Finitely dependent processes on $\mathbb{Z}$: a human-readable companion

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Contents

Scope. This document is intended to be a readable companion to the Lean 4 development in the repository root. It presents the same headline results as the public Lean API `FiniteDependence/API/Definitions.lean` and `FiniteDependence/API/MainTheorems.lean`. In particular, it proves:

- equivalence of the cut and noncontiguous notions of finite dependence on $\mathbb{Z}$;
- proper colorings: existence of stationary 1-dependent 4-colorings, but no stationary 0-dependent proper 4-coloring;
- proper colorings: existence of stationary 2-dependent 3-colorings, but no stationary 1-dependent proper 3-coloring;
- weak 2-colorings (forbidden `000`, `111`): existence at dependence range 3, impossibility at range 2;

- greedy proper 3-colorings: existence at dependence range 6, impossibility at range 5;
- MIS: existence at dependence range 6, impossibility at range 5.

The existence results are short reductions from Holroyd–Liggett [HL16]. For early examples separating finite dependence from block-factor structure, see Burton–Goulet–Meester [BGM93]. The MIS $k = 5$ lower bound is presented at the level of finite cylinder probabilities, with exact verification scripts in `proof/scripts/`.

Main statements (API order). The public Lean API exposes exactly the following headline theorems, proved here as:

- cut/noncontiguous dependence equivalence: Proposition 2;
- 4-colorings ($k = 1$ exists, $k = 0$ impossible): Corollary 2;
- 3-colorings ($k = 2$ exists, $k = 1$ impossible): Corollary 6;
- weak 2-colorings ($k = 3$ exists, $k = 2$ impossible): Corollary 4;
- greedy proper 3-colorings ($k = 6$ exists, $k = 5$ impossible): Corollary 5;
- MIS ($k = 6$ exists, $k = 5$ impossible): Corollary 3.

1 Problem statement and basic definitions

We view $\mathbb{Z}$ as the vertex set of the integer line graph, with edges between i and $i + 1$.

Definition 1 (MIS on $\mathbb{Z}$)**.** For a configuration $x \in \{0, 1\}^{\mathbb{Z}}$ we say:

- x is an *independent set* if there are no adjacent 1's, i.e. $x_i = x_{i+1} = 1$ never occurs;
- x is a *maximal independent set* (MIS) if it is an independent set and moreover one cannot add any additional vertex: for every i with $x_i = 0$, at least one of x_{i-1}, x_{i+1} is 1.

It is convenient to express MIS as a shift of finite type.

Proposition 1 (MIS as forbidden patterns)**.** *A configuration $x \in \{0, 1\}^{\mathbb{Z}}$ is an MIS if and only if it forbids the two finite patterns*

$$\texttt{11}, \qquad \texttt{000},$$

meaning that neither word appears as a contiguous subword of x.

Proof. If `11` occurs then x is not independent. If `000` occurs at sites $i - 1, i, i + 1$, then the middle site i has both neighbors 0, so adding i preserves independence, contradicting maximality.

Conversely, if `11` never occurs then x is independent. If $x_i = 0$ then the triple (x_{i-1}, x_i, x_{i+1}) is not `000`, hence at least one neighbor is 1; thus no vertex can be added and x is maximal. ☐

1.1 Stationary processes and cylinders

Let $(X_i)_{i\in\mathbb{Z}}$ be a random process taking values in $\{0,1\}$. For a finite binary word $w = w_0 \cdots w_{L-1} \in \{0,1\}^L$, define the origin cylinder probability

$$P(w) := \mathbb{P}(X_0 \cdots X_{L-1} = w).$$

When the law is stationary, this equals the probability of seeing w starting at *any* site.

Definition 2 (Stationarity)**.** Let μ be a probability measure on $\{0,1\}^{\mathbb{Z}}$ and let T be the left shift $(Tx)_i := x_{i+1}$. We call μ *stationary* if $\mu \circ T^{-1} = \mu$. Equivalently, all cylinder probabilities $\mathbb{P}(X_a \cdots X_{a+L-1} = w)$ are independent of a.

Stationarity implies the basic "prefix = suffix" identities: for every word $u \in \{0,1\}^{L-1}$,

$$\sum_{b\in\{0,1\}} P(ub) = \sum_{b\in\{0,1\}} P(bu). \tag{1}$$

(Both sides equal $\mathbb{P}(X_0 \cdots X_{L-2} = u)$, computed either as a prefix marginal or a suffix marginal.)

1.2 Two notions of k-dependence

We now define two standard notions of finite dependence for $\mathbb{Z}$-indexed processes. Let $\mathcal{F}_A := \sigma(X_i : i \in A)$ be the coordinate σ-field generated by indices in $A \subseteq \mathbb{Z}$.

Definition 3 (Distance between index sets)**.** For $A, B \subseteq \mathbb{Z}$ define

$$\mathrm{dist}(A,B) := \inf\{|a-b| : a \in A, b \in B\} \in \{0,1,2,\dots\} \cup \{\infty\}.$$

Definition 4 (k-dependence: noncontiguous (index-set) form)**.** The process $(X_i)_{i\in\mathbb{Z}}$ is *k-dependent* in the noncontiguous form if for all $A, B \subseteq \mathbb{Z}$ with $\mathrm{dist}(A,B) > k$, the σ-fields $\mathcal{F}_A$ and $\mathcal{F}_B$ are independent.

Definition 5 (k-dependence: cut form)**.** The process is *k-dependent* in the cut form if for every $i \in \mathbb{Z}$ the "past" and "future" σ-fields

$$\mathcal{F}_{(-\infty,i]} \quad \text{and} \quad \mathcal{F}_{[i+k+1,\infty)}$$

are independent.

Proposition 2 (Cut and noncontiguous definitions agree on $\mathbb{Z}$)**.** *Definitions 4 and 5 are equivalent.*

Proof. Noncontiguous $\Rightarrow$ cut: apply Definition 4 with $A = (-\infty, i]$ and $B = [i+k+1, \infty)$, whose distance is $k+1$.

