VE281

Data Structures and Algorithms

Comparison Sort

Learning Objectives:

- Know the difference between comparison sort and non-comparison sort
- Know the procedures of merge sort and quick sort
- Know the master theorem
- Know different characteristics of sorting algorithms, such as time complexity, stability, etc.

Outline

- Sorting Basics
- Merge Sort
- Quick Sort
- Comparison Sort Summary

Sorting

- Given array A of size N, reorder A so that its elements are in order.
  - "In order" with respect to a consistent comparison function, such as “≤” or “≥”.

- Sorting order
  - Ascending order
  - Descending order
- Unless otherwise specified, we consider sorting in ascending order.

Characteristics of Sorting Algorithms

- Average-case time complexity
- Worst-case time complexity
- Space usage: **in place** or not?
  - **in place**: requires $O(1)$ additional memory
  - Don't forget the stack space used in recursive calls
  - **In place is better**
    - Why? The data can fit into cache, not main memory
  - Real example: quick sort versus merge sort. Both have average-case time complexity of $O(n \log n)$. Quick sort is faster, due to in place

Characteristics of Sorting Algorithms

- **Stability**: whether the algorithm maintains the relative order of records with equal keys

(4, b), (3, e), (3, b), (5, b) → (3, e), (3, b), (4, b), (5, b)

Sort on the first number

Stable!

- Usually there is a secondary key whose ordering you want to keep. Stable sort is thus useful for sorting over multiple keys
- Example: sort complex numbers a+bi
  - Ordering rule: first compare a; when there is a tie, compare b
  - One sorting method: first sort b, then sort a

3+5i, 2+6i, 3+4i, 5+2i → Sort on b → 5+2i, 3+4i, 3+5i, 2+6i

… sort on a → 2+6i, 3+4i, 3+5i, 5+2i

Stability is important!

Types of Sorting Algorithms

- Sorting algorithms can be classified as **comparison sort** and **non-comparison sort**.

- **Comparison sort**: each item is compared against others to determine its order.

- **Non-comparison sort**: each item is put into predefined "bins" independent of the other items presented.
  - No comparison with other items needed.
  - It is also known as **distribution-based sort**.

Types of Sorting Algorithms

- General types of comparison sort
  - Insertion-based: insertion sort
  - Selection-based: selection sort, heap sort
  - Exchange-based: bubble sort, quick sort
  - Merging-based: merge sort

- Non-comparison sort:
  counting sort, bucket sort, radix sort

Insertion Sort

- **`A[0]`** alone is a sorted array.
- For **`i=1`** to **`N-1`**
  - **Insert** **`A[i]`** into the appropriate location in the sorted array **`A[0], …, A[i-1]`**, so that **`A[0], …, A[i]`** is sorted.
  - To do so, save **`A[i]`** in a temporary variable **`t`**, shift sorted elements greater than **`t`** right, and then insert **`t`** in the gap.
- Time comlexity? $O(N^2)$
- In place? Yes. O(1) additional memory.
- Stable?
  - Yes, because elements are visited in order and equal elements are inserted after its equals.

Insertion Sort

Best Case Time Complexity

- For **`i=1`** to **`N-1`**
  - **Insert** **`A[i]`** into the appropriate location in the sorted array **`A[0], …, A[i-1]`**, so that **`A[0], …, A[i]`** is sorted.

- The **best case** time complexity is $O(N)$.
  - It happens when the array is already sorted.
  - For other sorting algorithms we will talk, their best case time complexity is $\Omega(N \log N)$.

Selection Sort

- For **`i=0`** to **`N-2`**
  - Find the smallest item in the array **`A[i], …, A[N-1]`**. Then, swap that item with **`A[i]`**.

- Finding the smallest item requires **linear scan**.

Which Statements Are Correct for Selection Sort?

For **`i=0`** to **`N-2`**

Find the smallest item in the array **`A[i], …, A[N-1]`**. Then, swap that item with **`A[i]`**.

- **A.** Its worse-case time complexity is $O(N^2)$
- **B.** Its best-case time complexity is $\Omega(N^2)$
- **C.** It is not in-place
- **D.** It is stable

Bubble Sort

```
For i=N-2 downto 0
  For j=0 to i
    If A[j]>A[j+1] swap A[j] and A[j+1]
```

- Compares two adjacent items and swap them to keep them in ascending order.
  - From the beginning to the end. The last item will be the largest.

- Time complexity? $O(N^2)$
- In place? Yes.
- Stable?
  - Yes, because equal elements will not be swapped.

Two Problems with Simple Sorts

- They learn only one piece of information per comparison and hence might compare every pair of elements.
  - Contrast with binary search: learns N/2 pieces of information with first comparison.
- They often move elements one place at a time (bubble sort and insertion sort), even if the element is "far" from its **final place**.
  - Contrast with selection sort, which moves each element exactly to its final place.

