Problem Set #1.

4. (a)

$\eta \in \mathbb{R}, T(y) = y$

$p(y;\eta) = b(y)\exp(\eta y - a(\eta))$

Derive an expression for the mean of the distribution. Show that $E[Y|X;\theta]$ can be represented as the gradient of the log-partition function a with respect to the natural parameter η.

$\int p(y;\eta)dy = 1$

$$\frac{\partial}{\partial \eta}\int p(y;\eta)dy$$

$$= \frac{\partial}{\partial \eta}\int \{b(y)\exp(\eta y - a(\eta))\}dy$$

$$= \int \frac{\partial}{\partial \eta} p(y;\eta)\, dy$$

$$= \int \frac{\partial}{\partial \eta}\{b(y)\exp(\eta y - a(\eta))\}dy$$

$$= \int b(y)\exp(\eta y - a(\eta)) \cdot \{y - \frac{\partial}{\partial \eta}a(\eta)\}dy$$

$$= \int p(y;\eta)\cdot(y - \frac{\partial}{\partial \eta}a(\eta))dy$$

$$= \int (y - \frac{\partial}{\partial \eta}a(\eta))p(y;\eta)dy$$

$$= \int y p(y;\eta)dy - \int \frac{\partial}{\partial \eta}a(\eta)p(y;\eta)dy$$

$$= E[Y;\eta] - \frac{\partial}{\partial \eta}a(\eta) \times \int p(y;\eta)dy$$

$$= E[Y;\eta] - \frac{\partial}{\partial \eta}a(\eta)$$

$$= 0$$

$$\boxed{E[Y;\eta] = E[Y|x;\theta]} = \frac{\partial a(\eta)}{\partial \eta}$$