Using Bayes theorem, one can calculate the posterior $q(x_{t-1}|x_t, x_0)$ in terms of $\tilde{\beta}_t$ and $\tilde{\mu}_t(x_t, x_0)$ which are defined as follows:

$$\tilde{\beta}_t := \frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\beta_t \tag{10}$$

$$\tilde{\mu}_t(x_t, x_0) := \frac{\sqrt{\bar{\alpha}_{t-1}}\beta_t}{1-\bar{\alpha}_t}x_0 + \frac{\sqrt{\alpha_t}(1-\bar{\alpha}_{t-1})}{1-\bar{\alpha}_t}x_t \tag{11}$$

$$q(x_{t-1}|x_t, x_0) = \mathcal{N}(x_{t-1}; \tilde{\mu}(x_t, x_0), \tilde{\beta}_t\mathbf{I}) \tag{12}$$

- Overview
  - $q(X_{t-1} \mid X_t) \approx p_\theta(X_{t-1} \mid X_t)$
    - $N\big(X_{t-1};\ \mu_\theta(X_t, t),\ \Sigma_\theta(X_t, t)\big)$
  - $q(X_{t-1} \mid X_t) \to q(X_{t-1} \mid X_t, X_0)$
    - $N\Big(X_{t-1};\ \tilde{\mu}(X_t, X_0), \tilde{\Sigma}(X_t, X_0)\Big)$

Note

$$\boldsymbol{q(x_{t-1} \mid x_t, x_0) = q(x_t \mid x_{t-1})\frac{q(x_{t-1} \mid x_0)}{q(x_t \mid x_0)}}$$

$\alpha_t : 1-\beta_t \qquad \bar{\alpha}_t : \prod_{s=0}^{t} \alpha_s$

$q(x_t|x_0) = N(x_t; \sqrt{\bar{\alpha}_t}x_0, (1-\bar{\alpha}_t)I)$

$x_t = \sqrt{\bar{\alpha}_t}x_0 + \sqrt{1-\bar{\alpha}_t}\epsilon$

$N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$N(x_{t-1}; \sqrt{\bar{\alpha}_{t-1}}x_0, (1-\bar{\alpha}_{t-1})I)$

$q(x_{t-1}|x_t, x_0) = q(x_t|x_{t-1}) \times \frac{q(x_{t-1}|x_0)}{q(x_t|x_0)}$

$N(x_t, \sqrt{\bar{\alpha}_t}x_0, (1-\bar{\alpha}_t)I)$

$\sim N(x_t; \sqrt{1-\beta_t}x_{t-1}, \beta_t I)$

$= \sim N(x_t; \sqrt{\alpha_t}x_{t-1}, (1-\alpha_t)I)$

$$= \frac{1}{\sqrt{2\pi(1-\alpha_t)}} e^{\frac{-(x-\sqrt{\alpha_t}x_{t-1})^2}{2(1-\alpha_t)}} \times \frac{\frac{1}{\sqrt{2\pi(1-\bar{\alpha}_{t-1})}} e^{\frac{-(x_{t-1}-\sqrt{\bar{\alpha}_{t-1}}x_0)^2}{2(1-\bar{\alpha}_{t-1})}}}{\frac{1}{\sqrt{2\pi(1-\bar{\alpha}_t)}} e^{\frac{-(x-\sqrt{\bar{\alpha}_t}x_0)^2}{2(1-\bar{\alpha}_t)}}}$$

$$= \frac{1}{\sqrt{2\pi(1-\alpha_t)}} e^{\frac{-(x-\sqrt{\alpha_t}x_{t-1})^2}{2(1-\alpha_t)}} \times \frac{\sqrt{2\pi(1-\bar{\alpha}_t)}}{\sqrt{2\pi(1-\bar{\alpha}_{t-1})}} \exp\left\{-\frac{(x_{t-1}-\sqrt{\bar{\alpha}_{t-1}}x_0)^2}{2(1-\bar{\alpha}_{t-1})} + \frac{(x-\sqrt{\bar{\alpha}_t}x_0)^2}{2(1-\bar{\alpha}_t)}\right\}$$

$$= \underbrace{\boxed{\frac{\sqrt{2\pi(1-\bar{\alpha}_t)}}{\sqrt{2\pi(1-\alpha_t)} \times \sqrt{2\pi(1-\bar{\alpha}_{t-1})}}}}_{①} \underbrace{\boxed{\exp\left\{\frac{-(x_t-\sqrt{\alpha_t}x_{t-1})^2}{2(1-\alpha_t)} - \frac{(x_{t-1}-\sqrt{\bar{\alpha}_{t-1}}x_0)^2}{2(1-\bar{\alpha}_{t-1})} + \frac{(x_t-\sqrt{\bar{\alpha}_t}x_0)^2}{2(1-\bar{\alpha}_t)}\right\}}}_{②}$$

