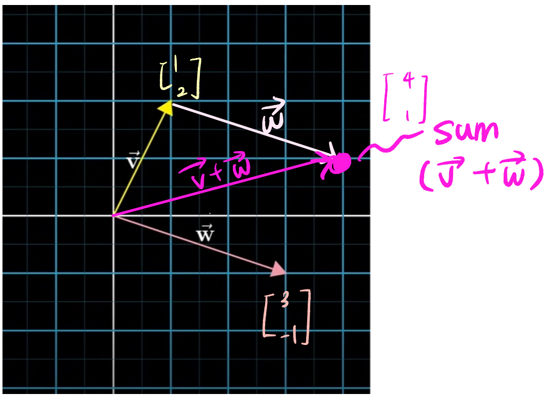
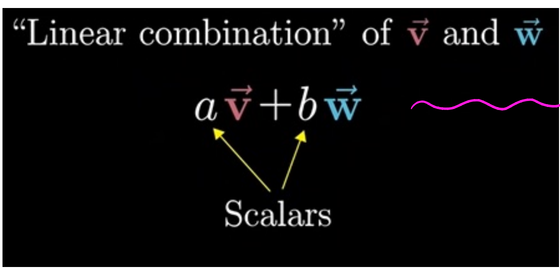
[Essence of Linear Algebra]

source:
https://www.youtube.com/watch?v=fNk_zzaMoSs

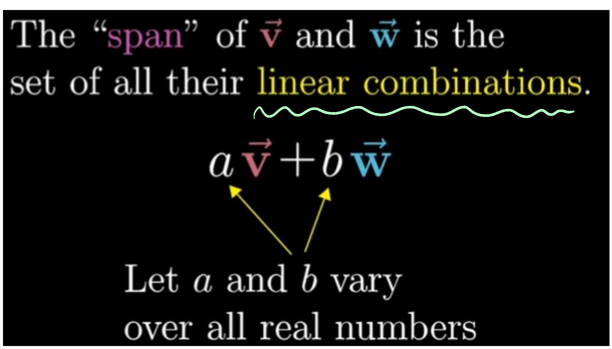
- vector : arrow

$: \begin{bmatrix}1\\2\end{bmatrix}$

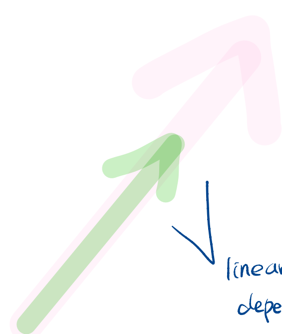




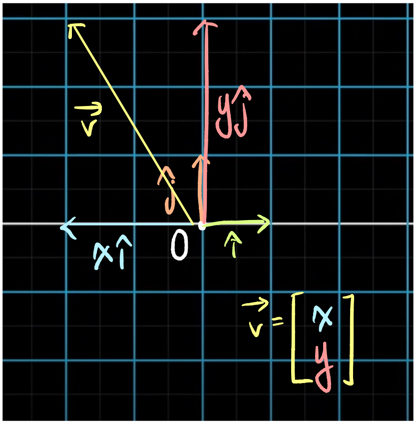
→ can access every coordinate on grid



⟺ what are all the points you can reach with these two fixed vectors



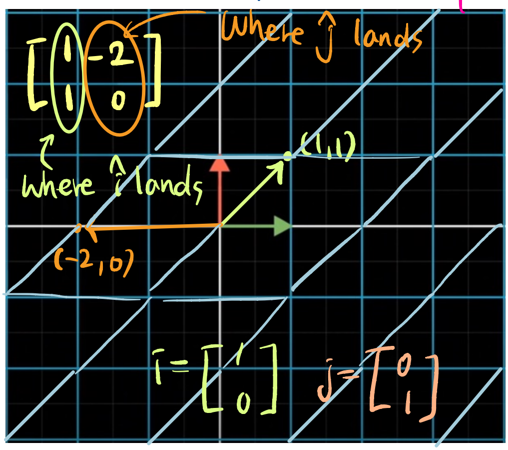
linearly dependent



"2x2 Matrix"

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}}_{\text{Where all the intuition is}} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

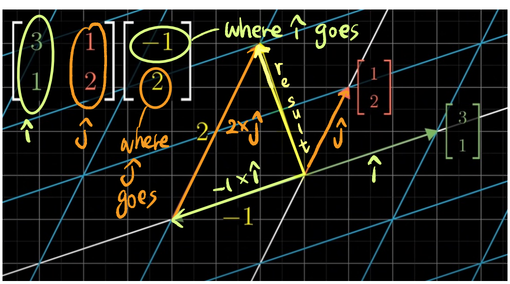
Transformation

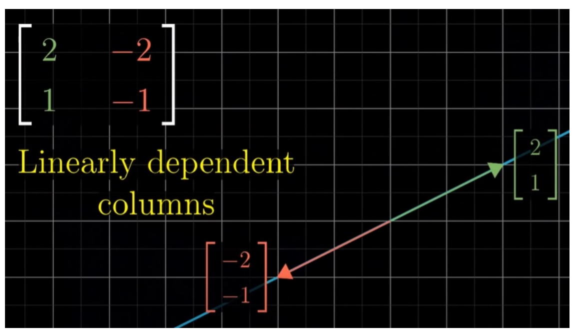


<Linear Transformation>

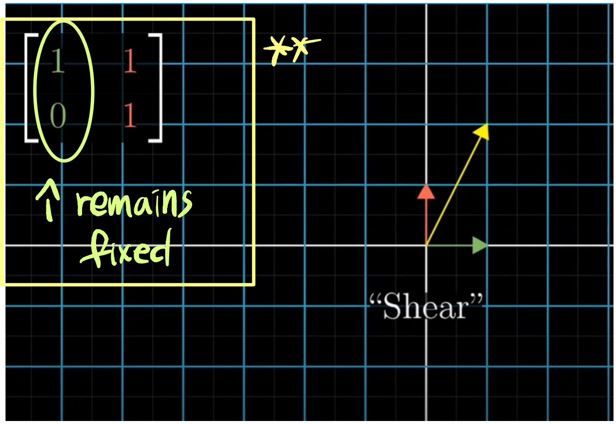
$L(\vec{v}) = \vec{w}$

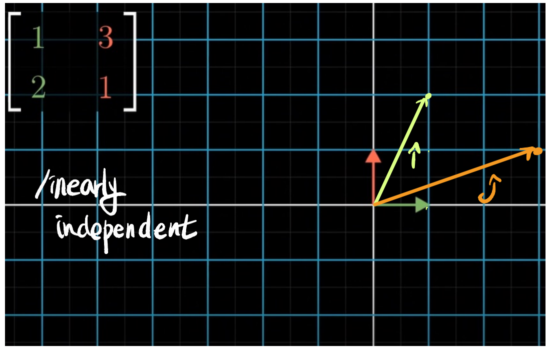
- lines remain lines
- origin remains fixed
- Grid lines remain parallel and evenly spaced

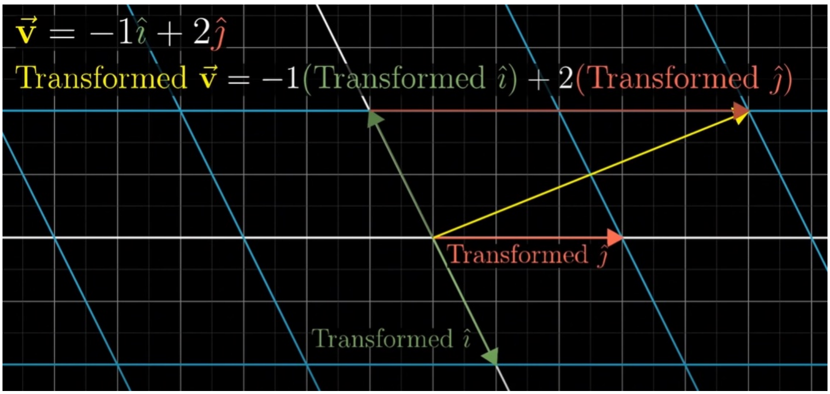




Shear







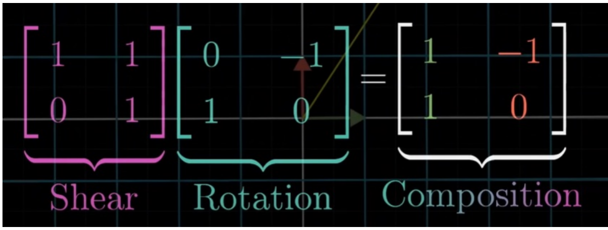
Matrix - Vector Multiplication : what the transformation (matrix) does to given vector

(→ transformation of space)

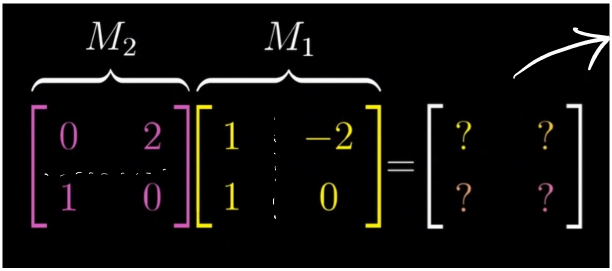
$$\overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{\text{transformation}} \begin{bmatrix} x \\ y \end{bmatrix}$$

