Mathematical Induction

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1 Mathematical Induction

1.1 Principle

Suppose $P(n)$ is a statement involving an integer n. Then to prove that $P(n)$ is true for every $n \geq n_0$, it is sufficient to show these two things:

- $P(n_0)$ is true.
- For any $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true.

1.2 Example

Let $P(n)$ be the statement $\sum_{i=0}^{n} i = n(n+1)/2$. Prove that $P(n)$ is true for every $n \geq 0$.

Proof: We prove $P(n)$ is true for $n \geq 0$ by induction.

Basis step. $P(0)$ is $0 = 0(0+1)/2$, and it is obviously true.

Induction Hypothesis. Assume $P(k)$ is true for some $k \geq 0$. Then $0+1+2+\cdots+k = k(k+1)/2$.

Proof of Induction Step. Now let us prove that $P(k+1)$ is true.

$$\begin{aligned} 0+1+2+\cdots+k+(k+1) &= k(k+1)/2+(k+1) \\ &= (k+1)(k/2+1) \\ &= (k+1)(k+2)/2 \qquad \square \end{aligned}$$

1.3 An Example for the Weakness of Mathematical Induction

Prove that $\forall n \in \mathbb{N}$ with $n \geq 2$, it has prime factorizations.

Proof: Define $P(n)$ be the statement that "n is either prime or the product of two or more primes". We will try to prove that $P(n)$ is true for every $n \geq 2$.

Basis step. $P(2)$ is true, since 2 is a prime. ✓

Induction hypothesis. $P(k)$ for $k \geq 2$. **(as usual process)**

Proof of induction step. Let's prove $P(k+1)$.

If $P(k+1)$ is prime, ✓

If $P(k+1)$ is not a prime, then we should prove that $k+1 = r \times s$, where r and s are positive integers greater than 1 and less than $k+1$.

However, from $P(k)$ we know nothing about r and s $\longrightarrow$???

2 The Strong Principle of Mathematical Induction

2.1 Principle

Suppose $P(n)$ is a statement involving an integer n. Then to prove that $P(n)$ is true for every $n \geq n_0$, it is sufficient to show these two things:

- $P(n_0)$ is true.
- For any $k \geq n_0$, if $P(n)$ is true for every n satisfying $n_0 \leq n \leq k$, then $P(k+1)$ is true.

Also called **the principle of complete induction**, or **course-of-values induction**.

2.2 Example

Prove that $\forall n \in \mathbb{N}$ with $n \geq 2$, it has prime factorizations.

Induction hypothesis. For $k \geq 2$ and $2 \leq n \leq k$, $P(n)$ is true. **(Strong Principle)**

Proof of induction step. Let's prove $P(k+1)$.

If $P(k+1)$ is prime, ✓
If $P(k+1)$ is not a prime, by definition of a prime, $k+1 = r \times s$, where r and s are positive integers greater than 1 and less than $k+1$.

It follows that $2 \leq r \leq k$ and $2 \leq s \leq k$. Thus by induction hypothesis, both r and s are either prime or the product of two or more primes. Then their product $k+1$ is the product of two or more primes. $P(k+1)$ is true.

3 The Minimal Counterexample Principle

Prove $\forall n \in \mathbb{N}$, $5^n - 2^n$ is divisible by 3.

Proof: If $P(n) = 5^n - 2^n$ is not true for every $n \geq 0$, then there are values of n for which $P(n)$ is false, and there must be a smallest such value, say $n = k$.

Since $P(0) = 5^0 - 2^0 = 0$, which is divisible by 3, we have $k \geq 1$, and $k - 1 \geq 0$.

Since k is the smallest value for which $P(k)$ false, $P(k-1)$ is true. Thus $5^{k-1} - 2^{k-1}$ is a multiple of 3, say $3j$.
However, we have

$$\begin{aligned} 5^k - 2^k &= 5 \times 5^{k-1} - 2 \times 2^{k-1} \\ &= 5 \times (5^{k-1} - 2^{k-1}) + 3 \times 2^{k-1} \\ &= 5 \times 3j + 3 \times 2^{k-1} \end{aligned}$$

This expression is divisible by 3. We have derived a contradiction, which allows us to conclude that our original assumption is false. □