2.1 We consider $(\mathbb{R}\backslash\{-1\}, \star)$, where

$$a \star b := ab + a + b, \qquad a, b \in \mathbb{R}\backslash\{-1\} \tag{2.134}$$

a. Show that $(\mathbb{R}\backslash\{-1\}, \star)$ is an Abelian group.
b. Solve

$$3 \star x \star x = 15$$

in the Abelian group $(\mathbb{R}\backslash\{-1\}, \star)$, where $\star$ is defined in (2.134).

2.1 a. Abelian group: $x \otimes y = y \otimes x$

$x \star y = 0$

$xy + x + y = 0$

$y(x+1) = -x$

$y = \frac{-x}{x+1} \quad (x \neq -1)$

$a \star b = b \star a$

$ab + a + b = ba + b + a$

b. $3 \star x \star x = 15$

$3 \star x = 3x + 3 + x$

$= 4x + 3$

$(4x+3) \star x = 15$

$(4x+3)x + 4x + 3 + x = 15$

$4x^2 + 3x + 4x + 3 + x = 15$

$4x^2 + 8x - 12 = 0$

$x^2 + 2x - 3 = 0$

± 3
$ 1$

$(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

2.2 Let n be in $\mathbb{N}\backslash\{0\}$. Let k, x be in $\mathbb{Z}$. We define the congruence class $\bar{k}$ of the integer k as the set

agreement

$$\begin{aligned}\bar{k} &= \{x \in \mathbb{Z} \mid x - k = 0 \pmod{n}\} \\ &= \{x \in \mathbb{Z} \mid \exists a \in \mathbb{Z}\colon (x - k = n \cdot a)\}\,.\end{aligned}$$

We now define $\mathbb{Z}/n\mathbb{Z}$ (sometimes written $\mathbb{Z}_n$) as the set of all congruence classes modulo n. Euclidean division implies that this set is a finite set containing n elements:

n으로 나눈 나머지

$$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$$

For all $\bar{a}, \bar{b} \in \mathbb{Z}_n$, we define

$$\bar{a} \oplus \bar{b} := \overline{a+b}$$

a. Show that $(\mathbb{Z}_n, \oplus)$ is a group. Is it Abelian?

b. We now define another operation $\otimes$ for all $\bar{a}$ and $\bar{b}$ in $\mathbb{Z}_n$ as

$$\bar{a} \otimes \bar{b} = \overline{a \times b}\,, \tag{2.135}$$

where $a \times b$ represents the usual multiplication in $\mathbb{Z}$.
Let $n = 5$. Draw the times table of the elements of $\mathbb{Z}_5\backslash\{\bar{0}\}$ under $\otimes$, i.e., calculate the products $\bar{a} \otimes \bar{b}$ for all $\bar{a}$ and $\bar{b}$ in $\mathbb{Z}_5\backslash\{\bar{0}\}$.
Hence, show that $\mathbb{Z}_5\backslash\{\bar{0}\}$ is closed under $\otimes$ and possesses a neutral element for $\otimes$. Display the inverse of all elements in $\mathbb{Z}_5\backslash\{\bar{0}\}$ under $\otimes$. Conclude that $(\mathbb{Z}_5\backslash\{\bar{0}\}, \otimes)$ is an Abelian group.

c. Show that $(\mathbb{Z}_8\backslash\{\bar{0}\}, \otimes)$ is not a group.

d. We recall that the Bézout theorem states that two integers a and b are relatively prime (i.e., $gcd(a, b) = 1$) if and only if there exist two integers u and v such that $au + bv = 1$. Show that $(\mathbb{Z}_n\backslash\{\bar{0}\}, \otimes)$ is a group if and only if $n \in \mathbb{N}\backslash\{0\}$ is prime.

2.a

- $\forall x, y \in Z_n : x \otimes y \in Z_n$ (closure of Z_n)
- Associativity
  $\forall x, y, z \in Z_n$:
  $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
  $\overline{x+y+z} = \overline{x+y+z}$
  ex.) $x=1$ $y=2$ $z=3$ $n=2$
  $\overline{1+3} = 2$ $\quad \overline{1+1} = 2$
- Neutral element
  $x \otimes e = x$ $\quad e \otimes x = x$
  ex.) $x=1$ $y=3$ $n=3$
  $1 \oplus 3 = 1$ $\quad 3 \oplus 1 = 1$

$\left.\begin{aligned}&\bar{a} \oplus \bar{b} = \bar{b} \oplus \bar{a} \\ &= \overline{a+b} = \overline{b+a}\end{aligned}\right\}$ Yes, it is Abelian.