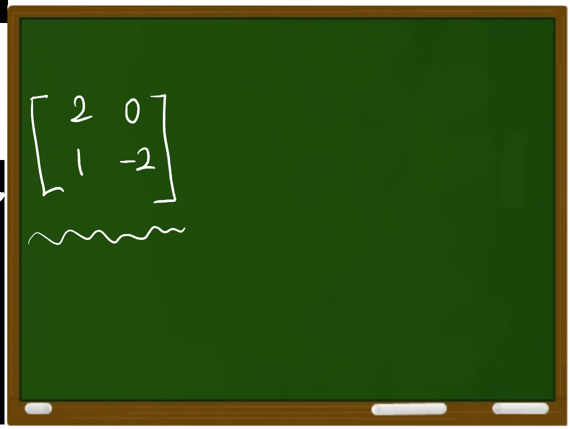
$\hat{\imath}$ $\hat{\jmath}$ ↑ vector

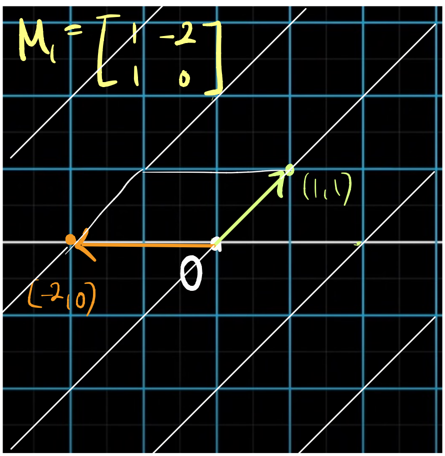
Matrix Multiplication



read right from left



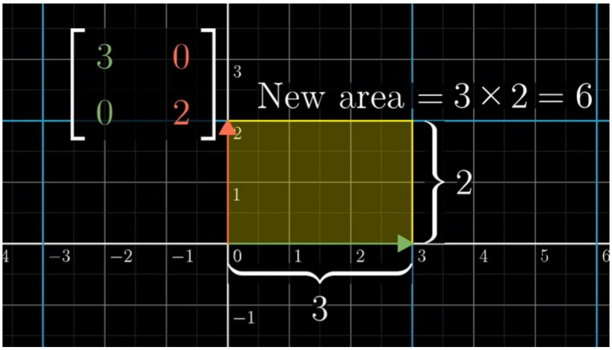




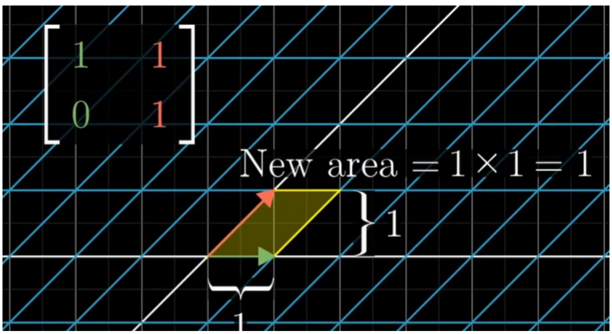
[Three-dimensional transformation]

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \quad L(\vec{v}) \quad \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

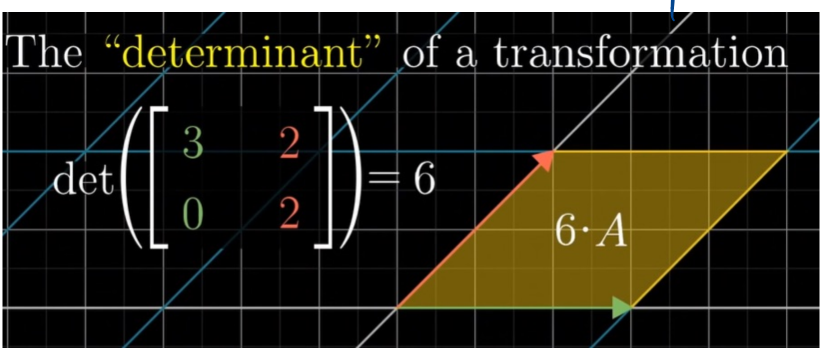
[The determinant]



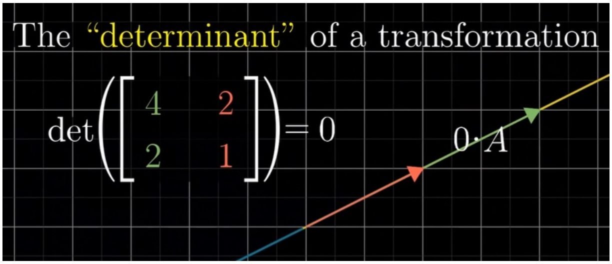
scale area by
a factor of 6



det : by how much
the area is increased

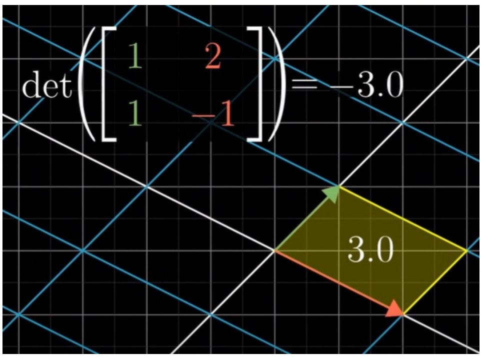


if det = 0

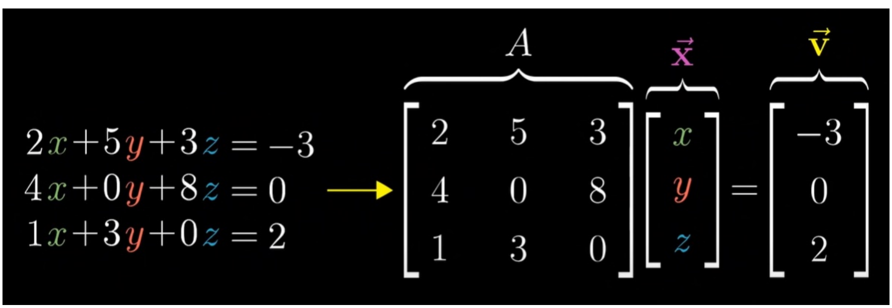


det(Matrix) = 0 if squished to a line/point

if det < 0



$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$



Inverse Matrix: Transformation that gets you back to the origin

Identity Matrix: Transformation that does nothing

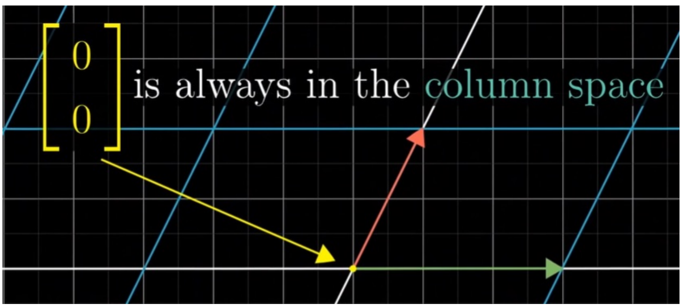
Rank: Number of dimensions in the output of transformation

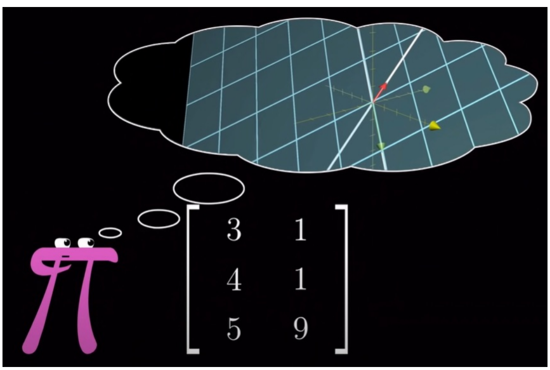
$\det(A) \neq 0 \rightarrow A^{-1}$ exists

$$\underbrace{\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}}_{\text{Span of columns}}$$