- Fast sorts attack these two problems.
  - Two famous ones: **merge sort** and **quick sort**.

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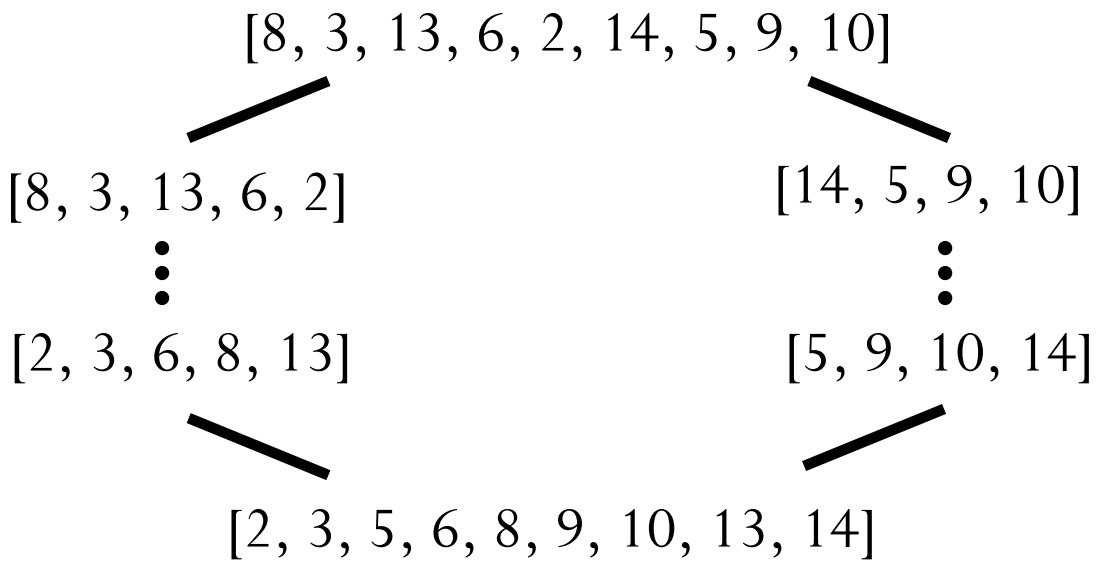
Merge Sort

Algorithm

- Spilt array into two (roughly) equal subarrays.
- Merge sort each subarray recursively.
  - The two subarrays will be sorted.
- Merge the two sorted subarrays into a sorted array.



Merge Sort

Pseudo-code

```
void mergesort(int *a, int left, int
  right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```

Merge Two Sorted Arrays

- For example, merge A = (2, 5, 6) and B = (1, 3, 8, 9, 10).

- Compare the smallest element in the two arrays A and B and move the smaller one to an additional array C.
- Repeat until one of the arrays becomes empty.
- Then append the other array at the end of array C.

Merge Two Sorted Arrays

Implementation

- We actually do not "remove" element from arrays A and B.
  - We just keep a pointer indicating the smallest element in each array.
  - We "remove" element by incrementing that pointer.

```
i = j = k = 0;
while(i < sizeA && j < sizeB) {
  if(A[i]<=B[j]) C[k++]=A[i++];
  else C[k++]=B[j++];
}
if(i == sizeA) append(C, B);
else append(C, A);
```

Time complexity?

Time complexity is $O(sizeA + sizeB)$

Merge Sort

Time Complexity

```
void mergesort(int *a, int left, int
  right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```

$T(N/2)$

$T(N/2)$

$O(N)$

- Let $T(N)$ be the time required to merge sort N elements.
- Merge two sorted arrays with total size N takes $O(N)$.

Recursive relation: $T(N) = 2T(N/2) + O(N)$

How to solve the recurrence?

Solve Recurrence: Master Method

- A "black box" for solving recurrence.
- However, there is an important assumption: all sub-problems have roughly **equal** sizes.
  - E.g., merge sort
  - Not apply to unbalanced division.

Solve Recurrence: Master Method

- Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$
  - Base case: $T(n) \leq constant$ for all sufficiently small n.
  - a = number of recursive calls (integer ≥ 1)
  - b = input size shrinkage factor (integer > 1)
  - $O(n^d)$: the runtime of merging solutions. d is real value ≥ 0.
  - a, b, d are independent of n.
- Claim:

$$T(n) = \begin{cases} O(n^d \log n) & if\ a = b^d \\ O(n^d) & if\ a < b^d \\ O(n^{\log_b a}) & if\ a > b^d \end{cases}$$

base doesn't matter

base matters!