Cut $\Rightarrow$ noncontiguous is more subtle: one first proves independence for finite index sets by repeatedly applying cut-independence across a suitable sequence of cuts, then extends to arbitrary sets by directed-σ-field (equivalently monotone-class) arguments. This is exactly what is formalized in Lean in `FiniteDependence/Coloring/DependenceEquivalence.lean`. $\square$

1.3 A cylinder identity from k-dependence

The lower-bound proofs use k-dependence only through a finite-word identity.

Lemma 1 (Prefix/suffix independence identity)**.** *Assume (X_i) is k-dependent. Let $x \in \{0,1\}^m$ and $y \in \{0,1\}^n$ and set $L = m + k + n$. Then*

$$\sum_{g \in \{0,1\}^k} P(xgy) = P(x)\,P(y). \tag{2}$$

Proof. Let A be the event $\{X_0 \cdots X_{m-1} = x\}$ and let B be the event $\{X_{m+k} \cdots X_{m+k+n-1} = y\}$. The index sets $\{0, \ldots, m-1\}$ and $\{m+k, \ldots, m+k+n-1\}$ have distance $k+1 > k$, so A and B are independent and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = P(x)P(y)$.

On the other hand, $A \cap B$ is the disjoint union (over $g \in \{0,1\}^k$) of the cylinder events $\{X_0 \cdots X_{L-1} = xgy\}$. Summing their probabilities gives (2). $\square$

2 Coloring and MIS existence results

The core existence input is Theorem 1 of Holroyd–Liggett [HL16].

Theorem 1 (Holroyd–Liggett)**.** *There exist stationary proper colorings on $\mathbb{Z}$ with:*

- *a 1-dependent 4-coloring;*
- *a 2-dependent 3-coloring.*

Corollary 1 (API existence statements for proper colorings)**.** *There exists a stationary proper 4-coloring with dependence range 1, and there exists a stationary proper 3-coloring with dependence range 2.*

Proof. This is exactly Theorem 1. $\square$

Remark 1. Beyond Theorem 1, explicit finitely dependent colorings on $\mathbb{Z}$ include the symmetric 1-dependent q-colorings for all $q \geq 4$ constructed by Holroyd–Liggett [HL15]. We do not use these additional families in the proofs below.

2.1 No stationary 0-dependent proper 4-coloring

Theorem 2. *There is no stationary 0-dependent proper 4-coloring process on $\mathbb{Z}$.*

Proof. Suppose $(C_i)_{i\in\mathbb{Z}}$ is stationary, proper, and 0-dependent, with values in $\{1,2,3,4\}$. For each color a, let

$$A_a := \{C_0 = a\}, \qquad B_a := \{C_1 = a\}.$$

Properness implies $A_a \cap B_a = \varnothing$, hence $\mathbb{P}(A_a \cap B_a) = 0$.

By 0-dependence, $\sigma(C_0)$ and $\sigma(C_1)$ are independent, so

$$0 = \mathbb{P}(A_a \cap B_a) = \mathbb{P}(A_a)\mathbb{P}(B_a).$$

Stationarity gives $\mathbb{P}(A_a) = \mathbb{P}(B_a)$, therefore $\mathbb{P}(A_a)^2 = 0$ and $\mathbb{P}(A_a) = 0$ for each a. Summing over $a \in \{1,2,3,4\}$,

$$1 = \sum_{a=1}^{4} \mathbb{P}(A_a) = 0,$$

contradiction. $\square$

Corollary 2 (Threshold for proper 4-colorings). *For proper* 4*-colorings on* $\mathbb{Z}$*, stationary finite dependence is possible at range* 1 *and impossible at range* 0*.*

Proof. Combine Corollary 1 (the 4-coloring existence part) with Theorem 2. $\square$

2.2 A greedy local map from a 3-coloring to an MIS

Given a proper coloring $c \in \{0,1,2\}^{\mathbb{Z}}$ we define a 0/1 configuration $x = \mathrm{GreedyMIS}(c) \in \{0,1\}^{\mathbb{Z}}$ by processing colors in order $0,1,2$:

- accept every vertex of color 0;
- accept a vertex of color 1 if neither neighbor was accepted (equivalently, if neither neighbor has color 0);
- accept a vertex of color 2 if neither neighbor was accepted (equivalently, if neither neighbor has color 0 and neither neighbor is a color-1 vertex that gets accepted).

Concretely, acceptance at site i depends only on the colors in the window $\{i-2, i-1, i, i+1, i+2\}$ (radius 2). This is exactly the local rule implemented in the Lean file `FiniteDependence/Coloring/MIS.lean`.

Lemma 2 (Greedy map produces an MIS). *If c is a proper coloring of $\mathbb{Z}$, then $x = \mathrm{GreedyMIS}(c)$ is a maximal independent set. Equivalently, x forbids* `11` *and* `000`*.*

Proof sketch. By construction, whenever a vertex is accepted, its two neighbors are rejected at the moment they are processed, so adjacent accepted vertices never occur; hence `11` is forbidden.

For maximality, if $x_i = 0$ then i was rejected because at least one neighbor was accepted earlier in the greedy order (a neighbor of color 0 is always accepted; a neighbor of color 1 is accepted precisely when it has no adjacent 0; etc.). Thus every rejected vertex has an accepted neighbor, so `000` is forbidden.

A fully formal proof appears in `FiniteDependence/Coloring/MIS.lean`. $\square$

2.3 Dependence range

Local maps preserve stationarity, and they enlarge the dependence range by at most twice the local radius.

