Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.9

Proof

(i)

$u_0 = \frac{S_1(H)}{S_0} = 2, d_0 = \frac{S_1(T)}{S_0} = \frac{1}{2},$

$u_1(H) = \frac{S_2(HH)}{S_1(H)} = \frac{3}{2}, d_1(H) = \frac{S_2(HT)}{S_1(H)} = 1,$

$u_1(T) = \frac{S_2(TH)}{S_1(T)} = 4, d_1(T) = \frac{S_2(TT)}{S_1(T)} = 1,$

So,

$\tilde{p}_0 = \frac{1+r_0-d_0}{u_0-d_0} = \frac{1}{2}, \tilde{q}_0 = \frac{1}{2},$

$\tilde{p}_1(H) = \frac{1+r_1(H)-d_1(H)}{u_1(H)-d_1(H)} = \frac{1}{2}, \tilde{q}_1(H) = \frac{1}{2},$

$\tilde{p}_1(T) = \frac{1+r_1(T)-d_1(T)}{u_1(T)-d_1(T)} = \frac{1}{6}, \tilde{q}_1(T) = \frac{5}{6},$

Therefore,

$\tilde{p}(HH) = \tilde{p}_0\tilde{p}_1(H) = \frac{1}{4}, \tilde{p}(HT) = \tilde{p}_0\tilde{q}_1(H) = \frac{1}{4},$

$\tilde{p}(TH) = \tilde{q}_0\tilde{p}_1(T) = \frac{1}{12}$, $\tilde{p}(TT) = \tilde{q}_0\tilde{q}_1(T) = \frac{5}{12}$,

Confirm that probabilities sum to 1. These probabilities ensure that the expected discounted stock prices equal current prices at each node, satisfying no-arbitrage conditions and the time-zero value V_0 of an option paying V_2 at time 2 is given by the formula from (i).

(ii)

$V_2(HH) = 5, V_2(HT) = 1, V_2(TH) = 1, V_2(TT) = 0.$

So,

$$V_1(H) = \frac{\tilde{p}_1(H)V_2(HH) + \tilde{q}_1(H)V_2(HT)}{1 + r_1(H)} = \frac{12}{5}$$

$$V_1(T) = \frac{\tilde{p}_1(T)V_2(TH) + \tilde{q}_1(T)V_2(TT)}{1 + r_1(T)} = \frac{1}{9}$$

$$V_0 = \frac{\tilde{p}_0 V_1(H) + \tilde{q}_0 V_1(T)}{1 + r_0} = \frac{226}{225}$$

(iii)

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{\frac{12}{5} - \frac{1}{9}}{8 - 2} = \frac{103}{270}$$

(iv)

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{5 - 1}{12 - 8} = 4$$