Stochastic Calculus for Finance I, Solution for Exercises

by Sun Yufei

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.5

Proof

(i)

$2I_n$

$$= 2\sum_{j=0}^{n-1} M_j\,(M_{j+1} - M_j)$$

$$= 2\sum_{j=0}^{n-1} M_j\,M_{j+1} - 2\sum_{j=0}^{n-1} M_j\,M_j$$

$$= 2\sum_{j=0}^{n-1} M_j\,M_{j+1} - \sum_{j=0}^{n-1} M_j\,M_j - \sum_{j=0}^{n-1} M_j\,M_j$$

$$= 2\sum_{j=0}^{n-1} M_j\,M_{j+1} + {M_n}^2 - \sum_{j=0}^{n-1} M_{j+1}\,M_{j+1} - \sum_{j=0}^{n-1} M_j\,M_j$$

$={M_n}^2 - \sum_{j=0}^{n-1}(M_{j+1} - M_j)^2$

$=M_n{}^2 - \sum_{j=0}^{n-1} X_{j+1}{}^2$

$=M_n{}^2 - n$

(ii)

We start by calculating the conditional expectation of $f(I_{n+1})$given I_n:

$$E_n[f(I_{n+1})] = E_n[f(I_n + M_n(M_{n+1} - M_n))]$$

Since $M_{n+1} - M_n = X_{n+1}$ (where X_{n+1} is the increment in the random walk), we can rewrite the equation as:

$$E_n[f(I_{n+1})] = E_n[f(I_n + M_n X_{n+1})]$$

The increment X_{n+1} takes values ± 1 with equal probability $\frac{1}{2}$, so we can split the expectation into two cases based on the possible values of X_{n+1}:

$$E_n[f(I_{n+1})] = \frac{1}{2}[f(I_n + M_n) + f(I_n - M_n)]$$

The expectation simplifies to a function $g(I_n)$, which is a symmetric average of the two possibilities:

$$g(I_n) = \frac{1}{2}[f(I_n + M_n) + f(I_n - M_n)]$$

This shows that the conditional expectation of $f(I_{n+1})$ given I_n is a function of I_n, making the process $\{I_n\}$ a Markov process.

From the earlier part of the problem, we have the relationship:

$$M_n{}^2 = 2I_n + n$$

so:

$$M_n = \pm\sqrt{2I_n + n}$$

Thus, the function $g(x)$ becomes:

$$g(x) = \frac{1}{2}\left[f\left(x + \sqrt{2I_n + n}\right) + f\left(x - \sqrt{2I_n + n}\right)\right]$$

This formula encapsulates the dependence of $g(I_n)$ on I_n and demonstrates that $E_n[f(I_{n+1})]$ can be written purely in terms of I_n.