浙江大学爱丁堡大学联合学院
ZJU-UoE Institute

Conditional Probabilities

ADS 2, Lecture 22

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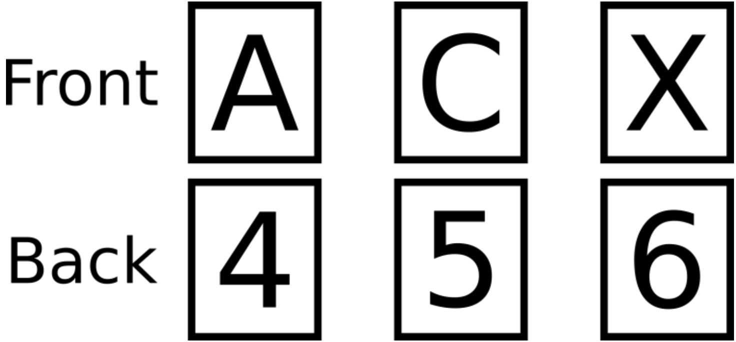
Semester 2, 2022/23

Based on Prof MI Stefan's slides

A card game (1)

I have a (large) number of cards. Each card has a letter on one side and a number on the other side



A card game (1)

Hypothesis: Every card that has a vowel on the front side has an even number on the back side.

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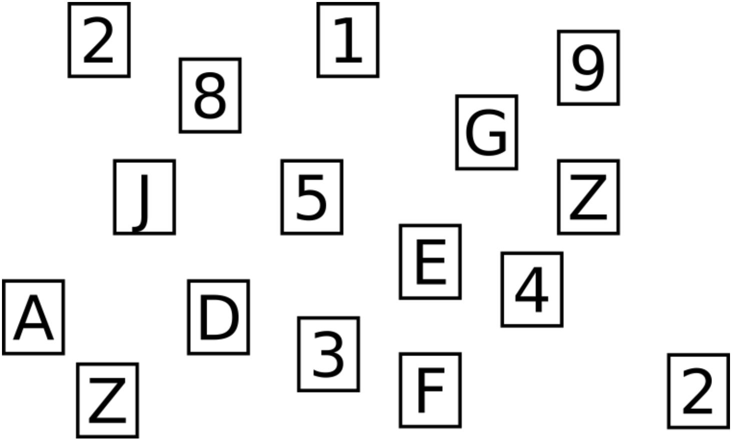
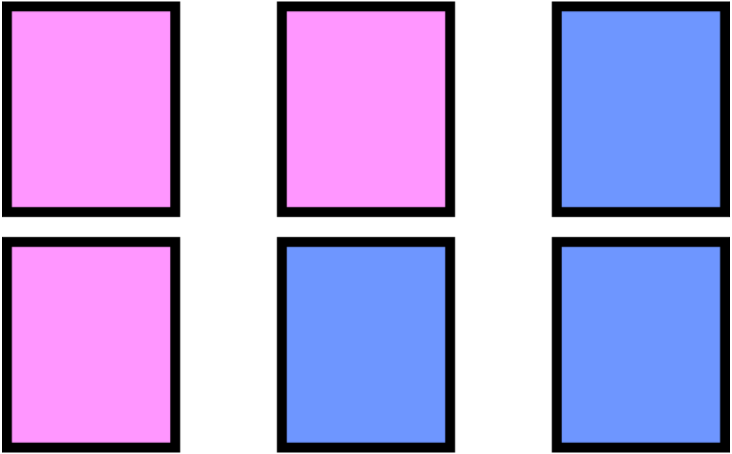
Question: Which cards do I have to turn around to test this hypothesis?

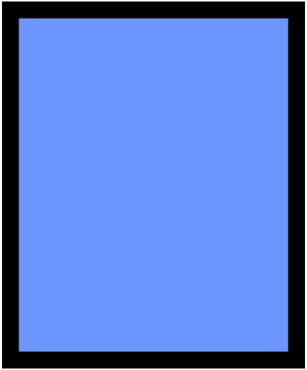


A card game (2)

I have three cards: one pink on both sides, one blue on both sides, one with one pink and one blue side



A card game (2)



Question: I drew a card and am showing you one side. It's blue. What is the probability that the other side is also blue?