2.b $Z_5 \backslash \{\bar{0}\} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ $n=5$

$\bar{a}$	$\bar{b}$	$\bar{a}\otimes\bar{b}$
$\bar{1}$	$\bar{1}$	1
$\bar{1}$	$\bar{2}$	2
$\bar{1}$	$\bar{3}$	3
$\bar{1}$	$\bar{4}$	4

$\bar{a}$	$\bar{b}$	$\bar{a}\otimes\bar{b}$
$\bar{2}$	$\bar{1}$	2
$\bar{2}$	$\bar{2}$	4
$\bar{2}$	$\bar{3}$	1
$\bar{2}$	$\bar{4}$	3

$\bar{a}$	$\bar{b}$	$\bar{a}\otimes\bar{b}$
$\bar{3}$	$\bar{1}$	3
$\bar{3}$	$\bar{2}$	1
$\bar{3}$	$\bar{3}$	4
$\bar{3}$	$\bar{4}$	2

$\bar{a}$	$\bar{b}$	$\bar{a}\otimes\bar{b}$
$\bar{4}$	$\bar{1}$	4
$\bar{4}$	$\bar{2}$	3
$\bar{4}$	$\bar{3}$	2
$\bar{4}$	$\bar{4}$	1

- Closed because all values for $\bar{a} \otimes \bar{b}$ are in $Z_5 \backslash \{\bar{0}\}$
- example of posessing a neutral element
- Abelian bc all $\bar{a} \otimes \bar{b} = \bar{b} \otimes \bar{a}$

2.d

ex) $gcd(2,3) = 1$
$2u + 3v = 1$
$u = -1$ $v = 1$

2.c $\bar{2} \otimes \bar{4} \otimes \bar{1} \notin Z_8 \backslash \{\bar{0}\}$

bc

$\bar{2} \otimes \bar{4} \otimes \bar{1} = \bar{0}$

2.~~3~~ Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

틀림

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ as the standard matrix multiplication.
Is $(\mathcal{G}, \cdot)$ a group? If yes, is it Abelian? Justify your answer.

Closure: $\mathcal{G}\cdot\mathcal{G} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2x & 2z+xy \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{bmatrix}$

$\mathcal{G}\cdot\mathcal{G} \notin \mathcal{G}$

it is not a group.

2.4 Compute the following matrix products, if possible:

a.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

a. not possible

b.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$

c.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

c. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$

d.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}$$

2×4 $\quad$ 4×2

d. $\begin{bmatrix} 14 & 6 \\ -21 & 2 \end{bmatrix}$

e.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}$$

4×2 $\quad$ 2×4

e. $\begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$

2.5 Find the set $\mathcal{S}$ of all solutions in $\boldsymbol{x}$ of the following inhomogeneous linear systems $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ and $\boldsymbol{b}$ are defined as follows:

a.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

a. $[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$ impossible!

No solution

b. 풀이를 외웠기 때문에 모르는듯 (못 푸는듯)

⊕ 계산 잘못함

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

b. $[A|b] = \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$

$$\begin{array}{c} \\ \rightsquigarrow \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 - x_5 = 3$
$-\ x_4 - x_5 = -1$
$x_1 - x_4 = 4$

λ

2.6 Using Gaussian elimination, find all solutions of the inhomogeneous equation system $\boldsymbol{Ax} = \boldsymbol{b}$ with

$b \neq 0$

reduced 될 때까지 연산

(f)

계속 틀림

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

$$[A|b] = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ -\; 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & -2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \end{array}$$

0 0 0 0 1 -1 1 가능

$x_2 + x_6 = 1$

$x_4 + x_6 = -2$

$-x_5 + x_6 = -1$

$x_1 = \alpha$

$x_6 = \beta$

x_1	α
x_2	$1-\beta$
x_3	γ
x_4	$-2-\beta$
x_5	$\beta+1$
x_6	β

2.7 Find all solutions in $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $\boldsymbol{A}\boldsymbol{x} = 12\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} x_i = 1$.

