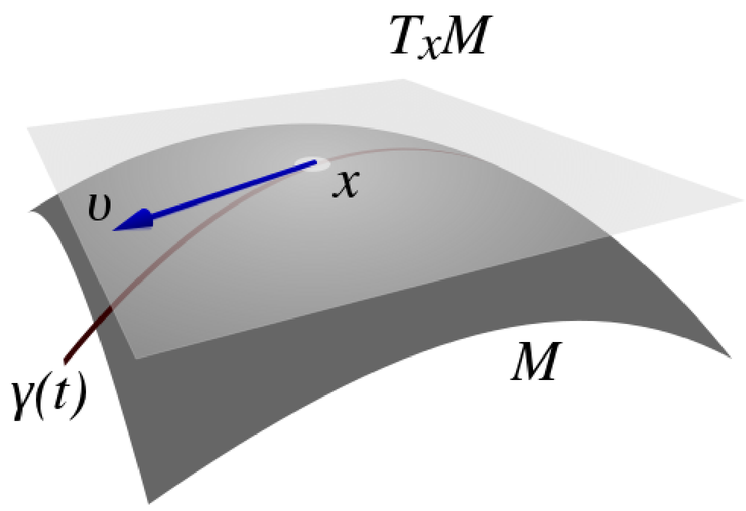
Prerequisites in Mathematics for Hyperbolic Embedding

ref:

- https://shorturl.at/qxLMZ
- https://arxiv.org/pdf/2101.04562.pdf
- https://arxiv.org/pdf/1909.05946.pdf



Manifold and Tangent Space around x

Manifold

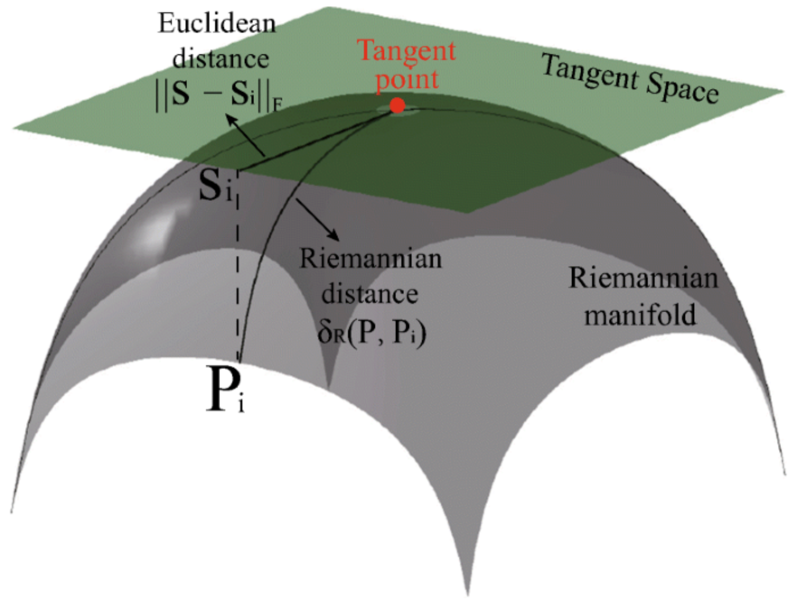
A manifold $\mathcal{M}$ of dimension n is a topological space of which each point's neighborhood can be locally approximated by the Euclidean space $\mathbb{R}^n$

Tangent Space

For each point $x \in \mathcal{M}$, the tangent space $\mathcal{T}_x\mathcal{M}$ of $\mathcal{M}$ at x is defined as an n-dimensional vector-space approximating $\mathcal{M}$ around x at a first order

Riemannian Metric

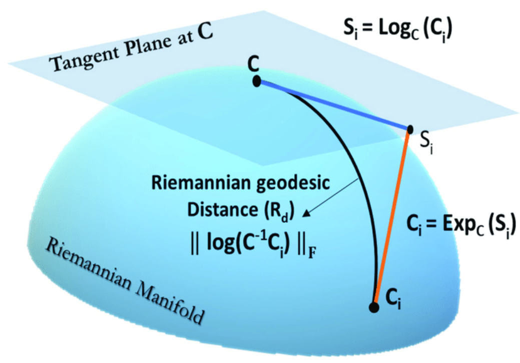
The metric tensor gives a local notion of angle, length of curves, surface area, and volume. For a manifold $\mathcal{M}$, a Riemannian metric g is a smooth family of inner products on the associated tangent space $< ., . >_x$: $\mathcal{T}_x\mathcal{M}$X$\mathcal{T}_x\mathcal{M} \to \mathbb{R}$. A given smooth manifold can be equipped with many different Riemannian metrics.