$$\Updownarrow$$

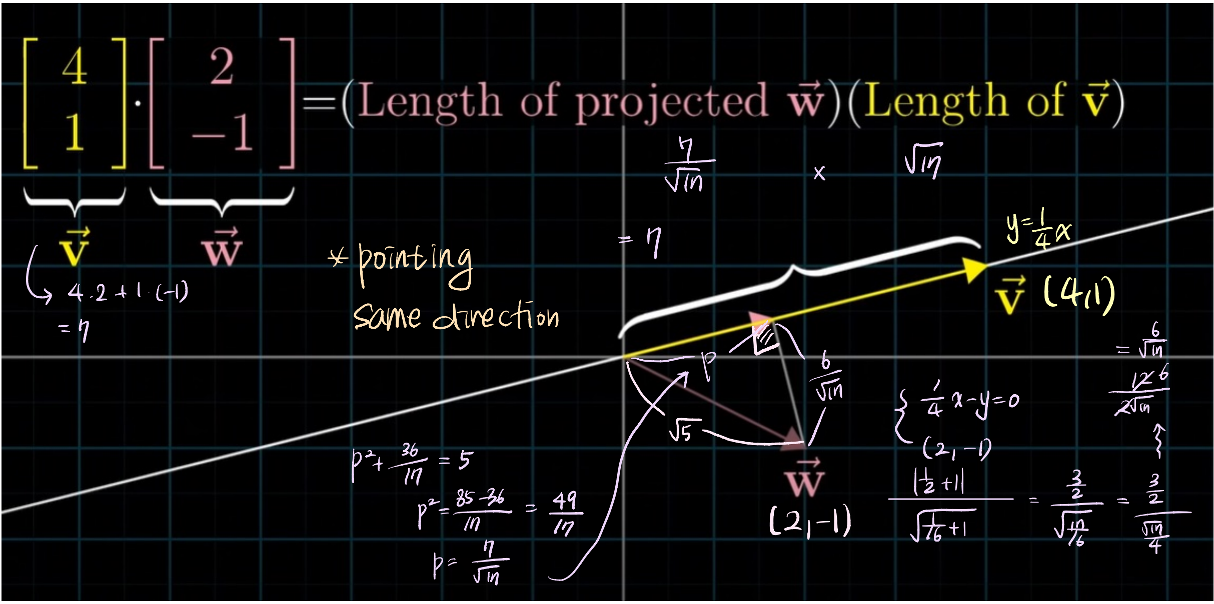
Column space



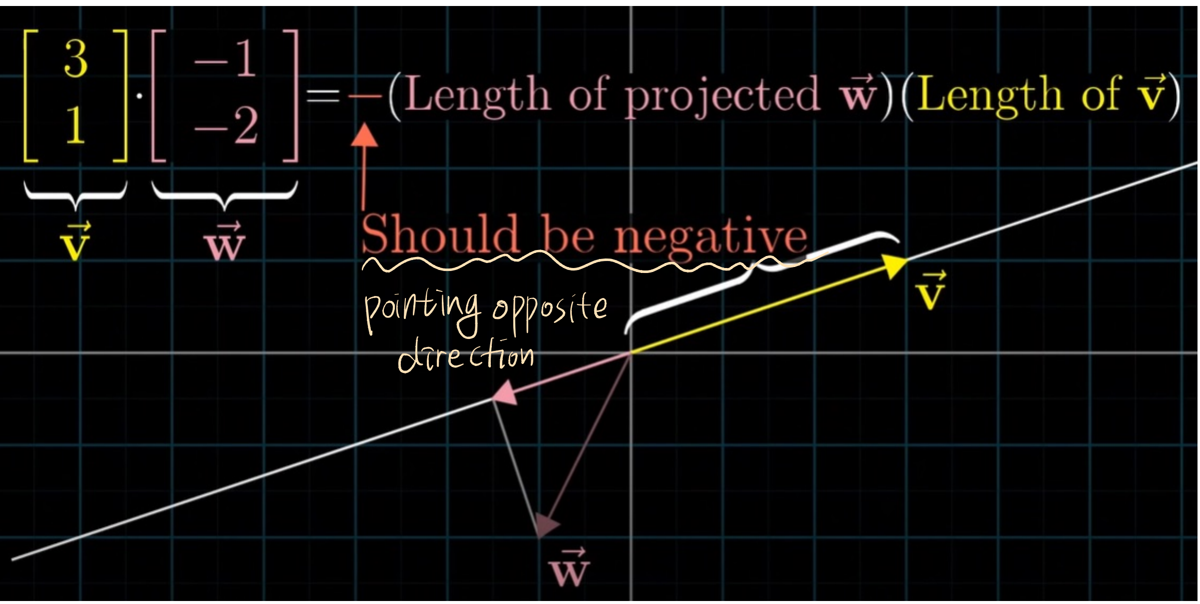


map 2 dimensions to 3 dimensions

Dot product

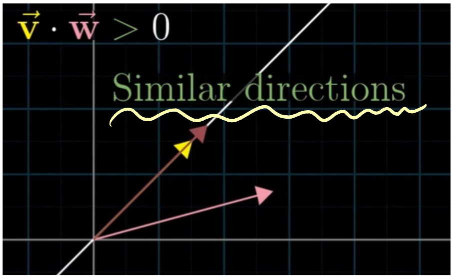


$\begin{bmatrix}4\\1\end{bmatrix}\cdot\begin{bmatrix}2\\-1\end{bmatrix}$ = (Length of projected $\vec{w}$)(Length of $\vec{v}$)
$\frac{7}{\sqrt{17}} \times \sqrt{17}$
$= 7$
$\vec{v}$
$\vec{w}$
$4\cdot 2+1\cdot(-1)$
$=7$
* pointing
same direction
$y=\frac{1}{4}x$
$\vec{v}$ (4,1)
p
$\frac{6}{\sqrt{17}}$
$\sqrt{5}$
$p^2+\frac{36}{17}=5$
$p^2=\frac{85-36}{17}=\frac{49}{17}$
$p=\frac{7}{\sqrt{17}}$
$\vec{w}$
(2,-1)
$\frac{1}{4}x-y=0$
(2,-1)
$\frac{|\frac{1}{2}+1|}{\sqrt{\frac{1}{16}+1}}=\frac{\frac{3}{2}}{\sqrt{\frac{17}{16}}}=\frac{\frac{3}{2}}{\frac{\sqrt{17}}{4}}$
$=\frac{6}{\sqrt{17}}$
$\frac{12}{2\sqrt{17}}$



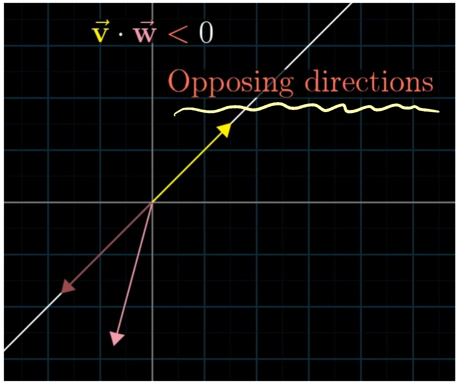
$\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\-2\end{bmatrix}$ = −(Length of projected $\vec{w}$)(Length of $\vec{v}$)
$\vec{v}$
$\vec{w}$
Should be negative
pointing opposite
direction
$\vec{v}$
$\vec{w}$

dot product >0

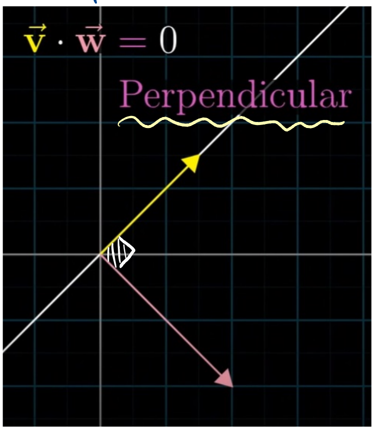


dot product → for understanding projections
→ for telling directions of two vectors

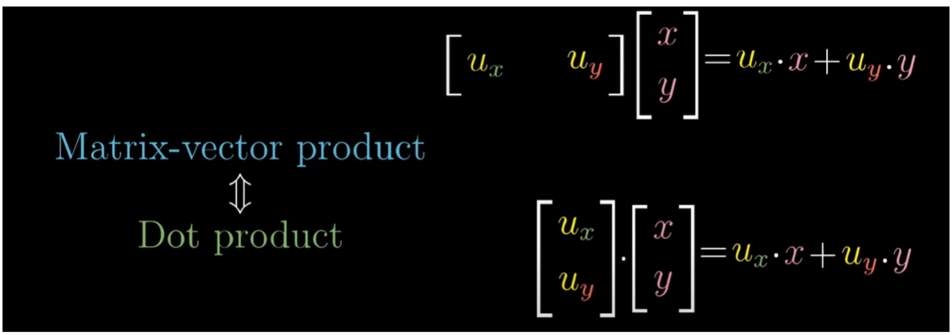
dot product <0



dot product = 0

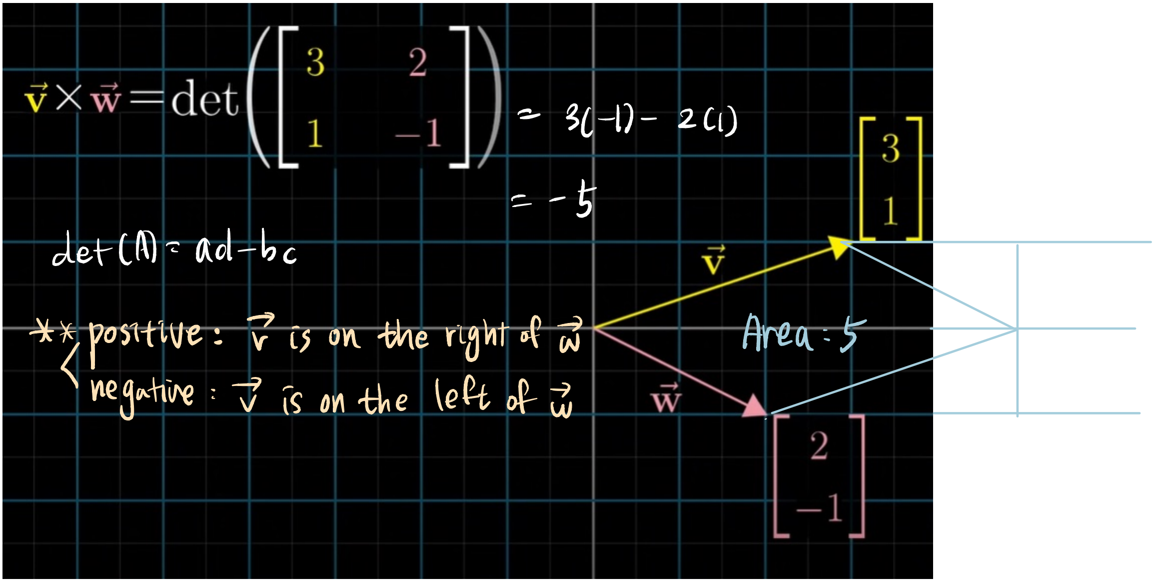


$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$



↑ duality

Cross products - (1)