$$Ax - 12x = 0$$

$$(A - 12I)x = 0$$

$$12I = 12 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$A - 12 = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} - \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 4 & 3 \\ 2 & -4 & 3 \\ 0 & 2 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -12 & 15 \\ 2 & -4 & 3 \\ 0 & 2 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -12 & 15 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$x_1 - \frac{3}{2}x_3 = 0$$

$$x_2 - \frac{3}{2}x_3 = 0 \qquad x : \begin{bmatrix} \frac{3}{2}\alpha \\ \frac{3}{2}\alpha \\ \alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$

$$x_3 = \alpha$$

$$\frac{3}{2}\alpha + \frac{3}{2}\alpha + \alpha = 4\alpha = 1$$

$$\alpha = \frac{1}{4}$$

$$x : \left[\frac{3}{8}, \frac{3}{8}, \frac{1}{4}\right]^T \in \mathbb{R}^3$$

2.8 Determine the inverses of the following matrices if possible: * determinant 3x3 구하는 법 기억하기!

a.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

a.

$\det(A) = 2(24-25) - 3(18-20) + 4(15-16)$
$= 2(-1) - 3(-2) + 4(-1)$
$= -2 + 6 - 4$
$= 0$ ⇝ inverse matrix doesn't exist

b.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Ex 2.9
Calculating an Inverse Matrix by Gaussian Elimination

b. $[A|I]$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

2.9 Which of the following sets are subspaces of $\mathbb{R}^3$?

O a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$

X b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

O c. Let γ be in $\mathbb{R}$.

$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$

X d. $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

a set of integers

a. subspaces of $\mathbb{R}^3$

b. subspaces of $\mathbb{R}^3$

c. $\xi_1 = 2\xi_2 - 3\xi_3 + \gamma$ ← not subspaces of $\mathbb{R}^3$

$C = (2\xi_2 - 3\xi_3 + \gamma, \xi_2, \xi_3)$

$\xi_2 = 1 \quad \xi_3 = 1 \qquad \xi_2 = 0 \quad \xi_3 = 0$

$(-1+\gamma, 1, 1) + (\gamma, 0, 0)$

$= (2r-1, 1, 1)$

$\notin$

C

d. subspaces of $\mathbb{R}^3$

2.9 Which of the following sets are subspaces of $\mathbb{R}^3$?

a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$

b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

c. Let γ be in $\mathbb{R}$.
$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$

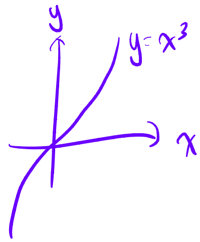
d. $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

$ax+by+cz=0$

↳ 평면

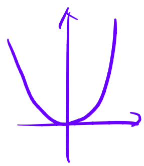
$(x,y,z)\cdot(1,-2,3)=0$

a.



b. $\lambda^2 = \lambda_1^2 + \lambda_2^2 \geq 0$

$(+)$



$(\cdot)$ $c=-1$

$(-1, 1, 0)$

$= (\lambda^2, -\lambda^2, 0)$ ≠

c. $\gamma = 0$

d. $c = \sqrt{2}$

2.9 Which of the following sets are subspaces of $\mathbb{R}^3$?

a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$

b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

c. Let γ be in $\mathbb{R}$.
$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$

d. $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$ a set of integers

〈풀이보고옴〉 아직도 모르겠음

a. $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

O subspace

b. $B: \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ X subspace

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ missing.

c. $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

2.10 Are the following sets of vectors linearly independent?

a.

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

b.

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

a. $$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ dependent

b. $$[x_1, x_2, x_3] \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$0 \quad 0 \quad 0$$

⇝ 3 pivot columns

independent

2.11 Write

$$y = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

as linear combination of

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

$$-6x_1 + 3x_2 + 2x_3 = y$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

2.12 Consider two subspaces of $\mathbb{R}^4$:

$$U_1 = \text{span}[\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}], \quad U_2 = \text{span}[\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}].$$

Determine a basis of $U_1 \cap U_2$.

$$U_1 = \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \end{array} \qquad U_2 = \begin{array}{c} \begin{array}{ccc} V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rightsquigarrow \begin{bmatrix} -1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[V_1 | V_2 | V_4 | V_5]$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \end{array} \qquad \rightsquigarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} V_4 & V_5 & V_6 \end{array} \end{array}$$

$$V_6 = (-V_4 - 2V_5)$$

$$V_3 = \frac{1}{3}(V_1 - 2V_2)$$

2.13 Consider two subspaces U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{0}$ and U_2 is the solution space of the homogeneous equation system $\boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{0}$ with

$$\underset{4\times3}{\boldsymbol{A}_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \underset{4\times3}{\boldsymbol{A}_2} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

Basis: Minimal generating set and a maximal linearly independent set of vectors.

a. Determine the dimension of U_1, U_2.
b. Determine bases of U_1 and U_2.
c. Determine a basis of $U_1 \cap U_2$.

dimension: number of basis vectors of vectorspace.

rank = # of independent rows/columns

a.

