Probset #3

4.(h)

$K(x,z) = p(K_1(x,z))$

$p(x)$: a polynomial over x with positive coefficients

K_1: Kernels over $\mathbb{R}^n \times \mathbb{R}^n$

$\hookrightarrow p\{\phi_1(x)^T \phi_1(z)\}$

For any vector z,

$$z^T K z = \sum_i \sum_j z_i K_{ij} z_j$$

$$= \sum_i \sum_j z_i \, p\{\phi_1(x^{(i)})^T \phi_1(z^{(j)})\} z_j$$

$$= \sum_i \sum_j z_i \, p\{\phi_1(x^{(i)}) \phi_1(x^{(j)})\} z_j$$

$$p(x^{(i)}) = \sum_{k=0}^{m} c_k x_k^{(i)} \quad (c_k > 0 \quad \forall k \in \{0, \dots, m\})$$

$$p(\phi_1(x^{(i)})\phi_1(x^{(j)})) = \sum_{k=0}^{m} c_k \phi_1(x_k^{(i)}) \phi_1(x_k^{(j)})$$

$$\rightarrow \sum_i \sum_j z_i \sum_k c_k \phi_1(x_k^{(i)}) \phi_2(x_k^{(j)}) \geq 0$$

$\rightarrow$ positive semi definite

$\rightarrow$ is necessarily a kernel.