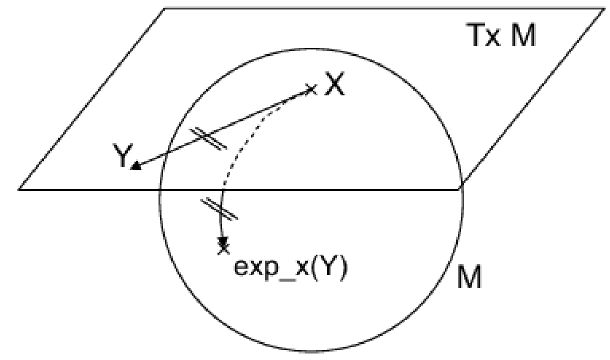
Riemanian Manifold, Tangent Point, Tangent Space

Riemannian Manifold

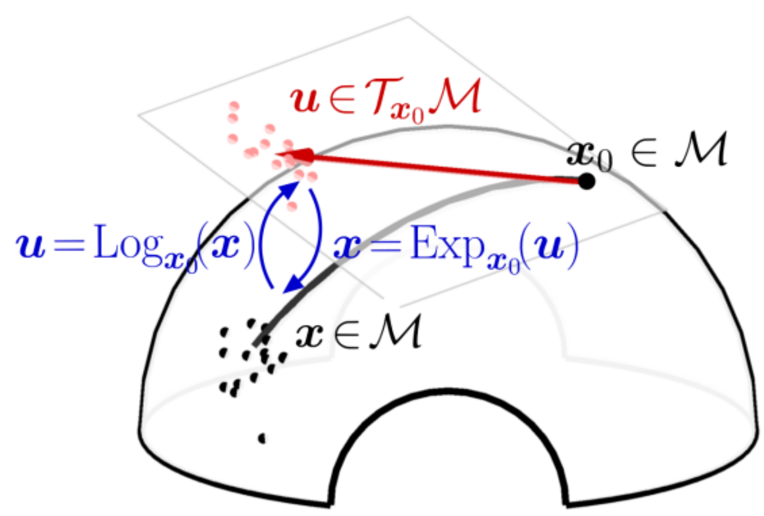
A Riemannian manifold is then defined as a manifold equipped with a group of Riemannian metrics g, which is formulated as a tuple ($\mathcal{M}, g)$



Geodesics, Exponential Map



Geodesics, Exponential Map



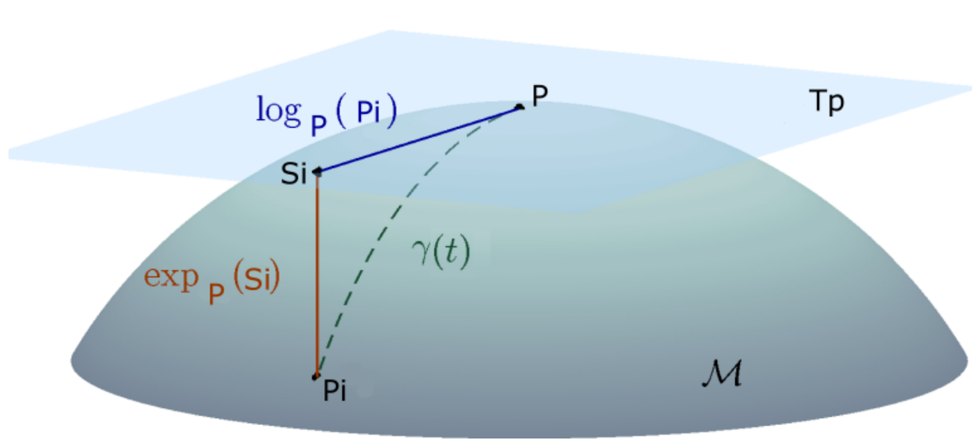
Tangent space – manifold mappings

Geodesics

Geodesics is the generalization of a straight line in the Euclidean space. It is the constant speed curve giving the shortest (straightest) path between pairs of points.

Exponential Map

The exponential map takes a vector $v \in \mathcal{T}_x\mathcal{M} \to \mathcal{M}$ of a point $x \in \mathcal{M}$ to a point on the manifold $\mathcal{M}$, i.e., $\mathrm{Exp}_x : \mathcal{T}_x\mathcal{M} \to \mathcal{M}$ by moving a unit length along the geodesic uniquely defined by $\gamma(0) = x$ with direction $\gamma'(0) = v$. Different manifolds have their own way to define
the exponential maps. Generally, this is very useful when computing the gradient, which provides an update that the parameter moves along the geodesic emanating from the current parameter position.