Example of Merge Sort

Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

Claim: $T(n) = \begin{cases} O(n^d \log n) & if\ a = b^d \\ O(n^d) & if\ a < b^d \\ O(n^{\log_b a}) & if\ a > b^d \end{cases}$

- $a = 2, b = 2, d = 1 \quad \Rightarrow b^d = a$
- $T(n) = O(n \log n)$

What are a, b, d for Binary Search?

Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

Claim: $T(n) = \begin{cases} O(n^d \log n) & if\ a = b^d \\ O(n^d) & if\ a < b^d \\ O(n^{\log_b a}) & if\ a > b^d \end{cases}$

A. a =2, b = 2, d = 0 **B.** a =1, b = 2, d =0

C. a =2, b = 2, d = 1 **D.** a =1, b = 2, d =1

Merge Sort

Characteristics

- Not in-place
  - For efficient merging two sorted arrays, we need an auxiliary $O(N)$ space.
  - Recursion needs up to $O(\log N)$ stack space.

- Stable if **`merge()`** **maintains** the relative order of equal keys.

Divide-and-Conquer Approach

- Merge sort uses the **divide-and-conquer** approach.

- Recursively **breaking** down a problem into two or more sub-problems of the same (or related) type, until these become simple enough to be solved directly.
  - For merge sort, split an array into two and sort them respectively.

- The solutions to the sub-problems are then **combined** to give a solution to the original problem.
  - For merge sort, merge two sorted arrays.

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Quick Sort

Algorithm

Another divide-and-conquer approach to sort

- Choose an array element as **pivot**.
- Put all elements $<$ pivot to the left of pivot.
- Put all elements $\geq$ pivot to the right of pivot.
- Move pivot to its correct place in the array.

} **partition()**

- Sort left and right subarrays recursively (not including pivot).

```
void quicksort(int *a, int left,
  int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

Choice of Pivot

- If your input is random, you can choose the **first** element.
  - But this is very bad for presorted input.

- A better strategy: **randomly** pick an element from the array as pivot.
  - **Claim**: **for any input**, the average running time is $O(n \log n)$.
    - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.

Partitioning the Array

- Once pivot is chosen, swap pivot to the beginning of the array.
- When another array B is available, scan original array A from left to right.
  - Put elements < pivot at the left end of B.
  - Put elements ≥ pivot at the right end of B.
  - The pivot is put at the remaining position of B.
  - Copy B back to A.

A	6	2	8	5	11	10	4	1	9	7	3

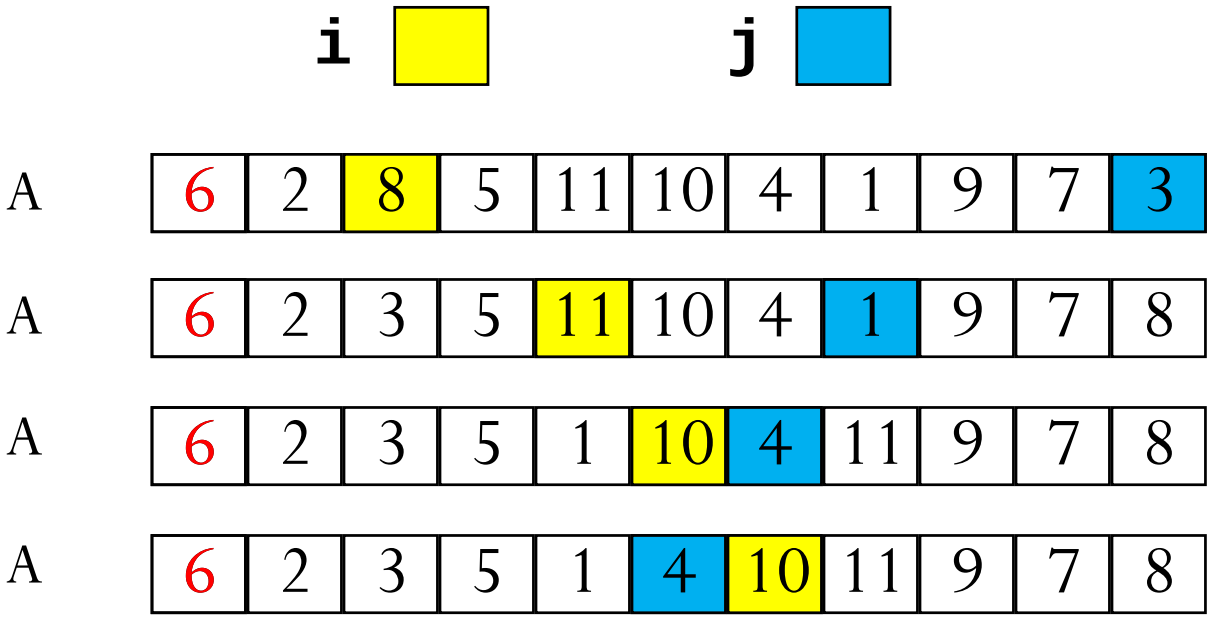
B	2	5	4	1	3	6	7	9	10	11	8

In-Place Partitioning the Array

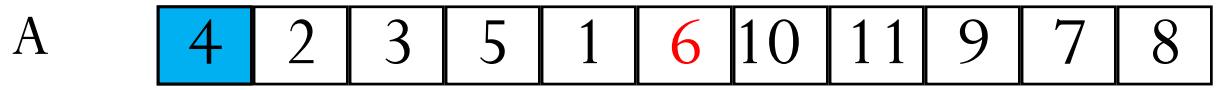
1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **`i=1`** and **`j=N-1`**.
3. Increment **`i`** until we find element **`A[i]>=pivot`**.
   - **`A[i]`** is the leftmost item ≥ pivot.
4. Decrement **`j`** until we find element **`A[j]<pivot`**.
   - **`A[j]`** is the rightmost item < pivot.
5. If **`i<j`**, swap **`A[i]`** with **`A[j]`**. Go back to step 3.
6. Otherwise, swap the first element (pivot) with **`A[j]`**.

In-Place Partitioning the Array

Example



- Now, **`j < i`**, swap the first element (pivot) with **`A[j]`**.



In-Place Partitioning the Array

Time Complexity

1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **`i=1`** and **`j=N-1`**.
3. Increment **`i`** until we find element **`A[i]>=pivot`**.
4. Decrement **`j`** until we find element **`A[j]<pivot`**.
5. If **`i<j`**, swap **`A[i]`** with **`A[j]`**. Go back to step 3.
6. Otherwise, swap the first element (pivot) with **`A[j]`**.

- Scan the entire array no more than twice.
- Time complexity is $O(N)$, where N is the size of the array.

Quick Sort

Time Complexity

