Probset #3

4. (e)

$$K(x,z) = K_1(x,z)K_2(x,z)$$

$$K_1(x,z) = \phi_1(x)^T\phi_1(z), \quad K_2(x,z) = \phi_2(x)^T\phi_2(z)$$

$$K(x,z) = \phi_1(x)^T\phi_1(z)\,\phi_2(x)^T\phi_2(z)$$

Given any vector z,

$$z^TKz = \sum_i \sum_j z_i K_{ij} z_j$$

$$= \sum_i \sum_j z_i \{\phi_1(x^{(i)})^T\phi_1(z^{(j)})\,\phi_2(x^{(i)})^T\phi_2(z^{(j)})\} z_j$$

$$= \sum_i \sum_j z_i \{\sum_k (\phi_1(x^{(i)}))_k (\phi_1(x^{(j)}))_k (\phi_2(x^{(i)}))_k (\phi_2 x^{(j)}))_k\} z_j$$

$$= \sum_k \left(\sum_i z_i\, \phi_1(x^{(i)})_k \phi_2(x^{(i)})\right)^2 \geq 0$$

$\rightarrow$ positive semi definite

$\rightarrow$ is a kernel.