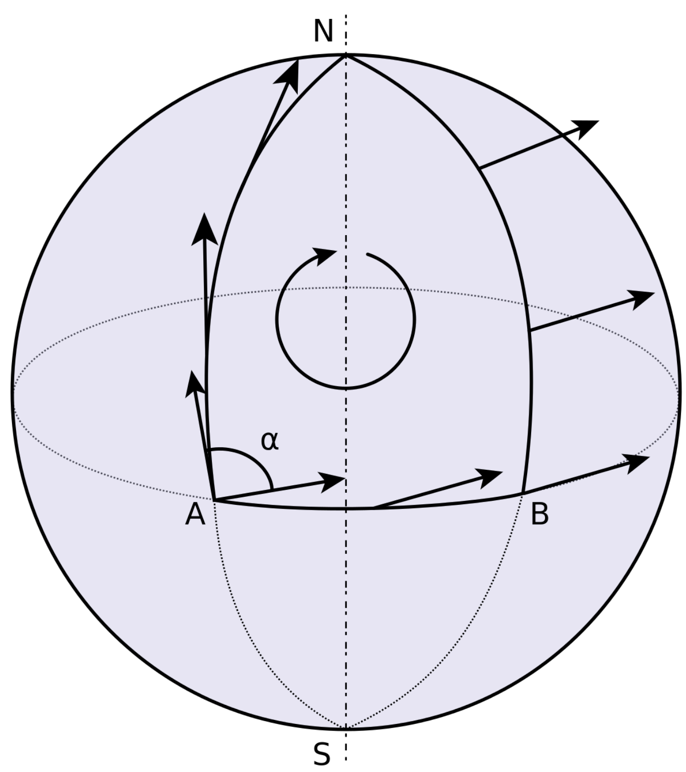


Logarithmic Map

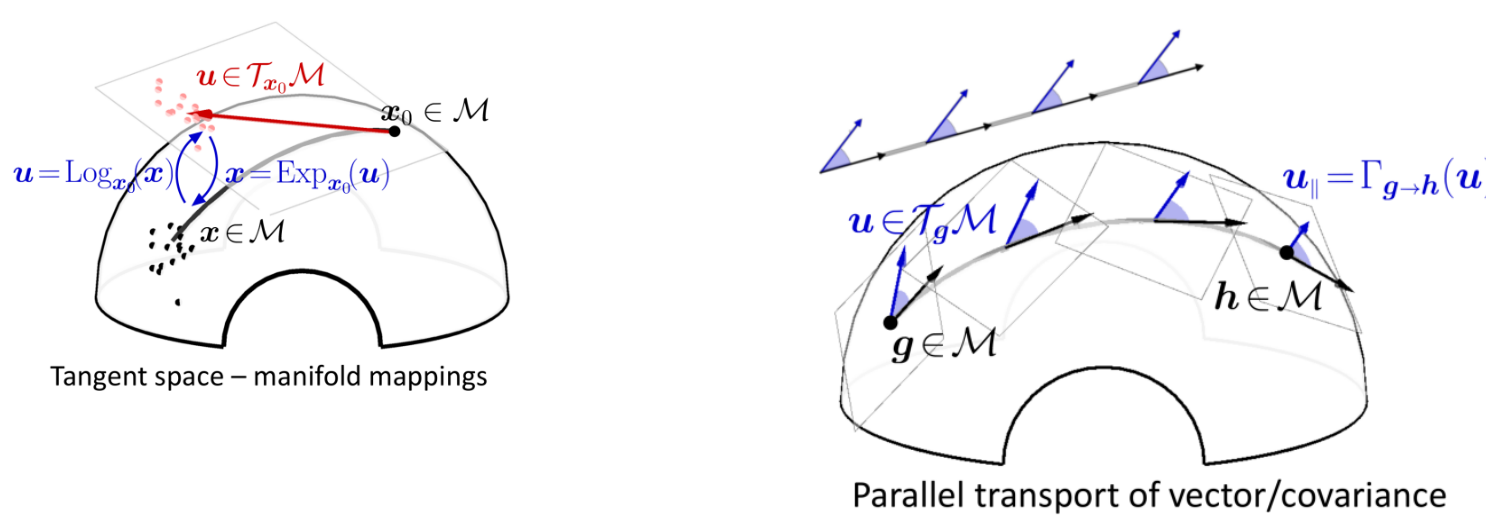
Logarithmic Map

As the inverse of the aforementioned exponential map, the logarithmic map projects a point $x \in \mathcal{M}$ on the manifold to the tangent space of another point $x \in \mathcal{M}$, which is $\mathrm{Log}_x : \mathcal{M} \to \mathcal{T}_x\mathcal{M}$.

Like the exponential map, there are different logarithmic maps for different manifolds



Parallel Transport



Parallel Transport

Parallel transport $\Gamma_{g \to h} : \mathcal{T}_g\mathcal{M} \to \mathcal{T}_h\mathcal{M}$ moves vectors between tangent spaces such that the inner product between two vectors in a tangent space is conserved. It employs the notion of connection, defining how to associate vectors between infinitesimally close tangent spaces. This connection allows the smooth transport of a vector from one tangent space to another by sliding it (with infinitesimal moves) along a curve.

Gromov δ-Hyperbolicity.

Gromov δ-Hyperbolicity is used to evaluate the hyperbolicity of a dataset/space. Normally, it is defined under four-point condition, say points a, b, c, v. A metric space (X, d) is δ-hyperbolic if there exists a δ >0 such that these four points in $X :< a, b >_v \gtrsim \min\{< a, c >_v, < b, c >_v\} - \delta$, where the $<, >_v$ with respect to a third point v is the Gromov product of two points and it is defined as $< a, b >_v = \frac{1}{2}(d(a, v) + d(b, v) - d(a, b))$ with $d(,)$ as the distance function. For instance, euclidean space $\mathbb{R}^n$ is not δ-hyperbolic, Poincare disc ($\mathbb{B}^2$) is (log3)-hyperbolic.