Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 1 The Binomial No-Arbitrage Pricing Model

Exercise 1.2

Analysis of the exercise

The task is to show that if there is a positive probability that X_1 is positive, then there must also be a positive probability that X_1 is negative. This means that under the condition that both events H and T have positive probabilities, there can be no arbitrage when the option price at time zero is 1.20.

Proof

According to the Example 1.1.1, Given that $S_1(H) = 8$ and $S_1(T) = 2$, we can proceed with a specific analysis of the expression for X_1.

Case 1: Market State H ($S_1(H) = 8$)

In this state:

- $S_1 = 8$
- $(S_1 - 5)^+ = 8 - 5 = 3$

Thus, the expression forX_1 becomes:

$$X_1(H) = \varDelta_0 \cdot 8 + \varGamma_0 \cdot 3 - \frac{5}{4}(4\varDelta_0 + 1.20\varGamma_0)$$

Simplifying:

$$X_1(H) = 3\mathit{\Delta}_0 + 1.5\mathit{\Gamma}_0$$

Case 2: Market State $\boldsymbol{T}$ ($\boldsymbol{S_1(T) = 2}$)

In this state:

- $S_1 = 2$
- $(S_1 - 5)^+ = 0$ (since $S_1 - 5 < 0$)

Thus, the expression for X_1 becomes:

$$X_1(H) = \mathit{\Delta}_0 \cdot 2 + \mathit{\Gamma}_0 \cdot 0 - \frac{5}{4}(4\mathit{\Delta}_0 + 1.20\mathit{\Gamma}_0)$$

Simplifying:

$$X_1(H) = -3\mathit{\Delta}_0 - 1.5\mathit{\Gamma}_0$$

For no arbitrage, we need to ensure that if X_1 can be positive, it can also be negative under the different market states.

Positive Probability of $\boldsymbol{X_1 > 0}$

To ensure$X_1 > 0$ in state H, we require:

$$3\mathit{\Delta}_0 + 1.5\mathit{\Gamma}_0 > 0$$

This implies:

$$\mathit{\Delta}_0 + 0.5\mathit{\Gamma}_0 > 0$$

Positive Probability of $\boldsymbol{X_1 < 0}$

For $X_1 < 0$ in state T, we require:

$$-3\mathit{\Delta}_0 - 1.5\mathit{\Gamma}_0 < 0$$

This implies:

$$\mathit{\Delta}_0 + 0.5\mathit{\Gamma}_0 > 0$$

Interestingly, the condition for $X_1 > 0$ in one state is the same as the condition for $X_1 < 0$ in the other state. This means that if the parameters $\mathit{\Delta}_0$ and $\mathit{\Gamma}_0$ are chosen such that X_1 is positive in one market state, it will be negative in the other state, satisfying the no-arbitrage condition.