Chapter 11

Comparing the Means of Many Independent Samples

11.1 Introduction

Comparing the Means of <u>Two</u> Independent Samples (Review of Chapter 7)

- **Hypothesis testing for 2 independent samples: t test**
  - Null hypothesis H_0: $\mu_1 = \mu_2$
    - H_0: the hypothesis that μ_1 and μ_2 are <u>equal</u> (no difference)
  - Alternative hypothesis H_A: $\mu_1 \neq \mu_2$
    - H_A : the hypothesis that μ_1 and μ_2 are <u>NOT equal</u> (differ)
  - t test: $t_s = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{SE_{(\bar{Y}_1 - \bar{Y}_2)}}$ → P-value → P-value vs. α

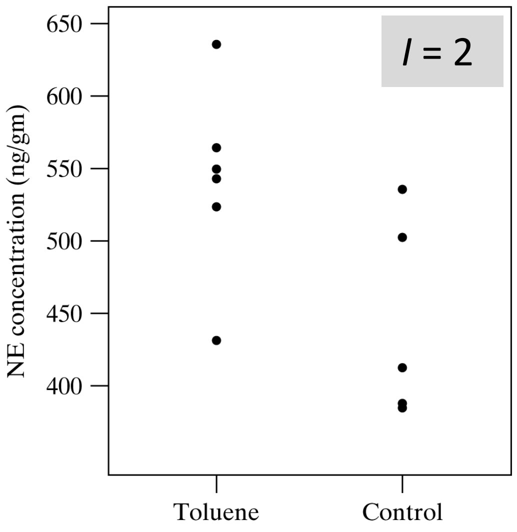

Figure 7.2.1 Parallel dotplots of NE concentration

What if we need to compare more than two independent samples?

11.1 Introduction

Comparing the Means of Many Independent Samples

- **Hypothesis testing for *I* independent samples:**
  - Global null hypothesis H_0: $\mu_1 = \mu_2 = \ldots = \mu_I$
    - H_0: all the population means are equal (no difference)
  - Alternative hypothesis H_A: $\mu_m \neq \mu_n$
    - H_A: at least one pair of the population means are NOT equal (differ)

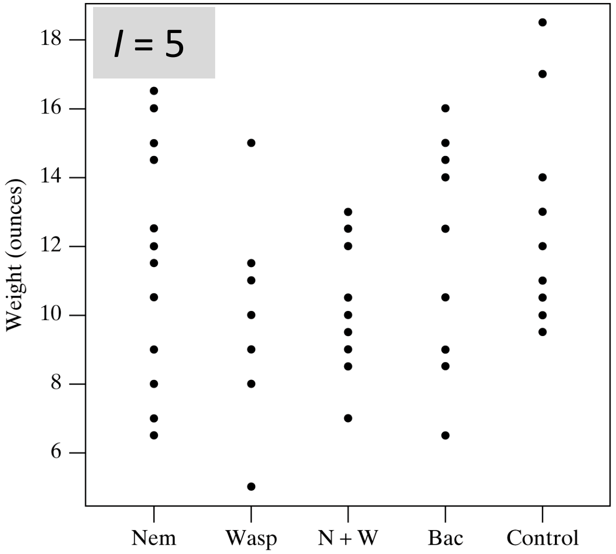

Figure 11.1.1 Weights of ears of corn receiving five different treatments

What statistical test should I use?

11.1 Introduction

A Graphical perspective on ANOVA

- In order to find compelling evidence for a difference in population means (H_A),
  - (1) not only must there be variation among the group means,
  - (2) but variation among the group means must be large relative to the inherent variability in the groups

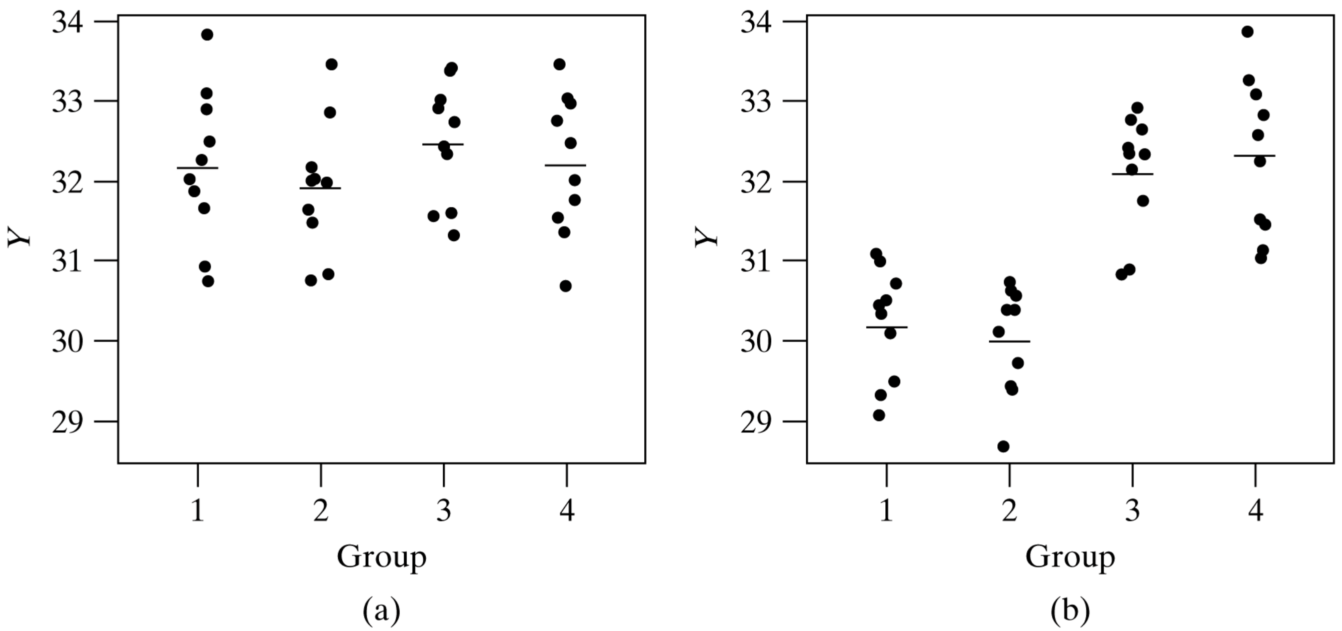

Figure 11.1.3
(a) H_0 true, (b) H_0 false, with small SDs for the groups

11.1 Introduction

A Graphical perspective on ANOVA

- “analysis of variance” → make an inference about means.
- If the between-group mean variability is large relative to within-group variability, we will take this as evidence against the null hypothesis (the population means are all equal).