$$① : \frac{\sqrt{2\pi(1-\bar{\alpha}_t)}}{\sqrt{2\pi\beta_t} \times \sqrt{2\pi(1-\bar{\alpha}_{t-1})}}$$

$$= \frac{\sqrt{1-\bar{\alpha}_t}}{\sqrt{2\pi\beta_t(1-\bar{\alpha}_{t-1})}}$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{\frac{1-\bar{\alpha}_t}{\beta_t(1-\bar{\alpha}_{t-1})}}$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{\frac{\beta_t(1-\bar{\alpha}_{t-1})}{1-\bar{\alpha}_t}}} = \frac{1}{\sqrt{2\pi\left(\frac{\beta_t(1-\bar{\alpha}_{t-1})}{1-\bar{\alpha}_t}\right)}} = \frac{1}{\sqrt{2\pi\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}}$$

2 :

$$\boxed{\exp\left\{\frac{-(x_t-\sqrt{\alpha_t}x_{t-1})^2}{2(1-\alpha_t)} - \frac{(x_{t-1}-\sqrt{\bar{\alpha}_{t-1}}x_0)^2}{2(1-\bar{\alpha}_{t-1})} + \frac{(x_t-\sqrt{\bar{\alpha}_t}x_0)^2}{2(1-\bar{\alpha}_t)}\right\}}$$

$$= \exp\left\{-\frac{(x_t-\sqrt{\alpha_t}x_{t-1})^2}{2\beta_t} - \frac{(x_{t-1}-\sqrt{\bar{\alpha}_{t-1}}x_0)^2}{2(1-\bar{\alpha}_{t-1})} + \frac{(x_t-\sqrt{\bar{\alpha}_t}x_0)^2}{2(1-\bar{\alpha}_t)}\right\}$$

$$= \exp\left\{-\frac{(x_t-\sqrt{1-\beta_t}x_{t-1})^2}{2\beta_t} - \frac{(x_{t-1}-\sqrt{\bar{\alpha}_{t-1}}x_0)^2}{2(1-\bar{\alpha}_{t-1})} + \frac{(x_t-\sqrt{\bar{\alpha}_t}x_0)^2}{2(1-\bar{\alpha}_t)}\right\}$$

$$= \exp\Big[-\frac{1}{2\beta_t}\left\{x_t^2 - 2\sqrt{1-\beta_t}x_t x_{t-1} + (1-\beta_t)x_{t-1}^2\right\} - \frac{1}{2(1-\bar{\alpha}_{t-1})}\left\{x_{t-1}^2 - 2\sqrt{\bar{\alpha}_{t-1}}x_{t-1}x_0 + \bar{\alpha}_{t-1}x_0^2\right\}$$
$$+ \frac{1}{2(1-\bar{\alpha}_t)}\left\{x_t^2 - 2\sqrt{\bar{\alpha}_t}x_0 x_t + \bar{\alpha}_t x_0^2\right\}\Big]$$

$$= \exp\left[x_{t-1}^2\left\{\frac{-(1-\beta_t)}{2\beta_t} - \frac{1}{2(1-\bar{\alpha}_{t-1})}\right\} + x_{t-1}\left\{\frac{1}{2\beta_t}\times 2\sqrt{1-\beta_t}\,x_t + \frac{\sqrt{\bar{\alpha}_{t-1}}x_0}{(1-\bar{\alpha}_{t-1})}\right\} + C\right]$$

$$= \exp\left[-\left\{\frac{(1-\beta_t)(1-\bar{\alpha}_{t-1})+\beta_t}{2\beta_t(1-\bar{\alpha}_{t-1})}\right\}x_{t-1}^2 + \left(\frac{\sqrt{\alpha_t}x_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}x_0}{(1-\bar{\alpha}_{t-1})}\right)x_{t-1} + C\right]$$

$$= \exp\left[-\left\{\boxed{\frac{1-\bar{\alpha}_{t-1}-\cancel{\beta_t}+\beta_t\bar{\alpha}_{t-1}+\cancel{\beta_t}}{2\beta_t(1-\bar{\alpha}_{t-1})}}\right\}x_{t-1}^2 + \left(\frac{\sqrt{\alpha_t}x_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}x_0}{(1-\bar{\alpha}_{t-1})}\right)x_{t-1} + C\right]$$

