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Exercises

2.1 We consider $(\mathbb{R}\backslash\{-1\}, \star)$, where

$$a \star b := ab + a + b, \qquad a, b \in \mathbb{R}\backslash\{-1\} \tag{2.134}$$

a. Show that $(\mathbb{R}\backslash\{-1\}, \star)$ is an Abelian group.
b. Solve

$$3 \star x \star x = 15$$

in the Abelian group $(\mathbb{R}\backslash\{-1\}, \star)$, where $\star$ is defined in (2.134).

2.2 Let n be in $\mathbb{N}\backslash\{0\}$. Let k, x be in $\mathbb{Z}$. We define the congruence class $\bar{k}$ of the integer k as the set

$$\begin{aligned} \bar{k} &= \{x \in \mathbb{Z} \mid x - k = 0 \pmod n\} \\ &= \{x \in \mathbb{Z} \mid \exists a \in \mathbb{Z} \colon (x - k = n \cdot a)\} . \end{aligned}$$

We now define $\mathbb{Z}/n\mathbb{Z}$ (sometimes written $\mathbb{Z}_n$) as the set of all congruence classes modulo n. Euclidean division implies that this set is a finite set containing n elements:

$$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$$

For all $\bar{a}, \bar{b} \in \mathbb{Z}_n$, we define

$$\bar{a} \oplus \bar{b} := \overline{a + b}$$

a. Show that $(\mathbb{Z}_n, \oplus)$ is a group. Is it Abelian?
b. We now define another operation $\otimes$ for all $\bar{a}$ and $\bar{b}$ in $\mathbb{Z}_n$ as

$$\bar{a} \otimes \bar{b} = \overline{a \times b} , \tag{2.135}$$

where $a \times b$ represents the usual multiplication in $\mathbb{Z}$.
Let $n = 5$. Draw the times table of the elements of $\mathbb{Z}_5\backslash\{\bar{0}\}$ under $\otimes$, i.e., calculate the products $\bar{a} \otimes \bar{b}$ for all $\bar{a}$ and $\bar{b}$ in $\mathbb{Z}_5\backslash\{\bar{0}\}$.
Hence, show that $\mathbb{Z}_5\backslash\{\bar{0}\}$ is closed under $\otimes$ and possesses a neutral element for $\otimes$. Display the inverse of all elements in $\mathbb{Z}_5\backslash\{\bar{0}\}$ under $\otimes$. Conclude that $(\mathbb{Z}_5\backslash\{\bar{0}\}, \otimes)$ is an Abelian group.
c. Show that $(\mathbb{Z}_8\backslash\{\bar{0}\}, \otimes)$ is not a group.
d. We recall that the Bézout theorem states that two integers a and b are relatively prime (i.e., $gcd(a, b) = 1$) if and only if there exist two integers u and v such that $au + bv = 1$. Show that $(\mathbb{Z}_n\backslash\{\bar{0}\}, \otimes)$ is a group if and only if $n \in \mathbb{N}\backslash\{0\}$ is prime.

2.3 Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ as the standard matrix multiplication.
Is $(\mathcal{G}, \cdot)$ a group? If yes, is it Abelian? Justify your answer.

2.4 Compute the following matrix products, if possible:

a.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

c.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

d.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}$$

e.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}$$

2.5 Find the set $\mathcal{S}$ of all solutions in $\boldsymbol{x}$ of the following inhomogeneous linear systems $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ and $\boldsymbol{b}$ are defined as follows:

a.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

b.

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

2.6 Using Gaussian elimination, find all solutions of the inhomogeneous equation system $\boldsymbol{Ax} = \boldsymbol{b}$ with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

2.7 Find all solutions in $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $\boldsymbol{A}\boldsymbol{x} = 12\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} x_i = 1$.

2.8 Determine the inverses of the following matrices if possible:

a.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

b.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

2.9 Which of the following sets are subspaces of $\mathbb{R}^3$?

a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$
b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$
c. Let γ be in $\mathbb{R}$.
$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$
d. $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

2.10 Are the following sets of vectors linearly independent?

a.

$$\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

b.

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

2.11 Write

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

as linear combination of

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

2.12 Consider two subspaces of $\mathbb{R}^4$:

$$U_1 = \text{span}[\begin{bmatrix}1\\1\\-3\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}], \quad U_2 = \text{span}[\begin{bmatrix}-1\\-2\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\6\\-2\\-1\end{bmatrix}].$$

Determine a basis of $U_1 \cap U_2$.

2.13 Consider two subspaces U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{0}$ and U_2 is the solution space of the homogeneous equation system $\boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{0}$ with

$$\boldsymbol{A}_1 = \begin{bmatrix}1 & 0 & 1\\1 & -2 & -1\\2 & 1 & 3\\1 & 0 & 1\end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix}3 & -3 & 0\\1 & 2 & 3\\7 & -5 & 2\\3 & -1 & 2\end{bmatrix}.$$

a. Determine the dimension of U_1, U_2.
b. Determine bases of U_1 and U_2.
c. Determine a basis of $U_1 \cap U_2$.

