Expanding VAE Loss (ELBO : Evidence Lower Bound)

Note: KL Divergence

$$D_{KL}(p(x)\|q(x)) = \int p(x) \log \frac{p(x)}{q(x)} dx$$

$$D_{KL}(q_\phi(z|x) \| p_\theta(z|x))$$
$$= \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(z|x)} dz$$
$$= \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(z,x)/p_\theta(x)} dz$$
$$= \int q_\phi(z|x) \log \frac{q_\phi(z|x)\, p_\theta(x)}{p_\theta(z,x)} dz$$
$$= \int q_\phi(z|x) \left\{ \log p_\theta(x) + \log \frac{q_\phi(z|x)}{p_\theta(z,x)} \right\} dz$$
$$= \int q_\phi(z|x) \log p_\theta(x)\, dz + \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(z,x)} dz$$

Note: Conditional Expectation

$$\mathbb{E}(X|Y=y) = \int x f_{X|Y}(x|y) dx$$
$$= \frac{1}{f_Y(y)} \int x f_{X,Y}(x,y) dx$$

$$= \log p_\theta(x) + \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(z,x)} dz$$
$$= \log p_\theta(x) + \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(x|z)\, p_\theta(z)} dz \rightarrow D_{KL}(q_\phi(z|x) \| p_\theta(z))$$
$$= \log p_\theta(x) + \underbrace{\int q_\phi(z|x) \left\{ \log \frac{q_\phi(z|x)}{p_\theta(z)}\right.} - \left.\log p_\theta(x|z) \right\} dz$$
$$= \log p_\theta(x) + \mathbb{E}_{z\sim q_\phi(z|x)} \left[\log \frac{q_\phi(z|x)}{p_\theta(z)} - \log p_\theta(x|z) \right\} dz$$
$$= \log p_\theta(x) + D_{KL}(q_\phi(z|x) \| p_\theta(z)) - \mathbb{E}_{z\sim q_\phi(z|x)} [\log p_\theta(x|z)]$$

$$\therefore D_{KL}(q_\phi(z|x) \| p_\theta(z|x)) = \log p_\theta(x) + D_{KL}(q_\phi(z|x) \| q_\theta(z)) - \mathbb{E}_{z\sim q_\phi(z|x)} \log p_\theta(x|z)$$

Objective

Minimize: $\underbrace{D_{KL}(q_\phi(z|x) \| p_\theta(z|x))}_{\text{= minimize difference between approx \& true posterior distributions}} \underbrace{- \log p_\theta(x)}_{\text{= maximize likelihood of generating } x}$

$D_{KL} \geq 0$

$$-L_{VAE} = +\log p_\theta(x) - D_{KL}(q_\phi(z|x) \| p_\theta(z|x)) \leq \log p_\theta(x)$$

Lower Bound!