$$\tilde{\eta} = \eta \frac{\beta}{\sigma}$$

$$\tilde{\varphi} = \varphi \frac{\beta}{\sigma}$$

NOTE: $\tilde{K} = a^{\frac{\alpha\beta}{\sigma}} \left(\tilde{H}^{T'}\right)^{-1} \dfrac{H^{0'}}{\int_0^\infty f^0(a)\,da} \cdot \hat{\hat{K}}$

$$e^* = \left[\frac{1+\tau^e}{\hat{K}\, a^{\frac{\alpha\beta}{\sigma}} \left(\tilde{H}^{T'}\right)^{-1} \tilde{\eta}\, (s^*)^{\tilde{\varphi}}} \right]^{\frac{1}{\tilde{\eta}-1}}$$

$$\hat{K} = (1-t)(1-\tau^w) \left(\frac{2\tilde{H}^T}{M}\right)^{\beta} \frac{H^{0'} A^{0'}}{\underbrace{\frac{M}{2}\int_0^\infty f^0(a)\,da}_{= X'}}$$

$$= (1-t)(1-\tau^w)\left(\tilde{H}^T\right)^{\beta}\left(\frac{2}{M}\right)^{\beta-1} \frac{H^{0'}A^{0'}}{\int_0^\infty f^0(a)\,da}$$

$$= \underbrace{(1-t)(1-\tau^w)\left(\tilde{H}^T\right)^{\beta}\left(\frac{2}{M}\right)^{\beta-1} A^{0'}}_{\equiv \hat{\hat{K}}} \frac{H^{0'}}{\int_0^\infty f^0(a)\,da}$$

$$\hat{K} = \hat{\hat{K}} \frac{H^{0'}}{\int_0^\infty f^0(a)\,da}$$

$$e^* = \left[\frac{1+\tau^e}{\hat{\hat{K}}\, \tilde{\eta}\, (s^*)^{\tilde{\varphi}}\, a^{\frac{\alpha\beta}{\sigma}} \frac{H^{0'}}{\int_0^\infty f^0(a)\,da} \left(\tilde{H}^{T'}\right)^{-1}} \right]^{\frac{1}{\tilde{\eta}-1}}$$

$$s^* = \frac{1}{1 + \frac{\frac{\sigma}{\beta}\eta}{M\varphi}}$$

$$H^{T'} = a^{\alpha} (s^*)^{\varphi} (e^*)^{\eta} \left(\frac{2}{M}\right)^{\sigma} \left(\tilde{H}^T\right)^{\sigma},$$

where superscript T is omitted on s^* and e^*

$$h^{T?} = \cancel{a^{\alpha}} (s^*)^{\varphi} \left(\frac{2}{M}\right)^{\sigma} \left[\frac{1+\tau^e}{\hat{K}\,\tilde{\eta}\,(s^*)^{\hat{\varphi}}}\right]^{\frac{\eta}{\tilde{\eta}-1}} \cancel{a^{-\frac{\alpha\beta}{\sigma}\frac{\eta}{\tilde{\eta}-1}}} \left(\tilde{H}^{T?}\right)^{\frac{\eta}{\tilde{\eta}-1}} \left(\frac{H^{o?}}{\int_0^{\infty} f^o(a)\,da}\right)^{-\frac{\eta}{\tilde{\eta}-1}} \left(\frac{2}{M}\right)^{\sigma} \left(\tilde{H}^{T}\right)^{\sigma}$$

$$= a^{\frac{\alpha(\sigma(\tilde{\eta}-1)-\beta\eta)}{\sigma(\tilde{\eta}-1)}} \underbrace{(s^*)^{\varphi}\left(\frac{2}{M}\right)^{\sigma}\left(\frac{1+\tau^e}{\hat{K}\,\tilde{\eta}\,(s^*)^{\hat{\varphi}}}\right)^{\frac{\eta}{\tilde{\eta}-1}}\left(\frac{2}{M}\right)^{\sigma}}_{\equiv \chi_1} \underbrace{\left(\tilde{H}^{T?}\right)^{\frac{\eta}{\tilde{\eta}-1}} \left(\tilde{H}^{T}\right)^{\sigma} \left(\frac{\int_0^{\infty} f^o(\hat{a})\,d\hat{a}}{H^{o?}}\right)^{\frac{\eta}{\tilde{\eta}-1}}}_{\equiv \chi_2}$$

$$\int h^{T?} f^T(a)\,da = H^{T?} =$$

$$= \underbrace{\int a^{\frac{\alpha(\sigma(\tilde{\eta}-1)-\beta\eta)}{\sigma(\tilde{\eta}-1)}} f^T(a)\,da}_{\equiv \chi_3} \cdot \chi_1 \cdot \chi_2$$

for given occupational choice

$$\int (\eta^{Tg})^{\beta/\sigma} f^{T}(a)\,da = \tilde{H}^{Tg}$$

$$= \underbrace{\int a^{\frac{\alpha\beta(\sigma(\hat{\eta}-1)-\beta\eta)}{\sigma^2(\hat{\eta}-1)}} f^{T}(a)\,da}_{\equiv\, \tilde{X}_3} \quad x_1^{\beta/\sigma} \quad x_2^{\beta/\sigma}$$

