ADS2 - Bayesian Inference, Problem set notes

ADS2 (based on Simon's and Dmytro's solutions)

Semester 2, 2022/23

These solutions are just model ones. Thus, you might have developed your own strategy that gives comparable results but uses more optimal code. Or your code may be very different from the one suggested here. It is absolutely fine as soon as our results are comparable.

In vitro **fertilization (IVF) and pregnancy tests**

This task will help to demonstrate the use of the Bayes formula that is shown below.

$$P_{(A \mid B)} = \frac{P_{(B \mid A)} \cdot P_{(A)}}{P_{(B)}}$$

$$P_{(B)} = \sum P_{(B_i \mid A_i)} \cdot P_{(A_i)}$$

As A and B do not look clear, we can assume B in the Bayes formula as the test$^+$ and A as the girl being pregnant. Thus, $P_{(A \mid B)}$ represents the chance of the girl being pregnant given the pregnancy test is positive. We do not know it. But we know $P_{(B \mid A)}$, the chance of the test to be positive (test$^+$) if the person is pregnant (the correct test$^+$).

Let's summarize what we have.

We have a system with two events each having two possible outcomes: pregnancy (pregnant vs non-pregnant) and test result (test is positive or negative). Thus, the formula becomes easier to grasp:

$$P_{(A \mid B)} = \frac{P_{(B \mid A)} \cdot P_{(A)}}{P_{(B)}}$$

$$P_{(pregnant \mid test^+)} = \frac{P_{(test^+ \mid pregnant)} \cdot P_{(pregnant)}}{P_{(test+)}}$$

We can build a table to visualize the information we have (Table 1):

Table 1: The data we have initially.

	Not pregnant	Pregnant	$P_{(\text{test})}$
Test $-$	$(0.8 \mid \text{not pregnant})$		
Test $+$		$(0.95 \mid 0.2222)$	
$\mathbf{P_{(preg)}}$		0.2222	1

What is the chance that a girl with a positive test is really pregnant?

So, we know $P_{(A)}$, the chances of pregnancy, $P_{(B \mid A)}$, the chance of the test to be positive (test$^+$) if the person is pregnant (the correct test$^+$), $P_{(B^C \mid A^C)}$, the chance of the test to be positive (test$^-$) while the person is not pregnant (correct test$^-$). Not bad. For this task, we need to use the Bayes formula. The only element we miss is $P_{(B)}$, the chances of test$^+$, either if the girl is pregnant (correct test$^+$) or not (*wrong* test$^+$, false positive result). We need to dive into the conditional probability.

Let's do it!

The probability of being pregnant *AND* tested positive can be determined by the intersection formula: $P_{(test^+ \cap preg)} = P_{(test^+ \mid preg)} \cdot P_{(preg)} = 0.2111$. To find out the probability of not being pregnant *AND* still tested positive, we need to use the formula: $P_{(test^+ \cap not\ preg)} = P_{(test^+ \mid not\ preg)} \cdot P_{(not\ preg)}$. But we do not know this term: $P_{(test^+ \mid not\ preg)}$. Luckily, we know its complement, $P_{(test^- \mid not\ preg)} = 0.8$. In this condition, being *NOT* pregnant, we disregard another state (*being pregnant*) and consider the chosen condition as 100%. Thus, the complement to $P_{(test^- \mid not\ preg)}$ is $P_{(test^+ \mid not\ preg)} = 1 - 0.8$. Our table now looks as follows (Table 2):

Table 2: We determined several terms.

	Not pregnant	Pregnant	$P_{(\text{test})}$
Test $-$	(0.8 | not pregnant)		
Test $+$	(0.2 | not pregnant)	0.2111 (0.95 | 0.2222)	
$\mathbf{P_{(preg)}}$		0.2222	1

The overall chance of being not pregnant after IVF, $P_{(A^C)} = 1 - P_{(A)} = 1 - 0.2222$. $P_{(test^+)} = P_{(test^+ \mid preg)} + P_{(test^+ \mid not\ preg)}$. Now, $P_{(test^+ \mid not\ preg)}$ is easy to determine. Finally, our table will look as follows (Table 3):

Table 3: All elements for the first task are determined!

	Not pregnant	Pregnant	$P_{(\text{test})}$
Test $-$	(0.8 | 0.7778)		
Test $+$	0.1556 (0.2 | 0.7778)	0.2111 (0.95 | 0.2222)	0.3667
$\mathbf{P_{(preg)}}$	0.7778	0.2222	1

Finally, the Bayes formula would look as follows:

$$P_{(pregnant \mid test^+)} = \frac{P_{(test^+ \mid pregnant)} \cdot P_{(pregnant)}}{P_{(test^+)}} = \frac{0.95 \cdot 0.2222}{0.3667} \approx 0.5756$$

Is this result convincing?

