Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.8

Analysis of the exercise

To solve this problem, we need to use the Equation 1.2.16, which is $V_n = \frac{1}{1+r}[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)]$. So, the equation of v_n will be $v_n(s,y) = \frac{1}{1+r}[\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)]$.

Where:

- $v_n(s,y)$: The value of the option at time step n, when the current stock price is s and the cumulative stock price is y.
- r: The risk-free interest rate.
- $\tilde{p}, \tilde{q}$: The risk-neutral probabilities, representing the probabilities of the stock price increasing and decreasing, respectively.
- u, d: The stock price movement factors, representing the stock price increasing by a factor of u or decreasing by a factor of d.
- $v_{n+1}(us, y+us)$: The value of the option at the next time step $n+1$, when the stock price becomes us and the cumulative stock price becomes $y+us$.
- $v_{n+1}(ds, y+ds)$: The value of the option at the next time step $n+1$, when the stock price becomes ds and the cumulative stock price becomes $y+ds$.

Proof

(i)

For n = 3, $v_3(s,y) = (\frac{1}{4}y - 4)^+$.

Thus, we have the algorithm:

$$\boldsymbol{v_n(s,y) == \frac{2}{5}\left[v_{n+1}(2s, y+2s) + v_{n+1}\left(\frac{1}{2}s, y+\frac{1}{2}s\right)\right]}$$

For n = 2,

$v_2(s,y)$

$= \frac{2}{5}\left[v_3(2s, y+2s) + v_3\left(\frac{1}{2}s, y+\frac{1}{2}s\right)\right]$

(ii)

We have $S_3(HHH) = 32$, $S_3(HHT) = 8$, $S_3(HTH) = 8$, $S_3(HTT) = 2$, $S_3(THH) = 8$, $S_3(THT) = 2$, $S_3(TTH) = 2$, and $S_3(TTT) = 0.5$. $S_2(HH) = 16$, $S_2(HT) = 4$, $S_2(TH) = 4$ and$S_2(TT) = 1$. $S_1(H) = 8$ and $S_1(T) = 2$. $S_0 = 4$.

Then applying the above algorithm, we have for n = 3:

$v_3(32,60) = (\frac{1}{4}60 - 4)^+ = 11$

where 60 is from the sum of $S_0 = 4, S_1(H) = 8, S_2(HH) = 16, S_3(HHH) = 32$. For the rest, they have the same logic.

$v_3(8,36) = (\frac{1}{4}36 - 4)^+ = 5$

$v_3(8,24) = (\frac{1}{4}24 - 4)^+ = 2$

$v_3(2,18) = (\frac{1}{4}18 - 4)^+ = 0.5$

$v_3(8,18) = (\frac{1}{4}18 - 4)^+ = 0.5$

$v_3(2,12) = (\frac{1}{4}12 - 4)^+ = 0$

$v_3(2,9) = (\frac{1}{4}9 - 4)^+ = 0$

$v_3(0.5,7.5) = (\frac{1}{4}7.5 - 4)^+ = 0$

For n = 2:

$$v_2(16, 28) = \frac{2}{5}[v_3(32, 60) + v_3(8, 36)] = \frac{2}{5}(11 + 5) = 6.4$$

$$v_2(4, 16) = \frac{2}{5}[v_3(8, 24) + v_3(2, 18)] = \frac{2}{5}(2 + 0.5) = 1$$

$$v_2(4, 10) = \frac{2}{5}[v_3(8, 18) + v_3(2, 12)] = \frac{2}{5}(0.5 + 0) = 0.2$$

$$v_2(1, 7) = \frac{2}{5}[v_3(2, 9) + v_3(0.5, 7.5)] = \frac{2}{5}(0 + 0) = 0$$

For n = 1:

$$v_1(8, 12) = \frac{2}{5}[v_2(16, 18) + v_2(4, 16)] = \frac{2}{5}(6.4 + 1) = 2.96$$

$$v_1(2, 6) = \frac{2}{5}[v_2(4, 10) + v_2(1, 7)] = \frac{2}{5}(0.2 + 0) = 0.08$$

$$\boldsymbol{v_0(4, 4) = \frac{2}{5}[v_2(8, 12) + v_2(2, 6)] = \frac{2}{5}(2.96 + 0.08) = 1.216}$$

(iii)

According to the Equation 1.2.17, which is $\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}$.

Then

$$\boldsymbol{\Delta_n(s, y) = \frac{v_{n+1}(us, y + us) - v_{n+1}(ds, y + ds)}{us - ds} = \frac{v_{n+1}(2s, y + 2s) - v_{n+1}\left(\frac{1}{2}s, y + \frac{1}{2}s\right)}{\frac{3}{2}s}}$$