Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.1

Proof

(i)

According to the Definition 2.1.1,

$$\sum_{\omega \epsilon \Omega} P(\omega) = 1$$

$$P(A) = \sum_{\omega \epsilon A} P(\omega)$$

Thus,

$$P(A) + P(A^c) = \sum_{\omega \epsilon A} P(\omega) + \sum_{\omega \epsilon A^c} P(\omega)$$

By definition, the complement of an event A, denoted as A^c, is the set of outcomes that are not in A. In other words:

$$A \cup A^c = \Omega$$

Since A and A^c are disjoint and their union is the entire sample space Ω:

$$P(A \cup A^c) = P(A) + P(A^c) = 1$$

we can combine the two sums with Definition 2.1.1 formulas:

$$\sum_{\omega \epsilon A} P(\omega) + \sum_{\omega \epsilon A^C} P(\omega) = \sum_{\omega \epsilon \Omega} P(\omega) = 1$$

Thus,

$$P(A) + P(A^c) = 1$$

(ii)

This inequality is an application of the union bound (or Boole's inequality), which states that the probability of the union of events is at most the sum of their individual probabilities.

Let's write the union of the events:

$$P(\cup_{n=1}^{N} A_N)$$

Intuitively, the union of events $A_1, A_2, \dots, A_N$ contains all the outcomes that are in at least one of the events. The reason the inequality holds (rather than equality) is that the events might overlap, meaning that some outcomes might be counted multiple times when summing the probabilities of individual events.

Let's consider $N = 2$, then we have:

$$P(A_1 \cup A_2) = P\big((A_1 - A_2) \cup A_2\big) = P(A_1 - A_2) + P(A_2) \leq P(A_1) + P(A_2)$$

Thus, when $N = n$, we have:

$$P(\cup_{n=1}^{N} A_N) \leq \sum_{n=1}^{N} P(A_N)$$

The inequality arises because when we sum $P(A_N)$, we are counting all outcomes in A_N, even those that might appear in other events. The probability of the union only counts each outcome once.