Greedy Algorithms*

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Algorithm Course: Shanghai Jiao Tong University

* Special thanks is given to Prof. Kevin Wayne@Princeton for sharing his lecture notes, and also given to Mr. Hongjian Cao from CS2015@SJTU for producing this slide.

Interval Scheduling: An Introductory Example

- Job j starts at s_j and finishes at f_j.
- Two jobs are compatible if they don't overlap.

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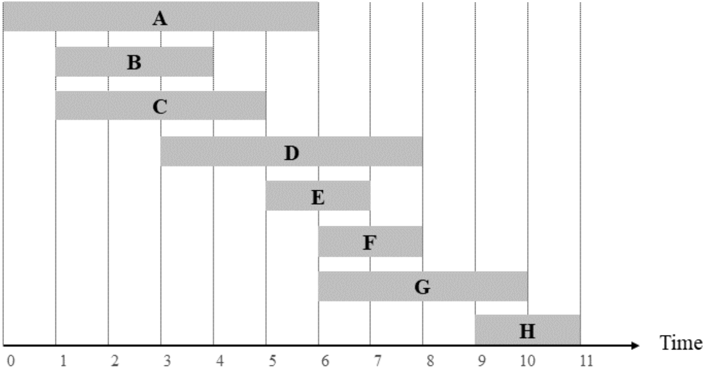

Greedy Strategy

Optimization Problem: Given a problem Π with domain X, choose a subset or determine a sequence according to some maximization or minimization objective.

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Interval Scheduling Problem: Consider jobs in some natural order. Take each job provided to judge its compatibility with the ones already taken.

[Earliest start time] Consider jobs in ascending order of s_j.

[Earliest finish time] Consider jobs in ascending order of f_j.

[Shortest interval] Consider jobs in ascending order of $f_j - s_j$.

[Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Attempts

[Earliest start time] Consider jobs in ascending order of s_j.

Counter Example:

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Greedy Interval Scheduling Algorithm

Algorithm 1: Greedy Interval Scheduling

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Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n;
A ← ∅ ;                                  // set of jobs selected
for j = 1 to n do
    if job j is compatible with A then
        A ← A ∪ {j};
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```

Implementation: $O(n \log n)$.

- After each iteration, set job j^* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

Correctness Analysis

Theorem. Greedy Interval Scheduling algorithm is optimal.

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Let $i_1, i_2, \cdots, i_k$ denote set of jobs selected by greedy.

Let $j_1, j_2, \cdots, j_m$ denote set of jobs in an optimal solution with $i_1 = j_1$, $i_2 = j_2, \cdots, i_r = j_r$ for the largest possible value of r.

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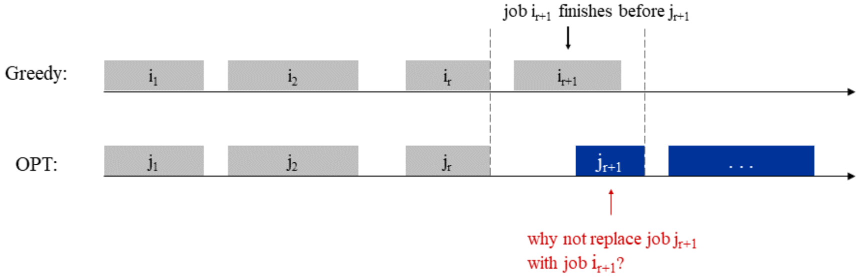

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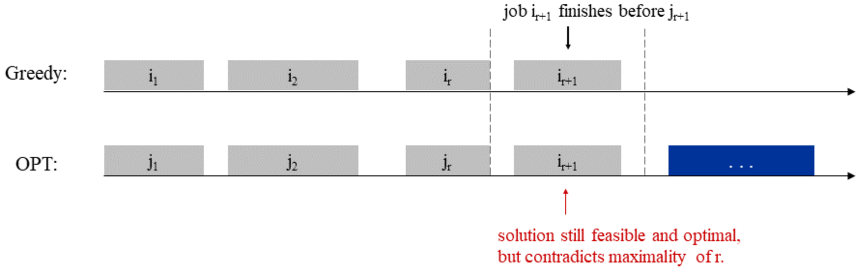

Interval Partitioning

Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

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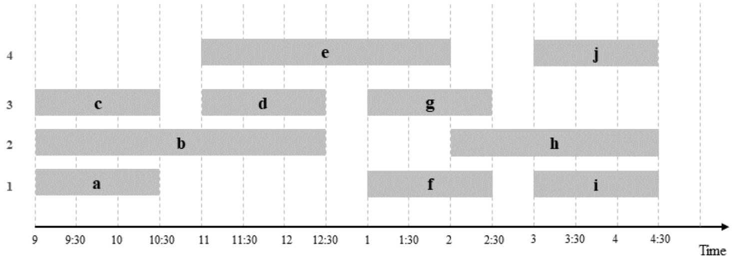
Example: This schedule uses 4 classrooms to schedule 10 lectures.