2.14 Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of $\boldsymbol{A}_1$ and U_2 is spanned by the columns of $\boldsymbol{A}_2$ with

$$\boldsymbol{A}_1 = \begin{bmatrix}1 & 0 & 1\\1 & -2 & -1\\2 & 1 & 3\\1 & 0 & 1\end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix}3 & -3 & 0\\1 & 2 & 3\\7 & -5 & 2\\3 & -1 & 2\end{bmatrix}.$$

a. Determine the dimension of U_1, U_2
b. Determine bases of U_1 and U_2
c. Determine a basis of $U_1 \cap U_2$

2.15 Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x+y-z = 0\}$ and $G = \{(a-b, a+b, a-3b) \mid a, b \in \mathbb{R}\}$.

a. Show that F and G are subspaces of $\mathbb{R}^3$.
b. Calculate $F \cap G$ without resorting to any basis vector.
c. Find one basis for F and one for G, calculate $F \cap G$ using the basis vectors previously found and check your result with the previous question.

2.16 Are the following mappings linear?

a. Let $a, b \in \mathbb{R}$.

$$\begin{aligned}\Phi : L^1([a, b]) &\to \mathbb{R}\\ f \mapsto \Phi(f) &= \int_a^b f(x)dx\,,\end{aligned}$$

where $L^1([a, b])$ denotes the set of integrable functions on $[a, b]$.

b.

$$\begin{aligned}\Phi : C^1 &\to C^0\\ f &\mapsto \Phi(f) = f'\,,\end{aligned}$$

where for $k \geqslant 1$, C^k denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.

c.

$$\Phi : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \Phi(x) = \cos(x)$$

d.

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^2$$
$$\boldsymbol{x} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \boldsymbol{x}$$

e. Let θ be in $[0, 2\pi[$ and

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\boldsymbol{x} \mapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \boldsymbol{x}$$

2.17 Consider the linear mapping

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$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$
$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

- Find the transformation matrix $\boldsymbol{A}_\Phi$.
- Determine $\mathrm{rk}(\boldsymbol{A}_\Phi)$.
- Compute the kernel and image of Φ. What are $\dim(\ker(\Phi))$ and $\dim(\mathrm{Im}(\Phi))$?

2.18 Let E be a vector space. Let f and g be two automorphisms on E such that $f \circ g = \mathrm{id}_E$ (i.e., $f \circ g$ is the identity mapping id_E). Show that $\ker(f) = \ker(g \circ f)$, $\mathrm{Im}(g) = \mathrm{Im}(g \circ f)$ and that $\ker(f) \cap \mathrm{Im}(g) = \{\boldsymbol{0}_E\}$.

2.19 Consider an endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix (with respect to the standard basis in $\mathbb{R}^3$) is

$$\boldsymbol{A}_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

a. Determine $\ker(\Phi)$ and $\mathrm{Im}(\Phi)$.

b. Determine the transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ with respect to the basis

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$$B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

i.e., perform a basis change toward the new basis B.

2.20 Let us consider $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}'_1, \boldsymbol{b}'_2$, 4 vectors of $\mathbb{R}^2$ expressed in the standard basis of $\mathbb{R}^2$ as

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b}'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{b}'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let us define two ordered bases $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $B' = (\boldsymbol{b}'_1, \boldsymbol{b}'_2)$ of $\mathbb{R}^2$.

a. Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors.
b. Compute the matrix $\boldsymbol{P}_1$ that performs a basis change from B' to B.
c. We consider $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$, three vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}^3$ as

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$.

(i) Show that C is a basis of $\mathbb{R}^3$, e.g., by using determinants (see Section 4.1).
(ii) Let us call $C' = (\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3)$ the standard basis of $\mathbb{R}^3$. Determine the matrix $\boldsymbol{P}_2$ that performs the basis change from C to C'.

d. We consider a homomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that

$$\begin{aligned} \Phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) &= \boldsymbol{c}_2 + \boldsymbol{c}_3 \\ \Phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) &= 2\boldsymbol{c}_1 - \boldsymbol{c}_2 + 3\boldsymbol{c}_3 \end{aligned}$$

where $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.
Determine the transformation matrix $\boldsymbol{A}_\Phi$ of Φ with respect to the ordered bases B and C.

e. Determine $\boldsymbol{A}'$, the transformation matrix of Φ with respect to the bases $\boldsymbol{B}'$ and $\boldsymbol{C}'$.
f. Let us consider the vector $\boldsymbol{x} \in \mathbb{R}^2$ whose coordinates in B' are $[2, 3]^\top$. In other words, $\boldsymbol{x} = 2\boldsymbol{b}'_1 + 3\boldsymbol{b}'_2$.

(i) Calculate the coordinates of $\boldsymbol{x}$ in B.
(ii) Based on that, compute the coordinates of $\Phi(\boldsymbol{x})$ expressed in C.
(iii) Then, write $\Phi(\boldsymbol{x})$ in terms of $\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3$.
(iv) Use the representation of $\boldsymbol{x}$ in B' and the matrix $\boldsymbol{A}'$ to find this result directly.

Basis Change에 대해 모르는듯.