Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.8

Proof

(i)

In a standard binomial tree model under the risk-neutral probability measure P, the value of a random variable at any time n can be represented as the conditional expectation of its value at time $n+1$. Formally, for M_n and M_n':

$$M_n = E[M_{n+1}],$$

$$M_n' = E[M_{n+1}']$$

At time $n+1$, we have:

$M_{n+1} = M_{n+1}'$for all possible coin toss sequences. This implies that the conditional expectations at time n will also be equal:

$$E[M_{n+1}] = E[M_{n+1}'],$$

Therefore, it follows that:

$$M_n = M_n'$$

By the principle of induction, starting from the initial base case at $n = N$ and moving backwards through the tree, we conclude that for each $n = 0,1, \dots, N$:

$M_n = M_n'$ for all possible coin toss sequences.

(ii)

The given sequence is defined as:

$$M_n = \frac{V_n}{(1+r)^n}, n = 0, 1, 2, \dots, N,$$

Under the risk-neutral probability measure P, the value of the derivative security at any time n is given by the expected discounted value of its future payoff. This can be mathematically expressed as:

$$V_n = E[\frac{V_{n+1}}{1+r}]$$

This equation states that the value at time n is equal to the conditional expectation of the discounted value at time $n+1$.

Thus,

$$M_n = \frac{1}{(1+r)^n} E[\frac{V_{n+1}}{1+r}]$$

Simplifying,

$$M_n = E[\frac{V_{n+1}}{(1+r)^{n+1}}]$$

Thus,

$$M_n = E[M_{n+1}].$$

The sequence {Mn} is a martingale under the risk-neutral measure P.

(iii)

We follow the same logic as in (ii).

$$V_n' = E[\frac{V_N}{(1+r)^{N-n}}]$$

The given sequence is defined as:

$$M_n = \frac{V_n'}{(1+r)^n}$$

Thus,

$$M_n = \frac{1}{(1+r)^n}[\frac{V_N}{(1+r)^{N-n}}]$$

Simplifying,

$$M_n = \frac{V_N}{(1+r)^N}$$

Now, we need to check if:

$$E[M_{n+1}] = M_n$$

For M_{n+1}, we have:

$$M_{n+1} = \frac{V_{n+1}'}{(1+r)^{n+1}}$$

Using the risk-neutral pricing formula again:

$$V_{n+1}' = E\left[\frac{V_N}{(1+r)^{N-(n+1)}}\right]$$

$$M_{n+1} = \frac{1}{(1+r)^{n+1}} E\left[\frac{V_N}{(1+r)^{N-(n+1)}}\right] = E\left[\frac{V_N}{(1+r)^N}\right]$$

Thus,

$$E(M_{n+1}) = E\left[E\left[\frac{V_N}{(1+r)^N}\right]\right] = M_n$$

The sequence {Mn} is a martingale under the risk-neutral measure P.

(iv)

To show that the recursive formula V_n and the risk-neutral formula V_n' yield the same values, we will start from the risk-neutral formula and use the Tower Property to show that it simplifies to the recursive formula.

By definition, at the final time N, we have:

$$V_n = V_n'$$

This is the starting point because both formulas depend on the terminal value V_n.

For $n = N - 1$, the risk-neutral formula is:

$$V'_{N-1} = E_{N-1}[\frac{V_N}{1+r}]$$

This is identical to the recursive formula for V_{N-1}:

$$V_{N-1} = E_{N-1}[\frac{V_N}{1+r}]$$

Thus, $V_{N-1} = V'_{N-1}$.

Suppose for some time $n+1$, we have $V_{n+1} = V'_{n+1}$. Then, using the recursive formula, the value at the preceding time n can be written as:

$$V_n = E_n[\frac{V_{n+1}}{1+r}]$$

Substituting $V_{n+1} = V'_{n+1}$, we get:

$$V_n = E_n[\frac{E_{n+1}[\frac{V_N}{(1+r)^{N-(n+1)}}]}{1+r}]$$

By applying the Tower Property:

$$V_n = E_n[\frac{V_n}{(1+r)^{N-n}}]$$

This expression is precisely the risk-neutral formula for V'_N.

Thus,

$$V_n = V'_n$$