Lemma 3 (Local maps preserve finite dependence). *Let $(Y_i)_{i\in\mathbb{Z}}$ be a k-dependent process and suppose X_i is a deterministic function of $(Y_{i-r}, \ldots, Y_{i+r})$ for some radius r. Then (X_i) is $(k+2r)$-dependent.*

Proof. If $A, B \subseteq \mathbb{Z}$ have $\mathrm{dist}(A,B) > k+2r$, then the r-neighborhoods $A^{+r} := \{a-r, \ldots, a+r : a \in A\}$ and B^{+r} satisfy $\mathrm{dist}(A^{+r}, B^{+r}) > k$. The σ-field generated by $(X_i)_{i\in A}$ is contained in that generated by $(Y_j)_{j\in A^{+r}}$, and similarly for B. Independence of the latter (by k-dependence of Y) implies independence of the former. $\square$

Combining Theorem 1 (its 2-dependent 3-coloring part) with the radius-2 greedy map gives:

Theorem 3 (Existence of a stationary 6-dependent MIS on $\mathbb{Z}$). *There exists a stationary* 6*-dependent probability measure on MIS configurations in $\{0,1\}^{\mathbb{Z}}$.*

Proof. Let (C_i) be the stationary 2-dependent proper 3-coloring from Theorem 1. Define $X = \mathrm{GreedyMIS}(C)$ by the greedy radius-2 map. By Lemma 2, X is almost surely an MIS. By Lemma 3 with $(k,r) = (2,2)$, the process X is 6-dependent, and it is stationary because it is a shift-commuting factor of a stationary process. $\square$

3 Lower bound: no stationary k-dependent MIS for $k \le 5$

Throughout this section, $(X_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ is assumed stationary, supported on MIS configurations, and k-dependent.

Write

$$p := \mathbb{P}(X_0 = 1).$$

From stationarity and the forbidden word `11` we have

$$P(\texttt{01}) = P(\texttt{10}) = p, \qquad P(\texttt{00}) = 1 - 2p.$$

Also `000` forbidden implies $p \ge 1/3$ (each triple contains a 1), and `11` implies $p \le 1/2$.

3.1 No k-dependent MIS for $k \le 2$

Proposition 3. *There is no stationary k-dependent MIS on $\mathbb{Z}$ for $k \le 2$.*

Proof. Assume $k \le 2$. Let

$$A := \{X_0X_1 = \texttt{00}\}, \qquad B := \{X_4X_5 = \texttt{00}\}.$$

These events depend on disjoint index sets at distance $3 > k$, hence are independent: $\mathbb{P}(A \cap B) = \mathbb{P}(A)^2$.

But A forces $X_2 = 1$ (else $X_0X_1X_2 = \texttt{000}$), and B forces $X_3 = 1$, so $A \cap B$ forces $X_2X_3 = \texttt{11}$, impossible for an MIS. Thus $\mathbb{P}(A \cap B) = 0$ and hence $\mathbb{P}(A) = 0$, i.e. $P(\texttt{00}) = 0$.

If `00` never occurs, the only MIS configurations are the two alternating sequences $\cdots\texttt{1010}\cdots$ and $\cdots\texttt{0101}\cdots$. Hence for every n, X_n is a deterministic function of X_0 (equal to X_0 for even n, equal to $1 - X_0$ for odd n). In particular, X_0 and X_{k+1} are not independent for any finite k, contradicting k-dependence. □

3.2 No stationary 3-dependent MIS

Proposition 4. *There is no stationary 3-dependent MIS process on $\mathbb{Z}$.*

Proof. Assume $k = 3$. The allowed length-5 words (forbidding `11` and `000`) are exactly:

$$\texttt{00100}, \texttt{00101}, \texttt{01001}, \texttt{01010}, \texttt{10010}, \texttt{10100}, \texttt{10101}.$$

Write their probabilities as

$$\begin{aligned} &a := P(\texttt{00100}), \quad b := P(\texttt{00101}), \quad c := P(\texttt{01001}), \quad d := P(\texttt{01010}), \\ &e := P(\texttt{10010}), \quad f := P(\texttt{10100}), \quad g := P(\texttt{10101}). \end{aligned}$$

Step 1: determine length-5 cylinders in terms of p. Since $|0 - 4| = 4 > 3$, 3-dependence implies $X_0 \perp X_4$, hence

$$\mathbb{P}(X_0 = 1, X_4 = 1) = p^2.$$

The only allowed length-5 word starting and ending with 1 is `10101`, hence $g = p^2$.

Stationarity at length 4 (Equation (1)) gives the five linear constraints:

$$\begin{aligned}
\text{for } \texttt{0010}: &\quad a + b = e, \\
\text{for } \texttt{0100}: &\quad c = a + f, \\
\text{for } \texttt{0101}: &\quad d = b + g, \\
\text{for } \texttt{1001}: &\quad e = c, \\
\text{for } \texttt{1010}: &\quad f + g = d.
\end{aligned}$$

Finally, $X_0 \perp X_4$ implies the endpoint constraints

$$a + d = \mathbb{P}(X_0 = 0, X_4 = 0) = (1-p)^2, \qquad e + f = \mathbb{P}(X_0 = 1, X_4 = 0) = p(1-p).$$

Solving these equations yields the explicit formulas:

$$\begin{aligned}
e = c &= 1 - 2p, \\
b = f &= 3p - 1 - p^2, \\
d &= 3p - 1, \\
g &= p^2, \\
a &= p^2 - 5p + 2.
\end{aligned}$$

Step 2: a length-6 split identity forces a quadratic for p. Because $\operatorname{dist}(\{0\}, \{4, 5\}) = 4 > 3$, we also have $X_0 \perp (X_4, X_5)$ and therefore

$$\mathbb{P}(X_0 = 1, X_4X_5 = \texttt{01}) = \mathbb{P}(X_0 = 1)\,\mathbb{P}(X_4X_5 = \texttt{01}) = p\,P(\texttt{01}) = p^2.$$