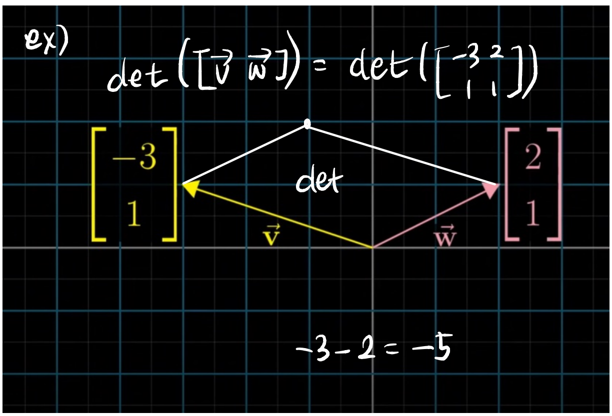
$= 3(-1) - 2(1)$

$= -5$

$\det(A) = ad - bc$

** positive : $\vec{v}$ is on the right of $\vec{w}$

negative : $\vec{v}$ is on the left of $\vec{w}$

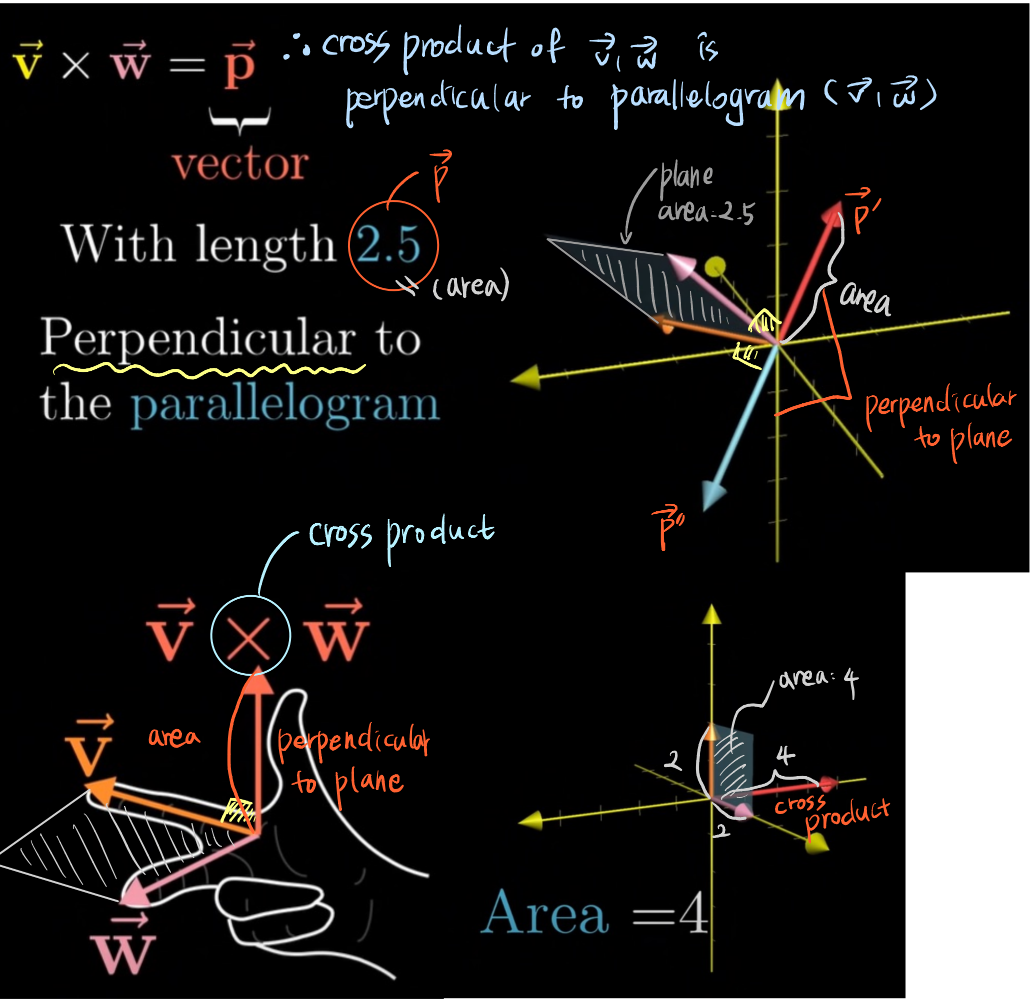


ex)

$\det([\vec{v}\ \vec{w}]) = \det\left(\begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}\right)$

$-3 - 2 = -5$

Cross products - (2)



Computation :

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \boxed{v_2 \cdot w_3 - w_2 \cdot v_3} \\ \boxed{v_3 \cdot w_1 - w_3 \cdot v_1} \\ \boxed{v_1 \cdot w_2 - w_1 \cdot v_2} \end{bmatrix}$$

ex)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9 \\ 0+1 \\ 3-0 \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \\ 3 \end{bmatrix}$$

Cross products - (3)

cross product

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\times\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=\det\left(\begin{bmatrix}\hat{\imath}&v_1&w_1\\\hat{\jmath}&v_2&w_2\\\hat{k}&v_3&w_3\end{bmatrix}\right)$$

$$\hat{\imath}(v_2w_3-v_3w_2)+\hat{\jmath}(v_3w_1-v_1w_3)+\hat{k}(v_1w_2-v_2w_1)$$

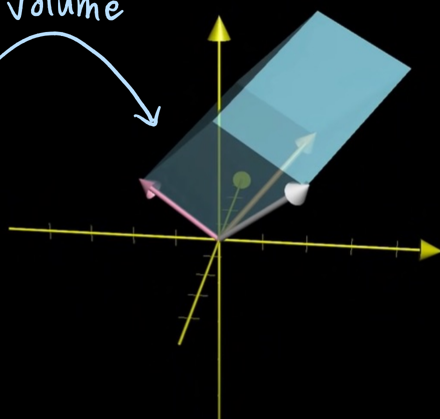
Duality

$$\underbrace{\begin{bmatrix}2\\1\end{bmatrix}\cdot\overbrace{\begin{bmatrix}x\\y\end{bmatrix}}^{\vec{\mathbf{v}}}}_{\text{Dot product}}=\overbrace{\begin{bmatrix}2&1\end{bmatrix}}^{\text{Transform}}\overbrace{\begin{bmatrix}x\\y\end{bmatrix}}^{\vec{\mathbf{v}}}$$

Not the real cross product

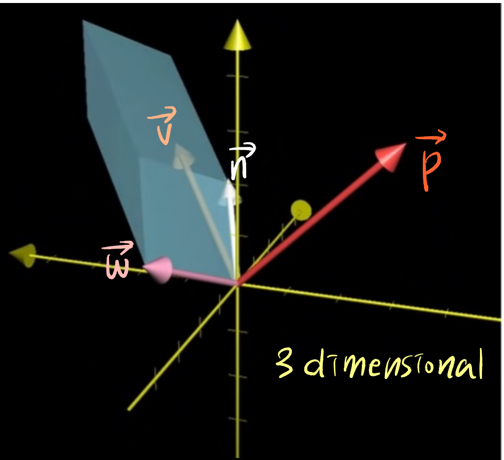
$$\vec{\mathbf{u}}\times\vec{\mathbf{v}}\times\vec{\mathbf{w}}=\underbrace{\det\left(\begin{bmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_3&v_3&w_3\end{bmatrix}\right)}_{\text{Number}}$$

volume



$$\underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}^{\vec{\mathbf{p}}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{array}{ccc} \vec{n} & \vec{\mathbf{v}} & \vec{\mathbf{w}} \\ \hline x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{array} \right)$$

dot product



$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}^{\vec{\mathbf{p}}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{array}{ccc} & \vec{\mathbf{v}} & \vec{\mathbf{w}} \\ x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{array} \right)$$

$$p_1 \cdot x + p_2 \cdot y + p_3 \cdot z = x\Big(\overbrace{\boxed{v_2 \cdot w_3 - v_3 \cdot w_2}}^{p_1}\Big) + y\Big(\overbrace{\boxed{v_3 \cdot w_1 - v_1 \cdot w_3}}^{p_2}\Big) + z\Big(\overbrace{\boxed{v_1 \cdot w_2 - v_2 \cdot w_1}}^{p_3}\Big)$$

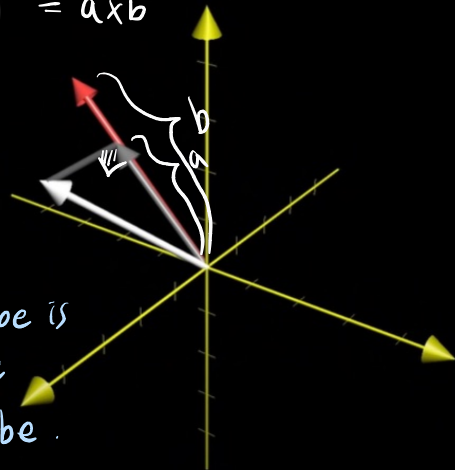
Coordinates of vector $\vec{p}$

$$\left\{ \begin{aligned} p_1 &= v_2 \cdot w_3 - v_3 \cdot w_2 \\ p_2 &= v_3 \cdot w_1 - v_1 \cdot w_3 \\ p_3 &= v_1 \cdot w_2 - v_2 \cdot w_1 \end{aligned} \right.$$

$$\vec{\mathbf{p}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{a}{(\text{Length of projection})} \times \underset{b}{(\text{Length of } \vec{\mathbf{p}})} = a \times b$$

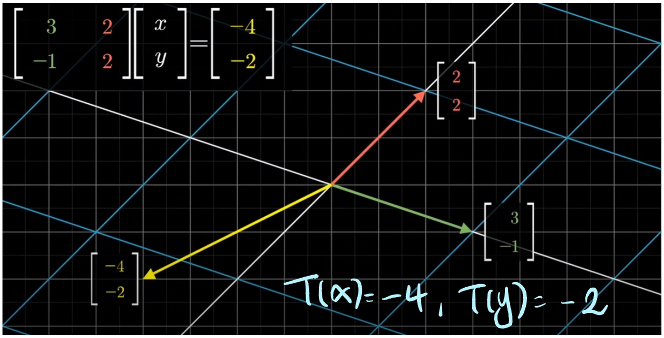
recap meaning of dot product

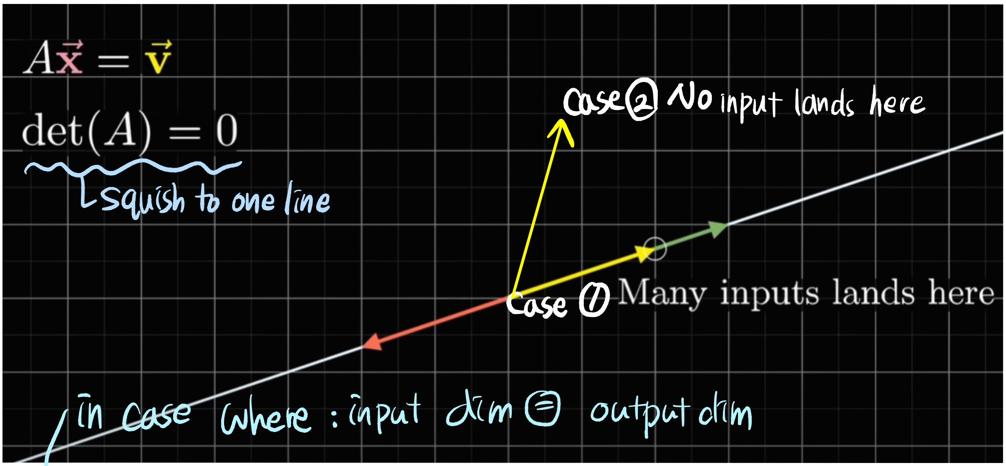
height of parallelogram cube is perpendicular to the surface of cube.



