Algebra

A course for Ph. D. students

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Contents

Chapter 1

Associative Algebras

We follow "Associative algebra" by R. S. Pierce. We do not claim any originality for the content.

1.1 Algebra over a commutative ring

We begin with a definition of *algebra* over a commutative ring.

Definition 1.1.1 Associative algebra. Let R be a commutative ring with unity 1_R. An R-**algebra** or algebra over R is a unital right R-module A together with R-bilinear map $A \times A \to A$ denoted by $(x, y) \mapsto xy$ that is associative, i.e., $(xy)z = x(yz)$ for all $x, y, z \in A$, and there is a unity $1_A \in A$ such that $1_R x = x 1_R = x$ for all $x \in A$. $\Diamond$

Observation 1.1.2 The bilinearlity of the map $A \times A \to A$ in Definition 1.1.1 implies the following relation for any $x, y \in A$ and any $r \in R$.

$$(xr)y = x(yr) = (xy)r \tag{1.1.1}$$

We have the following equivalent definition of R-algebra.

Proposition 1.1.3 *Following statements are equivalent.*

1. *A is an R-algebra*

2. *A is a ring and a right R-module such that for any $x, y \in A$ and $r \in R$, we have $(xr)y = x(yr) = (xy)r$.*

Proof. Assume that A is an R-algebra. Then A is a ring with the multiplication given by the R-bilinear map $A \times A \to A$. Indeed, the associativity of the multiplication follows from the definition. The distributivity of the multiplication over addition follows from the bilinearity of the map. The existence of unity is also given in the definition. The relation $(xr)y = x(ry) = (xy)r$ follows from Observation 1.1.2.

Conversely, assume that A is a ring and a right R-module such that for any $x, y \in A$ and $r \in R$, we have $(xr)y = x(yr) = (xy)r$. We consider a map $m \colon A \times A \to A$ as the multiplication in ring A. Therefore, $m(x + y, z) = m(x, z) + m(y, z)$ (resp., $m(x, y + z) = m(x, y) + m(x, z)$) for any $x, y, z \in A$ and also there is a unity in ring A. The R-linearity of the map m follows from the relation $(xr)y = x(yr) = (xy)r$, i.e., $m(xr, y) = m(x, yr) = (x, y)r$. ■

Lemma 1.1.4

1. *If A is an R algebra then, there is a ring homomorphism $f \colon R \to A$ such*

that $f(R)$ is contained in the center of A.

2. *Suppose that A is a ring and there is a ring homomorphism $f: R \to A$ such that $f(R)$ is contained in the center of A. Then, A is an R-algebra with the right R-module structure given by $xr = xf(r)$ for any $x \in A$ and $r \in R$.*

Proof. If A is an R-algebra then, we define a map $f: R \to A$ by $f(r) = 1_A r$ for any $r \in R$. It is easy to see that f is a ring homomorphism and $f(R)$ is contained in the center of A. Conversely, suppose that A is a ring and there is a ring homomorphism $f: R \to A$ such that $f(R)$ is contained in the center of A. We consider the right R-module structure on A given by $xr = xf(r)$ for any $x \in A$ and $r \in R$. Then, it is easy to see that the relation in Proposition 1.1.3(2) holds. Therefore, by Proposition 1.1.3, we conclude that A is an R-algebra. ■

1.2 Some examples of algebras

Example 1.2.1 Associative ring as $\mathbb{Z}$-algbera. Every associative ring A can be considered as a $\mathbb{Z}$-algebra. □

Example 1.2.2 Ring as an algebra over itself. Let R be a commutative ring with unity. Then R can be considered a right R-module using the multiplication in ring R. With this considerations, R is an R-algebra. □

Example 1.2.3 Ring over its center. Suppose that A is a ring (not necessarily commutative), and let $Z(A)$ be the center of A. Then A is a $Z(A)$-algebra. □

Example 1.2.4 Matrix algebra. Suppose that R is a commutative ring and n is a positive integer. Suppose that A is an R-algebra. Then the ring of $n \times n$ matrices over A, denoted by $M_n(A)$, is an R-algebra. □

Example 1.2.5 Endomorphism algebra. Let R be a commutative ring with unity. Suppose that M is a right R-module. The endomorphism ring $\mathrm{End}_R(M)$ is an R-algebra. □

Example 1.2.6 Polynomial ring as algebra. Let R be a commutative ring with unity. The polynomial ring $R[X]$ is an R-algebra. □

We discuss the example of 'Group algebra' in the next section.