Not really. Basically, it means that the girl may be pregnant or not with almost equal probability (0.5756 and 0.4244, respectively; OK, the probability of pregnancy is slightly higher). Still much better than before (0.2222) that is rather low.

What is the chance that a girl with a negative test is still pregnant?

We want to find out $\mathrm{P}_{(\mathrm{pregnant}\ |\ \mathrm{test}-)}$. We need to:

1. Calculate the chances of being pregnant (already known and is 0.2222) and *not* pregnant (1 - 0.2222);
2. Probability of tested negative given the person *is* pregnant or *is not* pregnant;
3. The overall, marginal, probability of the negative result of the test.

We can update Table 3 further and get the following result (Table 4):

Table 4: All the table elements are added.

	Not pregnant	Pregnant	$\mathrm{P}_{(\mathrm{test})}$
Test −	0.6222 (0.8 | 0.7778)	0.0111 (0.05 | 0.2222)	0.6333
Test +	0.1556 (0.2 | 0.7778)	0.2111 (0.95 | 0.2222)	0.3667
$\mathbf{P}_{\mathbf{(preg)}}$	0.7778	0.2222	1

Finally, the Bayes formula would look as follows:

$$P_{(pregnant\ |\ test-)} = \frac{P_{(test-\ |\ pregnant)} \cdot P_{(pregnant)}}{P_{(test-)}} = \frac{0.05 \cdot 0.2222}{0.6333} \approx 0.0175$$

Thus, if the person gets a negative test result, it is rather unlikely that they are pregnant. This posterior probability would be difficult to challenge even with several subsequent positive tests. Try it by yourself! Test^{+} may mean almost anything, but test^{-} means exactly what it is.

Suppose, the girl got a positive result. Did her beliefs get updated?

Sure! The prior belief in success (pregnancy) was rather low (0.2222). But after the first positive result, there is a good reason to think that the girl is actually pregnant. Although the test kit is not very good (we will discuss why later), it is still able to tell whether the person is pregnant or not. So, there is a good reason to consider pregnancy.

She decided to be double sure. After some time, she used the same test once again, and she got a positive result once again. What would be the chance that she is pregnant given she got positive results twice?

As our expectations changed (*we saw the results of the first test*), we may change our prior belief for the second test to the posterior belief from our previous step: from 0.2222 to 0.5756. Chances of the correct (or wrong) result of the test do not change (as she is using the same kit), but as the overall prior belief in the pregnancy changed, we need to recalculate the whole table (Table 5):

Table 5: Previous posteriors are our new priors.

	Not pregnant	Pregnant	$\mathrm{P}_{(\mathrm{test})}$
Test −			

	Not pregnant	Pregnant	$P_{(test)}$
Test +	0.0849 (0.2 | 0.4244)	0.5468 (0.95 | 0.5756)	0.6317
$\mathbf{P_{(preg)}}$	0.4244	0.5756	1

So, our Bayes formula will look as follows.

$$P_{(preg\ \mid\ test^{++})} = \frac{P_{(test^{+}\ \mid\ preg)} \cdot P_{(preg\ \mid\ test^{+})}}{P_{(test^{+}\ updated)}} \approx$$

$$\approx \frac{0.95 \cdot 0.5756}{0.6317} \approx 0.8656$$

The chances that the girl is really pregnant increase with each positive test! But they would increase faster if the test was better. How? See below!

What can you say about this test kit? Is it good? Is it specific? Is it sensitive? Support your claims with some calculations.

The test kit is not good. Why? The chances of the *correct results* are quite low for the positive test. How low? Let's calculate the odds of the correct result for each condition: for a positive test ($test^{+}$) and for a negative test ($test^{-}$).

$$Odds_{(pregnant\ vs\ non-pregnant\ \mid\ test^{+})} = \frac{0.211}{0.1556} \approx 1.356$$

$$Odds_{(non-pregnant\ vs\ pregnant\ \mid\ test^{-})} = \frac{0.6222}{0.0111} \approx 56.054$$

Thus, the odds of getting the correct results for the positive test are extremely low:

$$Odds\ ratio = \frac{1.356}{56.054} \approx 0.024$$

This test kit is too sensitive. Chances of false positive results are too high. It would be better to increase the chance of making the correct decision if the person is not pregnant == decrease the rate of false positive results.

A dicy game

Now, we can change $P_{(A\ \mid\ B)} = \frac{P_{(B\ \mid\ A)} \cdot P_{(A)}}{P_{(B)}}$ to $P_{(Hypothesis\ \mid\ Data)} = \frac{P_{(Data\ \mid\ Hypothesis)} \cdot P_{(Hypothesis)}}{P_{(Data)}}$.