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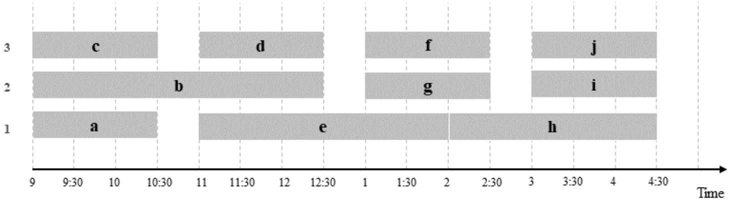
Example: This schedule uses only 3.



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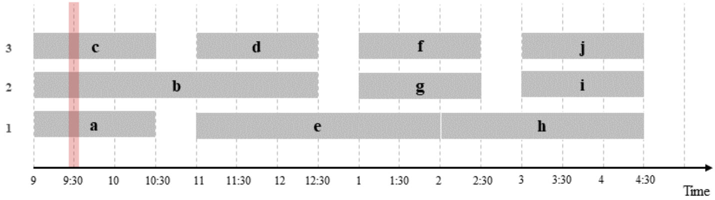

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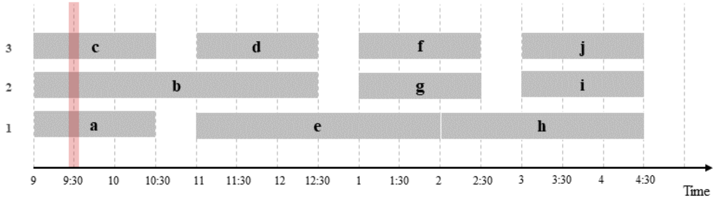

Question. Does there always exist a schedule equal to depth of intervals?

Greedy Interval Partitioning Algorithm

Algorithm 2: Interval Partitioning Greedy Algorithm

Sort intervals by starting time so that $s_1 \leq s_2 \leq \dots \leq s_n$;
$d \leftarrow 0$; `// number of allocated classrooms`
for $j = 1$ ***to*** n **do**
  if *lecture j is compatible with some classroom k* **then**
    schedule lecture j in classroom k;
  else
    allocate a new classroom $d + 1$;
    schedule lecture j in classroom $d + 1$;
    $d \leftarrow d + 1$;
return A;

