

The primal-dual algorithm is from [Chambolle-Pock2011], the algorithm is shown below. The algorithm in the paper [Zosso-Bustin 2014] is wrong. The dual variable update function is $\bar{p}^{k+1} = p^k - \sigma \nabla u^k$, but it should be $\bar{p}^{k+1} = p^k + \sigma \nabla u^k$. I waste a lot of time trying the first formula find it doesn't work and finally read the original paper [Chambolle-Pock2011] to find the right one.

$$\begin{cases} \text{Initialize } u^0 = u_0; p^0 = 0 \\ \bar{p}^{k+1} = p^k + \sigma \nabla u^k \\ p^{k+1} = \frac{\bar{p}^{k+1}}{\max(|\bar{p}^{k+1}|, 1)} \\ r = \lambda ((1 - c_1)^2 - (1 - c_2)^2) \\ u^{k+1} = \max(\min(u - \tau r + \text{div} p^{k+1}, 1), 0) \end{cases}$$

The problem here is I don't know how to choose sigma and tau to have the best performance.

From paper [Chambolle-Pock2011], it says $\tau \sigma L^2 < 1$, here $L^2 = |\nabla|^2 = |\text{div}|^2 \leq 8$, so $\tau \sigma \leq \frac{1}{L^2}$.

So the multiply of sigma and tau is a fixed value, but I don't know what the value is for each instance. I played with those parameters and done lots of experiments. I got the best result with tau = 1, sigma = 0.5.

If we want to use TV regularization for segmentation, we can use a minimum acceptable $\|u\|$ to plug in the gradient amplifier $z_{\max} = \frac{4(N-1)}{\Delta x^2 \min\|\nabla u\|}$. There always the case that some point on the surface will have a gradient $\|u\| = 0$. So z_{\max} goes to infinite and dt have to be zero. We can use a quantization level Q to represent digital u and the approximation bounds for $\|\nabla u\|$

$$\min\|\nabla u\| = \min_{\alpha} \sqrt{\sum_{k=1}^N \left(\frac{u_{\alpha+e_k} - u_{\alpha}}{\Delta x} \right)^2} \geq \sqrt{N \min_{\alpha,k} \left(\frac{u_{\alpha+e_k} - u_{\alpha}}{\Delta x} \right)^2} = \frac{\sqrt{N}}{\Delta x} \min |u_{\alpha+e_k} - u_{\alpha}|$$

Then we substitute the level set value between two pixels with the quantization interval Q. The quantization level will cause distortions between pixels, but it is not the same as the violation of CFL condition which will cause blowing up. Instead, the instability degree will be constrained by the quantization level. However, in that case, performance of the algorithm declines because the update of level does not strictly consistent with the gradient flow of the energy function.

We then plug

$$\min\|\nabla u\| = \frac{\sqrt{N}}{\Delta x} Q$$

into the gradient amplifier to obtain:

$$z_{\max} \leq \lambda + \frac{4(N-1)}{Q \Delta x \sqrt{N}} < \lambda + \frac{4\sqrt{N}}{Q \Delta x}$$

Then we substitute the gradient amplifier into the CFL condition we obtain the stability condition for acceleration with TV regularization.

For acceleration, the quantization level Q I use here is 1/255, the acceleration method is forward difference. And Dr.Yezzi told me that there is no optimal damping coefficient for TV, so I tried various of damping coefficients. The energy loglog plot is shown below. Line 1 to line 7 are acceleration, line 8 is primal-dual. All the experiments are done on noisy image.