The event $\{X_0 = 1, X_4X_5 = \texttt{01}\}$ is the disjoint union of the two length-6 cylinders

$$\texttt{100101}, \quad \texttt{101001},$$

so

$$P(\texttt{100101}) + P(\texttt{101001}) = p^2.$$

In the MIS SFT, each of these length-6 words is a *unique* extension of a length-5 word: `100101` is the unique length-6 word with suffix `00101`, and `101001` is the unique length-6 word with prefix `10100`. Hence

$$P(\texttt{100101}) = P(\texttt{00101}) = b, \qquad P(\texttt{101001}) = P(\texttt{10100}) = f = b,$$

so $2b = p^2$. Using $b = 3p - 1 - p^2$ gives

$$2(3p - 1 - p^2) = p^2, \qquad \text{i.e.} \qquad 3p^2 - 6p + 2 = 0.$$

Since $p \in [1/3, 1/2]$, the only admissible root is

$$p_\star = 1 - \frac{1}{\sqrt{3}}.$$

Step 3: one length-9 cylinder gets two incompatible values. From marginalization,

$$P(\texttt{010100100}) = P(\texttt{10100100})$$

(because 110100100 is forbidden).

Apply the split identity (Lemma 1) to 010100100 with split $1 + 3 + 5$:

$$P(\texttt{010100100}) = \mathbb{P}(X_0 = 0)\, P(\texttt{00100}) = (1 - p)(p^2 - 5p + 2).$$

Next compute $P(\texttt{10100100})$ directly. Among allowed words of the form 1___0100 there are exactly two:

$$\texttt{10010100}, \qquad \texttt{10100100}.$$

Hence, by $X_0 \perp (X_4, \ldots, X_7)$,

$$P(\texttt{10010100}) + P(\texttt{10100100}) = \mathbb{P}(X_0 = 1)\, P(\texttt{0100}) = p\, P(\texttt{0100}) = p(1 - 2p).$$

Also, 10010100 is the unique allowed word with prefix 100 and suffix 00, so the split $3 + 3 + 2$ gives

$$P(\texttt{10010100}) = P(\texttt{100})\, P(\texttt{00}) = (1 - 2p)^2.$$

Therefore

$$P(\texttt{10100100}) = p(1 - 2p) - (1 - 2p)^2 = (1 - 2p)(3p - 1).$$

So we get two expressions for the same cylinder:

$$P(\texttt{010100100}) = (1 - p)(p^2 - 5p + 2), \qquad P(\texttt{010100100}) = (1 - 2p)(3p - 1).$$

At $p = p_\star = 1 - 1/\sqrt{3}$, these are

$$1 - \frac{5\sqrt{3}}{9} \quad \text{and} \quad -4 + \frac{7\sqrt{3}}{3},$$

whose difference is

$$\left(-4 + \frac{7\sqrt{3}}{3}\right) - \left(1 - \frac{5\sqrt{3}}{9}\right) = -5 + \frac{26\sqrt{3}}{9} \neq 0.$$

Contradiction. $\square$

3.3 No stationary 4-dependent MIS

Proposition 5. *There is no stationary 4-dependent MIS process on $\mathbb{Z}$.*

Proof. Assume $k = 4$ and write $p = \mathbb{P}(X_0 = 1)$. As above, $P(\texttt{00}) = 1 - 2p$.

Step 1: two forced cylinders. We show that 4-dependence forces the exact formulas

$$P(\texttt{10100}) = \frac{p^2}{2}, \qquad P(\texttt{00100}) = 1 - 2p - \frac{p^2}{2}.$$

First, note that the word 10100 uniquely extends to the length-7 word 1010010, so

$$P(\texttt{10100}) = P(\texttt{1010010}).$$

Set

$$b := P(\texttt{0010100}).$$

Claim A: $P(\texttt{10100}) = p^2/2$. Split identities with gap 4 give

$$P(\texttt{0010101}) = p(1-2p), \qquad P(\texttt{1010100}) = p(1-2p).$$

Hence stationarity yields

$$P(\texttt{1001010}) = P(\texttt{0010100}) + P(\texttt{0010101}) = b + p(1-2p),$$

and

$$P(\texttt{1010010}) = P(\texttt{0101001}) = P(\texttt{0010100}) + P(\texttt{1010100}) = b + p(1-2p),$$

so $P(\texttt{1001010}) = P(\texttt{1010010})$. Now split $1+4+2$ with $(X_0, X_5X_6) = (1, 10)$. The only allowed words are 1001010 and 1010010, thus

$$P(\texttt{1001010}) + P(\texttt{1010010}) = p^2.$$

By equality of the two summands, each is $p^2/2$, therefore

$$P(\texttt{10100}) = P(\texttt{1010010}) = \frac{p^2}{2}.$$

Claim B: $P(\texttt{00100}) = 1 - 2p - p^2/2$. Again by unique extension, $P(\texttt{00100}) = P(\texttt{0010010})$. From Claim A and $P(\texttt{1010010}) = b + p(1-2p)$,

$$b = \frac{p^2}{2} - p(1-2p) = \frac{5p^2}{2} - p.$$

Now split $1+4+2$ with $(X_0, X_5X_6) = (0, 00)$: the only allowed words are 0010100 and 0100100, so

$$b + P(\texttt{0100100}) = (1-p)P(\texttt{00}) = (1-p)(1-2p),$$

hence $P(\texttt{0100100}) = 1 - 2p - p^2/2$. Finally stationarity gives

$$P(\texttt{0010010}) = P(\texttt{1001001}) = P(\texttt{0100100}),$$

therefore

$$P(\texttt{00100}) = P(\texttt{0010010}) = 1 - 2p - \frac{p^2}{2}.$$

Moreover, 4-dependence applied to the split $2+4+2 = 8$ and the SFT unique-middle fact

00____00is allowed only with middle 1001

forces

$$P(\texttt{00100100}) = P(\texttt{00})^2 = (1-2p)^2.$$

Step 2: a length-14 word identity gives $h(p) = 0$. Let

$$u := \texttt{00100100100100}.$$

This word admits two decompositions with the same unique middle word 1001:

$$\texttt{00100}, \texttt{1001}, \texttt{00100} \qquad \text{and} \qquad \texttt{00}, \texttt{1001}, \texttt{00100100}.$$