1.3 Group algebra

Definition 1.3.1 Group Ring. Let R be a commutative ring with unity and G a group. Consider the following set.

$$RG = \{\phi \in \mathrm{Hom}_{Sets}(G, R) : \phi(g) = 0 \quad \text{for all but finitely many } \ g \in G\}$$

Define 'addition' and 'scalar multiplication' in RG as follows.

$$(\phi r + \psi s)(g) = \phi(g)r + \psi(g)s$$

The 'multiplication' in RG is defined as convolution product.

$$(\phi\psi)(g) = \sum \phi(h)\psi(h^{-1}g) \quad \text{such that} \quad h \in G, \ \text{and } \phi(h)\psi(h^{-1}g) \neq 0$$

◊

Checkpoint 1.3.2 Show that RG is closed under addition, scalar multiplication, and it is an R-module. Furthermore, show that multiplication is bilinear.

We now proceed to check the associativity of multiplication and the existence of unity in RG. We first observe that RG is a free R-module.

Proposition 1.3.3 *Keep notations of Definition 1.3.1 above. Let for $g \in G$, χ_g be the characteristic function of g, i.e., $\chi_g(h) = 0$ if $h \neq g$ and $\chi_g(g) = 1$. Then, set $\{\chi_g : g \in G\}$ is an R-basis of RG. In particular, RG is a free R-module.*

Furthermore, $G \to RG$ given by $g \mapsto \chi_g$ is a monoid homomorphism. In particular, $\chi_g\chi_h = \chi_{gh}$ and if $e \in G$ is the identity then, χ_e is the unity in RG.

Proof. Suppose that $\sum \chi_{g_i} r_i = 0$ for some $g_i \in G$ and $r_i \in R$. Evaluating at g_j, we get $r_j = 0$. Thus, the set is linearly independent. Now, let $\phi \in RG$. Then, we can write $\phi = \sum \chi_g \phi(g)$ where the sum is over all $g \in G$ such that $\phi(g) \neq 0$. This shows that the set spans RG. Thus, it is an R-basis of RG.

Direct calculation shows that $\chi_g\chi_h = \chi_{gh}$ for all $g, h \in G$.

We show the associativity of multiplication. Let $\phi, \psi, \zeta \in RG$. Then, there exists $r_i, s_j, t_k \in R$ such that $\phi = \sum \chi_{g_i} r_i$, $\psi = \sum \chi_{g_j} s_j$, $\zeta = \sum \chi_{g_k} t_k$. Therefore, we obtain the following.

$$
\begin{aligned}
(\phi\psi)\,\zeta &= \left(\left(\sum \chi_{g_i} r_i\right)\left(\sum \chi_{g_j} s_j\right)\right)\sum \chi_{g_k} t_k \\
&= \sum \left((\chi_{g_i}\chi_{g_j})\chi_{g_k}\right)(r_i s_j t_k) \\
&= \sum \chi_{g_i}\left(\chi_{g_j}\chi_{g_k}\right)(r_i s_j t_k) \\
&= \phi\,(\psi\zeta)
\end{aligned}
$$

This shows the associativity of multiplication.

For any $\chi_{g_i} a_i \in RG$ we get the following.

$$
\left((\chi_{g_i} a_i)\chi_e\right)(h) = \sum \chi_{g_i} a_i(x)\chi_e(x^{-1}h)
$$

The above equation is nonzero only when $x = g_i$ and $x^{-1}h = e$. This implies that $h = g_i$. Thus, we get $\left((\chi_{g_i} a_i)\chi_e\right)(h) = \chi_{g_i} a_i(h)$. Similarly, we can show that $\left(\chi_e(\chi_{g_i} a_i)\right)(h) = \chi_{g_i} a_i(h)$. This shows that χ_e is the unity in RG. ■

