Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.3

Analysis of the exercise

In this scenario, we are using a one-period binomial model to price a derivative security that pays off at time 1 an amount equal to the stock price at that time, i.e., $V_1 = S_1$. This derivative can be considered as a European call option with a strike price $K = 0$, because at time 1, the holder of the option can "buy" the stock for nothing, effectively receiving the stock's value as the payoff.

Proof

In the one-period binomial model, the stock price at time 1, S_1, can take one of two possible values:

- **Upward Move**: The stock price increases to $S_u = u \times S_0$, where $u > 1$ represents the up factor.
- **Downward Move**: The stock price decreases to $S_d = d \times S_0$, where $0 < d < 1$ represents the down factor.

Since $V_1 = S_1$, the value of the derivative at time 1 is the same as the stock price:

- If the stock price goes up, $V_1(H) = S_u$.
- If the stock price goes down, $V_1(T) = S_d$.

The risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$ (where $\tilde{q} = 1 - \tilde{p}$) are defined as:

$$\tilde{p} = \frac{1 + r - d}{u - d}$$

$$\tilde{q} = \frac{u-(1+r)}{u-d}$$

These probabilities are not the actual probabilities but are used under the risk-neutral measure to compute the expected value of future cash flows.

The price of the derivative at time 0, V_0, is given by the risk-neutral pricing formula:

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}\, V_1(T)] \tag{1.1.10}$$

Substituting $V_1(H)$ and $V_1(T)$ into the formula 1.1.10:

$$V_0 = \frac{1}{1+r}[\tilde{p}S_u + \tilde{q}S_d]$$

Expanding further with the formulas $\tilde{p}$ and $\tilde{q}$:

$$V_0 = \frac{1}{1+r}[\frac{1+r-d}{u-d}S_u + \frac{u-(1+r)}{u-d}S_d]$$

Substituting $S_u = u \times S_0$ and $S_d = d \times S_0$ into the above formula:

$$V_0 = \frac{S_0}{1+r}\left[\frac{1+r-d}{u-d}u + \frac{u-(1+r)}{u-d}d\right] = S_0$$

This result confirms that the derivative, which pays the future stock price, is valued at the current stock price under the risk-neutral measure. The equality $V_0 = S_0$ is a fundamental result in the binomial model, showing that the value of a derivative that mirrors the underlying asset is equal to the value of the asset itself.