Canonical Response Functions (g)

giving the distribution's mean as a function of the natural parameter

⟨Gaussian distribution⟩

$$p(y;\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-\mu)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(y^2 - 2\mu y + \mu^2)\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y^2 + \mu y - \frac{1}{2}\mu^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \times \exp\left(\mu y - \frac{1}{2}\mu^2\right)$$

$\rightsquigarrow$

$T(y) = y$ ✓

$a(\eta) = \frac{1}{2}\mu^2$

$b(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)$ ✓

$\eta = \mu$ ✓

$a(\eta) = \frac{1}{2}\eta^2$ ✓

$$\left.\begin{aligned} g(\eta) &= \mu \\ \eta &= \mu \\ g(\eta) &= \eta \end{aligned}\right\} \rightarrow$$

canonical response function for gaussian distribution = identity function

⟨Bernoulli distribution⟩

ϕ : mean

$p(y=1;\phi) = \phi$

$p(y=0;\phi) = 1-\phi$

$$p(y;\phi) = \phi^y(1-\phi)^{1-y}$$

$$= \exp\{y \log\phi + (1-y)\log(1-\phi)\}$$

$$= \exp\{y \log\phi + \log(1-\phi) - y\log(1-\phi)\}$$

$$= \exp\{y(\log\phi - \log(1-\phi)) + \log(1-\phi)\}$$

$$= \exp\left\{y \times \log\frac{\phi}{1-\phi} + \log(1-\phi)\right\}$$

$$= \exp\left\{\log\left(\frac{\phi}{1-\phi}\right)y + \log(1-\phi)\right\}$$

$T(y) = y$ ✓

$\eta = \log\left(\frac{\phi}{1-\phi}\right)$ ✓

$e^\eta = \frac{\phi}{1-\phi}$

$(1-\phi)e^\eta = \phi$

$e^\eta = \phi(1+e^\eta)$

$\phi = \frac{e^\eta}{1+e^\eta}$

$\phi = \frac{1}{e^{-\eta}+1}$ ⇜ sigmoid

$a(\eta) = -\log(1-\phi)$

$= \log(1+e^\eta)$ ✓

$b(y) = 1$ ✓

$$\left.\begin{aligned} g(\eta) &= \phi \\ g(\eta) &= \frac{1}{e^{-\eta}+1} \end{aligned}\right\} \rightarrow$$

canonical response function for bernoulli distribution = sigmoid function