```
void quicksort(int *a, int left,
  int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

$O(N)$

$T(LeftSz)$

$T(RightSz)$

- Let $T(N)$ be the time needed to sort N elements.
  - $T(0) = c$, where c is a constant.
- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

  - $LeftSz + RightSz = N - 1$

Quick Sort

Worst Case Time Complexity

- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- Worst case happens when each time the pivot is the smallest item or the largest item
  - $T(N) = T(N-1) + T(0) + O(N)$

$$\leq T(N-1) + T(0) + dN \qquad \text{d is a constant}$$

$$\leq T(N-2) + 2T(0) + d(N-1) + dN$$

$$\cdots$$

$$\leq T(0) + NT(0) + d + 2d + \cdots + d(N-1) + dN$$

$$= O(N^2)$$

Quick Sort

Best Case Time Complexity

- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- Best case happens when each time the pivot divides the array into two equal-sized ones.
  - $T(N) = T((N-1)/2) + T((N-1)/2) + O(N)$
  - The recursive relation is similar to that of merge sort.
  - $T(N) = O(N \log N)$

Quick Sort

Average Time Complexity

- Average time complexity of quick sort can be proved to be $O(N \log N)$.
  - Assume **randomly** pick an element from the array as pivot.
  - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.
  - The claim holds for any input.

Quick Sort

Other Characteristics

- In-place?
  - In-place partitioning.
  - Worst case needs $O(N)$ stack space.
  - Average case needs $O(\log N)$ stack space.
    - "Weakly" in-place.
- Not stable.

Quick Sort

Summary

- Like merge sort, quick sort is a divide-and-conquer algorithm.

- Merge sort: easy division, complex combination.
- Quick sort: complex division (partition with pivot step), easy combination.

- Insertion sort is faster than quick sort for small arrays.
  - Terminate quick sort when array size is below a threshold. Do insertion sort on subarrays.

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Comparison Sorts

Summary

	Worst Case Time	Average Case Time	In Place	Stable
Insertion	$O(N^2)$	$O(N^2)$	Yes	Yes
Selection	$O(N^2)$	$O(N^2)$	Yes	No
Bubble	$O(N^2)$	$O(N^2)$	Yes	Yes
Merge Sort	$O(N \log N)$	$O(N \log N)$	No	Yes
Quick Sort	$O(N^2)$	$O(N \log N)$	Weakly	No

For comparison sort, is $O(N \log N)$ the best we can do in the **worst case**?

Comparison Sorts

Worst Case Time Complexity

- Theorem: A sorting algorithm that is based on pairwise comparisons must use $\Omega(N \log N)$ operations to sort in the worst case.