Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.1

Analysis of the exercise

The problem requires you to prove that if the portfolio X_1 has a strictly positive probability of being positive in one scenario, then it must have a possibility of being negative in another scenario.

In other words, if you can make money in one situation (i.e., $X_1 > 0$), then to avoid arbitrage opportunities, you must also have a possibility of losing money in another situation (i.e., $X_1 < 0$).

Proof

Given $X_0 = 0$, we have:

$$X_1 = \Delta_0(S_1 - (1+r)S_0)$$

where S_0 is the asset price at time 0, S_1 is the asset price at time 1, Δ_0 is the amount of the asset held at time 0, and r is the risk-free interest rate.

In a single-period binomial model, assume S_1 can take two possible values, S_1^H and S_1^T, corresponding to an 'up' state (Head, H) and a 'down' state (Tail, T). Specifically, we have:

$$S_1^H = uS_0 \text{ and } S_1^T = dS_0$$

where $u > 1 + r > d$, meaning the 'up' state results in the asset price exceeding the risk-free growth, while the 'down' state results in the asset price being lower than the risk-free growth.

Thus, the value of X_1 in these two cases is:

$$X_1^H = \varDelta_0(uS_0 - (1 + r)S_0) = \varDelta_0 S_0(u - (1 + r))$$

$$X_1^T = \varDelta_0(dS_0 - (1 + r)S_0) = \varDelta_0 S_0(d - (1 + r))$$

Here, X_1^H corresponds to the 'up' state of the market, and X_1^T corresponds to the 'down' state of the market.

If $\varDelta_0 > 0$ (long position), then:

- $X_1^H > 0$ if and only if $u > 1 + r$;
- $X_1^T < 0$ because $d < 1 + r$.

If $\varDelta_0 > 0$ (short position), then:

- $X_1^H < 0$ because $u > 1 + r$;
- $X_1^T > 0$ if and only if $d < 1 + r$.

Note that regardless of whether $\varDelta_0$ is positive or negative, X_1^H and X_1^T cannot both be positive at the same time.

According to the no-arbitrage condition, if X_1 is strictly positive in one scenario, then it must be negative in the other scenario. This means:

- If $X_1^H > 0$ occurs with positive probability, then $X_1^T < 0$ must also occur with positive probability;
- Similarly, if $X_1^T > 0$ occurs with positive probability, then $X_1^H < 0$ must also occur with positive probability.

Otherwise, if X_1 is always non-negative and has a possibility of being strictly positive, this would create an arbitrage opportunity, violating the no-arbitrage assumption.

Therefore, we have proven that unless X_1 has the possibility of being negative, X_1 cannot be strictly positive with positive probability.