Applying the split identity (gap 4) in each decomposition gives

$$P(\texttt{00100100100100}) = P(\texttt{00100})^2 = P(\texttt{00})\,P(\texttt{00100100}).$$

Substituting the forced expressions above yields a necessary polynomial identity $h(p) = 0$, namely

$$h(p) := P(\texttt{00100})^2 - P(\texttt{00})\,P(\texttt{00100100}) = \frac{p(p^3 + 40p^2 - 36p + 8)}{4} = 0.$$

Step 3: a length-10 solvability condition gives $g(p) = 0$. From Step 1 and 4-dependence:

$$P(\texttt{0010010010}) = (1-2p)^2, \tag{E1}$$

$$P(\texttt{0101010100}) = (1-2p)(3p-1), \tag{E2}$$

$$P(\texttt{0010010100}) + P(\texttt{0101010100}) = (1-p)\,P(\texttt{10100}) = (1-p)\frac{p^2}{2}, \tag{E3}$$

$$P(\texttt{0010010010}) + P(\texttt{0010010100}) = (1-p)\,P(\texttt{00100}) = (1-p)\Big(1-2p-\frac{p^2}{2}\Big). \tag{E4}$$

These are all short finite-word checks:

- (E1) `00100100` has unique two-step extension `10`.
- (E2) split $4+4+2$ with $(x, y) = (\texttt{0101}, \texttt{00})$ has unique allowed word `0101010100`, and $P(\texttt{0101}) = P(\texttt{101}) = 3p-1$.
- (E3) split $1+4+5$ with $(\texttt{0}, \texttt{10100})$ has exactly two allowed words: `0010010100`, `0101010100`.
- (E4) split $5+4+1$ with $(\texttt{00100}, \texttt{0})$ has exactly two allowed words: `0010010010`, `0010010100`.

Taking the linear combination $-(E1)+(E2)-(E3)+(E4)$ cancels all length-10 cylinders and yields

$$g(p) := p^3 - 9p^2 + 6p - 1 = 0,$$

so any stationary 4-dependent MIS must satisfy this cubic constraint.

Step 4: g and h are incompatible. Let

$$H(p) := 4h(p) = p(p^3 + 40p^2 - 36p + 8).$$

Any putative process must satisfy

$$g(p) = 0, \qquad H(p) = 0.$$

Euclidean division gives

$$H(p) = (p+49)g(p) + r(p), \qquad r(p) := 399p^2 - 285p + 49,$$

$$2793\,g(p) = (7p-58)\,r(p) + (49 - 115p).$$

Hence $g = H = 0$ implies $r = 0$, then $49 - 115p = 0$, so $p = 49/115$. But

$$r(49/115) = \frac{49}{13225} \neq 0,$$

contradiction. $\square$

3.4 No stationary 5-dependent MIS (computer-assisted, exact)

Theorem 4. *There is no stationary 5-dependent MIS process on $\mathbb{Z}$.*

Proof (with exact computation). Assume $k = 5$. Let

$$p := \mathbb{P}(X_0 = 1), \qquad t := P(\texttt{1010101}).$$

The proof is a finite, exact contradiction about cylinder probabilities. Conceptually, stationarity and 5-dependence impose a large but *linear* constraint system on finite-word probabilities; inconsistency of that system rules out the process.

Complexity bookkeeping. The MIS shift forbidding `11` and `000` has Fibonacci-many admissible words: the number of allowed length-L words is $2, 3, 4, 5, 7, 9, 12, \ldots$ (in particular, 12 at $L = 7$ and 1897 at $L = 25$). The argument below uses only sparse split-row cancellations and exact polynomial algebra over $\mathbb{Q}$.

Step 1: forced length-7 family. Stationarity, the MIS constraint, and independence of X_0 and X_6 (distance $6 > 5$) determine an affine 2-parameter family of length-7 cylinder probabilities, parameterized by (p, t). Concretely, the allowed length-7 words are the following 12 words, with:

$$
\begin{aligned}
P(\texttt{0010010}) &= p^2 - t,\\
P(\texttt{0010100}) &= -2p^2 - 7p + 3t + 3,\\
P(\texttt{0010101}) &= p^2 + 5p - 2t - 2,\\
P(\texttt{0100100}) &= p^2 - t,\\
P(\texttt{0100101}) &= -p^2 - 2p + t + 1,\\
P(\texttt{0101001}) &= -p^2 - 2p + t + 1,\\
P(\texttt{0101010}) &= p^2 + 5p - t - 2,\\
P(\texttt{1001001}) &= p^2 - t,\\
P(\texttt{1001010}) &= -p^2 - 2p + t + 1,\\
P(\texttt{1010010}) &= -p^2 - 2p + t + 1,\\
P(\texttt{1010100}) &= p^2 + 5p - 2t - 2,\\
P(\texttt{1010101}) &= t.
\end{aligned}
$$

(All other length-7 cylinders are 0 because they are not MIS-admissible.) This small linear system is solved exactly and printed by `proof/scripts/prove_k5_impossible.py`.

Step 2: two explicit compatibility certificates. Starting from the forced length-7 family, we first extend exactly to length 15. Now build two *fixed* overdetermined split systems whose RHS is computed from these length-15 marginals:

- length 16: 151 unknown cylinders, 727 equations;
- length 20: 465 unknown cylinders, 3002 equations.

For a split row, write

$$E_{m,x;n,y} : \sum_{g \in \{0,1\}^5 \,:\, xgy \text{ admissible}} P(\texttt{xgy}) = P(\texttt{x})P(\texttt{y}).$$

Every $E_{m,x;n,y}$ is one row of the corresponding fixed system.