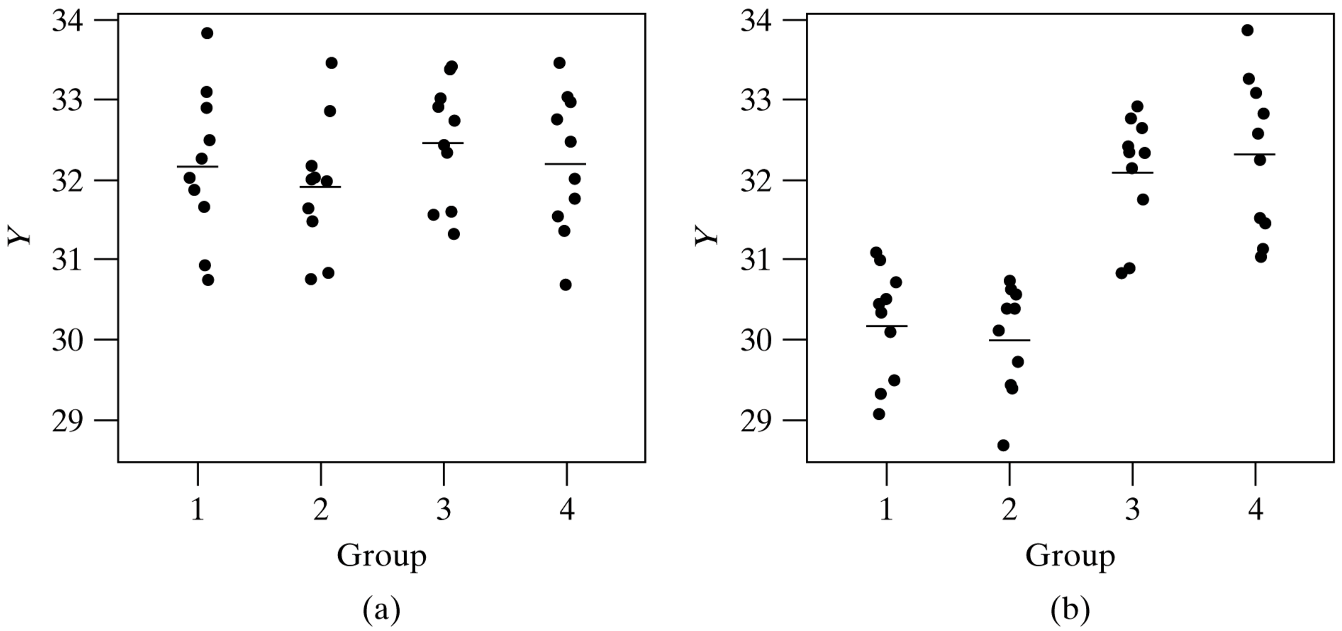

Figure 11.1.3
(a) H_0 true, (b) H_0 false, with small SDs for the groups

11.2 The Basic One-Way Analysis of Variance

One-way ANOVA

- The classical method of analyzing data from three or more (I>2) independent samples is called an ANalysis Of VAriance, or ANOVA.
- In applying analysis of variance (ANOVA), the data are regarded as random samples from I populations.
- The term "one-way" refers to the fact that there is one variable that defines the groups or treatments
  - e.g. in the sweet corn example the treatments were based on the type of harmful insect/bacteria.
    - Treatment 1: Nematodes
    - Treatment 2: Wasps
    - Treatment 3: Nematodes and wasps
    - Treatment 4: Bacteria
    - Treatment 5: Control

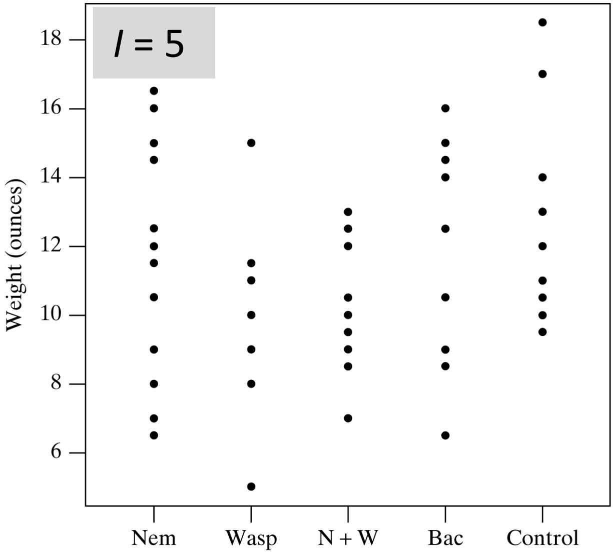

Figure 11.1.1 Weights of ears of corn receiving five different treatments

11.2 The Basic One-Way Analysis of Variance

One-way ANOVA

- **Notation**
  - Population means: μ_1 , μ_2 , … , μ_I
  - Population standard deviations: σ_1 , σ_2 , … , σ_I
  - To describe several groups of quantitative observations,
    - y_{ij} = observation j in group i
    - I = number of groups
    - n_i = number of observations in group i
    - $\bar{y}_i$ = mean for group i
    - s_i = standard deviation for group i

