Radiography and Tomography
Lab 2: Computing Radiographs

An Abbreviated View of Radiography

It is important to have an understanding of the following information before proceeding with the lab. Please read this information **carefully**.

Transmission Radiography

is the process of measuring and recording changes in a high-energy particle beam (X-rays, protons, neutrons, etc.) resulting from passage through an object of interest. In the below image, the high-energy particle beams are travelling along the x-axis (so y and z are fixed for a particular beam) with the intensity at a point x_0 given by $E(x_0, y, z)$. Here $x_0 = 0$ gives the initial intensity and $x_0 = D_x$ gives the final intensity once the beams have passed through the object. Also $\rho(x, y, z)$ is the mass density of the object at a given point. The reduction in intensity of a beam is proportional to the mass along the beam's path.

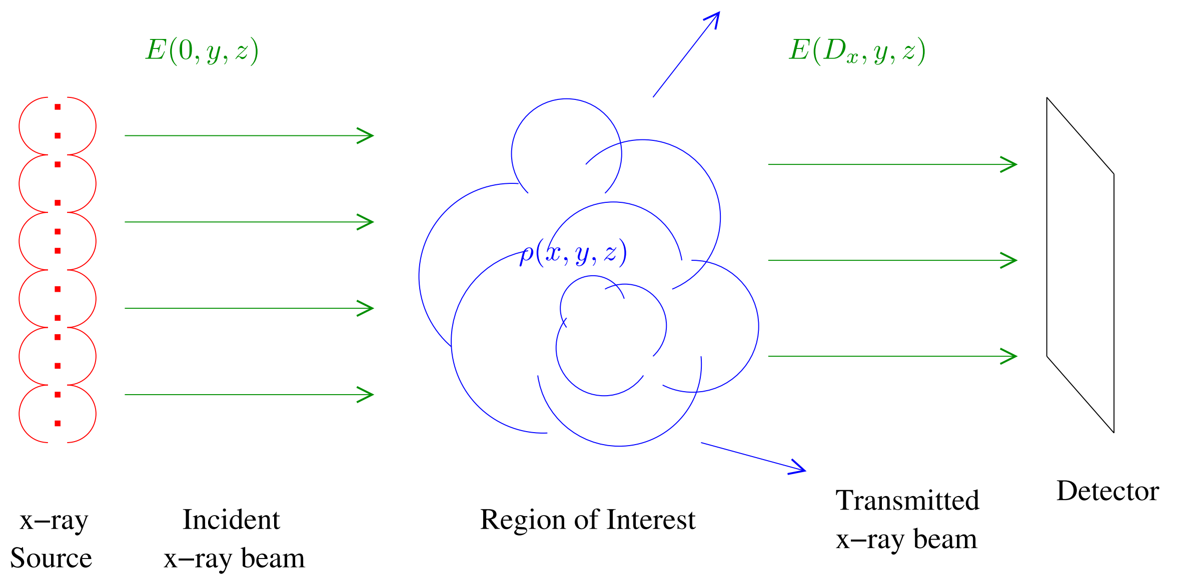


Figure 1: Typical radiographic experiment.

For our basic setup, the detector is divided into some fixed number of "bins" (numbered, say, from 1 to m). For the k-th bin, denote the initial number of photons sent by p_k^0 and the total detected (after passing through the object) by p_k. Then it turns out that

$$p_k = p_k^0 e^{-s_k/\alpha},$$

where s_k is the total mass in the path of the k-th bin portion of the beam, and α is a constant proportional to the bin area.

We consider a slice of the region of interest subdivided into N cubic voxels (three-dimensional pixels). Let x_j be the mass in object voxel j and T_{kj} the fraction of voxel j in beam path k (see Figure 2). (Note that this is a different context, in particular, x_j is **not** related to the direction that the x-ray beams are traveling!) Then the mass along beam path k is

$$s_k = \sum_{j=1}^{N} T_{kj} x_j,$$

and the expected photon count at radiograph pixel k, p_k, is given by

$$p_k = p_k^0 e^{-\frac{1}{\alpha}\sum_{j=1}^{N} T_{kj} x_j},$$

or equivalently,

$$b_k \equiv \left(-\alpha \ln \frac{p_k}{p_k^0}\right) = \sum_{j=1}^{N} T_{kj} x_j.$$

Do not worry about the intricacies of these formulae, the important thing to note is that b_k is a measure of the change in intensity of the particle beam along the beam path k. In other words, the new quantities b_k represent a variable change that allows us to formulate the matrix expression for the radiographic transformation $\mathbf{b} = T\mathbf{x}$, where T is the matrix with entries T_{kj} and $\mathbf{x} = (x_1, \ldots, x_n)^T$ (Note the use of the transpose here, so $\mathbf{x}$ is a column vector. This is used here just because it is more convenient to type!).

