f(P_t, L_t, G_t) - f(P^*, L^*, G^*))^{1/2} \|(P_t, L_t, G_t) - (P^*, L^*, G^*)\|.
\end{aligned}
$$

then

$$
\begin{aligned}
\nabla loss(K_t)^T \nabla loss(K_t) &\geq \frac{1}{\|u\|_F^2} (\nabla f(P_t, L_t, G_t)[(P_t, L_t, G_t) - (P^*, L^*, G^*)])^2 \\
&\geq \frac{\mu \|(P_t, L_t, G_t) - (P^*, L^*, G^*)\|^2}{\|u\|_F^2} (f(P_t, L_t, G_t) - f(P^*, L^*, G^*)) \\
&= \frac{\mu(\|L^* - L_t\|^2 + \|G^* - G_t\|^2 + \|P^* - P_t\|^2)}{\|(L^* - L_t)G_t^{-1} - L_t G_t^{-1}(G^* - G_t)G_t^{-1}\|_F^2} (f(P_t, L_t, G_t) - f(P^*, L^*, G^*)) \\
&\geq \frac{\mu(\|L^* - L_t\|^2 + \|G^* - G_t\|^2)}{\|(L^* - L_t)G_t^{-1} - L_t G_t^{-1}(G^* - G_t)G_t^{-1}\|_F^2} (f(P_t, L_t, G_t) - f(P^*, L^*, G^*)) \\
&\geq \frac{4\mu}{(\max\{\lambda_{min}^{-1}(G_t), \lambda_{min}^{-2}(G_t)\sigma_{\max}(L_t)\})^2} (f(P_t, L_t, G_t) - f(P^*, L^*, G^*)).
\end{aligned}
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References

[1] H. Mohammadi, A. Zare, M. Soltanolkotabi, and M. R. Jovanović, "Convergence and sample complexity of gradient methods for the model-free linear quadratic regulator problem," *arXiv preprint arXiv:1912.11899*, 2019.