$A_1x = 0$

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \boxed{1} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & \boxed{0} \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

non-pivot column

← rank = 2
← dim = 2 ?

$$0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times 1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \times 1 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \times (-1)$$

rank과 dimension 차이...?

solution space: $\{x \in \mathbb{R}^3 \mid x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \lambda_1, \lambda_1 \in \mathbb{R}\}$

← dim = 1

$A_2 x = 0$

$$A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

solution space: $\{x \in \mathbb{R}^3 \mid x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \lambda_1, \lambda_1 \in \mathbb{R}\}$ dim = 1

2.13 Consider two subspaces U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{0}$ and U_2 is the solution space of the homogeneous equation system $\boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{0}$ with

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

a. Determine the dimension of U_1, U_2.
b. Determine bases of U_1 and U_2.
c. Determine a basis of $U_1 \cap U_2$.

$U_1, U_2 = \{ x \in \mathbb{R}^3,\ x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \lambda,\ \lambda \in \mathbb{R} \}$

b.

bases of U_1 and U_2 = $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

c. $U_1 \cap U_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

basis = $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

2.14 Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of $\boldsymbol{A}_1$ and U_2 is spanned by the columns of $\boldsymbol{A}_2$ with

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

a. Determine the dimension of U_1, U_2
b. Determine bases of U_1 and U_2
c. Determine a basis of $U_1 \cap U_2$

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) $A_1, A_2 : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

both dimension = 2
(# of basis vectors)

b. bases of U_1 and U_2

$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

이렇게
표현해도
되는거
아닌가?

2.14 Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of $\boldsymbol{A}_1$ and U_2 is spanned by the columns of $\boldsymbol{A}_2$ with

$$\boldsymbol{A}_1 = \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \end{array}, \quad \boldsymbol{A}_2 = \begin{array}{c} \begin{array}{ccc} & c_1 & c_2 \end{array} \\ \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \end{array}.$$

a. Determine the dimension of U_1, U_2
b. Determine bases of U_1 and U_2
c. Determine a basis of $U_1 \cap U_2$

풀이에서 가져옴
↓
풀이 외울듯...

$$B = \left\{ \overset{b_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}}, \overset{b_2}{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} \right\} \qquad C = \left\{ \overset{c_1}{\begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix}}, \overset{c_2}{\begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}} \right\}$$

$$x \in U_1 \cap U_2 \Leftrightarrow \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} : (x = \lambda_1 b_1 + \lambda_2 b_2) \wedge (x = \lambda_3 c_1 + \lambda_4 c_2)$$

$$\Leftrightarrow \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} : (x = \lambda_1 b_1 + \lambda_2 b_2) \wedge (\lambda_1 b_1 + \lambda_2 b_2 = \lambda_3 c_1 + \lambda_4 c_2)$$

$$\Leftrightarrow \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} : (x = \lambda_1 b_1 + \lambda_2 b_2) \wedge (\lambda_1 b_1 + \lambda_2 b_2 - \lambda_3 c_1 - \lambda_4 c_2 = 0)$$

$$\lambda := [\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$$

$$A\lambda = 0$$

$$A = \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & -c_1 & -c_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & -2 & -2 & -3 \\ 2 & 1 & 5 & -2 \\ 1 & 0 & 1 & -2 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 5 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} x_1 - 3x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \\ x_4 = \alpha \end{array} \qquad x : \begin{bmatrix} 3\alpha \\ \alpha \\ -\alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \qquad \lambda : \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \alpha, \alpha \in \mathbb{R} \qquad \alpha = 1.$$

$$\lambda_1 = 3, \quad \lambda_2 = 1$$

$$x = 3 \times \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}$$

2.15 Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x+y-z = 0\}$ and $G = \{(a-b, a+b, a-3b) \mid a, b \in \mathbb{R}\}$.

a. Show that F and G are subspaces of $\mathbb{R}^3$.

b. Calculate $F \cap G$ without resorting to any basis vector.

c. Find one basis for F and one for G, calculate $F \cap G$ using the basis vectors previously found and check your result with the previous question.