Cramer's rule

(Gaussian Elimination is faster to compute)

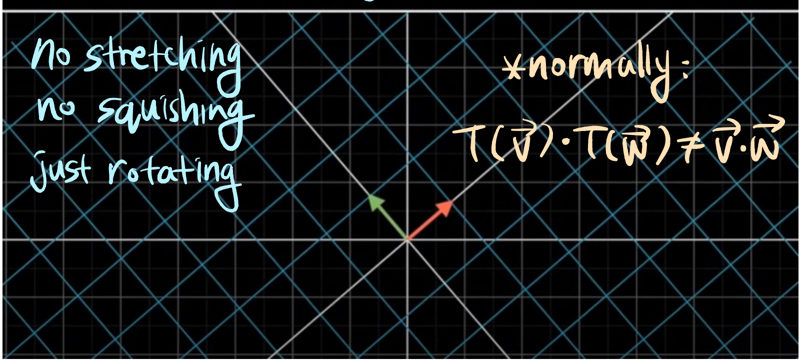


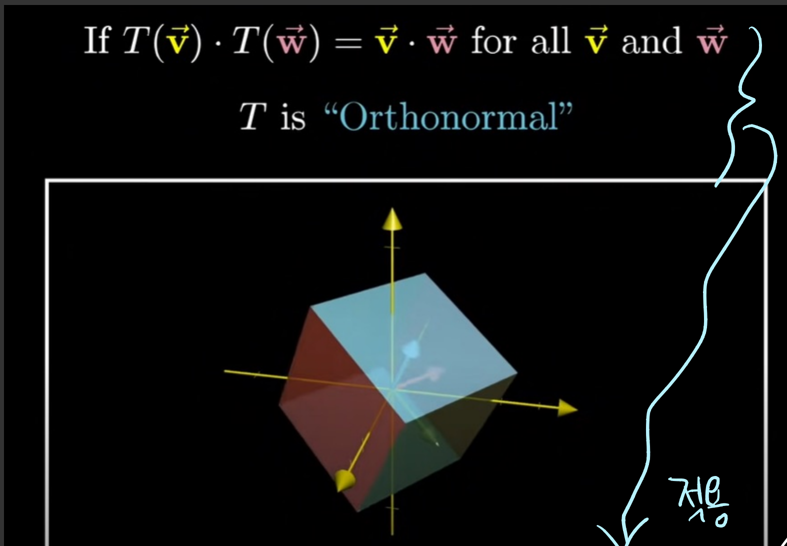


in case where : input dim = output dim

If $\boxed{T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}}$ for all $\vec{v}$ and $\vec{w}$

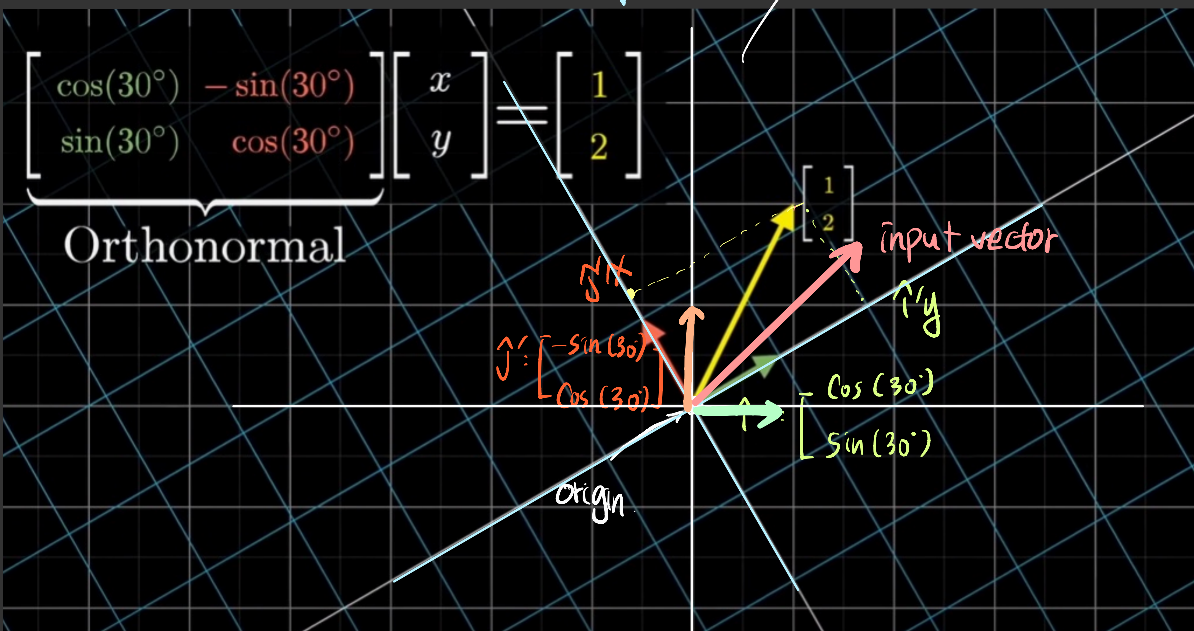
T is "Orthonormal"

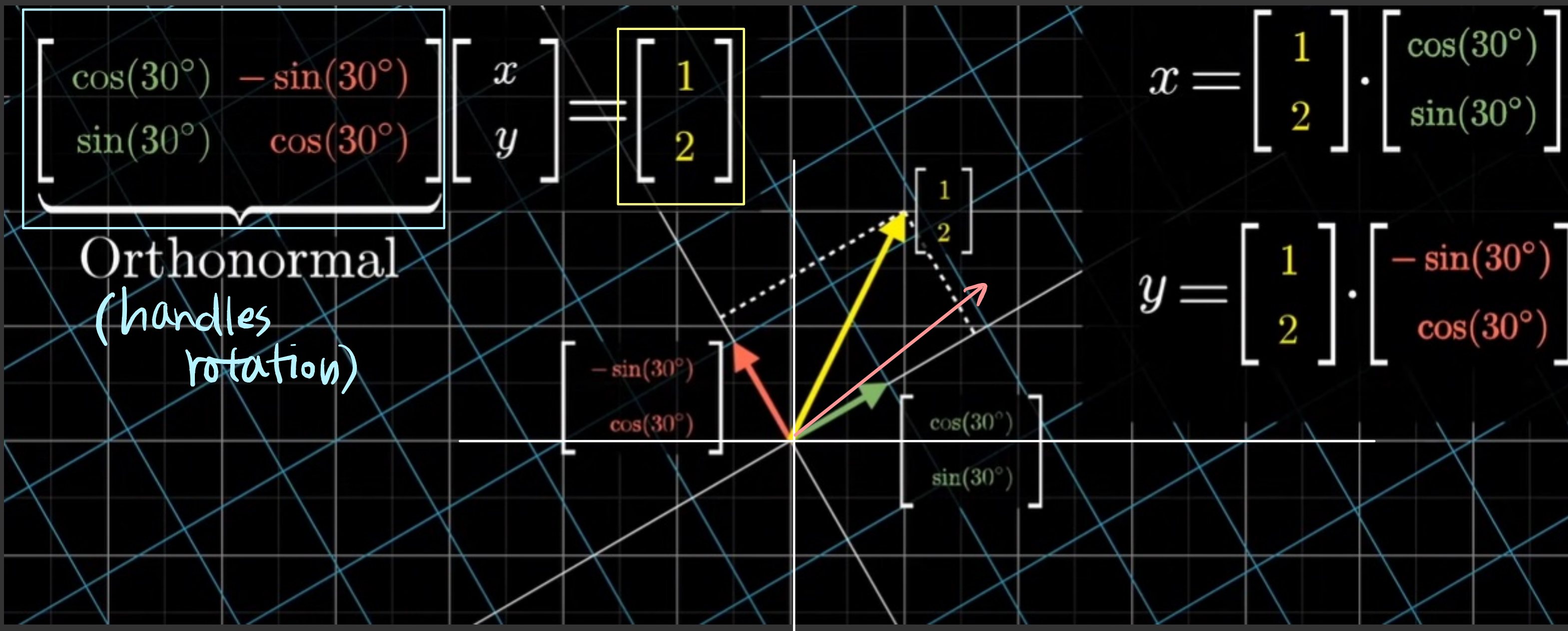




orthonormal

$$\begin{bmatrix}\cos 30^\circ \\ \sin 30^\circ\end{bmatrix} x + \begin{bmatrix}-\sin 30^\circ \\ \cos 30^\circ\end{bmatrix} y = \begin{bmatrix}1 \\ 2\end{bmatrix}$$





$\vec{v}\cdot\vec{w} = T(\vec{v})\cdot T(\vec{w})$, T is orthonormal

Since dot products are preserved,
taking the dot product between the output vector and all the columns of matrix will be the same as taking the dot products between the input vector and all the basis vectors

$\hat{i},\hat{j}: \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Downarrow$

output vector · columns of matrix = input vector · basis vectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} \text{orthonormal} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