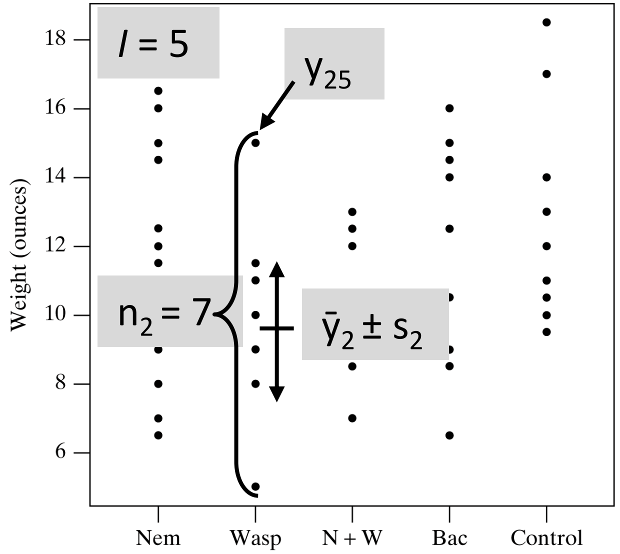

Figure 11.1.1 Weights of ears of corn receiving five different treatments

11.2 The Basic One-Way Analysis of Variance

One-way ANOVA

- **Notation**

  To describe several groups of quantitative observations,

  – The **total number of observations**

  $$n. = \sum_{i=1}^{I} n_i$$

  – The **grand mean** (the mean of all the observations)

  $$\bar{\bar{y}} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i} y_{ij}}{n.}$$

  - The grand mean can be expressed as a weighted average of the group means

  $$\bar{\bar{y}} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i} y_{ij}}{n.} = \frac{\sum_{i=1}^{I} n_i \bar{y}_i}{n.}$$

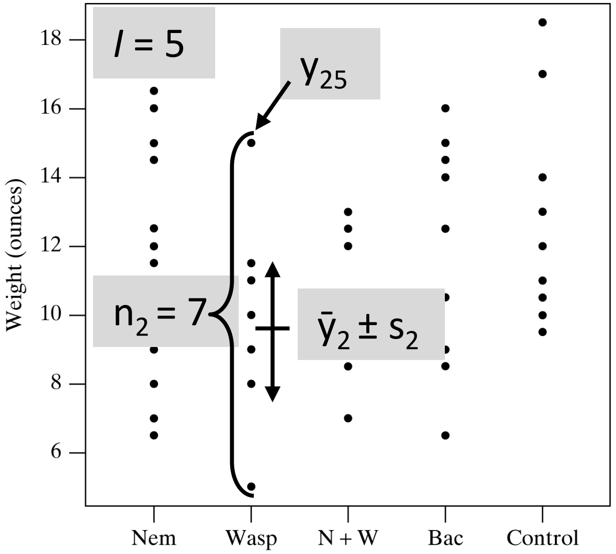

Figure 11.1.1 Weights of ears of corn receiving five different treatments

11.2 The Basic One-Way Analysis of Variance

One-way ANOVA

- **Notation**

Example 11.2.1 Weight Gain of Lambs

- What is the total number of observations?
- What is the grand mean?

Table 11.2.1 Weight gains of lambs (lb)*

	Diet 1	Diet 2	Diet 3
	8	9	15
	16	16	10
	9	21	17
		11	6
		18	
n_i	3	5	4
Sum $= \sum_{j=1}^{n_i} y_{ij}$	33	75	48
Mean $= \bar{y}_i$	11.000	15.000	12.000
SD $= s_i$	4.359	4.950	4.967

*Extra digits are reported for accuracy of subsequent calculations.

11.2 The Basic One-Way Analysis of Variance

Measuring Variation Within Groups

- **Pooled standard deviation**: a combined measure of variation within the I groups is the pooled standard deviation s_{pooled}, often simply denoted as just s, which is computed as follows.

Pooled Standard Deviation

$$s_{\text{pooled}} = s = \sqrt{\frac{\sum_{i=1}^{I}(n_i - 1)s_i^2}{\sum_{i=1}^{I}(n_i - 1)}} = \sqrt{\frac{\sum_{i=1}^{I}(n_i - 1)s_i^2}{n_{\bullet} - I}}$$

- **Pooled variance**: we call $s^2_{pooled} = s^2$ the pooled variance

11.2 The Basic One-Way Analysis of Variance

Measuring Variation Within Groups

Example 11.2.1 Weight Gain of Lambs

- What is the pooled variance?
- What is the pooled standard deviation?

Pooled Standard Deviation

$$s_{\text{pooled}} = s = \sqrt{\frac{\sum_{i=1}^{I}(n_i - 1)s_i^2}{\sum_{i=1}^{I}(n_i - 1)}} = \sqrt{\frac{\sum_{i=1}^{I}(n_i - 1)s_i^2}{n. - I}}$$

Table 11.2.1 Weight gains of lambs (lb)*

	Diet 1	Diet 2	Diet 3
	8	9	15
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*Extra digits are reported for accuracy of subsequent calculations.

11.2 The Basic One-Way Analysis of Variance

Measuring Variation Within Groups - ANOVA notation

- **Pooled variance**:

$$s^2_{pooled} = s^2 = \frac{\sum_{i=1}^{I}(n_i-1)s_i^{\,2}}{\sum_{i=1}^{I}(ni-1)} = \frac{\sum_{i=1}^{I}(n_i-1)s_i^{\,2}}{n.-I}$$

  - **SS (within)**: the numerator of s^2_{pooled} is known as the sum of squares within groups
  - **df (within)**: the denominator of s^2_{pooled} is known as the degrees of freedom within groups
  - **MS (within)**: their ratio is defined as the mean square within groups
    - Note that MS(within) is just another name for the pooled variance.

