Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.10

Proof

(i)

According to the theorem 2.4.5, we need to prove that $\frac{X_n}{(1+r)^n} = \tilde{E}_n[\frac{X_{n+1}}{(1+r)^{n+1}}]$ when $X_{n+1} = \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)$.

Therefore,

$$\tilde{E}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right]$$

$$= \tilde{E}_n\left[\frac{\Delta_n Y_{n+1} S_n}{(1+r)^{n+1}} + \frac{(1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}\right]$$

$$= \frac{\Delta_n S_n}{(1+r)^{n+1}} \tilde{E}_n[Y_{n+1}] + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$=\frac{\Delta_n S_n}{(1+r)^{n+1}}(u\tilde{p} + d\tilde{q}) + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$=\frac{\Delta_n S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$=\frac{\Delta_n S_n + X_n - \Delta_n S_n}{(1+r)^n}$$

$=\frac{X_n}{(1+r)^n}$

Additionally,

$u\tilde{p} + d\tilde{q} = u\frac{1+r-d}{u-d} + d\frac{u-1-r}{u-d} = 1 + r.$

(ii)

We aim to show that the risk-neutral pricing formula (Theorem 2.4.7) still holds in the dividend-paying stock model. Specifically, for a derivative security paying off V_N at time N, we need to show that its price at time n is given by:

$$V_N = (1+r)^{-(N-n)}E[V_N]$$

Even with the presence of dividends, the binomial model remains complete because at each time step, there are two possible outcomes, and we have one risky asset (the stock) and one risk-free asset (the bond). Therefore, any contingent claim V_N can be perfectly replicated by a self-financing trading strategy $\{\varDelta_n\}$.

In the dividend-paying stock model, the wealth process X_n of an agent who follows a self-financing trading strategy $\{\varDelta_n\}$ satisfies:

$$X_{n+1} = \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)$$

From part (i) of the exercise, we know that the discounted wealth process $\frac{X_n}{(1+r)^n}$ is a martingale under the risk-neutral measure. This means:

$$\tilde{E}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] = \frac{X_n}{(1+r)^n}$$

Consider a self-financing trading strategy $\{\varDelta_n\}$ that replicates the derivative security V_N, meaning $X_N = V_N$. Since the discounted wealth process is a martingale, we have:

$$\frac{X_n}{(1+r)^n} = \tilde{E}_n\left[\frac{X_N}{(1+r)^N}\right] = \tilde{E}_n\left[\frac{V_N}{(1+r)^N}\right]$$

Multiplying both sides by $(1+r)^n$, we obtain the price $V_N = X_N$ of the derivative security at time n:

$$V_N = (1+r)^n\tilde{E}_n\left[\frac{V_N}{(1+r)^N}\right] = (1+r)^{-(N-n)}E[V_N]$$

(iii)

$$\tilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]$$

$$=\tilde{E}_n\left[\frac{(1-A_{n+1})Y_{n+1}S_n}{(1+r)^{n+1}}\right]$$

$$=\frac{S_n}{(1+r)^{n+1}}(u\tilde{p}+d\tilde{q})(1-A_{n+1})$$

$$\neq\frac{S_n}{(1+r)^n}$$

If A_{n+1} is a constant α, then

$$\tilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]=\frac{S_n}{(1+r)^{n+1}}(u\tilde{p}+d\tilde{q})(1-\alpha)=\frac{S_n}{(1+r)^n}(1-\alpha)$$

So,

$$\tilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}(1+\alpha)^{n+1}}\right]=\frac{S_n}{(1+r)^n(1+\alpha)^n}$$