$$= \tilde{X}_3\, x_1^{\beta/\sigma} \left(\tilde{H}^{Tg}\right)^{\frac{\eta}{\hat{\eta}-1}\cdot\frac{\beta}{\sigma}} \left(\tilde{H}^{T}\right)^{\beta} \left(\frac{\int_0^\infty f^0(\hat{a})\,d\hat{a}}{H^{0g}}\right)^{\frac{\eta}{\hat{\eta}-1}\cdot\frac{\beta}{\sigma}}$$

$$\tilde{H}^{Tg\;1-\frac{\eta\beta}{\sigma(\hat{\eta}-1)}} = \tilde{X}_3\, x_1^{\beta/\sigma} \left(\frac{\int_0^\infty f^0(\hat{a})\,d\hat{a}}{H^{0g}}\right)^{\frac{\eta}{\hat{\eta}-1}\cdot\frac{\beta}{\sigma}} \left(\tilde{H}^{T}\right)^{\beta}$$

$$\tilde{H}^{Tg} = \left\{ \tilde{X}_3\, x_1^{\beta/\sigma} \left(\frac{\int_0^\infty f^0(\hat{a})\,d\hat{a}}{H^{0g}}\right)^{\frac{\eta}{\hat{\eta}-1}\cdot\frac{\beta}{\sigma}} \right\}^{\frac{\sigma(\hat{\eta}-1)}{\sigma(\hat{\eta}-1)-\eta\beta}}$$

$$\cdot\ \tilde{H}^{T\,\underbrace{\scriptstyle \frac{\beta\sigma(\hat{\eta}-1)}{\sigma(\hat{\eta}-1)-\eta\beta}}}$$

$$= \frac{\beta\sigma\left(\eta\frac{\beta}{\sigma}-1\right)}{\sigma\left(\eta\beta/\sigma-1\right)-\eta\beta}$$

$$= \frac{\beta\cancel{\sigma}\eta\beta/\cancel{\sigma}-\beta\sigma}{\cancel{\sigma\eta\beta/\sigma}-\sigma-\cancel{\eta\beta}} = \frac{\beta(\eta\beta-\sigma)}{-\sigma}$$

$$= \frac{\beta(\sigma-\eta\beta)}{\sigma}$$

$$\tilde{H}^{T\prime} = \left(\tilde{X}_3 \cdot X_1^{\beta/\sigma}\right)^{\frac{\sigma(\tilde{\eta}-1)}{\sigma(\tilde{\eta}-1)-\eta\beta}} \cdot \tilde{H}^{T\,\frac{\beta\sigma(\tilde{\eta}-1)}{\sigma(\tilde{\eta}-1)-\eta\beta}} \cdot \left(\frac{\int_0^\infty f^o(\hat{a})d\hat{a}}{H^{o\prime}}\right)^{\frac{\eta\beta}{\sigma(\tilde{\eta}-1)-\eta\beta}}$$

Let's simplify the exponents:

$$\frac{\beta\sigma(\tilde{\eta}-1)}{\sigma(\tilde{\eta}-1)-\eta\beta} = \frac{\beta\sigma\left(\eta\frac{\beta}{\sigma}-1\right)}{\sigma\left(\eta\frac{\beta}{\sigma}-1\right)-\eta\beta} = \frac{\beta^2\eta-\beta\sigma}{\cancel{\eta\beta}-\sigma-\cancel{\eta\beta}} = \frac{\beta(\beta\eta-\sigma)}{-\sigma}$$
$$= \frac{\beta(\sigma-\beta\eta)}{\sigma}$$

$$\frac{\eta\beta}{\sigma(\tilde{\eta}-1)-\eta\beta} = -\frac{\eta\beta}{\sigma}$$

$$\frac{\sigma(\tilde{\eta}-1)}{\sigma(\tilde{\eta}-1)-\eta\beta} = \frac{\sigma\left(\eta\frac{\beta}{\sigma}-1\right)}{-\sigma} = \frac{\sigma-\beta\eta}{\sigma}$$

Aggregate law of motion for $\tilde{H}^T$:

$$\tilde{H}^{T\prime} = \left(\tilde{X}_3 \cdot X_1^{\beta/\sigma}\right)^{\frac{\sigma-\beta\eta}{\sigma}} \left(\frac{H^{o\prime}}{\int_0^\infty f^o(\hat{a})d\hat{a}}\right)^{\frac{\beta\eta}{\sigma}} \left(\tilde{H}^T\right)^{\frac{\beta(\sigma-\beta\eta)}{\sigma}}$$

$$= \left(\tilde{X}_3 \cdot X_1^{\beta/\sigma}\right)^{1-\tilde{\eta}} \left(\frac{H^{o\prime}}{\int_0^\infty f^o(\hat{a})d\hat{a}}\right)^{\tilde{\eta}} \left(\tilde{H}^T\right)^{\beta(1-\tilde{\eta})}$$