Mean Square Within Groups

$$\text{MS(within)} = \frac{\text{SS(within)}}{\text{df(within)}}$$

Sum of Squares and df Within Groups

$$\text{SS(within)} = \sum_{i=1}^{I}(n_i - 1)s_i^2$$

$$\text{df(within)} = n. - I$$

11.2 The Basic One-Way Analysis of Variance

Measuring Variation Between Groups

- **MS(between)**
  - the mean square between groups, or MS(between), describes between-group variability for more than two groups.
  - In fact, the MS(between) would indeed be the sample variance of the group means.

Mean Square Between Groups

$$\text{MS(between)} = \frac{\sum_{i=1}^{I} n_i(\bar{y}_i - \bar{\bar{y}})^2}{I - 1} = \frac{\text{SS(between)}}{\text{df(between)}}$$

Sum of Squares and df Between Groups

$$\text{SS(between)} = \sum_{i=1}^{I} n_i(\bar{y}_i - \bar{\bar{y}})^2$$

$$\text{df(between)} = I - 1$$

11.2 The Basic One-Way Analysis of Variance

Measuring Variation Between Groups

Example 11.2.1 Weight Gain of Lambs (continued)

- What is the variation within groups?
- What is the variation between groups?

Table 11.2.1 Weight gains of lambs (lb)*

	Diet 1	Diet 2	Diet 3
	8	9	15
	16	16	10
	9	21	17
		11	6
		18	
n_i	3	5	4
Sum $= \sum_{j=1}^{n_i} y_{ij}$	33	75	48
Mean $= \bar{y}_i$	11.000	15.000	12.000
SD $= s_i$	4.359	4.950	4.967

*Extra digits are reported for accuracy of subsequent calculations.

11.2 The Basic One-Way Analysis of Variance

A Fundamental Relationship of ANOVA Between Groups

- Consider an individual observation y_{ij}, the deviation of an observation from the grand mean is

$$y_{ij} - \bar{\bar{y}} = \underline{(y_{ij} - \bar{y}_i)} + \underline{(\bar{y}_i - \bar{\bar{y}})}$$

1. a within-group deviation 2. a between-group deviation

- It is also true that the analogous relationship holds for the corresponding sums of squares; that is,

$$\sum_{i=1}^{I} \sum_{j=1}^{n_i} (y_{ij} - \bar{\bar{y}})^2$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 + \sum_{i=1}^{I} \sum_{j=1}^{n_i} (\bar{y}_i - \bar{\bar{y}})^2$$

$$= \sum_{i=1}^{I} (n_i - 1)\, si^2 + \sum_{i=1}^{I} n_i\, (\bar{y}_i - \bar{\bar{y}})^2$$

= SS(within) + SS(between)

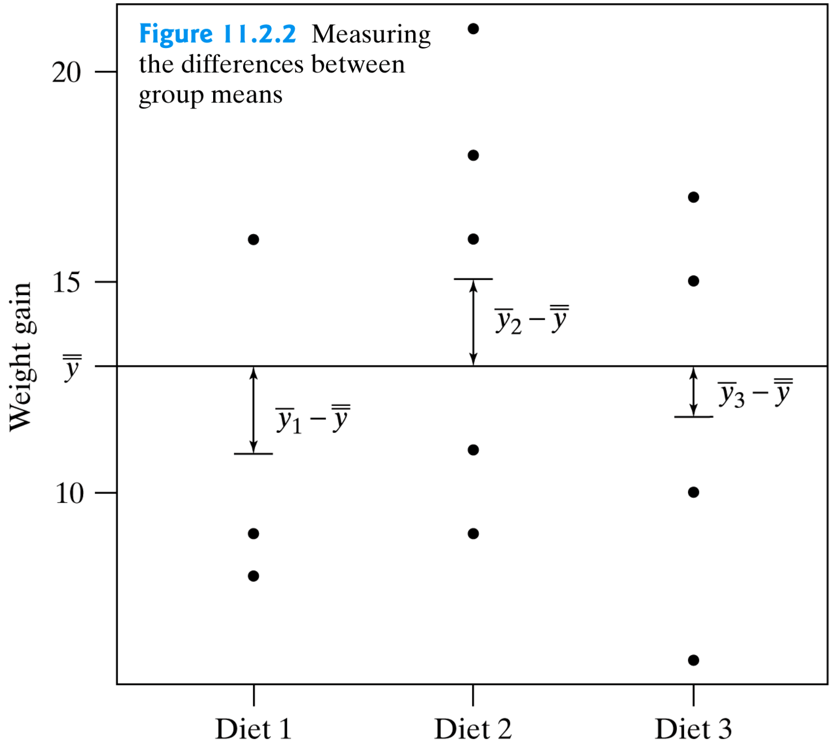

Figure 11.2.2 Measuring the differences between group means

- The quantity on the left-hand side of formula is called the total sum of squares, or SS(total).

11.2 The Basic One-Way Analysis of Variance

A Fundamental Relationship of ANOVA Between Groups

- The total sum of squares, or SS(total) is defined as

Definition of Total Sum of Squares

$$\text{SS(total)} = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij} - \bar{\bar{y}})^2$$

Relationship between Sums of Squares

$$\text{SS(total)} = \text{SS(between)} + \text{SS(within)}$$

- The total degrees of freedom, or df(total), is defined as follows:

Total df

$$\text{df(total)} = n_{\bullet} - 1$$

Relationship between df

$$\text{df(total)} = \text{df(between)} + \text{df(within)}$$

11.2 The Basic One-Way Analysis of Variance

The ANOVA table

- When working with the ANOVA quantities, it is customary to arrange them in a table.