$x \in \mathbb{R}^3, y \in \mathbb{R}^3, z \in \mathbb{R}^3$?

a. $x=0, y=0, z=0 \rightarrow \mathbb{R}^3$

2.15 Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x+y-z = 0\}$ and $G = \{(a-b, a+b, a-3b) \mid a, b \in \mathbb{R}\}$.

a. Show that F and G are subspaces of $\mathbb{R}^3$.

b. Calculate $F \cap G$ without resorting to any basis vector.

c. Find one basis for F and one for G, calculate $F \cap G$ using the basis vectors previously found and check your result with the previous question.

b. $F \cap G$

$= \{(a-b, a+b, a-3b) \mid a+ab=0\}$

linear mapping $\Leftrightarrow$ $\forall x, y \in V\ \forall \lambda, \psi \in \mathbb{R}$

$: \phi(\lambda x + \psi y) = \lambda \phi(x) + \psi \phi(y)$

2.16 Are the following mappings linear?

Linear a. Let $a, b \in \mathbb{R}$.

$a\int = \int a$

$$\Phi : L^1([a, b]) \to \mathbb{R}$$

$$f \mapsto \Phi(f) = \int_a^b f(x)dx\,,$$

$\int (a+b) = \int a + \int b$

where $L^1([a, b])$ denotes the set of integrable functions on $[a, b]$. $\int$

Linear b.

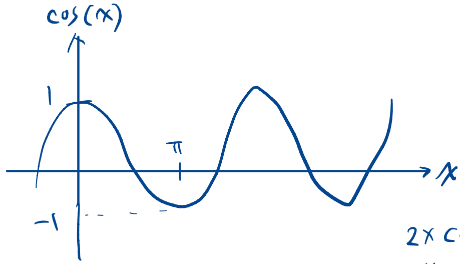
$$\Phi : C^1 \to C^0$$

$$f \mapsto \Phi(f) = f'\,,$$

- $\frac{\partial}{\partial x}(x + 2x) = \frac{\partial}{\partial x} x + \frac{\partial}{\partial x} 2x$

where for $k \geqslant 1$, C^k denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.

- $\frac{\partial}{\partial x} 2x = 2 \frac{\partial}{\partial x} x$



$2\times\cos(\frac{1}{2}\pi) = 2\times 0 = 0$

• $\neq$

c.

Non-linear

$x_1 = \frac{1}{2}\pi \quad x_2 = \frac{1}{2}\pi \quad \cos(2\times\frac{1}{2}\pi) = -1$

$$\Phi : \mathbb{R} \to \mathbb{R}$$

• $\cos(\frac{1}{2}\pi + \frac{1}{2}\pi) = \cos(\pi) = -1$

$$x \mapsto \Phi(x) = \cos(x)$$

$\neq$

$\cos(\frac{1}{2}\pi) + \cos(\frac{1}{2}\pi) = 0$

d.

Linear

• $x_1 = 1, \; x_2 = 2$

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^2$$

$\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$

$$\boldsymbol{x} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \boldsymbol{x}$$

• $2\times\phi(x_1) = \phi(2x_1)$

e. Let θ be in $[0, 2\pi[$ and

Linear.

이 친구는 뭘까...

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\boldsymbol{x} \mapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \boldsymbol{x}$$

Rotation

2.17 Consider the linear mapping

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix} \in \mathbb{R}^4$$

- Find the transformation matrix $\boldsymbol{A}_\Phi$.
- Determine $\mathrm{rk}(\boldsymbol{A}_\Phi)$.
- Compute the kernel and image of Φ. What are $\dim(\ker(\Phi))$ and $\dim(\mathrm{Im}(\Phi))$?