Example 1.3.4 Group algebra of the cyclic group of order 2. Suppose that $C_2 = \{\pm 1\}$ denotes the cyclic group of order two. The general element of $\mathbb{Z}C_2$ is of the form $a\chi_1 + b\chi_{-1}$ for some $a, b \in \mathbb{Z}$. Observe that $\chi_{-1}^k = \chi_{(-1)^k}$, and that $\mathbb{Z}C_2$ is commutative. Consider the map $\varphi = \mathbb{Z}[x] \to \mathbb{Z}C_2$ given as follows.

$$
a_0 + a_1 x + \cdots + a_n x^n \mapsto a_0\chi_1 + a_1\chi_{-1} + a_2\chi_{(-1)^2} + \cdots + a_n\chi_{(-1)^n}
$$

Check that the above map is a surjective homomorphism of rings. We compute its kernel.

$$
\ker\varphi = \left\{ \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x] : a_0\chi_1 + a_1\chi_{-1} + \cdots + a_n\chi_{(-1)^n} = 0 \right\}
$$

We have $x^2 - 1 \in \ker\varphi$. We claim that the kernel is generated by $x^2 - 1$. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \ker\varphi$. Then, $\left(\sum_{k \text{ even}} a_k\right)\chi_1 + \left(\sum_{\ell \text{ odd}} a_\ell\right)\chi_{-1} = 0$. As $\{\chi_1, \chi_{-1}\}$ is a $\mathbb{Z}$ basis we have that $\sum_{k \text{ even}} a_k = 0$ and $\sum_{\ell \text{ odd}} a_\ell = 0$. Note that $f(1) = \sum_{k \text{ even}} a_k + \sum_{\ell \text{ odd}} a_\ell = 0$, and that $\sum_{k \text{ even}} a_k - \sum_{\ell \text{ odd}} a_\ell = 0$. Hence, $x-1$ and $x+1$ divides $f(x)$. As $x-1$ and $x+1$ are coprime, we get that

$(x-1)(x+1) = x^2 - 1$ divides $f(x)$. This shows that $\ker \varphi = (x^2 - 1)$. By the first isomorphism theorem, φ induces isomorphism of rings $\mathbb{Z}[x]/(x^2-1) \cong \mathbb{Z}C_2$. □

Checkpoint 1.3.5 Let G be a group of order two. Show that $\mathbb{Q}G \simeq \mathbb{Q} \times \mathbb{Q}$ via the map

$$\chi_1 \cdot r + \chi_{-1} \cdot s \mapsto (r+s, r-s).$$

Checkpoint 1.3.6 Let G be a finite group. Show that in a group ring $\mathbb{Z}G$ following holds.

1. $f = \sum_{g \in G} \chi_g$ satisfies $f^2 = f$
2. $f \neq 0$ and $1 - f \neq 0$
3. $f(1-f) = 0$

Theorem 1.3.7 *Suppose that G is a group and that R is a commutative ring. Then, RG is an R-algebra, and it is a free R-module with basis $\{\chi_g \mid g \in G\}$.*

Let A be an R-algebra, and let $\phi\colon G \to A$ be a homomorphism to the multiplicative monoid of A. Then, ϕ extends uniquely to an R-algebra homomorphism $\phi\colon RG \to A$ of R-algebras.

In other words, the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "g \mapsto \chi_g"] \arrow[d, "\phi"'] & RG \arrow[dl, dashed] \\ A & \end{tikzcd}$$

Figure 1.3.8 *Universal property of group algebra*

Proof. In view of Proposition 1.3.3, it suffices to show the universal property. Let A be an R-algebra and let $\phi\colon G \to A$ be a homomorphism to the multiplicative monoid of A. Define $\tilde{\phi}\colon RG \to A$ as follows.

$$\tilde{\phi}\left(\sum \chi_g r_g\right) = \sum \phi(g) r_g$$

We leave it to the reader to check that $\tilde{\phi}$ is a well-defined homomorphism of R-algebras and that it is the unique extension of ϕ. ■

Definition 1.3.9 Augmentation Ideal. Let R be a commutative ring with unity and G a nontrivial group. The *augmentation map* is the R-algebra homomorphism $\epsilon : RG \to R$ given by $\epsilon(\sum \chi_g r_g) = \sum r_g$. The kernel of the augmentation map is called the *augmentation ideal* of RG, and it is denoted by $I(G)$. ◊

Checkpoint 1.3.10 Augmentation map is an R-algebra homomorphism. Show that the augmentation map $\epsilon : RG \to R$ is a well-defined surjective R-algebra homomorphism.