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        d ← d + 1;
return A;
```

Implementation: $O(n \log n)$.

- For classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Correctness Proof

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Key observation $\Rightarrow$ all schedules use $\geq d$ classrooms.

Scheduling to Minimize Lateness

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $l_j = \max\{0, f_j - d_j\}$.

Goal: schedule all jobs to minimize maximum lateness $L = \max l_j$.

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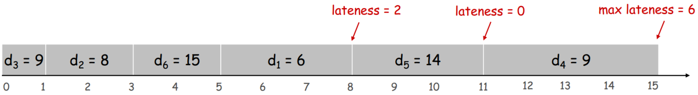
	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Attempt: Consider jobs in ascending order by some strategy

[Shortest processing time first] Sort by processing time t_j.

[Earliest deadline first] Sort by deadline d_j.

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	1	2
t_j	1	10
d_j	100	10

counterexample

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A Greedy Algorithm: Earliest Deadline First

Algorithm 3: Greedy Minimizing Lateness

Sort n jobs by deadline so that $d_1 \leq d_2 \leq ... \leq d_n$;
$t \leftarrow 0$;
for $j = 1$ ***to*** n **do**
  Assign job j to interval $[t, t + t_j]$;
  $s_j \leftarrow t, f_j \leftarrow t + t_j$;
  $t \leftarrow t + t_j$;
return intervals $[s_j, f_j]$;

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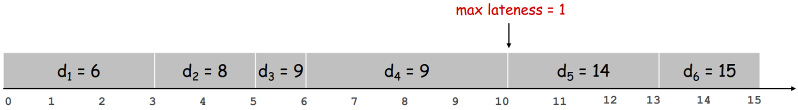
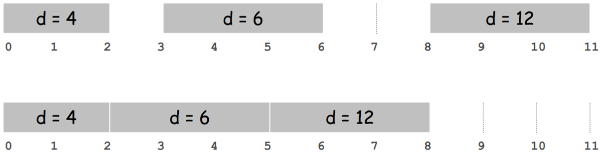


Correctness Proof: Reduce Optimal Solution

Observation. There exists an optimal schedule with no idle time.



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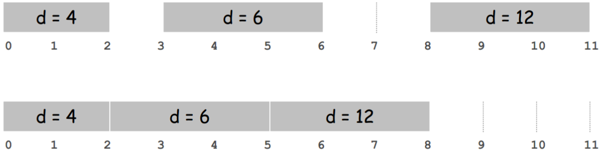
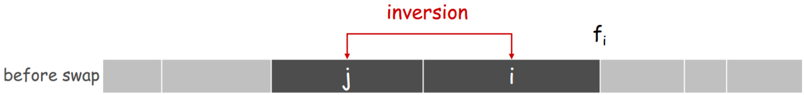


Observation. The greedy schedule has no idle time.

Correctness Proof: Optimal Solution vs Algorithm Solution

Definition. Given a schedule S, an inversion is a pair of jobs i and j such that: $i < j$ but j scheduled before i.



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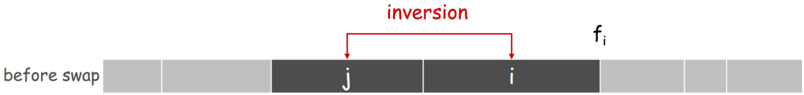

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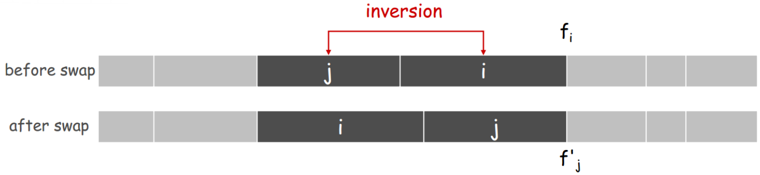
Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Correctness Proof: Inversions

Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.



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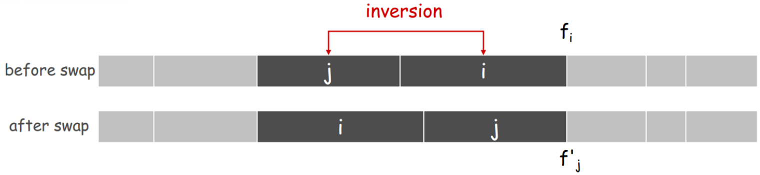

Proof. Let l be the lateness before the swap, and let l' be it afterwards.

- $l'_k = l_k$ for all $k \neq i, j$
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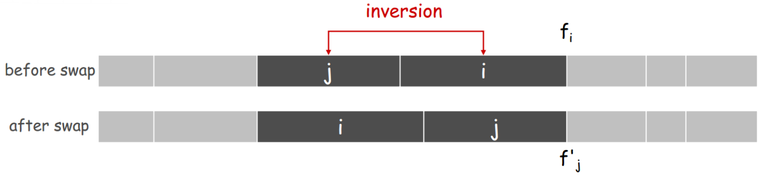

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$$
\begin{aligned}
l'_j &= f'_j - d_j && \text{(definition)} \\
&= f_i - d_j && (j \text{ finishes at time } f_i) \\
&\leq f_i - d_i && (i < j) \\
&\leq l_i && \text{(definition)}
\end{aligned}
$$

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  Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions.

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- If S^* has an inversion, let $i - j$ be an adjacent inversion.

  Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions. This contradicts definition of S^*. $\square$

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, ...

Coin Changing

Goal. Given US currency denominations:

1 (cent), 5 (nickel), 10 (dime), 25 (quarter), 100 (dollar),

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Example. $2.89.

Cashier's Algorithm

Algorithm 4: Cashier's Algorithm

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Sort coins denominations by value: c_1 < c_2 < ... < c_n;
S ← ∅ ;                                   // coins selected
while x ≠ 0 do
    let k be largest integer such that c_k < x;
    if k = 0 then
        return "no solution found";
    x ← x − c_k;
    S ← S ⋃ {k};
return S;
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Sort coins denominations by value: $c_1 < c_2 < \ldots < c_n$;
$S \leftarrow \emptyset$; // coins selected
while $x \neq 0$ **do**
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    if $k = 0$ **then**
        return "no solution found";
    $x \leftarrow x - c_k$;
    $S \leftarrow S \bigcup \{k\}$;
return S;

Question. Is cashier's algorithm optimal?

Properties of Optimal Solution

Property. Number of pennies ≤ 4.
Proof. Replace 5 pennies with 1 nickel.

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Property. Number of nickels ≤ 1.

Property. Number of quarters ≤ 3.

Property. Number of pennies ≤ 2.
Proof.

- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.
- Recall: at most 1 nickel.

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If not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x.

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, …, k-1 in any OPT
1	1	$P \le 4$	-
2	5	$N \le 1$	4
3	10	$N + D \le 2$	4 + 5 = 9
4	25	$Q \le 3$	20 + 4 = 24
5	100	no limit	75 + 24 = 99

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Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm.

Is Cashier's Algorithm Work for Any Denominations?

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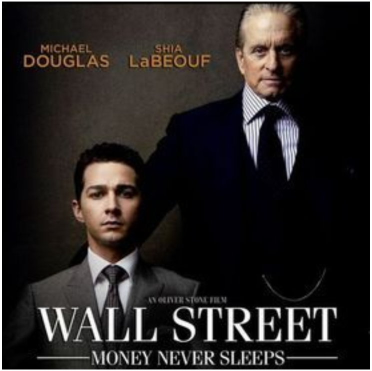
Observation 2. Even no feasible solution with system ¢={7, 8, 9}.

- Cashier's algorithm: 15¢$=$ 9+???
- Optimal: 15¢$= 7 + 8$.



†

Greed is good.

Greed is right.

Greed works.

Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.

— Gordon Gecko (Michael Douglas)

† Watch the movie segment at the class webpage.