Certificate for $f(p,t) = 0$ (length 16*).* In the length-16 system, consider

$$
\begin{aligned}
\mathcal{C}_f := {} & E_{1,\texttt{1};10,\texttt{0010100101}} - E_{3,\texttt{100};8,\texttt{10100101}} + E_{5,\texttt{10100};6,\texttt{001001}}\\
& - E_{5,\texttt{10100};6,\texttt{100101}} + E_{6,\texttt{100100};5,\texttt{00101}} + E_{8,\texttt{10100101};3,\texttt{101}} - E_{10,\texttt{1010010101};1,\texttt{1}}.
\end{aligned}
$$

The left-hand side of $\mathcal{C}_f$ is an exact telescoping cancellation: there are 12 split-word terms before cancellation and 0 after cancellation. Hence the same signed combination of right-hand sides must vanish, giving

$$f(p, t) = 0.$$

Certificate for $r(p,t)=0$ (length 20). In the length-20 system, consider

$$\begin{aligned}\mathcal{C}_r := & -E_{5,\mathtt{00100};10,\mathtt{1010010100}} + E_{8,\mathtt{00100100};7,\mathtt{0010100}} \\ & + E_{10,\mathtt{0010010100};5,\mathtt{10100}} - E_{13,\mathtt{0010010100100};2,\mathtt{00}}.\end{aligned}$$

Again the left-hand side cancels exactly (here: 6 split-word terms to 0), so the signed RHS combination vanishes:

$$r(p,t)=0.$$

Thus Step 2 uses only sparse row-combination checks (certificates), not solving the 727×151 or 3002×465 fixed systems.

Therefore any stationary 5-dependent MIS must satisfy

$$\begin{aligned} f(p,t) &= -3p^4 - 20p^3 + 6p^2t - 48p^2 + 30pt + 45p - 3t^2 - 12t - 9, \\ r(p,t) &= -16p^4 + 12p^2t + 84p^2 - 60pt - 60p + 9t^2 + 21t + 11. \end{aligned}$$

Hence

$$g(p,t) := 3f(p,t) + r(p,t) = -25p^4 - 60p^3 + 30p^2t - 60p^2 + 30pt + 75p - 15t - 16$$

must also vanish.

Step 3: eliminate t. The polynomial g is linear in t, so one can solve explicitly:

$$t = \frac{25p^4 + 60p^3 + 60p^2 - 75p + 16}{15(2p^2 + 2p - 1)}.$$

Substituting this into f and clearing denominators yields

$$A(p)B(p) = 0,$$

where

$$\begin{aligned} A(p) &= 5p^4 - 355p^3 + 460p^2 - 200p + 29, \\ B(p) &= 5p^4 - 5p^3 - 5p^2 + 5p - 1. \end{aligned}$$

Hence any putative process must satisfy either $A(p)=0$ or $B(p)=0$.

Step 4: a sparse length-19/25 certificate gives $q(p,t)=0$. Set

$$\begin{aligned} Y_{19} &:= P(\mathtt{0010101010010010100}), \\ S_{25} &:= P(\mathtt{0101010010101010010010100}), \\ A_{25} &:= P(\mathtt{0010010010101010010010100}), \\ T_{25} &:= P(\mathtt{0010010100101010010010100}). \end{aligned}$$

From two length-19 split rows (supports of sizes 1 and 2):

$$P(\mathtt{0010101001010010100}) = P(\mathtt{00})\,P(\mathtt{001010010100}),$$

$$P(\mathtt{0010101001010010100}) + Y_{19} = P(\mathtt{0010101})\,P(\mathtt{0010100}),$$

we get

$$Y_{19} = P(\mathtt{0010101})\,P(\mathtt{0010100}) - P(\mathtt{00})\,P(\mathtt{001010010100}).$$

From two length-25 split rows (again supports of sizes 1 and 2):

$$P(\mathtt{0101010010101001010010100}) = P(\mathtt{01010100})\,P(\mathtt{001010010100}),$$

$$P(\mathtt{0101010010101001010010100}) + S_{25} = P(\mathtt{0101010010101})\,P(\mathtt{0010100}),$$

we get

$$S_{25} = P(\mathtt{0101010010101})\,P(\mathtt{0010100}) - P(\mathtt{01010100})\,P(\mathtt{001010010100}).$$

Now use the three length-25 rows (supports of sizes $2, 2, 1$):

$$\begin{aligned}
&\text{(A)} \quad A_{25} + S_{25} = P(\mathtt{0})\,Y_{19} = (1-p)\,Y_{19},\\
&\text{(B)} \quad A_{25} + T_{25} = P(\mathtt{00100})\,P(\mathtt{101010010010100}),\\
&\qquad\text{(W)} \quad T_{25} = P(\mathtt{0010010100})^{\,2}.
\end{aligned}$$

Eliminating A_{25}, T_{25} gives

$$(1-p)Y_{19} - S_{25} - P(\mathtt{00100})\,P(\mathtt{101010010010100}) + P(\mathtt{0010010100})^{\,2} = 0.$$

Substituting the formulas above for Y_{19}, S_{25} and the forced length-15 marginals yields a third compatibility polynomial

$$\begin{aligned}
q(p,t) := {}& 4p^6 + 64p^5 - 12p^4t + 528p^4 - 240p^3t + 80p^3 + 12p^2t^2\\
&- 576p^2t - 552p^2 + 180pt^2 + 540pt + 260p - 4t^3 - 72t^2 - 108t - 35 = 0.
\end{aligned}$$

Step 5: eliminate t and compare univariate constraints. From Step 3:

$$A(p)B(p) = 0.$$

Substituting

$$t = \frac{25p^4 + 60p^3 + 60p^2 - 75p + 16}{15(2p^2 + 2p - 1)}$$

into q gives

$$q\big(p, t(p)\big) = \frac{P_{12}(p)}{3375(2p^2 + 2p - 1)^3},$$

where

$$\begin{aligned}
P_{12}(p) := {}& 500p^{12} - 306000p^{11} + 976500p^{10} + 6273000p^9 - 11736300p^8 - 1616400p^7\\
&+ 16300800p^6 - 13617000p^5 + 4003260p^4 + 377640p^3 - 541710p^2 + 129690p - 10579.
\end{aligned}$$

Hence any putative process must satisfy $P_{12}(p) = 0$ as well. Exact Euclidean-algorithm checks give

$$\gcd(A, P_{12}) = \gcd(B, P_{12}) = 1, \qquad \gcd\big(A, 2p^2 + 2p - 1\big) = \gcd\big(B, 2p^2 + 2p - 1\big) = 1.$$

So $A(p)B(p) = 0$ and $P_{12}(p) = 0$ are incompatible, contradiction.