ANOVA Quantities with Formulas

Source	df	SS (Sum of Squares)	MS (Mean Square)
Between groups	$I - 1$	$\sum_{i=1}^{I} n_i(\bar{y}_i - \bar{\bar{y}})^2$	SS/df
Within groups	$n. - I$	$\sum_{i=1}^{I} (n_i - 1)s_i^2$	SS/df
Total	$n. - 1$	$\sum_{i=1}^{I}\sum_{j=1}^{n_i} (y_{ij} - \bar{\bar{y}})^2$	

- Comment on terminology
  - While the terms "between-groups" and "within-groups" are not technical terms, they are useful in describing and understanding the ANOVA model.
  - Computer software and other texts commonly refer to these sources of variability as treatment (between groups) and error (within groups).

11.2 The Basic One-Way Analysis of Variance

The ANOVA table

Example 11.2.1 Weight Gain of Lambs (continued)

- Construct the ANOVA table.

Table 11.2.1 Weight gains of lambs (lb)*

	Diet 1	Diet 2	Diet 3
	8	9	15
	16	16	10
	9	21	17
		11	6
		18	
n_i	3	5	4
Sum $= \sum_{j=1}^{n_i} y_{ij}$	33	75	48
Mean $= \bar{y}_i$	11.000	15.000	12.000
SD $= s_i$	4.359	4.950	4.967

*Extra digits are reported for accuracy of subsequent calculations.

11.3 The Analysis of Variance Model

- It can be helpful to think of ANOVA in terms of the following model.

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

  - y_{ij}: jth observation in group i.
  - μ: represents the grand population mean
  - τ_i: represents the effect of group i
  - ε_{ij}: represents random error associated with observation j in group i
- The null hypothesis H_0: $\mu_1 = \mu_2 = \ldots = \mu_I$ is equivalent to H_0: $\tau_1 = \tau_2 = \ldots = \tau_I$
- Estimate the overall average, μ, with the grand mean of the data $\hat{\mu} = \bar{\bar{y}}$
  - estimate the population average for group i $\quad \hat{\mu}_i = \bar{y}_i$; $\hat{\mu} = \bar{\bar{y}}$
  - estimate the group effect $\quad \hat{\tau}_i = \bar{y}_i - \bar{\bar{y}}$; SS(between) = $\sum_{i=1}^{I} n_i \hat{\tau}_i^{\,2}$
  - estimate the random error $\quad \hat{\varepsilon}_{ij} = y_{ij} - \bar{y}_i$; SS(within) = $\sum_{i=1}^{I} \sum_{j=1}^{n_i} \hat{\varepsilon}_{ij}^{\,2}$
- We have $y_{ij} = \bar{\bar{y}} + (\bar{y}_i - \bar{\bar{y}}) + (y_{ij} - \bar{y}_i) = \hat{\mu} + \hat{\tau}_i + \hat{\varepsilon}_{ij}$

11.4 The Global F Test

Hypothesis

- The global null hypothesis is H_0: $\mu_1 = \mu_2 = \ldots = \mu_I$
  - H_0: all the population means are <u>equal</u> (no difference)
- The nondirectional alternative hypothesis H_A: The μi's are not all equal
  - H_A : at least one pair of the population means are <u>NOT equal</u> (differ)

The F distributions

- The F distributions named after the statistician and geneticist R. A. Fisher.
- F distribution depends on two parameters:
  - the numerator degrees of freedom: Numerator df = df(between)
  - the denominator degrees of freedom: Denominator df = df(within)
- Critical values for the F distribution are given in **Table 10** at the end of this book.

11.4 The Global F Test

The F test

- The F test is a classical test of the global null hypothesis.
- The test statistic, the **F statistic**, is calculated as follows:

$$F_s = \frac{\text{MS(between)}}{\text{MS(within)}}$$

  - Thus, large values of Fs tend to provide evidence against H_0.

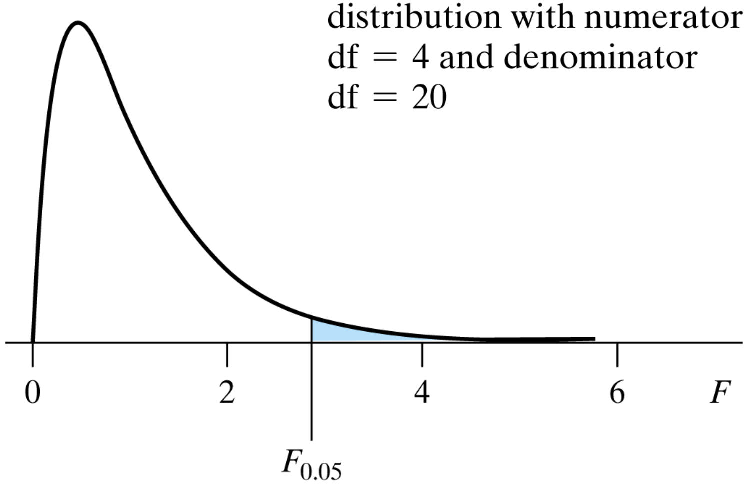

Figure 11.4.1 The *F* distribution with numerator df = 4 and denominator df = 20

Relationship Between F test And t test

- If only two groups are to be compared (I = 2), use either the F test or the t test
  - It can be shown that the F test and this "pooled" t test are actually equivalent procedures.
  - The test statistics is $t_s{}^2 = F_s$
  - Because of the equivalence of the tests, the application of the F test to compare the means of two samples will always give exactly the same P-value as the pooled t test applied to the same data.

11.4 The Global F Test

The F test

Example 11.4.1 Weight Gain of Lambs (continued)

- Is there any difference among the diets with respect to population mean weight gain?

