# Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

## Chapter 1 The Binomial No-Arbitrage Pricing Model

### Exercise 1.6

**Analysis of the exercise**

The problem is asking how a bank can hedge its long position in a European call option within a one-period binomial model. The bank owns the call option at time zero, with a time-zero price of $V_0 = 1.20$, and wants to ensure it has 1.50 at time one, regardless of the outcome of the coin toss that determines the stock price. The bank needs to decide how to invest in the stock and money markets to achieve this goal.

- The call option has a strike price $K = 5$ and expires at time one.
- The stock price at time zero is $S_0 = 4$, and at time one, it can either go up to $S_1(H) = 8$ or down to $S_1(T) = 2$.
- The bank wants to grow its capital from 1.20 to 1.50, which implies an interest rate of 25% on the capital.

**Proof**

The European call option payoff at time one depends on whether the stock price goes up or down:

- If the stock price goes up to $S_1(H) = 8$, the payoff of the call option is:

$$max(S_1(H) - K, 0) = max(8 - 5, 0) = 3$$

If the stock price goes down to $S_1(T) = 2$, the payoff of the call option is:

$$max(S_1(T) - K, 0) = max(2 - 5, 0) = 0$$

Thus, the call option pays 3 when the stock price goes up and 0 when the stock price goes down.

The goal of the bank is to ensure that after collecting the payoff from the option, it will have 1.50 at time one, regardless of the outcome. To achieve this, the bank will invest part of its capital in the stock and part in the money market. Let:

- $\Delta$ be the number of shares of stock held by the bank.
- B be the amount invested in the money market.

The initial value of the portfolio is:

$$V_0 = \Delta S_0 + B = 1.20$$

At time one, the value of the portfolio is:

- If the stock price goes up: $V_1(H) = \Delta S_1(H) + B(1 + r)$.
- If the stock price goes down: $V_1(T) = \Delta S_1(T) + B(1 + r)$.

So, the conditions at time one are:

**Up State:**

$$\Delta \times 8 + B \times 1.25 + 3 = 1.50$$

This simplifies to:

$$8\Delta + 1.25B = -1.50$$

**Down State:**

$$\Delta \times 2 + B \times 1.25 + 0 = 1.50$$

This simplifies to:

$$2\Delta + 1.25B = 1.50$$

Solve above the two simplified equations, we got:

$$\Delta = -0.5, B = 2$$

Thus, the optimal hedge strategy is:

- **Hold −0.5 shares of stock**: This means that the bank should **short sell** 0.5 shares of the stock.
- **Invest 2 in the money market**: This means that the bank should invest 2 units of capital in the money market, which will grow by 25% to 2.50 at time one.