delta Hyperbolic Metric Space

Hyperbolic metric space is a metric space satisfying certain metric relations (depending quantitatively on a nonnegative real number δ) between points.

Definition

A metric space is said to be (Gromov-) hyperbolic if it is δ-hyperbolic for some $\delta > 0$

Definition using Gromov Product

Let (X, d) be a metric space. The Gromov product of two points $y, z \in X$ with respect to a third one $x \in X$ is defined by the formula:

$$(y, z)_x = \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

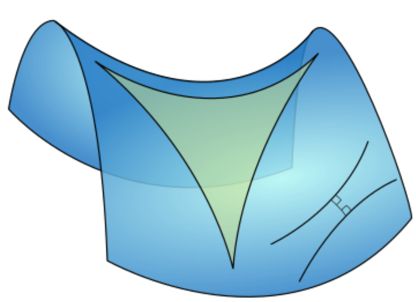
Gromov's definition of a hyperbolic metric space is then as follows: X is δ-hyperbolic if and only if all $x, y, z, w \in X$ satisfy the *four-point condition*

$$(x, z)_w \geq \min\left((x, y)_w, (y, z)_w\right) - \delta$$

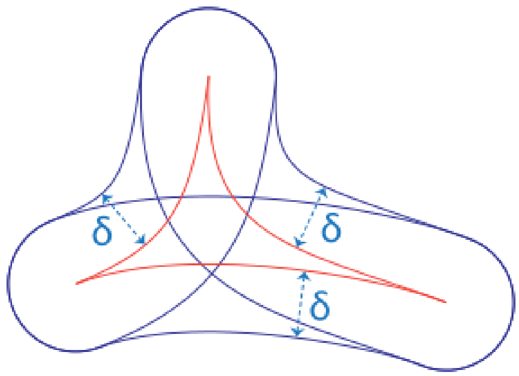
Note that if this condition is satisfied for all $x, y, z \in X$ and one fixed base point w_0, then it is satisfied for all with a constant 2δ.[1] Thus the hyperbolicity condition only needs to be verified for one fixed base point; for this reason, the subscript for the base point is often dropped from the Gromov product.

Definition using Triangles

Hyperbolic triangle



Hyperbolic triangle



The δ-slim triangle condition

δ-Hyperbolicity captures the basic common features of “negatively curved” spaces like the classical real-hyperbolic space $\mathbb{D}^n$ and of discrete spaces like trees.