This lecture is about . . .

Probabilities of combinations of events
(partly a review from last year, but with a bit more depth and rigour)

Learning Objectives

After this week, you will be able to . . .

- Recall how to compute conditional probabilities
- Visualise joint probabilities using Euler diagrams and probability trees
- State and apply Bayes' theorem
- Describe and use Markov chains

Outline

A tiny little bit of logic notation

$\neg A$	"not A"	True if A is false
$A \& B$	"A and B"	True if both A and B are true, false otherwise
$A \vee B$	"A or B"	True if A is true or B is true (or both)
$A \rightarrow B$	"If A then B"	

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$A \rightarrow B$	"If A then B"	?

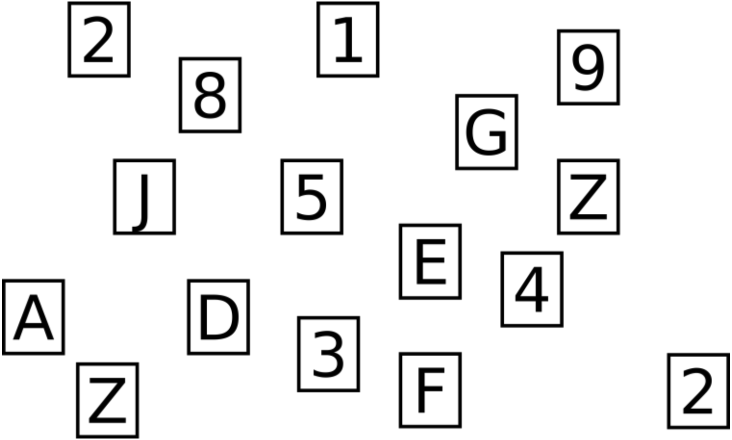
A	B	$A \rightarrow B$
T	T	
T	F	
F	T	
F	F	

Back to our card game

Hypothesis: Every card that has a vowel on the front side has an even number on the back side.

vowel $\rightarrow$ even number

Question: Which cards do I have to turn around to test this hypothesis?



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$A \leftrightarrow B$	"If and only if A then B" "Iff A then B"	

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Outline

Conditional probabilities: A bit of notation

$$P(A|B)$$

"Probability of A given B": Probability of A *if B is true*

Examples (reminder)

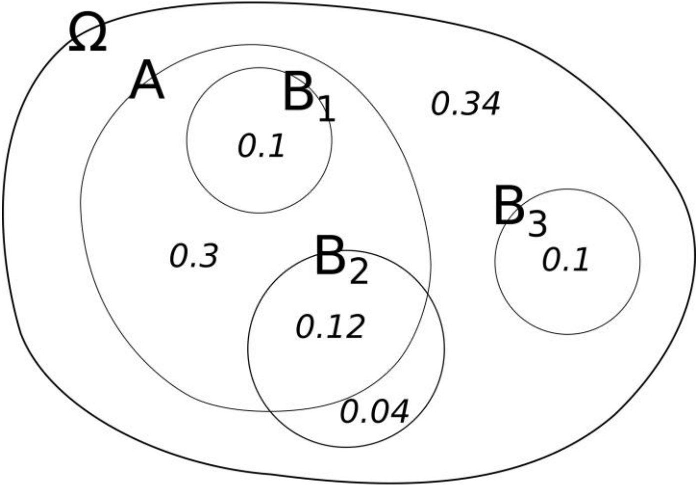
- Probability of having a disease given a positive test result
- Probability of a person getting a disease given they are a carrier for a specific allele variant
- Probability of cell survival given treatment with a toxic chemical
- Probability of seeing a result as or more extreme as the one in your experiment given the Null Hypothesis is true

Example: Lie detector test

Problem

In a big store, management finds out that around 10 % of employees must be stealing, but they don't know who.
In response to this, all employees have to go through a lie detector test.
The lie detector has 80 % accuracy in both directions: It correctly categorises 80 % of the people telling the truth as telling the truth, and 80 % of the people lying as lying.
Every employee was tested and everybody said they did not steal.
According to the lie detector, 50 employees were lying.
How many were thieves?