$x : [x_1, x_2, x_3]^T \in \mathbb{R}^3$

$$Ax = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

$$A_\phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{rk}(A_\phi) = 3$

$\mathrm{Im}(\phi) = \mathrm{span}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right] = \mathrm{span}\left[\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right]$

$\ker(\phi) = \cancel{\emptyset}$ $(0, 0, 0)$

$\dim(\ker(\emptyset)) = 0$

$\dim(\mathrm{Im}(\phi)) = 3$

~~2.18~~ Let E be a vector space. Let f and g be two automorphisms on E such that $f \circ g = \mathrm{id}_E$ (i.e., $f \circ g$ is the identity mapping id_E). Show that $\ker(f) = \ker(g \circ f)$, $\mathrm{Im}(g) = \mathrm{Im}(g \circ f)$ and that $\ker(f) \cap \mathrm{Im}(g) = \{\mathbf{0}_E\}$.

자기동형

they map linearly & bijectively from the space E to itself

$E \in \mathbb{R}^n$

$$f \circ g = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Kernel = null space

$\ker(f) = 0$

$\ker(g) = 0$

$\text{image}(f) = E$

$\text{image}(g) = E$

$$\begin{cases} \underbrace{\ker(f)}_{=0} = \underbrace{\ker(g \circ f)}_{=0} \\ \underbrace{\mathrm{Im}(g)}_{=E} = \underbrace{\mathrm{Im}(g \circ f)}_{=E} \end{cases}$$

2.19 Consider an endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix (with respect to the standard basis in $\mathbb{R}^3$) is

왜지?

$$A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

a. Determine $\ker(\Phi)$ and $\mathrm{Im}(\Phi)$.
b. Determine the transformation matrix $\tilde{A}_\Phi$ with respect to the basis

$$B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

i.e., perform a basis change toward the new basis B.

$$A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. $\ker(\Phi)$ = trivial, $\{(0,0,0)\}$

$\mathrm{Im}(\Phi) = \mathrm{span}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$

b. $$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{A} = T^{-1} A_\Phi S$$

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_\Phi S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

2.20 Let us consider $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}'_1, \boldsymbol{b}'_2$, 4 vectors of $\mathbb{R}^2$ expressed in the standard basis of $\mathbb{R}^2$ as

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b}'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{b}'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let us define two ordered bases $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $B' = (\boldsymbol{b}'_1, \boldsymbol{b}'_2)$ of $\mathbb{R}^2$.

a. Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors.
b. Compute the matrix $\boldsymbol{P}_1$ that performs a basis change from B' to B.
c. We consider $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$, three vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}^3$ as

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$.

   (i) Show that C is a basis of $\mathbb{R}^3$, e.g., by using determinants (see Section 4.1).
   (ii) Let us call $C' = (\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3)$ the standard basis of $\mathbb{R}^3$. Determine the matrix $\boldsymbol{P}_2$ that performs the basis change from C to C'.

d. We consider a homomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that

$$\begin{aligned} \Phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) &= \boldsymbol{c}_2 + \boldsymbol{c}_3 \\ \Phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) &= 2\boldsymbol{c}_1 - \boldsymbol{c}_2 + 3\boldsymbol{c}_3 \end{aligned}$$

where $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.
Determine the transformation matrix $\boldsymbol{A}_\Phi$ of Φ with respect to the ordered bases B and C.

e. Determine $\boldsymbol{A}'$, the transformation matrix of Φ with respect to the bases $\boldsymbol{B}'$ and $\boldsymbol{C}'$.
f. Let us consider the vector $\boldsymbol{x} \in \mathbb{R}^2$ whose coordinates in B' are $[2, 3]^\top$. In other words, $\boldsymbol{x} = 2\boldsymbol{b}'_1 + 3\boldsymbol{b}'_2$.

   (i) Calculate the coordinates of $\boldsymbol{x}$ in B.
   (ii) Based on that, compute the coordinates of $\Phi(\boldsymbol{x})$ expressed in C.
   (iii) Then, write $\Phi(\boldsymbol{x})$ in terms of $\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3$.
   (iv) Use the representation of $\boldsymbol{x}$ in B' and the matrix $\boldsymbol{A}'$ to find this result directly.

$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $b_1' = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $b_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B = (b_1, b_2)$ $B' = (b_1', b_2')$

$B = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right)$ $B' = \left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

a. $B: \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^2$ basis vectors

$B': \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^2$ basis vectors

~~b.~~ $B = PB'$

$B(B')^{-1} = P$

$(B')^{-1} = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 2 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

풀이 (왜?)