Therefore no stationary 5-dependent MIS can exist. □

Corollary 3 (Threshold for MIS)**.** *For MIS on $\mathbb{Z}$, stationary finite dependence is possible at range 6 and impossible at range 5.*

Proof. Combine Theorem 3 with Theorem 4. □

4 Weak 2-coloring (forbidden 000,111): threshold $k = 3$

Call a binary process $(X_i)_{i\in\mathbb{Z}}$ a *weak 2-coloring process* if it forbids

$$\texttt{000}, \qquad \texttt{111}.$$

Theorem 5 (Existence at dependence range 3)**.** *There exists a stationary 3-dependent weak 2-coloring process on $\mathbb{Z}$.*

Proof. Let $(C_i)_{i\in\mathbb{Z}}$ be the stationary 2-dependent proper 3-coloring from Theorem 1, with values in $\{0, 1, 2\}$. Define

$$Y_i := \mathbf{1}\{C_i < C_{i+1}\} \in \{0, 1\}.$$

If $Y_i = Y_{i+1} = Y_{i+2} = 1$, then

$$C_i < C_{i+1} < C_{i+2} < C_{i+3},$$

impossible in $\{0, 1, 2\}$. So 111 is forbidden.

If $Y_i = Y_{i+1} = Y_{i+2} = 0$, then

$$C_i > C_{i+1} > C_{i+2} > C_{i+3},$$

also impossible in $\{0, 1, 2\}$. So 000 is forbidden.

Thus Y is a weak 2-coloring process. It is stationary because it is a shift-commuting factor of C.

For dependence, note Y_i depends only on (C_i, C_{i+1}). Hence for index sets $A, B \subseteq \mathbb{Z}$, the σ-field generated by $(Y_i)_{i\in A}$ is contained in that generated by $(C_j)_{j\in A\cup(A+1)}$, and similarly for B. If $\mathrm{dist}(A, B) > 3$, then $\mathrm{dist}(A \cup (A + 1), B \cup (B + 1)) > 2$. By 2-dependence of C, these enlarged coordinate families are independent, so (Y_i) is 3-dependent. □

Theorem 6 (No dependence range 2)**.** *There is no stationary 2-dependent weak 2-coloring process on $\mathbb{Z}$.*

Proof. Suppose $X = (X_i)_{i\in\mathbb{Z}}$ is stationary, 2-dependent, and forbids $\texttt{000}, \texttt{111}$. Define a one-sided radius-3 local map $\Phi : \{0, 1\}^4 \to \{0, 1\}$ by

$$\Phi(u_0u_1u_2u_3) := \mathbf{1}\{u_0u_1u_2u_3 \in \{\texttt{0010}, \texttt{0011}, \texttt{1010}, \texttt{1011}, \texttt{1100}\}\}.$$

Set

$$M_i := \Phi(X_iX_{i+1}X_{i+2}X_{i+3}).$$

So M_i depends only on $(X_i, \dots, X_{i+3})$.

We claim M is always an MIS configuration (forbids 11 and 000) whenever X forbids $\texttt{000}, \texttt{111}$. This is a finite check on admissible source words:

- among admissible length-5 words of X, none yields $M_iM_{i+1} = \texttt{11}$;
- among admissible length-6 words of X, none yields $M_iM_{i+1}M_{i+2} = \texttt{000}$.

An explicit exhaustive verifier is included as `proof/scripts/prove_weak2_threshold.py`.

Therefore M is a stationary MIS process. Since M_i uses the one-sided window $[i, i + 3]$, a k-dependent input gives a $(k + 3)$-dependent output; with $k = 2$ this gives that M is 5-dependent. This contradicts Theorem 4 (no stationary 5-dependent MIS). So no stationary 2-dependent weak 2-coloring exists. □

Corollary 4 (Exact threshold)**.** *For weak 2-coloring on $\mathbb{Z}$ (forbidden $\texttt{000}, \texttt{111}$), finite dependence is possible at $k = 3$ and impossible at $k \le 2$.*

Proof. Combine Theorems 5 and 6. Impossibility at $k \le 2$ follows since k-dependence is monotone in k (if a process is k-dependent, then it is k'-dependent for every $k' \ge k$). □

5 Greedy proper 3-coloring: threshold $k = 6$

Definition 6 (Greedy proper 3-coloring on $\mathbb{Z}$). A process $G = (G_i)_{i\in\mathbb{Z}}$ with values in $\{0, 1, 2\}$ is a *greedy proper 3-coloring* if:

- it is proper: $G_i \neq G_{i+1}$ for all i;
- if $G_i = 1$, then $G_{i-1} = 0$ or $G_{i+1} = 0$;
- if $G_i = 2$, then it has a 0-neighbor and a 1-neighbor:

$$(G_{i-1} = 0 \text{ or } G_{i+1} = 0) \quad \text{and} \quad (G_{i-1} = 1 \text{ or } G_{i+1} = 1).$$

This is exactly the predicate `IsGreedyThreeColoring` in `FiniteDependence/API/Definitions.lean`.