A convenient tool: Euler diagrams



A convenient tool: Euler diagrams

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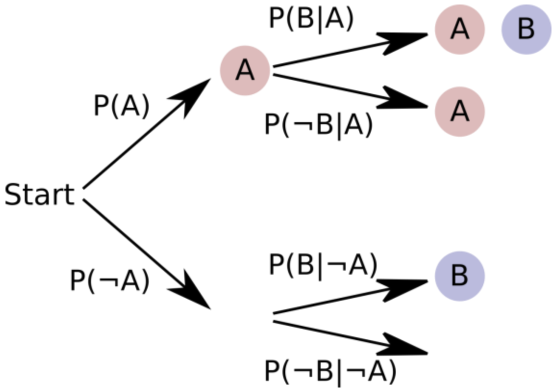
Every employee was tested and everybody said they did not steal.
According to the lie detector, 50 employees were lying.
Draw an Euler diagram of the situation

A convenient tool: Euler diagrams

Review: Probability trees

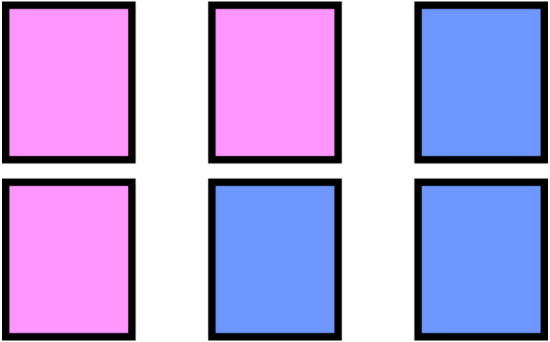
Another convenient way to visualise and compute joint probabilities.
Multiply probabilities along branches.



Looking back to our second card game

I have three cards: one pink on both sides, one blue on both sides, one with one pink and one blue side

Question: I drew a card and am showing you one side. It's blue. What is the probability that the other side is also blue?



Draw a probability tree for this problem

Outline

Bayes' theorem

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We already know that

$$P(A|B) \neq P(B|A)$$

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But how *are* $P(A|B)$ and $P(B|A)$ related?

Thomas Bayes

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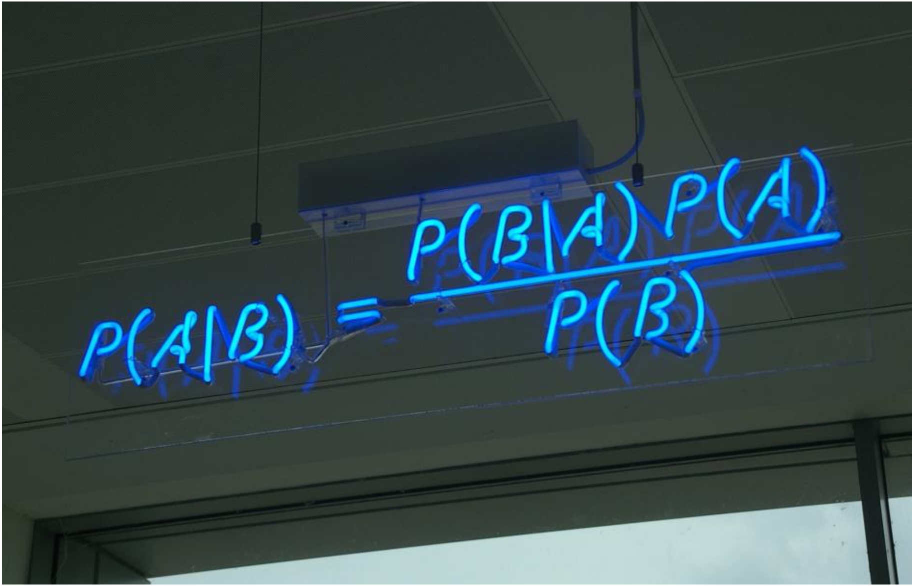
University of Edinburgh alumn!

Bayes' theorem



Let's prove it!

We already know

$$P(A\&B) = P(A) \times P(B|A)$$

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Similarly

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Let's prove it!

