Stochastic Calculus for Finance I, Solution for Exercises

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This is the solution for the textbook *Stochastic calculus for finance I*, by *Steven E. Shreve*. If you have any comments or suggestions, please email me at sunyufei814@gmail.com.

Chapter 2 Probability Theory on Coin Toss Space

Exercise 2.4

Proof

(i)

To prove that M_n is a martingale, we need to verify the martingale properties. Specifically, we need to show:

1. $E[\mid M_n \mid] < \infty$ for all n.
2. $E[M_{n+1} \mid M_0, M_1, \dots, M_n] = M_n$.

The first condition $E[\mid M_n \mid] < \infty$ is trivially satisfied because M_n is a finite sum of the random variables X_j and each X_j takes a finite value (either 1 or -1).

For the second condition, consider the conditional expectation:

$$E[M_{n+1} \mid M_0, M_1, \dots, M_n]$$

Since $M_{n+1} = M_n + X_{n+1}$ and the value of X_{n+1} depends only on the outcome of the next coin toss, we have:

$$E[\, M_{n+1} \mid M_0, M_1, \dots, M_n\,] = M_n + E[\, X_{n+1} \mid M_0, M_1, \dots, M_n\,]$$

Because the coin toss is fair, $E[X_{n+1}] = 0$. Therefore:

$$E[\,M_{n+1} \mid M_0, M_1, \dots, M_n\,] = M_n$$

This shows that M_n is a martingale.

(ii)

Since $M_{n+1} = M_n + X_{n+1}$, Thus we have:

S_{n+1}

$= e^{\sigma M_{n+1}} \left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right)^{n+1}$

$= e^{\sigma(M_n + X_{n+1})} (\frac{2}{e^{\sigma} + e^{-\sigma}})^{n+1}$

$= e^{\sigma M_n} e^{\sigma X_{n+1}} (\frac{2}{e^{\sigma} + e^{-\sigma}})^{n+1}$

$= e^{\sigma M_n} (\frac{2}{e^{\sigma} + e^{-\sigma}})^{n} \; e^{\sigma X_{n+1}} \frac{2}{e^{\sigma} + e^{-\sigma}}$

$= S_n e^{\sigma X_{n+1}} \frac{2}{e^{\sigma} + e^{-\sigma}}$

Now, let's calculate the conditional expectation $E[S_{n+1} \mid S_0, S_1, \dots, S_n]$. Since S_n is known at time n, we get:

$$E[\,S_{n+1} \mid S_0, S_1, \dots, S_n\,] = S_n E[e^{\sigma X_{n+1}}] \frac{2}{e^{\sigma} + e^{-\sigma}}$$

The random variable X_{n+1} can either be 1 or -1 with equal probability. Hence:

$$E[e^{\sigma X_{n+1}}] = \frac{1}{2} e^{\sigma} + \frac{1}{2} e^{-\sigma}$$

Thus, we have:

$$\boldsymbol{E[\,S_{n+1} \mid S_0, S_1, \dots, S_n\,] = S_n \left(\frac{1}{2} e^{\sigma} + \frac{1}{2} e^{-\sigma}\right) \frac{2}{e^{\sigma} + e^{-\sigma}} = S_n}$$

This shows that S_n is indeed a martingale.