Table 11.2.3 ANOVA table for lamb weight gains

Source	df	SS	MS
Between diets	2	36	18.00
Within diets	9	210	23.33
Total	11	246	

Summary

Chapter 11. Comparing the Means of Many Independent Samples

- 11.1 Introduction
- 11.2 The Basic One-Way Analysis of Variance
- 11.3 The Analysis of Variance Model
- 11.4 The Global F Test

Chapter 11

Comparing the Means of Many Independent Samples

11.5 Applicability of Methods

Standard Conditions

The ANOVA techniques, including the global F test, are valid if the following conditions hold.

1. Design conditions
   - (a) It must be reasonable to regard the groups of observations as <u>random samples</u> from their respective populations.
   - (b) The I samples must be <u>independent</u> of each other.

2. Population conditions
   - The I population distributions must be (approximately) <u>normal</u> with <u>equal standard deviations</u>:*

$$\sigma_1 = \sigma_2 = \ldots = \sigma_I$$

   - These conditions are extensions of the conditions given in Chapter 7 for the independent-samples t test with the added condition that the standard deviations be equal. The condition of normal populations with equal standard deviations is less crucial if the sample sizes (n_i) are large and approximately equal.

11.6 One-Way Randomized Blocks Design

Analyzing Data from a Randomized Block Experiment

- In the same way we cannot use a two-sample t test <u>when data are paired</u>, when an experiment has been blocked, we no longer can use our ANOVA methods of Section 11.4.
- Instead, we will use a randomized blocks ANOVA model.

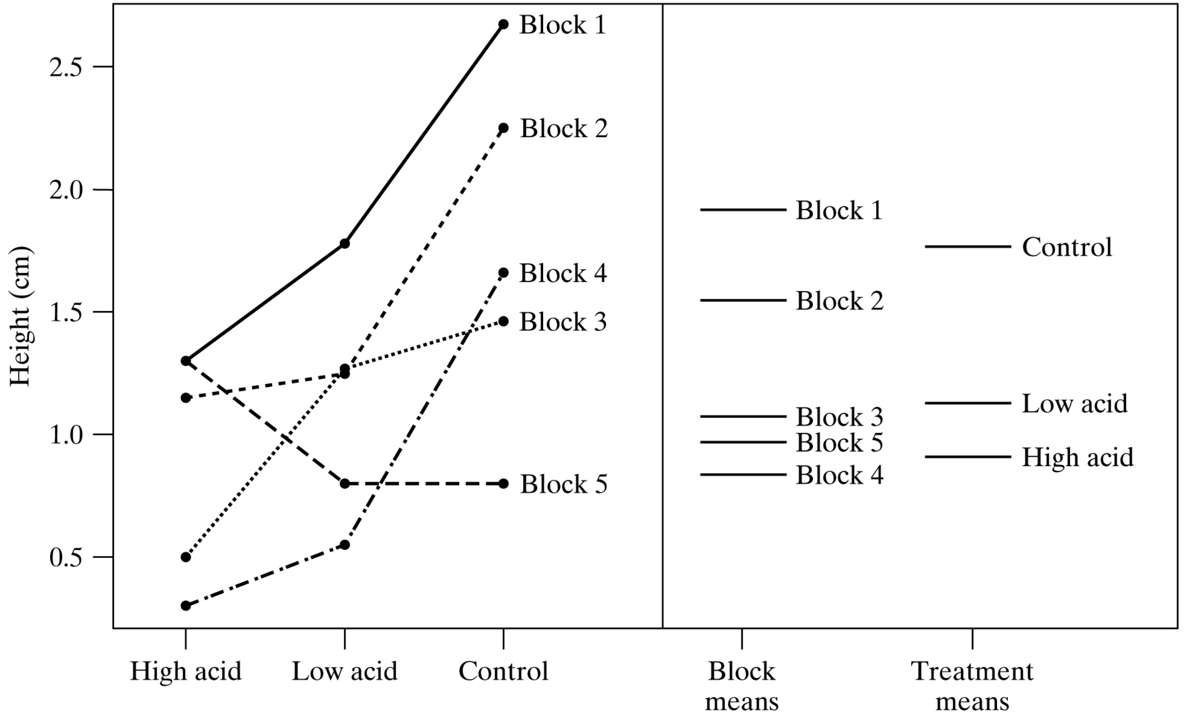

Figure 11.6.3 Dotplots of the alfalfa growth data with a summary of block and treatment means

11.6 One-Way Randomized Blocks Design

Analyzing Data from a Randomized Block Experiment

- Creating the blocks
  - create blocks that are as homogeneous within themselves as possible, so that the inherent variation between blocks becomes as far as possible
  - variation between blocks rather than within blocks.

- We extend the ANOVA model presented in Section 11.3 to the following model:

$$y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}$$

  - y_{ijk}: the kth observation when treatment i is applied in block j.
  - μ: represents the grand population mean
  - τ_i: represents the effect of group I
  - β_j: represents the effect of the jth block.
  - ε_{ijk}: represents random error associated with observation k in block j and group i

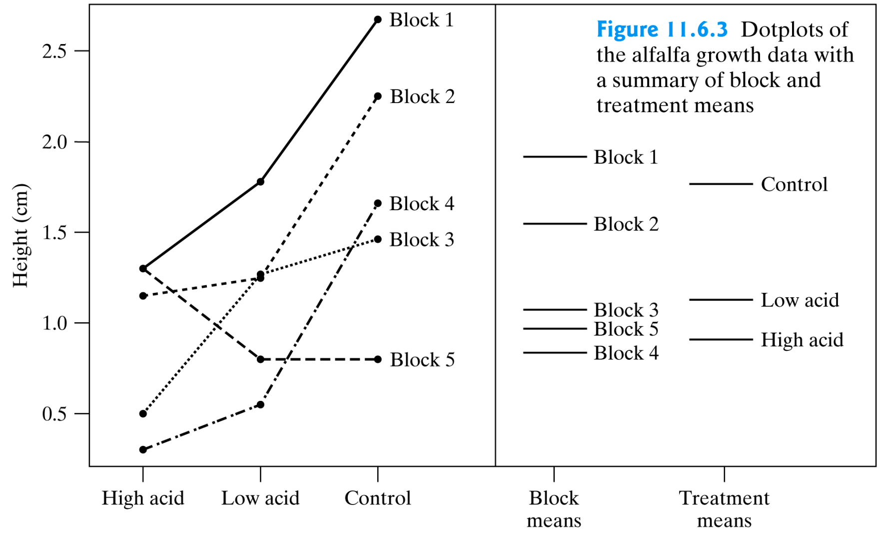

Figure 11.6.3 Dotplots of the alfalfa growth data with a summary of block and treatment means