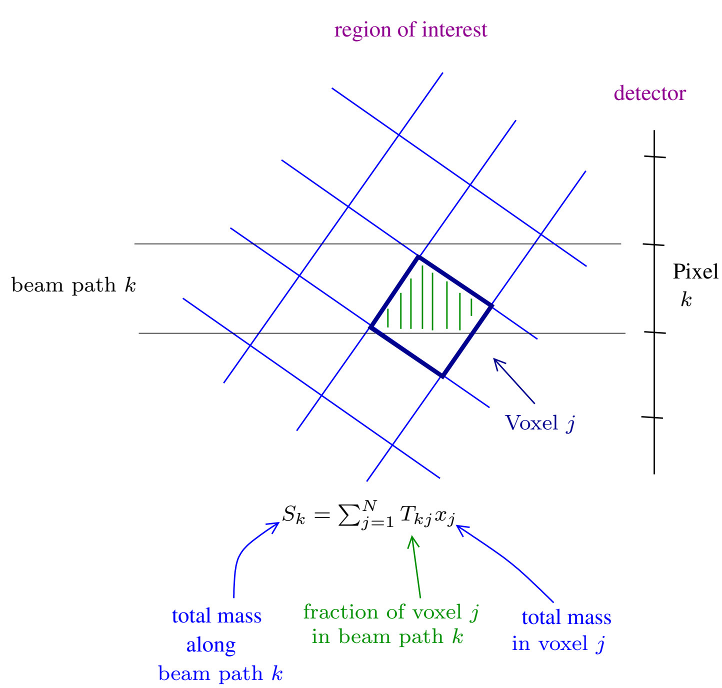


Figure 2: Object space and radiograph space discretization.

Radiographic Scenarios and Notation

A single-view radiographic setup consists of an area of interest where the object will be placed, and a single screen onto which the radiograph will be recorded. A multiple-view radiographic setup consists of a single area of interest experimentally designed so that radiographs of this area can be recorded for different locations about the object.

The geometry of radiographic scenarios is illustrated in figures 1 and 2 (Pay attention to subscripts in figure 2). The notation is as follows.

- Slice of region of interest: n by n array of voxels (each box represents a voxel).
- Total number of voxels in each slice is $N = n^2$.
- Each voxel has a width and height of 1 unit.
- For each radiographic view we record m pixels of data (each bin along the red/blue lines below represents a pixel).
- The width of each pixel is $ScaleFac$. If $ScaleFac = 1$ then pixel width is the same as voxel width.
- Number of radiographic angles (views): a.
- Total number of pixels in the radiograph: $M = am$
- Angle of the i^{th} view (measured in degrees east of south): θ_i
- Object mass at voxel j is x_j
- Recorded radiograph value at pixel k is b_k

In this Lab we will be constructing matrix representations of the radiographic transformation operators for example scenarios. Recall that an object can be represented by an N-dimensional vector $\mathbf{x} = (x_1, \ldots, x_{n^2})^T$ and a set of radiographs can be represented by an M-dimensional vector $\mathbf{b} = (b_1, \ldots, b_m)^T$. What will be the size of the corresponding matrix operator that maps object vector $\mathbf{x}$ to radiograph vector $\mathbf{b}$? Notice that particular values for $\mathbf{x}$ and $\mathbf{b}$ are **not necessary** for computing the matrix of the radiographic transformation!

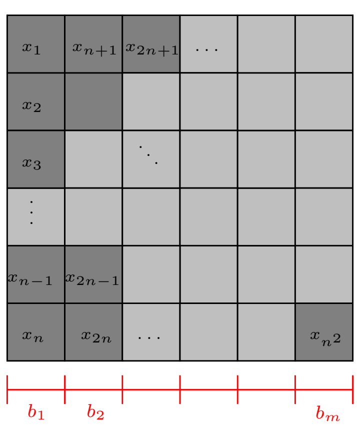


Figure 1: The geometry of a single view radiographic transformation.

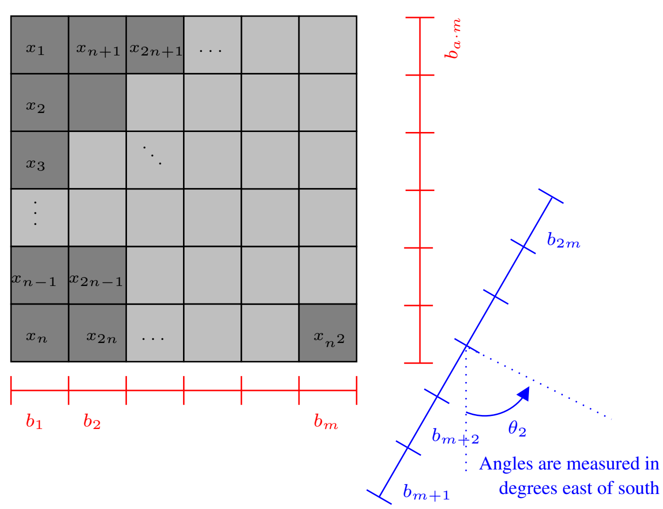
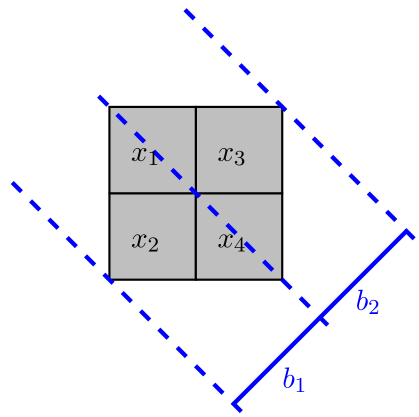


Figure 2: The Geometry of a multiple view radiographic transformation showing view 1, view 2, and view a.