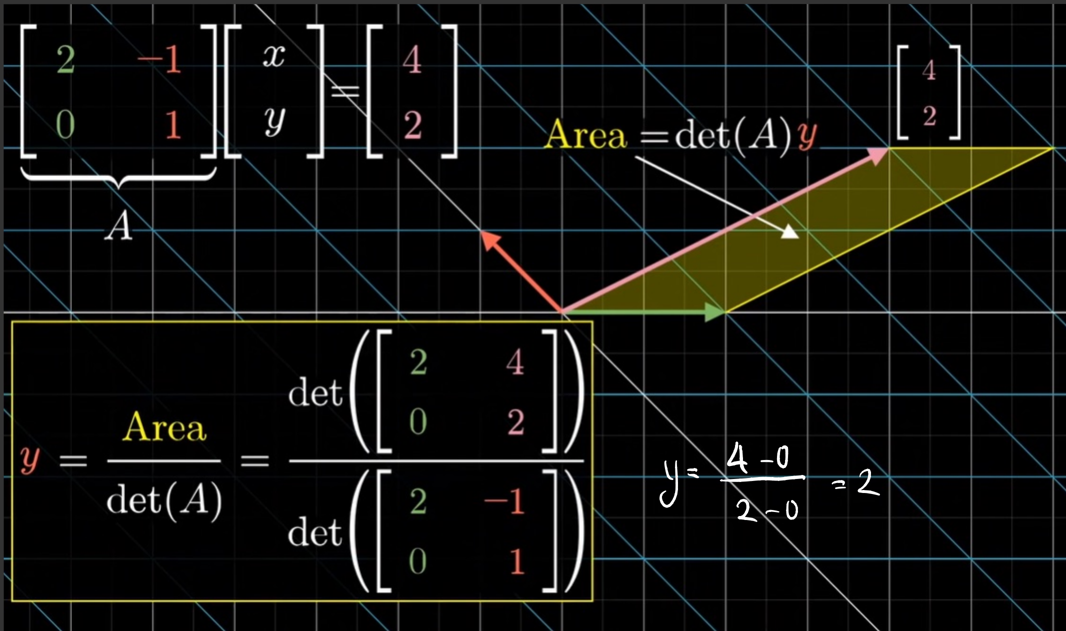
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

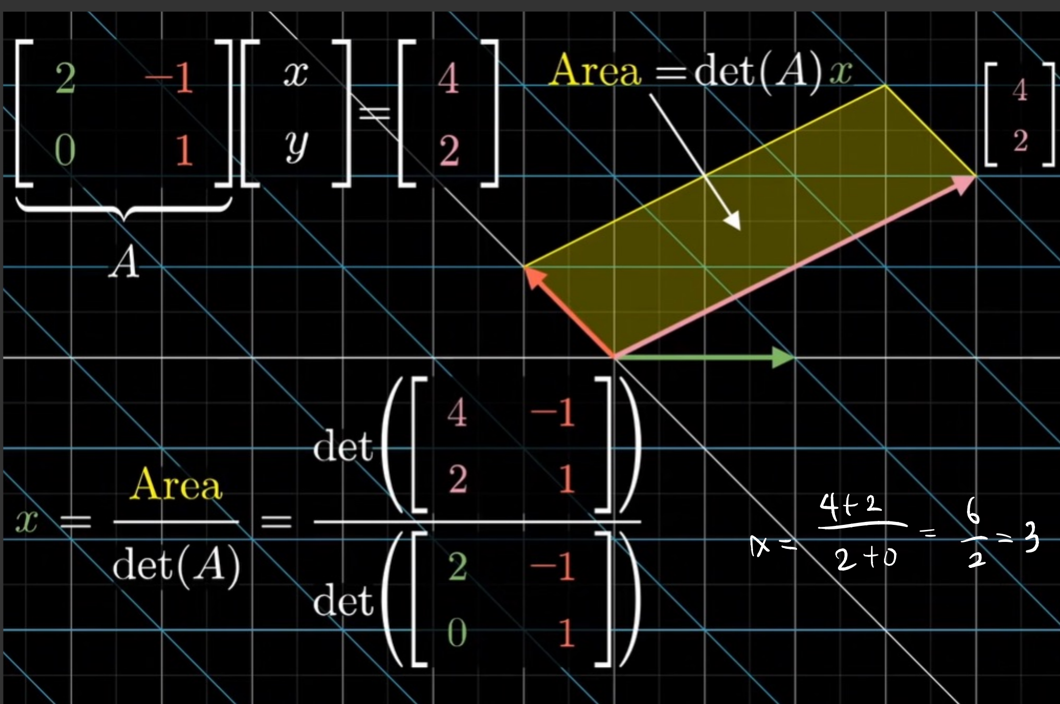
$\cos 30^\circ - \sin 30^\circ$

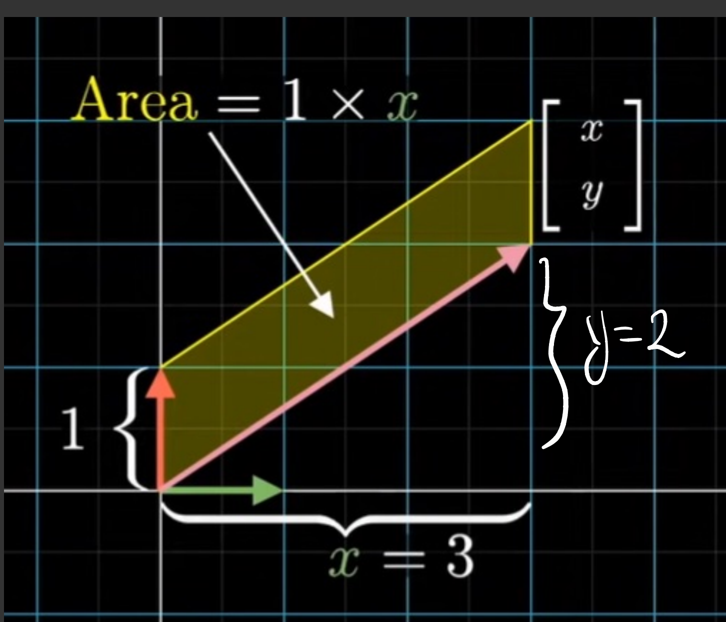
$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} \cos 30^\circ \\ \sin 30^\circ \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} -\sin 30^\circ \\ \cos 30^\circ \end{bmatrix}$$

Cramer's Rule







Apply Cramer's Rule to get mysterious input. (2D)

$$\begin{array}{cc} x & y \end{array}$$
$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad T: \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \qquad \det(T) = 2 - 0 = 2$$

$$y = \frac{\det\left(\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}\right)}{\det(T)} = \frac{4-0}{2} = 2$$

$$x = \frac{\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\right)}{\det(T)} = \frac{4+2}{2} = 3$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Apply Cramer's Rule to get the mysterious input (3D)

$$3x + 2y - 7z = 4$$
$$1x + 2y - 4z = 2$$
$$4x + 0y + 1z = 5$$

$$\begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{Mystery input}} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$T = \begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(T) = \det\left(\begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix}\right)$$
$$= 3 \times \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix} - 7\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$$
$$= 3\{2\} - 2\{1 + 16\} - 7\{0 - 8\}$$
$$= 6 - 2(17) + 56$$
$$= 62 - 34$$
$$= 28$$

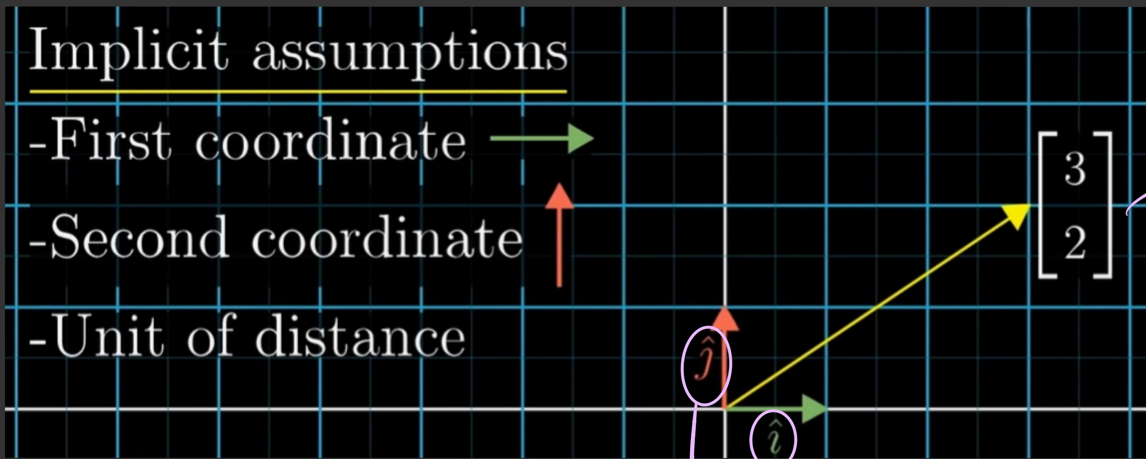
$$x = \frac{\det\left(\begin{bmatrix} 4 & 2 & -7 \\ 2 & 2 & -4 \\ 5 & 0 & 1 \end{bmatrix}\right)}{28} \qquad y = \frac{\det\left(\begin{bmatrix} 3 & 4 & -7 \\ 1 & 2 & -4 \\ 4 & 5 & 1 \end{bmatrix}\right)}{28} \qquad z = \frac{\det\left(\begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 2 \\ 4 & 0 & 5 \end{bmatrix}\right)}{28}$$

$$4\det\left(\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}\right) - 2d\left(\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}\right) - 7d\left(\begin{bmatrix} 2 & 2 \\ 5 & 0 \end{bmatrix}\right)$$
$$4\{2\} - 2\{2 + 20\} - 7\{0 - 10\}$$
$$8 - 2 \cdot 22 - 7(-10)$$
$$8 - 44 + 70$$
$$= 78 - 44$$
$$= 74$$
$$x = \frac{74}{28}$$

$$3\det\left(\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}\right) - 4d\left(\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}\right) - 7d\left(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\right)$$
$$3\{2 + 20\} - 4\{1 + 16\} - 7\{5 - 8\}$$
$$3 \cdot 22 - 4 \cdot 17 - 7(-3)$$
$$66 - 68 + 21$$
$$87 - 68$$
$$= 19$$
$$y = \frac{28}{19}$$

$$3d\left(\begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix}\right) - 2d\left(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\right) + 4d\left(\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}\right)$$
$$= 3 \cdot (10) - 2(5 - 8) + 4(-8)$$
$$= 30 + 6 - 32$$
$$= 4$$
$$z = \frac{4}{28}$$

