Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.13

Proof

(i)

Note first that $S_{n+1} = S_n \frac{S_{n+1}}{S_n}$, $Y_{n+1} = Y_n + S_n \frac{S_{n+1}}{S_n}$ and whereas S_n and Y_n depend only on the first n tosses, $\frac{S_{n+1}}{S_n}$ depends only on toss n + 1.

According to the Independence Lemma 2.5.3, for any function $g(s, y)$ of dummy variables s and y, we have

$$\tilde{E}_n[g_{n+1}(S_{n+1}, Y_{n+1})] = \tilde{E}_n\left[g_{n+1}\left(S_n \frac{S_{n+1}}{S_n}, Y_n + S_n \frac{S_{n+1}}{S_n}\right)\right] = g_n(S_n, Y_n),$$

Where $g_n(s, y) = \tilde{E} g_{n+1}\left(s \frac{S_{n+1}}{S_n}, y + s \frac{S_{n+1}}{S_n}\right) = \tilde{p} g_{n+1}(su, y + su) + \tilde{q} g_{n+1}(sd, y + sd)$

Because $\tilde{E}_n[g_{n+1}(S_{n+1}, Y_{n+1})]$ can be written as a function of (S_n, Y_n), the two-dimensional process (S_n, Y_n), n = 0, 1, …, N, is a Markov process.

(ii)

Set $v_N(s, y) = f(\frac{y}{N+1})$.

Then $v_N(\, S_N, Y_N) = \; f\left(\frac{\sum_{n=0}^{N} S_n}{N+1}\right) = V_N$.

Suppose v_{n+1} is given, then

$$V_N = \tilde{E}_n\left[\frac{V_{N+1}}{1+r}\right] = \tilde{E}_n\left[\frac{v_{n+1}(S_{n+1}, Y_{n+1})}{1+r}\right]$$
$$= \frac{1}{1+r}\left[\tilde{p}v_{n+1}(uS_n, Y_n + uS_n) + \tilde{q}v_{n+1}(dS_n, Y_n + dS_n)\right] = \; v_n(\, S_n, Y_n)$$

Where,

$$v_n(s,y) = \frac{\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)}{1+r}$$