In the practical, you formulated five hypotheses about the number of sixes on your friend's die: the die may have 1-5 sixes. Thus, chances of getting sixes are $\frac{1}{6}$, ..., $\frac{5}{6}$. In the in-class materials, you gave equal weight to each of these five hypotheses. Probably, you will think that this prior belief is unreasonable. And this opinion is makes sense! Whatever many people we ask that many different opinions we get. Let's try different priors! Now, you need to set different prior hypotheses: three students suggest their own prior hypotheses. So, let's incorporate their priors in our code too.

```
Sixes <- 1:5
P_six <- Sixes / 6 # Statistical model

P_data_hyp <- dbinom(x = 7, # Likelihood of each model
```

```r
                                  size = 20,
                                  prob = P_six)

# Prioirs
P_hyp_Aditi <- c(0.99, rep((1 - 0.99) / 4, 4))
P_hyp_Bo <- rep(0.2, 5)
P_hyp_Charlie <- c(0.01, rep((1 - 0.01) / 4, 4))
dice <- data.frame(
  Sixes,
  P_six,
  P_data_hyp,
  P_hyp_Bo,
  P_hyp_Aditi,
  P_hyp_Charlie
)
```

Our results and priors can be summarized in a table. See Table 6 for details.

```r
dice %>%
  round(digits = 4) %>%
  kable(caption = "Summary of our data, statistical models,
        and prior beliefs of each student")
```

Table 6: Summary of our data, statistical models, and prior beliefs of each student

Sixes	P_six	P_data_hyp	P_hyp_Bo	P_hyp_Aditi	P_hyp_Charlie
1	0.1667	0.0259	0.2	0.9900	0.0100
2	0.3333	0.1821	0.2	0.0025	0.2475
3	0.5000	0.0739	0.2	0.0025	0.2475
4	0.6667	0.0028	0.2	0.0025	0.2475
5	0.8333	0.0000	0.2	0.0025	0.2475

Everything is possible

Bo's beliefs (everything is possible) are exactly the same as ours from the practical class – it is quite unlikely that the die has only one sixes. The die may be anything. Thus, the chances that the die has only 1 sixes are pretty low:

```r
P_givenData_Bo <- sum(P_hyp_Bo * dice[, 3])
(P_data_hyp[1] * P_hyp_Bo[1]) / P_givenData_Bo
```

```
## [1] 0.09088213
```

The die is likely normal

According to Aditi's beliefs (it is highly unlikely that the die has more than one sixes), it is highly likely that the die has only one sixes. After all, unfair dice are not that commonly sold, so your friend would have to spend a lot of effort to find them to only win you this time. Thus, the chances that the die has only 1 sixes are still pretty high:

```r
P_givenData_Aditi <- sum(P_hyp_Aditi * dice[, 3])
(P_data_hyp[1] * P_hyp_Aditi[1]) / P_givenData_Aditi
```

```
## [1] 0.9753616
```

For sure, the die is biased!

According to Charlie's beliefs (it is highly unlikely that the die has only one sixes), it is highly unlikely that the die has only one sixes. Most probably, it had two or three sixes. Thus, the chances that the die has only 1 sixes are also pretty low:

```r
P_givenData_Charlie <- sum(P_hyp_Charlie * dice[, 3])
(P_data_hyp[1] * P_hyp_Charlie[1]) / P_givenData_Charlie
```

```
## [1] 0.004022837
```

Now, you can see how the observed data changed the opinions of each of the students about their hypotheses. You can try yourself and see how posterior beliefs about the other hypotheses change for each student as a result of observing the data. For example, Bo's beliefs underwent quite a dramatic transformation (Table 8):

```r
P_poster_Bo <- ((P_data_hyp * P_hyp_Bo) / P_givenData_Bo)
dice_poster <- data.frame(Sixes,
                          P_six,
                          P_data_hyp,
                          P_hyp_Bo,
                          P_poster_Bo)
dice_poster %>%
  round(digits = 4) %>%
  kable(caption = "Prior and posterior beliefs of Bo (everything is possible)")
```

Table 7: Prior and posterior beliefs of Bo (everything is possible)

Sixes	P_six	P_data_hyp	P_hyp_Bo	P_poster_Bo
1	0.1667	0.0259	0.2	0.0909
2	0.3333	0.1821	0.2	0.6395
3	0.5000	0.0739	0.2	0.2596
4	0.6667	0.0028	0.2	0.0100
5	0.8333	0.0000	0.2	0.0000

Check how the results changed for the other two students or try your own beliefs!

Ref: https://allendowney.github.io/BiteSizeBayes/04_dice.html

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Created by Dmytro Shytikov in 2023 with suggestions from Simon (Jingyuan Chen).

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