11.6 One-Way Randomized Blocks Design

Analyzing Data from a Randomized Block Experiment

- The one-way randomized complete block F test
  - One-way ANOVA:

    SS(total) = SS(within) + **SS(between)**

  - One-way ANOVA with blocks:

    SS(total) = SS(within) + SS(blocks) + **SS(treatments)**

    - The F test split the one-way ANOVA SS(within) into two parts: SS(blocks) - variability among the block means; SS(within) - the remaining unexplained variation in the data.
    - write SS(treatments) rather than SS(between) to describe the variability between treatment
  - The F test
    - F_s = MS(treatments)/MS(within)
    - Numerator df = df(treatments), Denominator df = df(within)

11.6 One-Way Randomized Blocks Design

Analyzing Data from a Randomized Block Experiment

- Mean squares between blocks

Mean Squares Between Blocks

$$\text{MS(blocks)} = \frac{\sum_{j=1}^{J} m_j(\bar{y}_{\bullet j} - \bar{\bar{y}})^2}{J - 1}$$

- The sum of squares, or SS(total), and the total degrees of freedom, or df(total), are defined as follows:

Sum of Squares and df Between Blocks

$$\text{SS(blocks)} = \sum_{j=1}^{J} m_j(\bar{y}_{\bullet j} - \bar{\bar{y}})^2$$

$$\text{df(blocks)} = J - 1$$

11.6 One-Way Randomized Blocks Design

Analyzing Data from a Randomized Block Experiment

ANOVA Table – One-way ANOVA with blocks

– The F statistic: F_s = MS(treatments)/MS(within)

ANOVA Quantities with Formulas

Source	df	SS (Sum of Squares)	MS (Mean Square)
Between treatments	$I - 1$	$\sum_{i=1}^{I} n_i(\bar{y}_i - \bar{\bar{y}})^2$	SS/df
Between blocks	$J - 1$	$\sum_{j=1}^{J} m_j(\bar{y}_{\cdot j} - \bar{\bar{y}})^2$	SS/df
Within groups	$n. - I - J + 1$	SS(within) = SS(total) – SS(treatment) – SS(blocks)	SS/df
Total	$n. - 1$	$\sum_{i=1}^{I}\sum_{j=1}^{n_j}(y_{ij} - \bar{\bar{y}})^2$	

11.6 One-Way Randomized Blocks Design

Analyzing Data from a Randomized Block Experiment

Example 11.6.5 Agricultural Field Study

- Effect of acid on the growth rate of alfalfa plants.
- 3 treatment groups: low acid, high acid, and control.
- 5 cups for each of the 3 treatments, for a total of 15 observations.
- However, the cups were arranged near a window and researchers wanted to account for the effect of differing amounts of sunlight.
- Will acid affect the growth of alfalfa plants?

Figure 11.6.1 Design of the alfalfa experiment

	Block 1	Block 2	Block 3	Block 4	Block 5
Window	high	control	control	control	high
	control	low	high	low	low
	low	high	low	high	control

Organization of blocks for alfalfa experiment

Table 11.6.3 Alfalfa plant height after 5 days (cm)

	High acid	Low acid	Control	Block mean
Block 1	1.30	1.78	2.67	1.917
Block 2	1.15	1.25	2.25	1.550
Block 3	0.50	1.27	1.46	1.077
Block 4	0.30	0.55	1.66	0.837
Block 5	1.30	0.80	0.80	0.967
Treatment mean $= \bar{y}_i$	0.910	1.130	1.768	
n	5	5	5	

11.7 Two-Way ANOVA

Factorial ANOVA

- Some analysis of variance settings involve the simultaneous study of two or more factors.
  - two factors do NOT interact
    - two factors are additive in their effects, if the joint influence of two factors is equal to the sum of their separate influences.
  - two factors interact
    - the effect that one factor has on a response variable depends on the level of a second factor.