A First Example

Let's look at a specific scenario. For the setup pictured below, we have:

- Total number of voxels: $N = 4$ $(n = 2)$.
- Total number of pixels: $M = m = 2$
- $ScaleFac = \sqrt{2}$ (the length of b_1 and b_2 is $\sqrt{2}$ times the length of one of the square sides using Pythagoras' theorem).
- Number of views: $a = 1$
- Angle of the single view: $\theta_1 = 45°$



Recalling that T_{kj} is the fraction of voxel j which projects perpendicularly onto pixel k, the matrix associated with this radiographic setup is

$$T = \begin{bmatrix} 1/2 & 1 & 0 & 1/2 \\ 1/2 & 0 & 1 & 1/2 \end{bmatrix}.$$

Be sure and check this to see if you agree! Hence, for *any* input vector $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, the radiographic output is $\mathbf{b} = T\mathbf{x}$. So the output when the slice of the object is the vector $\mathbf{x} = (10, 0, 5, 10)^T$ is given by

$$\mathbf{b} = T\mathbf{x} = \begin{bmatrix} 1/2 & 1 & 0 & 1/2 \\ 1/2 & 0 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Now it's your turn!

Notes:

1. To do the following tasks, you will need to first download the function `tomomap.m` from the **Create** section of the peerScholar site and load it into MatLab. This is done by following the file path above the command window in MatLab through the series of drop down arrows to the folder where you saved `tomomap.m`, and then selecting the file. You then calculate T using the code $T =$ `tomomap`$(n, m, [\theta_1, \ldots, \theta_a], Scalefac)$ where you replace the variables as appropriate. This output only gives the non-zero values of T. For the full matrix, follow this with the code `full(T)`.

2. If there is more than one angle specified, the matrix for the radiographic transformation is obtained by placing the matrix for the first angle "on top" of the matrix for the second (i.e. the matrix for the first angle forms the first m rows of T, the second forms the next m rows etc.).

3. It is possible that for certain combinations of scale factor and angle that certain portions of the voxels are outside of all the bins. In this case we find the transformation matrix as normal and simply ignore the regions of the voxels not contained in any bin.

Task 1: Answer the following questions.

1. Suppose you have the setup where

    - Height and width of image in voxels: $n = 2$ (Total voxels $N = 4$)
    - Pixels per view in radiograph: $m = 2$
    - $ScaleFac = 1$
    - Number of views: $a = 2$
    - Angle of the views: $\theta_1 = 0°, \theta_2 = 90°$

    (a) Sketch this setup.

    (b) Calculate the matrix associated with the setup.

    (c) Verify your answer for (b) using MatLab (specifically the `tomomap` function).

    (d) Find the radiographs of the following objects.

4	3
6	1

0	1
1	0

2. Suppose you have the setup where

    - Height and width of image in voxels: $n = 2$ (Total voxels $N = 4$)
    - Pixels per view in radiograph: $m = 4$
    - $ScaleFac = \sqrt{2}/2$
    - Number of views: $a = 1$
    - Angle of the views: $\theta_1 = 45°$

    (a) Sketch this setup.

    (b) Calculate the matrix associated with the setup.

    (c) Repeat step (b) using the code `tomomap`.

3. Suppose you have the setup where

    - Height and width of image in voxels: $n = 4$ (Total voxels $N = 16$)
    - Pixels per view in radiograph: $m = 2$
    - $ScaleFac = \sqrt{2}$
    - Number of views: $a = 2$
    - Angle of the views: $\theta_1 = 0°$, $\theta_2 = 45°$

    (a) Sketch this setup.

    (b) Calculate the matrix associated with the setup. (this is a big matrix, you can use MatLab)

    (c) Find the radiographs of images A, B, and C from lab 1 under this transformation (you can again use MatLab).

Task 2: Radiographs of Linear Combinations of Objects

Take the two objects defined in Question 1 of Task 1 above to be $\mathbf{x}$ (left object) and $\mathbf{y}$ (right object). For the transformation T from this same question answer the following questions.

1. What is the radiograph of the object $3\mathbf{x}$?
2. What is the radiograph of the object $0.5\mathbf{y}$?
3. What is the radiograph of the object $3\mathbf{x} + 0.5\mathbf{y}$?
4. How do these compare with taking the radiographs of $\mathbf{x}$ and $\mathbf{y}$ first and then performing the same operations on the result? What does this tell you about T?