Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.4

Analysis of the exercise

To solve this exercise, we need to collect all the formulas that given and substitute into the Equation (1.2.14).

Given:

1. $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ from Equation (1.2.14).

2. $\tilde{p} = \frac{1+r-d}{u-d}$, $\tilde{q} = \frac{u-(1+r)}{u-d}$ from Equation (1.2.15)

3. $V_n = \frac{1}{1+r}[\,\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)]$ from Equation (1.2.16)

4. $\Delta_n = \frac{V_{n+1}(H)-V_{n+1}(T)}{S_{n+1}(H)-S_{n+1}(T)}$ from Equation (1.2.17)

5. $X_n = V_n$ from Equation (1.2.18)

Proof

Note that the asset may go up or down at time N+1.

- **Upward Move**: The stock price increases to $S_{n+1}(H) = uS_n$, where $u > 1$ represents the up factor.

- **Downward Move**: The stock price decreases to $S_{n+1}(T) = dS_n$, where $0 < d < 1$ represents the down factor.

Let's consider the upward move:

$X_{n+1}(H)$

$= \Delta_n S_{n+1}(H) + (1 + r)(X_n - \Delta_n S_n)$

$= \Delta_n S_{n+1}(H) + (1 + r)(V_n - \Delta_n S_n)$ (substitute into 1.2.18)

$= \Delta_n u S_n + (1 + r)(V_n - \Delta_n S_n)$ $(S_{n+1}(H) = uS_n)$

$= \Delta_n (u - 1 - r)S_n + (1 + r)V_n$

$= \frac{V_{n+1}(H)-V_{n+1}(T)}{S_{n+1}(H)-S_{n+1}(T)}(u - 1 - r)S_n + (1 + r)V_n$ (substitute into 1.2.17)

$= \frac{V_{n+1}(H)-V_{n+1}(T)}{uS_n-dS_n}(u - 1 - r)S_n + (1 + r)V_n$ $(S_{n+1}(H) = uS_n, S_{n+1}(T) = dS_n)$

$= \frac{V_{n+1}(H) - V_{n+1}(T)}{u - d}(u - 1 - r) + (1 + r)V_n$

$= \tilde{q}[V_{n+1}(H) - V_{n+1}(T)] + (1 + r)V_n$ (substitute into 1.2.15)

$= \tilde{q}[V_{n+1}(H) - V_{n+1}(T)] + (1 + r)\frac{1}{1+r}[\,\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)]$ (substitute into 1.2.16)

$= \tilde{q}[V_{n+1}(H) - V_{n+1}(T)] + [\,\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)]$

$= \tilde{q}V_{n+1}(H) + \tilde{p}V_{n+1}(H)$

$= V_{n+1}(H)$

Now Let's consider the downward move:

$X_{n+1}(T)$

$= \Delta_n S_{n+1}(T) + (1 + r)(X_n - \Delta_n S_n)$

$= \Delta_n S_{n+1}(T) + (1 + r)(V_n - \Delta_n S_n)$ (substitute into 1.2.18)

$= \Delta_n d S_n + (1 + r)(V_n - \Delta_n S_n)$ $(S_{n+1}(H) = uS_n)$

$= \Delta_n (d - 1 - r)S_n + (1 + r)V_n$

$= \frac{V_{n+1}(H)-V_{n+1}(T)}{S_{n+1}(H)-S_{n+1}(T)}(d - 1 - r)S_n + (1 + r)V_n$ (substitute into 1.2.17)

$= \frac{V_{n+1}(H)-V_{n+1}(T)}{uS_n-dS_n}(d - 1 - r)S_n + (1 + r)V_n$ $(S_{n+1}(H) = uS_n, S_{n+1}(T) = dS_n)$

$$= \frac{V_{n+1}(H) - V_{n+1}(T)}{u - d}(d - 1 - r) + (1 + r)V_n$$

$$= -\widetilde{p}[V_{n+1}(H) - V_{n+1}(T)] + (1 + r)V_n$$ (substitute into 1.2.15)

$$= -\widetilde{p}[V_{n+1}(H) - V_{n+1}(T)] + (1 + r)\frac{1}{1+r}[\,\widetilde{p}V_{n+1}(H) + \widetilde{q}V_{n+1}(T)]$$ (substitute into 1.2.16)

$$= -\widetilde{p}[V_{n+1}(H) - V_{n+1}(T)] + [\,\widetilde{p}V_{n+1}(H) + \widetilde{q}V_{n+1}(T)]$$

$$= \widetilde{p}V_{n+1}(T) + \widetilde{q}V_{n+1}(T)$$

$$= V_{n+1}(T)$$

After finish the proof, I deeply feel the beauty of math.