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Similarly

$$P(B\&A) = P(B) \times P(A|B)$$

But of course,

$$P(A\&B) = P(B\&A)$$

Therefore,

$$P(A) \times P(B|A) = P(B) \times P(A|B)$$

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)} \quad \bullet$$

Bayes' theorem: Example

What is $P(\text{thief}|\text{lied according to the lie detector})$?

Bayes' theorem: Example

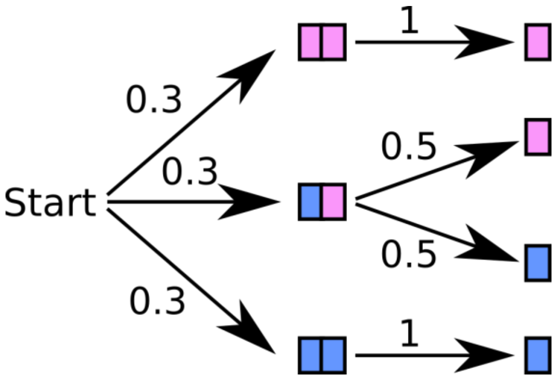
What is P(thief|lied according to the lie detector)?
Practical!

Outline

What are Markov chains?

- Stochastic model
- A system is represented as being in a number of possible states
- Transitions between states happen with specified probabilities
- Probabilities of state transitions depend on the state the system is currently in, not its history
- Probabilities going out of any one node should add up to 1

Markov chains: Simple example



Markov chains: More complicated example

If tossing a fair coin (H=Head, T=Tail), how long would it take to get the sequence H-T-T-H?

Start

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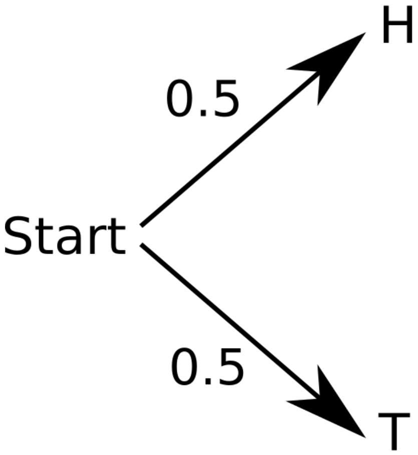

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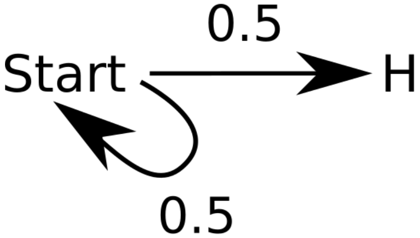

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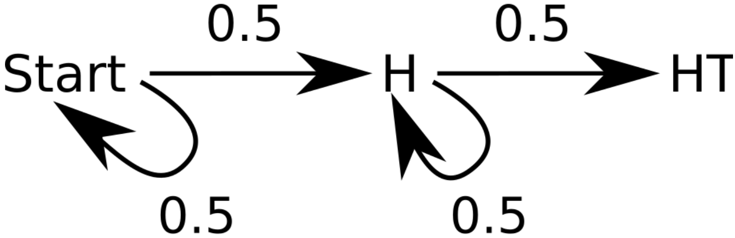

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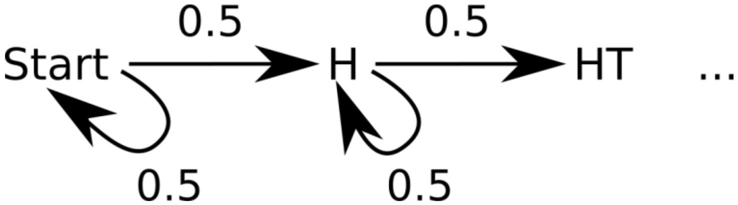

Practical!

Markov chains

- There is a mathematical theory of Markov chains, with ways to compute probabilities to reach states, path lengths etc.
- For this course, we ask you to do 2 things
  - Draw up a Markov chain for a new problem
  - Write code to allow you to simulate a Markov chain many times

What questions do you have?

After this week, you will be able to ...

- Recall how to compute conditional probabilities
- Visualise joint probabilities using Euler diagrams and probability trees
- State and apply Bayes' theorem
- Describe and use Markov chains

Image credits

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