Probset #2

3.(b)

Show that MAP estimation with a zero-mean Gaussian prior over θ, specifically $\theta \sim \mathcal{N}(0, \eta^2 I)$, is equivalent to applying L_2 regularization with MLE estimation.

Show that: $\theta_{MAP} = \underset{\theta}{\operatorname{argmin}} \{ -\log p(y|x,\theta) + \lambda \|\theta\|_2^2 \}$

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}}\ p(\theta|x,y)$$

$$= \underset{\theta}{\operatorname{argmax}}\ p(y|\theta,x)p(\theta)$$

prior : $\theta \sim \mathcal{N}(0, \eta^2 I)$

$$p(\theta) = \frac{1}{\eta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta}{\eta}\right)^2}$$

$$\underset{\theta}{\operatorname{argmax}}\ p(y|\theta,x) \times \frac{1}{\eta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\theta}{\eta}\right)^2\right)$$

$$= \underset{\theta}{\operatorname{argmax}}\ \log \left\{ p(y|\theta,x) \times \frac{1}{\eta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\theta}{\eta}\right)^2\right) \right.$$

$$= \underset{\theta}{\operatorname{argmin}} - \left\{ \log p(y|\theta,x) + \log \frac{1}{\eta\sqrt{2\pi}} + \log \exp\left(-\frac{1}{2}\left(\frac{\theta}{\eta}\right)^2\right) \right\}$$

$$= \underset{\theta}{\operatorname{argmin}} - \left\{ \log p(y|\theta,x) - \frac{1}{2}\left(\frac{\theta}{\eta}\right)^2 \right\}$$

$$= \underset{\theta}{\operatorname{argmin}} \left\{ -\log p(y|\theta,x) + \frac{1}{2}\left(\frac{\theta}{\eta}\right)^2 \right\}$$

$$\therefore\ \lambda = \frac{1}{2\eta^2}$$