Checkpoint 1.3.11 Generators of augmentation ideal. Let R be a commutative ring with unity and G a nontrivial group. Show that the augmentation ideal $I(G)$ is generated as an R-module by the set $\{\chi_g - \chi_e : g \in G, g \neq e\}$, where e is the identity of G.

1.4 Morphisms of algebras

Definition 1.4.1 Homomorphism R-algebras. Let R be a commutative ring with unity. Let A and B be R-algebras. An *R-algebra homomorphism* (or simply, a *morphism of R-algebras*) $\phi : A \to B$ is a map which is both a ring homomorphism and an R-module homomorphism.

An R-algebra homomorphism from A to itself is called an *endormorphism of R-algebras.* The set of all endomorphisms of R-algebras is denoted by $\mathrm{End}_R\,(A)$. ♢

Example 1.4.2 Regular representation. Let F be a field, and let A be an F-algebra. For $a \in A$ we define an F-linear map $L_a \colon A \to A$ by $L_a(x) = ax$ for all $x \in A$. The following map is called the *regular representation* of A.

$$A \to \mathrm{End}_F\,(A)\,, \quad \text{given by} \quad a \mapsto L_a.$$

We leave it to the reader to verify that the regular representation is a homomorphism of F-algebras.

Note that as A is unital $L_a \in \mathrm{End}_F\,(A)$ is nonzero for all nonzero $a \in A$. Thus, the regular representation is injective. □

Checkpoint 1.4.3 Quotient algebra. Let R be a commutative ring with unity. Let A be an R-algebra. Show that the quotient A/I is an R-algebra for any ideal I of A. Furthermore, show that the natural projection $\pi : A \to A/I$ is an R-algebra homomorphism.

If $\phi\colon A \to B$ is homomorphism of R-algebras, then show that $\ker(\phi)$ is an ideal of A and that the image of ϕ is an R-subalgebra of B. Furthermore, show that $A/\ker(\phi) \simeq \mathrm{Im}(\phi)$ as R-algebras.

Checkpoint 1.4.4 Factorization criterion. Let R be a commutative ring with unity. Let $\phi\colon A \to B$ and $\psi\colon A \to C$ be homomorphisms of R-algebras with ϕ surjective. Show that ψ factors through ϕ (i.e., there exists a homomorphism of R-algebras $\theta\colon B \to C$ such that $\psi = \theta \circ \phi$ if and only if $\ker(\phi) \subseteq \ker(\psi)$.

$$\begin{tikzcd} & A \arrow[dl, two heads, "\phi"'] \arrow[dr, "\psi"] & \\ B \arrow[rr, dashed, "\theta"] & & C \end{tikzcd}$$

Figure 1.4.5 Factorization criterion

In other words, there exists a dotted arrow (R-algebra homomorphism θ) making the above diagram commutative if and only if $\ker(\phi) \subseteq \ker(\psi)$.

Definition 1.4.6 Isomorphism of R-algebras. Let R be a commutative ring with unity. Let A and B be R-algebras. An R-algebra homomorphism $\phi : A \to B$ is an *isomorphism of R-algebras* if there exists an R-algebra homomorphism $\psi : B \to A$ such that $\psi \circ \phi = \mathbb{1}_A$ and $\phi \circ \psi = \mathbb{1}_B$. In this case, we say that A and B are *isomorphic as R-algebras* and we write $A \simeq B$.

An R-algebra isomorphism from A to itself is called an *automorphism of R-algebras.* The set of all automorphisms of R-algebras is denoted by $\mathrm{Aut}_R\,(A)$. ♢

Definition 1.4.7 Subalgebra. Let R be a commutative ring with unity. Let A be an R-algebra. An R-algebra B is called an *R-subalgebra* of A if $B \subseteq A$ and the inclusion map $B \hookrightarrow A$ is an R-algebra homomorphism. ♢

Colophon

This book was authored in PreTeXt.