Law of motion for H^o:

$$h^{o\prime} = a^{\alpha}\left(s^{o*}\right)^{\phi}\left(e^{o*}\right)^{\eta}\left(\frac{2}{M}\right)^{\sigma}\left(\tilde{H}^{T}\right)^{\sigma},$$

where

$$s^{o*} = \frac{\mu\phi}{\mu\phi + 1 - \eta}$$

$$e^{o*} = \left(\frac{1+\tau^{e}}{\hat{K}\eta a^{\alpha}\left(s^{o*}\right)^{\phi}\left(\tilde{H}^{T}\right)^{-\sigma}}\right)^{\frac{1}{\eta-1}}, \text{ and}$$

$$\hat{K} = (1-t)\left(1-\tau^{w}\right)\cdot A\,(2M)^{\sigma}$$

$$h^{o\prime} = a^{\alpha}\left(s^{o*}\right)^{\phi} a^{-\frac{\alpha\eta}{\eta-1}}\left(\frac{1+\tau^{e}}{\hat{K}\eta\left(s^{o*}\right)^{\phi}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{2}{M}\right)^{\sigma}\left(\tilde{H}^{T}\right)^{\frac{\sigma\eta}{\eta-1}+\sigma}$$

$$= a^{\frac{\alpha(\eta-1)-\alpha\eta}{\eta-1}}\left(s^{o*}\right)^{\phi}\left(\frac{1+\tau^{e}}{\hat{K}\eta\left(s^{o*}\right)^{\phi}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{2}{M}\right)^{\sigma}\left(\tilde{H}^{T}\right)^{\frac{\sigma(2\eta-1)}{\eta-1}}$$

$$= a^{\frac{\alpha}{1-\eta}}\left(s^{o*}\right)^{\phi}\left(\frac{1+\tau^{e}}{\hat{K}\eta\left(s^{o*}\right)^{\phi}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{2}{M}\right)^{\sigma}\left(\tilde{H}^{T}\right)^{\frac{\sigma(2\eta-1)}{\eta-1}}$$

$$H^{o\prime} = \int_{0}^{\infty} h^{o\prime} f^{o}(a)\,da$$

$$= \underbrace{\int_{0}^{\infty} a^{\frac{\alpha}{1-\eta}} f^{o}(a)\,da}_{\chi_4}\;\underbrace{\left(s^{o*}\right)^{\phi}\left(\frac{1+\tau^{e}}{\hat{K}\eta\left(s^{o*}\right)^{\phi}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{2}{M}\right)^{\sigma}}_{\chi_5}\left(\tilde{H}^{T}\right)^{\frac{\sigma(2\eta-1)}{\eta-1}}$$

$$H^{o\prime} = \chi_4 \cdot \chi_5 \left(\tilde{H}^T\right)^{\frac{\sigma(2\eta-1)}{\eta-1}}$$

Now, substitute $H^{o\prime}$ into (aggregate) law of motion for $\tilde{H}^T$:

$$\tilde{H}^{T\prime} = \left(\tilde{\chi}_3 \cdot \chi_1^{\beta/\sigma}\right)^{1-\tilde{\eta}} \left(\int_0^\infty f^o(\hat{a})\,d\hat{a}\right)^{-\tilde{\eta}} \cdot \left(\chi_4 \cdot \chi_5\right)^{\tilde{\eta}} \left(\tilde{H}^T\right)^{\frac{\tilde{\eta}(2\eta-1)}{\eta-1} + \beta(1-\tilde{\eta})}$$

$$= \chi_6 \cdot \left(\tilde{H}^T\right)^{\frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})}{\eta-1}}$$

Simplify exponent on $\tilde{H}^T$:

$$\frac{\tilde{\eta}(2\eta-1)}{\eta-1} + \beta(1-\tilde{\eta}) = \frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})}{\eta-1}$$

Not sure how to simplify further! :(
Would be nice if we had a more intuitive expression for this exponent.

Sufficient condition(s) for stable fixed point:

(1) $$\left.\frac{\partial \tilde{H}^{T\prime}}{\partial \tilde{H}^T}\right|_{\tilde{H}^T=0} = \chi_6 \frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})}{\eta-1} \tilde{H}^{T\,\frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})-(\eta-1)}{\eta-1}}$$

$$> 1$$

$$\Rightarrow \frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})-(\eta-1)}{\eta-1} < 0$$

$$\Leftrightarrow \frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})}{\eta-1} < 1$$

(2) Increasing function of $\tilde{H}^T$:

$$\frac{\tilde{\eta}(2\eta-1)+(\eta-1)\beta(1-\tilde{\eta})}{\eta-1} > 0$$

COMMENTS:

Equation for H^{og} at bottom of page 5 implies that $\eta \in (0, \frac{1}{2})$. If $\eta > \frac{1}{2}$, then H^{og} is decreasing in $\tilde{H}^T$.
I don't have much intuition for this result (or the restriction).