Probset #3

4. (a)

$K(x,z) = \Phi(x)^T\Phi(z)$

<Mercer's theorem>

any finite set : $\{x^{(1)}, \dots, x^{(m)}\}$

square matrix : $K \in \mathbb{R}^{m\times m}$ is symmetric and positive semidefinite.

whose entries are given by $K_{ij} = K(x^{(i)}, x^{(j)})$

K_1, K_2 : Kernels over $\mathbb{R}^n \times \mathbb{R}^n$

$a \in \mathbb{R}^+$: positive real number

$f : \mathbb{R}^n \to \mathbb{R}$: a real-valued function

$\phi : \mathbb{R}^n \to \mathbb{R}^d$

K_3 : Kernel over $\mathbb{R}^d \times \mathbb{R}^d$

$p(x)$: polynomial over x with positive coefficients

(a) $K(x,z) = K_1(x,z) + K_2(x,z)$

$K_1(x,z) = \phi_1(x)^T\phi_1(z)$

$K_2(x,z) = \phi_2(x)^T\phi_2(z)$

$\phi_1(x)^T\phi_1(z) + \phi_2(x)^T\phi_2(z)$

(a) $K(x,z) = K_1(x,z) + K_2(x,z)$

$K_1(x,z) = \phi_1(x)^T\phi_1(z)$

$K_2(x,z) = \phi_2(x)^T\phi_2(z)$

$\rightarrow \phi_1(x)^T\phi_1(z) + \phi_2(x)^T\phi_2(z)$

for any vector z,

$z^T K z = \sum_i \sum_j z_i K_{ij} z_j$

$K_{ij} = K(x^{(i)}, x^{(j)})$

$= \phi_1(x^{(i)})^T\phi_1(z^{(j)}) + \phi_2(x^{(i)})^T\phi_2(z^{(j)})$

$\rightarrow \sum_i \sum_j z_i \{\phi_1(x^{(i)})^T\phi_1(z^{(j)}) + \phi_2(x^{(i)})^T\phi_2(z^{(j)})\} z_j$

$= \sum_i \sum_j z_i \left\{ \sum_k (\phi_1(x^{(i)}))_k (\phi_1(z^{(j)}))_k + \sum_k (\phi_2(x^{(i)}))_k (\phi_2(z^{(j)}))_k \right\} z_j$

$= \sum_i \sum_j z_i \sum_k (\phi_1(x^{(i)}))_k (\phi_1(z^{(j)}))_k z_j + \sum_i \sum_j z_i \sum_k (\phi_2(x^{(i)}))_k (\phi_2(z^{(j)}))_k z_j$

$\left(\sum_k \sum_i z_i (\phi_1(x^{(i)}))_k \right)^2 \geq 0$ $\quad\quad$ $\left(\sum_k \sum_i z_i (\phi_2(x^{(i)}))_k \right)^2 \geq 0$

$\rightarrow$ Kernel matrix is positive semi-definite.