$b_1' = 4b_1 + 6b_2$

$b_2' = 0b_1 - b_2$

$(B')^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$B(B')^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{4} & -\frac{3}{4} \end{bmatrix}$

$P_1 \Rightarrow \begin{array}{cc} b_1' & b_2' \end{array} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$

$Pb \neq b'$

$Pb' \neq b$

c. We consider $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$, three vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}^3$ as

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$.

(i) Show that C is a basis of $\mathbb{R}^3$, e.g., by using determinants (see Section 4.1).

(ii) Let us call $C' = (\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3)$ the standard basis of $\mathbb{R}^3$. Determine the matrix $\boldsymbol{P}_2$ that performs the basis change from C to C'.

$$C = \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

(i) $C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

$$\det(C) = 1 \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 1 + 4 - 1$$

$$= 4 \neq 0$$

(ii) $P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$

ใช?

d. We consider a homomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that

$$\begin{aligned} \Phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) &= \boldsymbol{c}_2 + \boldsymbol{c}_3 \\ \Phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) &= 2\boldsymbol{c}_1 - \boldsymbol{c}_2 + 3\boldsymbol{c}_3 \end{aligned}$$

where $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Determine the transformation matrix $\boldsymbol{A}_\Phi$ of Φ with respect to the ordered bases B and C.

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b}_1' = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{b}_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$b_1 + b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[\Phi]{} c_2 + c_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$b_1 - b_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \xrightarrow[\Phi]{} 2c_1 - c_2 + 3c_3 = \begin{bmatrix} 5 \\ 5 \\ -7 \end{bmatrix}$$

$$A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad A\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \\ -7 \end{bmatrix}$$

$$2\Phi(b_2) = -2c_1 + 2c_2 - 2c_3$$

$$\boxed{\Phi(b_2) = -c_1 + c_2 - c_3}$$

$$2\Phi(b_1) = 2c_1 + 4c_3$$

$$\boxed{\Phi(b_1) = c_1 + 2c_3}$$

$$A_\Phi = \begin{array}{cc} \Phi(b_1) & \Phi(b_2) \\ \end{array}$$
$$A_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

e. Determine A', the transformation matrix of Φ with respect to the bases B' and C'.

$\Phi(b_1'+b_2') = c_2'+c_3'$

$\Phi(b_1'-b_2') = 2c_1'-c_2'+3c_3'$

$\tilde{B} \xrightarrow{P_1} B$

$\tilde{C} \xrightarrow{P_2} C$

$\boxed{\tilde{A}_\Phi = P_2^{-1} A_\Phi P_1}$

왜 P_2^{-1}이 아니지?

$P_1 : \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$

$A_\Phi : \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

$P_2 : \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -4 & -3 & 2 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 4 & 3 & -2 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

$P_2^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & -1 \end{bmatrix}$

$$A_\Phi P_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 6 & -1 \\ -2 & 1 \end{bmatrix}$$

$3\times 3 \quad 3\times 2 \to 3\times 2$

$$P_2^{-1} A_\Phi P_1 = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 6 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 & 0 \\ -8 & 4 \\ -16 & 4 \end{bmatrix} = \cancel{\begin{bmatrix} 2 & 0 \\ -2 & 1 \\ 4 & 1 \end{bmatrix}}$$

풀이 맞는데 틀림

$\boxed{A' = P_2^{-1} A_\phi P_1 \text{이}}$ 아닌 $A' = P_2 A_\phi P_1$로 풀기.

왜 이식이 아니지? $\tilde{A}_\phi = T^{-1} A_\phi S$

$P_1 = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$ $P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$ $A_\phi : \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$

$$A_\phi P_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 6 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P_2 A_\phi P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 6 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$$

f. Let us consider the vector $\boldsymbol{x} \in \mathbb{R}^2$ whose coordinates in B' are $[2, 3]^\top$. In other words, $\boldsymbol{x} = 2\boldsymbol{b}_1' + 3\boldsymbol{b}_2'$.

(i) Calculate the coordinates of $\boldsymbol{x}$ in B.

(ii) Based on that, compute the coordinates of $\Phi(\boldsymbol{x})$ expressed in C.

(iii) Then, write $\Phi(\boldsymbol{x})$ in terms of $\boldsymbol{c}_1', \boldsymbol{c}_2', \boldsymbol{c}_3'$.

(iv) Use the representation of $\boldsymbol{x}$ in B' and the matrix $\boldsymbol{A}'$ to find this result directly.

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b}_1' = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{b}_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

아예모르겠음..