Change of basis

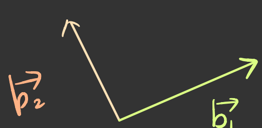


Coordinate system

Basis vectors

"How do you translate between coordinate systems"

Jennifer's alternate basis vectors



$$\begin{bmatrix}2&-1\\1&1\end{bmatrix} \neq \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

"Different language"

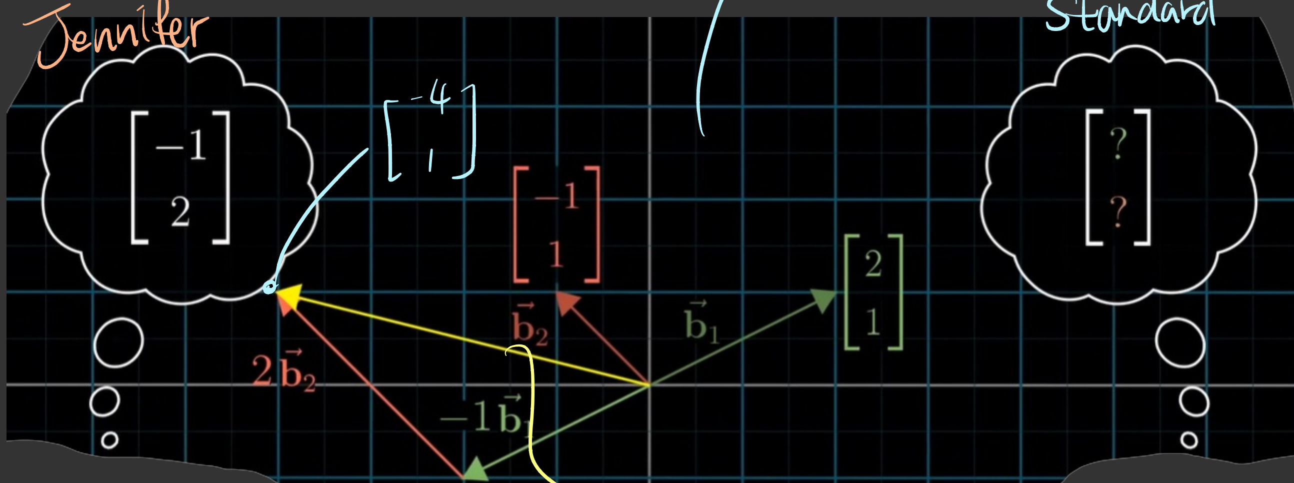
but, same origin

Jennifer → Our

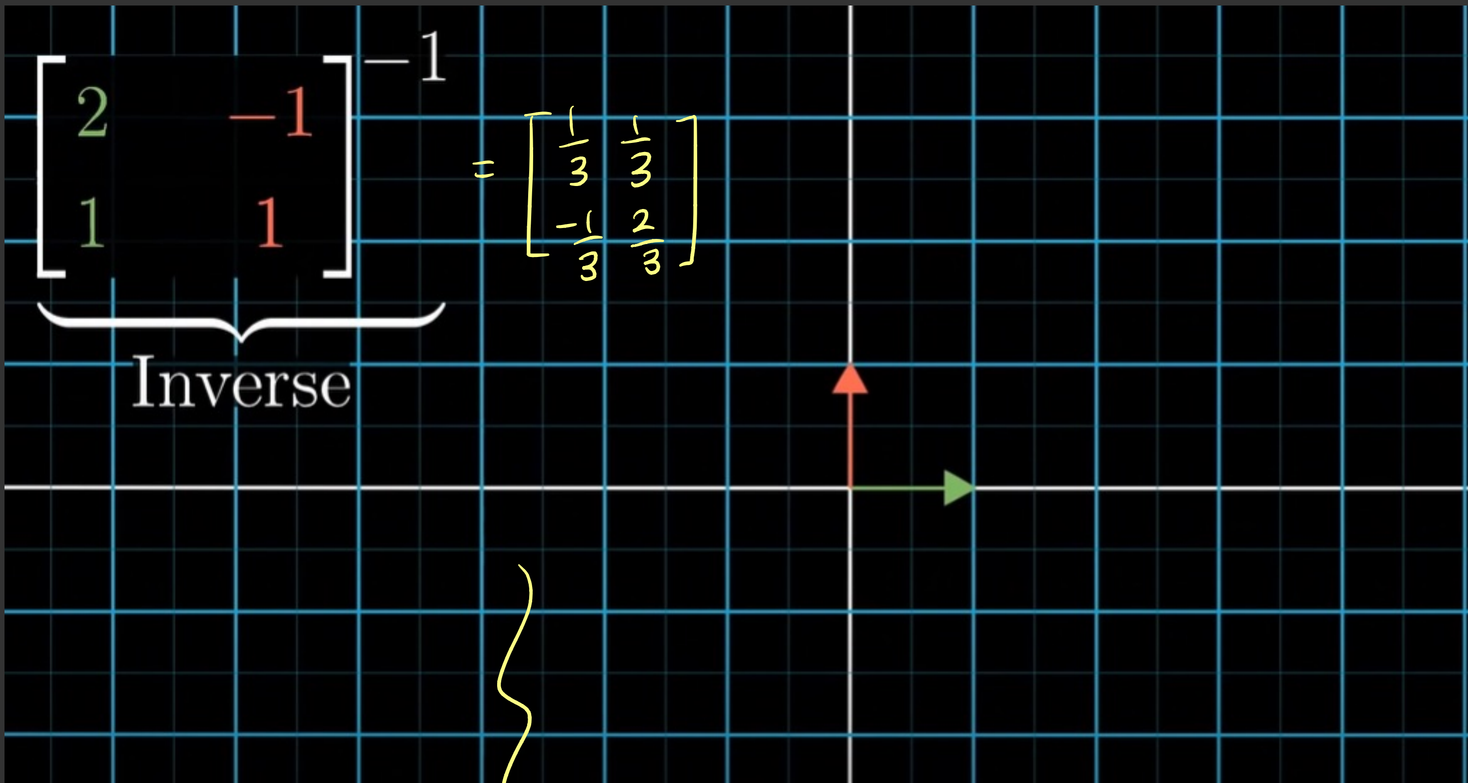
$$-1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$$

Jennifer

Standard



$$\begin{bmatrix}2&-1\\1&1\end{bmatrix}$$



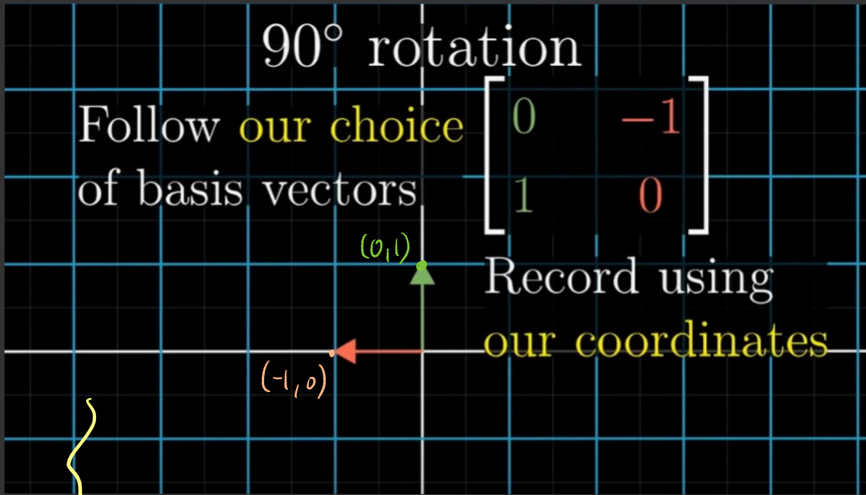
$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

our $\longrightarrow$ Jennifer

Q. what is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in Jennifer's language?

A.

$$\underbrace{\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}}_{\substack{\text{Inverse} \\ \text{change of basis} \\ \text{matrix}}} \overset{\text{our}}{\underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\substack{\text{Written in} \\ \text{our language}}}} = \overbrace{\begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}}^{\substack{\text{Same vector} \\ \text{in her language}}}_{\text{Jennifer's}}$$



How would Jennifer describe 90° rotation of space?