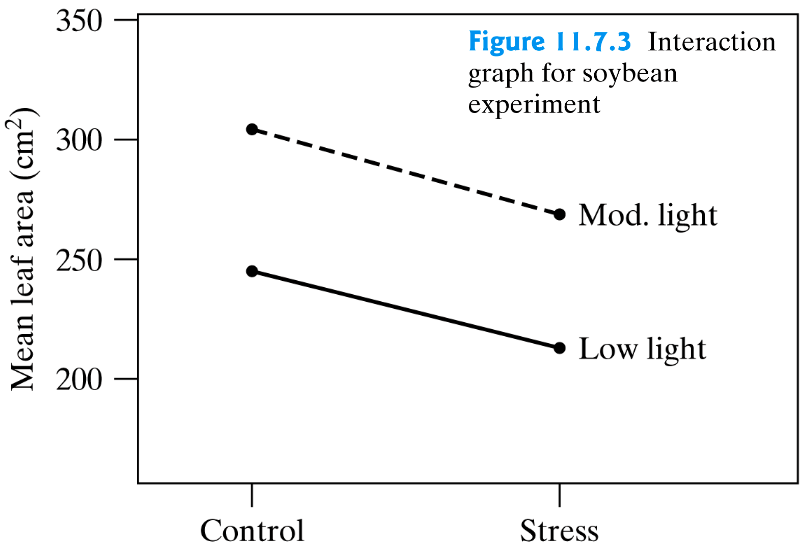

Figure 11.7.3 Interaction graph for soybean experiment

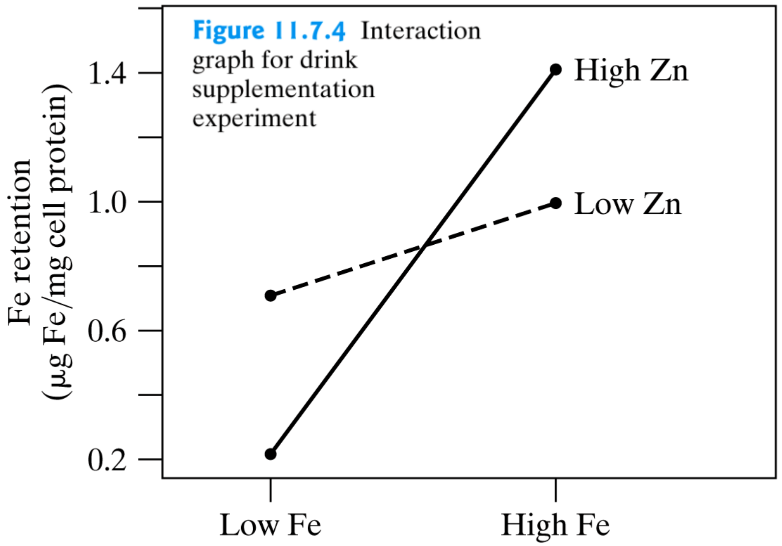

Figure 11.7.4 Interaction graph for drink supplementation experiment

11.7 Two-Way ANOVA

Hypothesis

- The global null hypothesis is H_0: $\gamma_{11} = \gamma_{12} = \ldots = \gamma_{IJ} = 0$
- The nondirectional alternative hypothesis H_A: The γ_{ij} ’s are not all equal

Factorial ANOVA

- We extend the ANOVA model presented in Section 11.3 to the following model:

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$$

  - where y_{ijk} is the kth observation of level i of the first factor and level j of the second factor.
  - τ_i represents the effect of level i of the first factor
  - β_j represents the effect of level j of the second factor
  - $\boldsymbol{\gamma_{ij}}$ is the effect of the interaction between level i of the first factor and level j of the second factor

11.7 Two-Way ANOVA

The F test

- The F statistic: F_s = MS(**interaction**)/MS(within)

ANOVA Quantities with Formulas

Source	df	SS (Sum of Squares)	MS (Mean Square)
Between i treatments	$I - 1$	$\sum_{i=1}^{I} n_i(\bar{y}_i - \bar{\bar{y}})^2$	SS/df
Between j treatments	$J - 1$	$\sum_{j=1}^{J} m_j(\bar{y}_{\cdot j} - \bar{\bar{y}})^2$	SS/df
Interaction	$(I - 1) \times (J - 1)$	*Can be calculated by computer*	SS/df
Within groups	$n. - IJ$	**SS(within)** = SS(total) – SS(treatment) – SS(interaction)	SS/df
Total	$n. - 1$	$\sum_{i=1}^{I}\sum_{j=1}^{n_j}(y_{ij} - \bar{\bar{y}})^2$	

11.7 Two-Way ANOVA

Factorial ANOVA

Example 11.7.3 Iron Supplements in Milk-Based Fruit Beverages

- Effects of drink fortification on the cellular retention of iron
- Researchers conducted an experiment by fortifying milk-based fruit drinks with low and high levels of iron (Fe) and zinc (Zn).
- There were 8 observations at each combination of Fe and Ze supplementation level.
- Whether Fe and Zn supplementation levels interact?

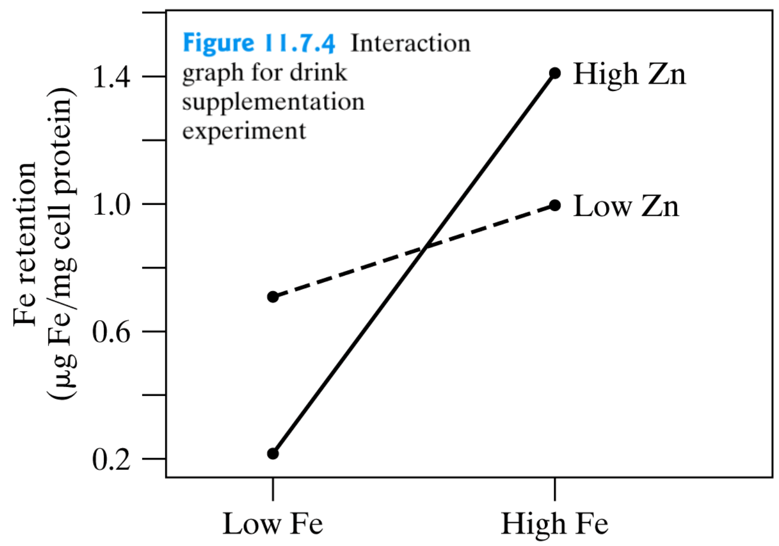

Figure 11.7.4 Interaction graph for drink supplementation experiment

Table 11.7.4 ANOVA table for drink supplement experiment

Source	df	SS	MS	*F* ratio
Between Fe levels	1	4.4023	4.4023	2317.0
Between Zn levels	1	0.0109	0.0109	5.74
Interaction	1	1.6555	1.6555	871.3
Within groups	28	0.0523	0.0019	
Total	31	6.1210		

Summary

Chapter 11. Comparing the Means of Many Independent Samples

- 11.1 Introduction
- 11.2 The Basic One-Way Analysis of Variance
- 11.3 The Analysis of Variance Model
- 11.4 The Global F Test

- 11.5 Applicability of Methods
- 11.6 One-Way Randomized Blocks Design
- 11.7 Two-Way ANOVA