Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.9

Proof

(i)

To determine the price of the derivative at time 0, we use backward induction.

At time N, the value of the derivative is known, and it is simply $V_N(\omega_1, \dots, \omega_N)$.

At time $n = N - 1$, the value of the derivative will be the expected value of the payoff at time N, discounted by the random interest rate $r_{N-1}(\omega_1, \dots, \omega_{N-1})$.

Let V_N represent the value of the derivative at time n, given the outcomes of the coin tosses up to time n. Then:

$$V_{N-1}(\omega_1, \dots, \omega_{N-1}) = \frac{1}{1 + r_{N-1}(\omega_1, \dots, \omega_{N-1})} [\tilde{p} V_N(\omega_1, \dots, \omega_{N-1}, H) + \tilde{q} V_N(\omega_1, \dots, \omega_{N-1}, T)]$$

Continuing this process back to time 0, the price of the derivative at time 0 is given by:

$$V_0 = \frac{1}{1 + r_0} [\tilde{p} V_1(H) + \tilde{q} V_1(T)]$$

We can generalize the backward induction process. At each node, the value of the derivative is given by the risk-neutral expected value of the derivative at the next time step, discounted by the appropriate random interest rate.

Thus, the algorithm for determining the price of the derivative at time N is as follows:

$$V_N(\omega_1, \dots, \omega_N) = \frac{1}{1 + r_N(\omega_1, \dots, \omega_N)} [\tilde{p}V_{N+1}(\omega_1, \dots, \omega_N, H) + \tilde{q}V_{N+1}(\omega_1, \dots, \omega_N, T)]$$

(ii)

According to the wealth process of the replicating portfolio, we have:

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}$$

(iii)

According to the no-arbitrage condition, the expected return on the stock should equal the risk-free rate. Thus, the stock price at time n should be equal to the expected value of the discounted future stock price:

$$S_n(\omega) = \frac{E_{RN}[S_{n+1}]}{1 + r_n(\omega)}$$

where:

- $S_n(\omega)$ is the stock price at time n,
- $E_{RN}[S_{n+1}]$ is the expected stock price at time n+1n+1n+1 under the risk-neutral probability,
- $r_n(\omega)$ is the risk-free interest rate at time n.

We express the expected value $E_{RN}[S_{n+1}]$ using the up and down moves in the binomial tree:

$$E_{RN}[S_{n+1}] = \tilde{p}_n(\omega)S_{n+1}(\omega H) + \tilde{q}_n(\omega)S_{n+1}(\omega T)$$

Using the no-arbitrage condition, we have:

$$S_n(\omega) = \frac{\tilde{p}_n(\omega)S_{n+1}(\omega H) + \tilde{q}_n(\omega)S_{n+1}(\omega T)}{1 + r_n(\omega)}$$

Since $\tilde{q}_n(\omega) = 1 - \tilde{p}_n(\omega)$, we can rewrite the equation as:

$$S_n(\omega) = \frac{\tilde{p}_n(\omega)S_{n+1}(\omega H) + [1 - \tilde{p}_n(\omega)]S_{n+1}(\omega T)}{1 + r_n(\omega)}$$

Thus, we have:

$$\tilde{p}_n(\omega) = \frac{\big(1 + r_n(\omega)\big)S_n(\omega) - S_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)}$$

Given this special setting, we have:

$\tilde{p}_n(\omega)$

$= \dfrac{(1 + r_n(\omega))S_n(\omega) - S_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)}$

$= \dfrac{(1 + 0)S_n(\omega) - [S_n(\omega) - 10]}{[S_n(\omega) - 10] - [S_n(\omega) - 10]}$

$= \dfrac{1}{2}$

$\tilde{q}_n(\omega) = 1 - \tilde{p}_n(\omega) = \dfrac{1}{2}$

According to the Equation 1.2.16$V_n = \frac{1}{1+r}[\,\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)]$., we have

$v_n(s)$

$= \dfrac{1}{1+r}[\tilde{p}v_{n+1}(s + 10) + \tilde{q}v_{n+1}(s - 10)]$

$= \dfrac{1}{2}v_{n+1}(s + 10) + \dfrac{1}{2}v_{n+1}(s - 10)$

For $n = 0,1,2, \dots, N - 1$, where $v_N(s) = (s - K)^+$.

Thus, when $t = 5$,

$v_5(130) = (130 - 80)^+ = 50$

$v_5(110) = (110 - 80)^+ = 30$

$v_5(90) = (90 - 80)^+ = 10$

$v_5(70) = (70 - 80)^+ = 0$

$v_5(50) = (50 - 80)^+ = 0$

$v_5(30) = (30 - 80)^+ = 0$

When $t = 4$,

$v_4(120) = \frac{1}{2}v_5(130) + \frac{1}{2}v_5(110) = \frac{1}{2}(50 + 30) = 40$

$v_4(100) = \frac{1}{2}v_5(110) + \frac{1}{2}v_5(90) = \frac{1}{2}(30 + 10) = 20$

$v_4(80) = \frac{1}{2}v_5(90) + \frac{1}{2}v_5(70) = \frac{1}{2}(10 + 0) = 5$

$v_4(60) = \frac{1}{2}v_5(70) + \frac{1}{2}v_5(50) = \frac{1}{2}(0 + 0) = 0$

$v_4(40) = \frac{1}{2}v_5(50) + \frac{1}{2}v_5(30) = \frac{1}{2}(0 + 0) = 0$

When $t = 3$,

$v_3(110) = \frac{1}{2}v_4(120) + \frac{1}{2}v_4(100) = \frac{1}{2}(40 + 20) = 30$

$v_3(90) = \frac{1}{2}v_4(100) + \frac{1}{2}v_4(80) = \frac{1}{2}(20 + 5) = 12.5$

$v_3(70) = \frac{1}{2}v_4(80) + \frac{1}{2}v_4(60) = \frac{1}{2}(5 + 0) = 2.5$

$v_3(50) = \frac{1}{2}v_4(60) + \frac{1}{2}v_4(40) = \frac{1}{2}(0 + 0) = 0$

When $t = 2$,

$v_2(100) = \frac{1}{2}v_3(110) + \frac{1}{2}v_3(90) = \frac{1}{2}(30 + 12.5) = 21.25$

$v_2(80) = \frac{1}{2}v_3(90) + \frac{1}{2}v_3(70) = \frac{1}{2}(12.5 + 2.5) = 7.5$

$v_2(60) = \frac{1}{2}v_3(70) + \frac{1}{2}v_3(50) = \frac{1}{2}(2.5 + 0) = 1.25$

When $t = 1$,

$v_1(90) = \frac{1}{2}v_2(100) + \frac{1}{2}v_2(80) = \frac{1}{2}(21.25 + 7.5) = 14.375$

$v_1(70) = \frac{1}{2}v_2(80) + \frac{1}{2}v_2(60) = \frac{1}{2}(7.5 + 1.25) = 4.375$

When $t = 0$,

$\boldsymbol{v_0(80) = \frac{1}{2}v_1(90) + \frac{1}{2}v_1(70) = \frac{1}{2}(14.375 + 4.375) = 9.375}$