$$\frac{1-\bar{\alpha}_{t-1}+(1-\alpha_t)\bar{\alpha}_{t-1}}{2\beta_t(1-\bar{\alpha}_{t-1})} = \frac{1-\bar{\alpha}_{t-1}+\bar{\alpha}_{t-1}-\bar{\alpha}_t}{2\beta_t(1-\bar{\alpha}_{t-1})} = \frac{1-\bar{\alpha}_t}{2\beta_t(1-\bar{\alpha}_{t-1})} = \frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}$$

$$\exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}\left\{x_{t-1}^2 - \frac{2\beta_t(1-\bar{\alpha}_{t-1})}{1-\bar{\alpha}_t}\left(\frac{\sqrt{\alpha_t}x_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}x_0}{(1-\bar{\alpha}_{t-1})}\right)x_{t-1} + C\right\}\right)$$

$$= \exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}\left\{x_{t-1}^2 - \left(\frac{2\beta_t(1-\bar{\alpha}_{t-1})(\sqrt{\alpha_t})}{(1-\bar{\alpha}_t)(1-\alpha_t)}x_t + \frac{2\beta_t(1-\bar{\alpha}_{t-1})(\sqrt{\bar{\alpha}_{t-1}})}{(1-\bar{\alpha}_t)(1-\bar{\alpha}_{t-1})}x_0\right)x_{t-1} + C\right\}\right)$$

$$= \exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}\left\{x_{t-1}^2 - \left(\frac{2\beta_t(1-\bar{\alpha}_{t-1})(\sqrt{\alpha_t})}{(1-\bar{\alpha}_t)(1-\alpha_t)}x_t + \frac{2\beta_t(1-\bar{\alpha}_{t-1})(\sqrt{\bar{\alpha}_{t-1}})}{(1-\bar{\alpha}_t)(1-\bar{\alpha}_{t-1})}x_0\right)x_{t-1} + c\right\}\right)$$

$$= \exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}\left\{x_{t-1}^2 - \left(\frac{2\cancel{(1-\bar{\alpha}_t)}(1-\bar{\alpha}_{t-1})(\sqrt{\alpha_t})}{(1-\alpha_t)\cancel{(1-\bar{\alpha}_t)}}x_t + \frac{2\beta_t(\sqrt{\bar{\alpha}_{t-1}})}{1-\bar{\alpha}_t}x_0\right)x_{t-1} + c\right\}\right)$$

$$= \exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}\right)}\left\{x_{t-1}^2 - \left(\frac{2(1-\bar{\alpha}_{t-1})(\sqrt{\alpha_t})}{1-\alpha_t}x_t + \frac{2\beta_t(\sqrt{\bar{\alpha}_{t-1}})}{1-\bar{\alpha}_t}x_0\right)x_{t-1} + c\right\}\right)$$

$$= \exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\alpha_t}\right)}\left\{x_{t-1} - \left(\frac{\beta_t(\sqrt{\bar{\alpha}_{t-1}})}{1-\bar{\alpha}_t}x_0 + \frac{\sqrt{\alpha_t}(1-\bar{\alpha}_{t-1})}{1-\alpha_t}x_t\right)\right\}^2\right)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\underbrace{\boxed{\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\alpha_t}\right)}}_{\sigma^2}}} \exp\left(-\frac{1}{2\beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\alpha_t}\right)}\left\{x_{t-1} - \underbrace{\left(\frac{\beta_t(\sqrt{\bar{\alpha}_{t-1}})}{1-\bar{\alpha}_t}x_0 + \frac{\sqrt{\alpha_t}(1-\bar{\alpha}_{t-1})}{1-\alpha_t}x_t\right)}_{\mu}\right\}^2\right)$$

$\therefore q(x_{t-1}|x_t, x_0) \sim \mathcal{N}(\mu, \sigma^2)$

$$\tilde{\mu}_t(x_0, x_t) = \frac{\beta_t(\sqrt{\bar{\alpha}_{t-1}})}{1-\bar{\alpha}_t}x_0 + \frac{\sqrt{\alpha_t}(1-\bar{\alpha}_{t-1})}{1-\alpha_t}x_t$$

$$\tilde{\beta}_t(x_0, x_t) = \beta_t\left(\frac{1-\bar{\alpha}_{t-1}}{1-\alpha_t}\right)$$

$$q(x_{t-1}|x_t, x_0) \sim \mathcal{N}(x_{t-1}; \tilde{\mu}_t(x_0, x_t), \tilde{\beta}_t(x_0, x_t)I)$$