Theorem 7 (Existence at dependence range 6). *There exists a stationary* 6*-dependent greedy proper* 3*-coloring process on* $\mathbb{Z}$*.*

Proof. Let $X = (X_i)_{i\in\mathbb{Z}}$ be the stationary 3-dependent weak 2-coloring from Theorem 5, so X forbids `000` and `111`. Define a local map $\Phi : \{0, 1\}^4 \to \{0, 1, 2\}$ by

$$\Phi(u) = \begin{cases} 2, & u \in \{\texttt{0010}, \texttt{1011}\}, \\ 1, & u \in \{\texttt{0110}, \texttt{1001}, \texttt{1010}\}, \\ 0, & \text{otherwise}, \end{cases}$$

and set

$$G_i := \Phi(X_i X_{i+1} X_{i+2} X_{i+3}).$$

So G_i depends only on the one-sided window $[i, i + 3]$.

A finite local check on admissible source windows (equivalently, the exhaustive `decide`-lemmas formalized in `FiniteDependence/Coloring/GreedyThree.lean`) shows: whenever X forbids `000`,`111`, the output G is always a greedy proper 3-coloring.

Stationarity is preserved by this shift-commuting local factor. For dependence, one-sided radius 3 enlarges range by at most 3, so from 3-dependence of X we get 6-dependence of G. □

Theorem 8 (No dependence range 5). *There is no stationary* 5*-dependent greedy proper* 3*-coloring process on* $\mathbb{Z}$*.*

Proof. Suppose $G = (G_i)_{i\in\mathbb{Z}}$ is stationary, 5-dependent, and a greedy proper 3-coloring. Define a binary process

$$M_i := \mathbf{1}\{G_i = 0\}.$$

This is a sitewise factor, so M is stationary and still 5-dependent.

Also M is an MIS configuration:

- no `11`: if $M_i = M_{i+1} = 1$, then $G_i = G_{i+1} = 0$, contradicting properness of G;
- no `000`: if $M_i = M_{i+1} = M_{i+2} = 0$, then $G_{i+1} \in \{1, 2\}$. The greedy condition at site $i+1$ forces a neighboring 0, i.e. $G_i = 0$ or $G_{i+2} = 0$, contradiction.

Hence M is a stationary 5-dependent MIS process, contradicting Theorem 4. □

Corollary 5 (Threshold for greedy proper 3-colorings). *For greedy proper* 3*-colorings on* $\mathbb{Z}$*, stationary finite dependence is possible at range* 6 *and impossible at range* 5*.*

Proof. Combine Theorems 7 and 8. □

6 No stationary 1-dependent proper 3-coloring

Theorem 9. *There is no stationary* 1*-dependent proper* 3*-coloring process on* $\mathbb{Z}$.

Proof. Suppose $(C_i)_{i\in\mathbb{Z}}$ is stationary, proper, and 1-dependent, with values in $\{0,1,2\}$. Define the ascent factor

$$Y_i := \mathbf{1}\{C_i < C_{i+1}\} \in \{0,1\}.$$

Exactly as in the proof of Theorem 5, Y forbids `000` and `111`, so Y is a weak 2-coloring process, and Y is stationary.

For dependence (cut form), fix a cut index i. The past σ-field generated by $(Y_j)_{j\le i}$ is contained in $\sigma(C_j : j \le i+1)$, while the future σ-field generated by $(Y_j)_{j\ge i+3}$ is contained in $\sigma(C_j : j \ge i+3)$. By 1-dependence of C at cut $i+1$, these are independent. Hence Y is 2-dependent.

So Y is a stationary 2-dependent weak 2-coloring, contradicting Theorem 6. □

Corollary 6 (Threshold for proper 3-colorings)**.** *For proper* 3*-colorings on* $\mathbb{Z}$*, stationary finite dependence is possible at range* 2 *and impossible at range* 1*.*

Proof. Combine Corollary 1 (the 3-coloring existence part) with Theorem 9. □

7 How to verify the computations

The $k=3$ and $k=4$ arguments above are now written as finite pen-and-paper derivations. Their scripts are optional independent audits.

The $k=5$ argument remains genuinely computer-assisted. The script `prove_k5_impossible.py` now has two modes:

- default mode: a rigorous exact-certificate check of the proof strategy in Steps 1–5: it verifies Step 1 identities, builds the forced length-15 family exactly, verifies the sparse Step 2 certificates by explicit LHS cancellation and exact RHS evaluation, verifies the sparse Step 4 length-19/25 certificate and polynomial $q(p,t)$, then performs exact univariate elimination/GCD checks showing incompatibility of $A(p)B(p)=0$ with $P_{12}(p)=0$;

- `--full-audit`: an additional slower check that also rebuilds the large fixed Step 2 systems (sizes 727×151 and 3002×465) before applying the same sparse certificates.

Independent verification checklist. For a minimal independent check of the final theorem:

- verify the displayed split identities and algebra in Steps 2–4;

- run `prove_k5_impossible.py` (default rigorous mode) and confirm it reproduces: the 7-row length-16 certificate $\mathcal{C}_f$ (LHS cancellation and RHS polynomial f), the 4-row length-20 certificate $\mathcal{C}_r$ (LHS cancellation and RHS polynomial r), the identity $g = 3f + r$, the factorization $A(p)B(p) = 0$ after eliminating t, the sparse Step 4 polynomial $q(p,t)$, the elimination numerator $P_{12}(p)$ from $q\big(p, t(p)\big)$, and the exact checks $\gcd(A, P_{12}) = \gcd(B, P_{12}) = 1$.

- optionally run `prove_k5_impossible.py --full-audit` for the extra large-system Step 2 rebuild check.

Run (from `proof/`):

```
uv run python scripts/prove_weak2_threshold.py
uv run --with sympy python scripts/prove_k3_impossible.py
uv run --with sympy python scripts/prove_k4_impossible.py
uv run --with sympy python scripts/prove_k5_impossible.py
uv run --with sympy python scripts/prove_k5_impossible.py --full-audit
```

References

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