Stochastic Calculus for Finance I, Solution for Exercises

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Last updated: August 16, 2024

This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.7

Analysis of the exercise

The bank's goal is to achieve a return of $\frac{5}{4}^3 \times 1.376 = 2.6875$ at time 3 regardless if the stock price goes up or down. The return is denoted by R, then we should have $R_3(HHH) = R_3(HHT) = R_3(HTH) = R_3(HTT) = R_3(THH) = R_3(THT) = R_3(TTH) = R_3(TTT) = 2.6875$

This means the bank should achieve a return of $\frac{5}{4}^2 \times 1.376 = 2.15$ at time 2 regardless if the stock price goes up or down. Then we should have $R_2(HH) = R_2(HT) = R_2(TH) = R_2(TT) = 2.15$

Also, the bank should achieve a return of $\frac{5}{4}^1 \times 1.376 = 1.72$ at time 1 regardless if the stock price goes up or down. Then we should have $R_1(H) = R_1(T) = 1.72$

Our final goal is to calculate the allocation of funds between the stock market and the currency market at each moment. Let:

• Δ be the number of shares of stock held by the bank.

• B be the amount invested in the money market.

Proof

At time 1, there are two states of stock price, up ($S_1(H)$=8) and down ($S_1(T) = 2$). Also, we have two values of option price, $V_1(H) = 2.24$ and $V_1(T) = 1.2$.

Up state:

$R_1(H)$

$= \Delta_0 S_1(H) + B_0(1 + r) + V_1(H)$

$= \Delta_0 8 + B_0(1 + 0.25) + 2.24$

$= 1.72$

Down state:

$R_1(T)$

$= \Delta_0 S_1(T) + B_0(1 + r) + V_1(T)$

$= \Delta_0 2 + B_0(1 + 0.25) + 1.2$

$= 1.72$

Then we have: $\boldsymbol{\Delta_0} = \mathbf{-0.1733}, \boldsymbol{B_0} = \mathbf{0.6933}$. Which means we should short sell 0.1733 share of stock, invest 0.6933 into money market account, to insure that at time 1 the return will be 1.72.

At time 2, there are four states of stock price, $S_2(HH) = 16$, $S_2(HT) = 4$, $S_2(TH) = 4$ and $S_2(TT) = 1$. Also, we have two values of option price, $V_2(HH) = 3.2$, $V_2(HT) = 2.4$, $V_2(TH) = 0.8$ and $V_2(TT) = 2.2$.

$R_2(HH)$

$= \Delta_1(H) S_2(HH) + B_1(1 + r) + V_2(HH)$

$= \Delta_1(H)16 + B_1(1 + 0.25) + 3.2$

$= 2.15$

$R_2(HT)$

$= \Delta_1(H) S_2(HT) + B_1(1 + r) + V_2(HT)$

$= \Delta_1(H)4 + B_1(1 + 0.25) + 2.4$

$= 2.15$

Then we have: $\boldsymbol{\Delta_1(H) = -0.0667, B_1 = \ 0.0133}$. Which means we should have short position with 0.0667 share of stock, cash investment position with 0.0133 into money market account, to insure that at time 2 the return will be 2.15.

$R_2(TH)$

$= \Delta_1(T)S_2(TH) + B_1(1 + r) + \ V_2(TH)$

$= \Delta_1(T)4 + B_1(1 + 0.25) + \ 0.8$

$= 2.15$

$R_2(TT)$

$= \Delta_1(T)S_2(TT) + B_1(1 + r) + \ V_2(TT)$

$= \Delta_1(T)1 + B_1(1 + 0.25) + \ 2.2$

$= 2.15$

Then we have: $\boldsymbol{\Delta_1(T) = 0.4667, B_1 = \ 0.4133}$. Which means we should have long position with 0.4667 share of stock, cash investment position with 0.4133 into money market account, to insure that at time 2 the return will be 2.15.

At time 3, there are eight states of stock price, $S_3(HHH) = 32$, $S_3(HHT) = 8$, $S_3(HTH) = 8$, $S_3(HTT) = 2$, $S_3(THH) = 8$, $S_3(THT) = 2$, $S_3(TTH) = 2$, and $S_3(TTT) = 0.5$.

Also, we have two values of option price, $V_3(HHH) = 0$, $V_3(HHT) = 8$, $V_3(HTH) = 0$, $V_3(HTT) = 6$, $V_3(THH) = 0$, $V_3(THT) = 2$, $V_3(TTH) = 2$ and $V_3(TTT) = 3.5$.

$R_3(HHH)$

$= \Delta_2(HH)S_3(HHH) + B_2(1 + r) + \ V_3(HHH)$

$= \Delta_2(HH)32 + B_2(1 + 0.25) + \ 0$

$= 2.6875$

$R_3(HHT)$

$= \Delta_2(HH)S_3(HHT) + B_2(1 + r) + \ V_3(HHT)$

$= \Delta_2(HH)8 + B_2(1 + 0.25) + \ 8$

$= 2.6875$

Then we have: $\Delta_{\mathbf{2}}(\boldsymbol{HH}) = \mathbf{0.3333}, \boldsymbol{B_1} = \mathbf{-6.3833}$. Which means we should have long position with 0.3333 share of stock, cash investment position with -6.3833 into money market account, to insure that at time 3 the return will be 2.6875.

$R_3(HTH)$

$= \Delta_2(HT)S_3(HTH) + B_2(1+r) + V_3(HTH)$

$= \Delta_2(HT)8 + B_2(1+0.25) + 0$

$= 2.6875$

$R_3(HTT)$

$= \Delta_2(HT)S_3(HTT) + B_2(1+r) + V_3(HTT)$

$= \Delta_2(HT)2 + B_2(1+0.25) + 6$

$= 2.6875$

Then we have: $\Delta_{\mathbf{2}}(\boldsymbol{HT}) = \mathbf{1}, \boldsymbol{B_1} = \mathbf{-4.25}$. Which means we should have long position with 1 share of stock, cash investment position with -4.25 into money market account, to insure that at time 3 the return will be 2.6875.

$R_3(THH)$

$= \Delta_2(TH)S_3(THH) + B_2(1+r) + V_3(THH)$

$= \Delta_2(TH)8 + B_2(1+0.25) + 0$

$= 2.6875$

$R_3(THT)$

$= \Delta_2(TH)S_3(THT) + B_2(1+r) + V_3(THT)$

$= \Delta_2(TH)2 + B_2(1+0.25) + 2$

$= 2.6875$

Then we have: $\Delta_{\mathbf{2}}(\boldsymbol{TH}) = \mathbf{0.3333}, \boldsymbol{B_1} = \mathbf{0.0167}$. Which means we should have long position with 0.3333 share of stock, cash investment position with 0.0167 into money market account, to insure that at time 3 the return will be 2.6875.

$R_3(TTH)$

$= \Delta_2(TT)S_3(TTH) + B_2(1+r) + V_3(TTH)$

$= \Delta_2(TT)2 + B_2(1+r) + 2$

$= 2.6875$

$R_3(TTT)$

$= \Delta_2(TT)S_3(TTT) + B_2(1+r) + V_3(TTT)$

$= \Delta_2(TT)0.5 + B_2(1+r) + 3.5$

$= 2.6875$

Then we have: $\boldsymbol{\Delta_2(TT) = 1, B_1 = -1.05}$. Which means we should have long position with 1 share of stock, cash investment position with -1.05 into money market account, to insure that at time 3 the return will be 2.6875.