< How to translate a matrix >

① Start with any vector written in Jennifer's language.
② Change of basis matrix (Jennifer's basis matrix)

→ Same vector in our language

③ Transformation matrix in our language

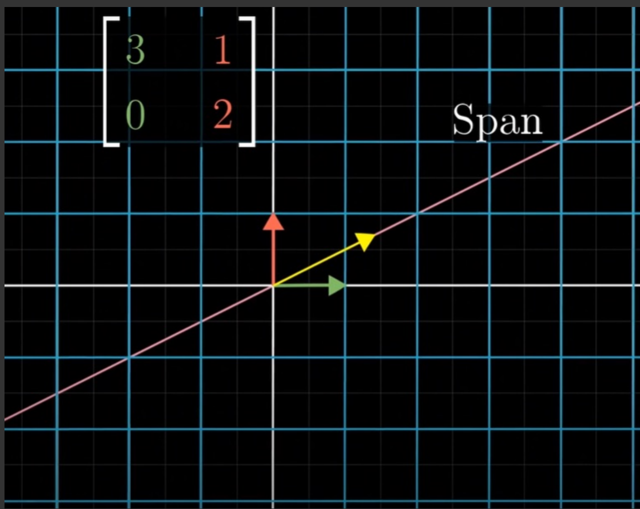
→ Transformed vector in our language

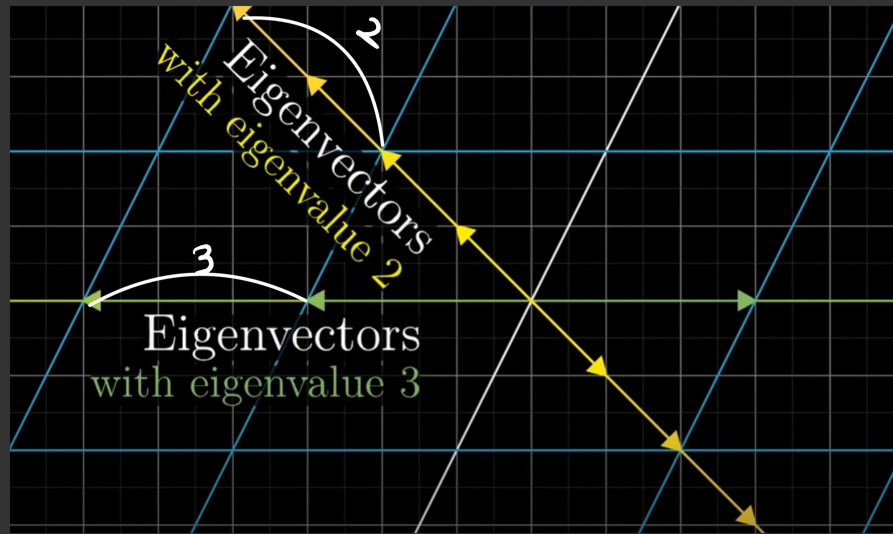
④ Inverse Change of basis matrix

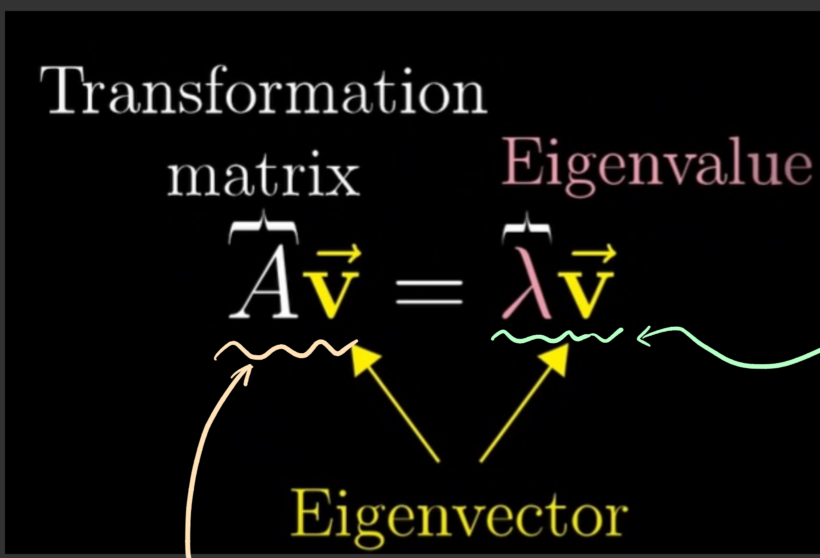
$$\overbrace{\underbrace{\begin{bmatrix}2&-1\\1&1\end{bmatrix}^{-1}}_{\text{Inverse change of basis matrix}}\underbrace{\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}_{\substack{\text{transformation}\\ \text{in our language}\\ (90^\circ)}}\overbrace{\underbrace{\begin{bmatrix}2&-1\\1&1\end{bmatrix}}_{\text{Change of basis matrix}}\vec{\mathbf{v}}}^{\text{Same vector written in our language}}}_{\text{transformed vector in our language}}}^{\text{Transformed vector in her language}}$$

$\vec{\mathbf{v}}$: vector written in Jennifer's language

Eigen vectors and Eigen values

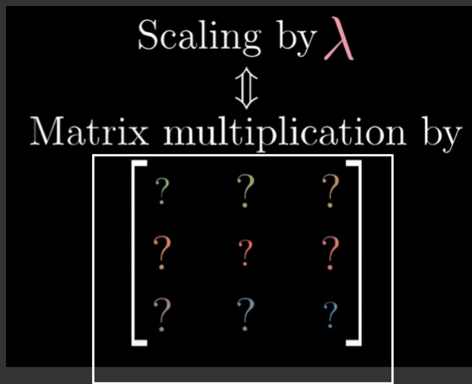






Scalar Multiplication

Matrix-vector multiplication



$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \Leftrightarrow \lambda I$$

$$A\vec{v} = \lambda I \vec{v}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\Downarrow$$

$$* \det(A - \lambda I) = 0 *$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

Seeking eigenvalue λ

$$(A - \lambda I)\vec{v} = 0$$

when $\vec{v} \neq 0$:

$$A - \lambda I = 0$$

$$A - \lambda I : \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)(2-\lambda) = 0$$

$\lambda = 3$ or $\lambda = 2$

(i) $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(ii) $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = 0$$

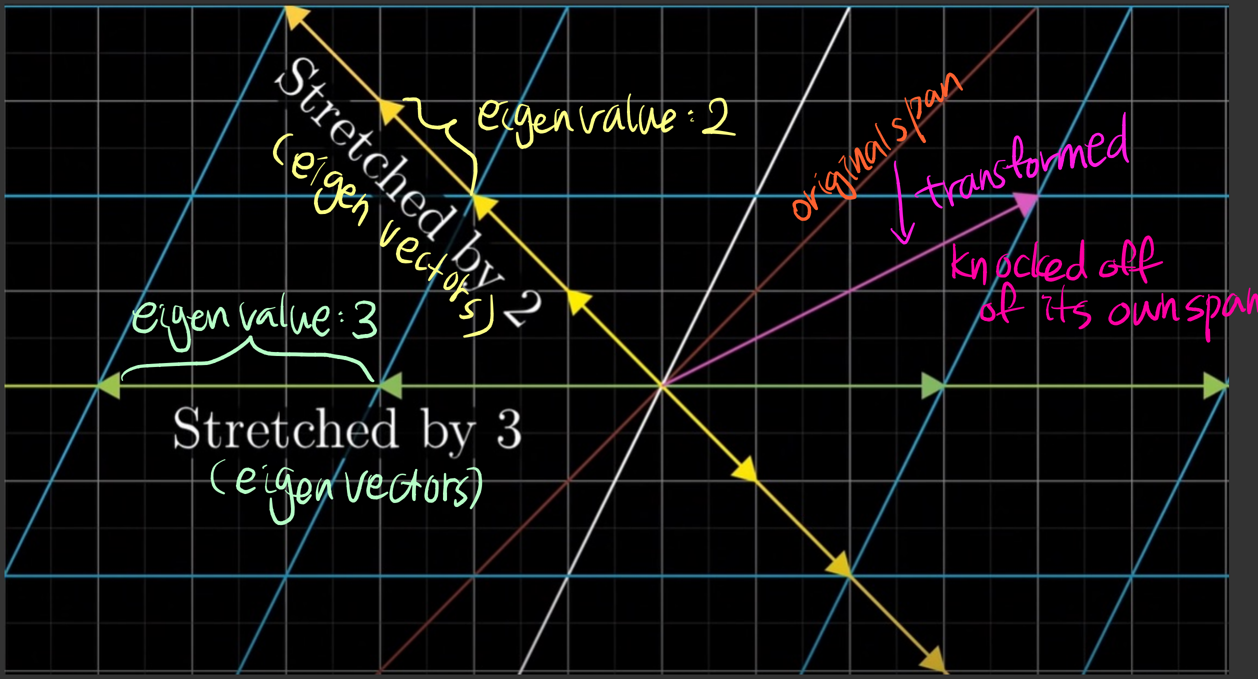
eigen value may not exist

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\boxed{\begin{array}{c} \lambda^2 + 1 = 0 \\ \lambda = \pm i \end{array}} \rightarrow \text{eigen vector } X$$



Eigen Basis

“Diagonal matrix”

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

← all basis vectors are eigen vectors

eigen values

Abstract vector spaces

determinant : how much the transformation scales area
eigenvector : stant on their span during transformation

Formal definition of linearity

Additivity: $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$

Scaling: $L(c\vec{\mathbf{v}}) = cL(\vec{\mathbf{v}})$

Linear transformations preserve addition and scalar multiplication

Derivative is linear

$$L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$$

$$\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

Derivative is linear

$$L(c\vec{\mathbf{v}}) = cL(\vec{\mathbf{v}})$$

$$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$$

vector-ish things : function

Our current space: All polynomials

$$\frac{d}{dx}(1x^3 + 5x^2 + 4x + 5) = \underbrace{3x^2 + 10x + 4}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\frac{d}{dx}} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1\cdot 4 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \end{bmatrix}$$

Basis functions

$b_0(x) = 1$

$b_1(x) = x$

$b_2(x) = x^2$

$b_3(x) = x^3$

$\vdots$

Linear transformations

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \frac{df}{dx}$$

Rules for vectors addition and scaling

1. $\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}$
2. $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \vec{\mathbf{w}} + \vec{\mathbf{v}}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{0} + \vec{\mathbf{v}} = \vec{\mathbf{v}}$ for all $\vec{\mathbf{v}}$
4. For every vector $\vec{\mathbf{v}}$ there is a vector $-\vec{\mathbf{v}}$ so that $\vec{\mathbf{v}} + (-\vec{\mathbf{v}}) = \mathbf{0}$
5. $a(b\vec{\mathbf{v}}) = (ab)\vec{\mathbf{v}}$
6. $1\vec{\mathbf{v}} = \vec{\mathbf{v}}$
7. $a(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = a\vec{\mathbf{v}} + a\vec{\mathbf{w}}$
8. $(a + b)\vec{\mathbf{v}} = a\vec{\mathbf{v}} + b\vec{\mathbf{v}}$

"Axioms"

= rules