Reparameterization Trick

$z \sim \mathcal{N}(\mu, \sigma^2)$

$z = \mu + \sigma\epsilon \quad \epsilon \sim \mathcal{N}(0,1)$

[Forward Diffusion Process]

$x_t = \sqrt{1-\beta_t}\, x_{t-1} + \sqrt{\beta_t}\, \epsilon_{t-1} \qquad \epsilon_{t}, \cdots \epsilon_{t-1} \sim \mathcal{N}(0,1)$

$\alpha_t = 1 - \beta_t$

$x_t = \sqrt{\alpha_t}\, x_{t-1} + \sqrt{1-\alpha_t}\, \epsilon_{t-1}$

$x_{t-1} = \sqrt{\alpha_{t-1}}\, x_{t-2} + \sqrt{1-\alpha_{t-1}}\, \epsilon_{t-2}$

$= \sqrt{\alpha_t}\left(\sqrt{\alpha_{t-1}}\, x_{t-2} + \sqrt{1-\alpha_{t-1}}\, \epsilon_{t-2}\right) + \sqrt{1-\alpha_t}\, \epsilon_{t-1}$

$= \sqrt{\alpha_t \alpha_{t-1}}\, x_{t-2} + \boxed{\boxed{\sqrt{\alpha_t(1-\alpha_{t-1})}\, \epsilon_{t-2}} + \boxed{\sqrt{1-\alpha_t}\, \epsilon_{t-1}}}$

$\epsilon_{t-1} \sim \mathcal{N}(0,1)$

$\sqrt{1-\alpha_t}\, \epsilon_{t-1}$

$= 0 + \sqrt{1-\alpha_t}\, \epsilon_{t-1} \sim \mathcal{N}(0, (1-\alpha_t)I)$

$\epsilon_{t-2} \sim \mathcal{N}(0,1)$

$\sqrt{\alpha_t(1-\alpha_{t-1})}\, \epsilon_{t-2}$

$= 0 + \sqrt{\alpha_t(1-\alpha_{t-1})}\, \epsilon_{t-2} \sim \mathcal{N}(0, \alpha_t(1-\alpha_{t-1})I)$

Note

$z = x + y \qquad x \sim \mathcal{N}(\mu_x, \sigma_x^2) \quad y \sim \mathcal{N}(\mu_y, \sigma_y^2)$

$z \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

$\underbrace{\underset{\sim \mathcal{N}(0, \alpha_t(1-\alpha_{t-1})I)}{\sqrt{\alpha_t(1-\alpha_{t-1})}\, \epsilon_{t-2}} + \underset{\sim \mathcal{N}(0, (1-\alpha_t)I)}{\sqrt{1-\alpha_t}\, \epsilon_{t-1}}}$

$\rightarrow \sim \mathcal{N}(0, (\alpha_t - \alpha_t\alpha_{t-1})I + (1-\alpha_t)I)$

$\sim \mathcal{N}(0, (\alpha_t - \alpha_t\alpha_{t-1} + 1 - \alpha_t)I)$

$\sim \mathcal{N}(0, (1-\alpha_t\alpha_{t-1})I)$

$\sqrt{\alpha_t\alpha_{t-1}}\, x_{t-2} + \sqrt{1-\alpha_t\alpha_{t-1}}\, \epsilon_{t-2}$

$$\sqrt{\alpha_t\alpha_{t-1}}\, \underbrace{x_{t-2}} + \sqrt{1-\alpha_t\alpha_{t-1}}\, \epsilon_{t-2}$$

$$\hookrightarrow x_{t-2} = \sqrt{\alpha_{t-2}}\, x_{t-3} + \sqrt{1-\alpha_{t-2}}\, \varepsilon_{t-3}$$

$\epsilon = \varepsilon$

$$= \sqrt{\alpha_t\alpha_{t-1}}\left(\sqrt{\alpha_{t-2}}\, x_{t-3} + \sqrt{1-\alpha_{t-2}}\, \varepsilon_{t-3}\right) + \sqrt{1-\alpha_t\alpha_{t-1}}\, \epsilon_{t-2}$$

$$= \sqrt{\alpha_t\alpha_{t-1}\alpha_{t-2}}\, x_{t-3} + \underline{\sqrt{\alpha_t\alpha_{t-1}(1-\alpha_{t-2})}\, \varepsilon_{t-3} + \sqrt{1-\alpha_t\alpha_{t-1}}\, \varepsilon_{t-2}}$$

$\sim N(0, \alpha_t\alpha_{t-1}(1-\alpha_{t-2}))$ $\quad$ $\sim N(0, 1-\alpha_t\alpha_{t-1})$

$\sim N(0, \alpha_t\alpha_{t-1} - \alpha_t\alpha_{t-1}\alpha_{t-2})$ $\quad$ $\sim N(0, 1-\alpha_t\alpha_{t-1})$

$\Rightarrow \sim N(0, 1-\alpha_t\alpha_{t-1}\alpha_{t-2})$

$$= \sqrt{\alpha_t\alpha_{t-1}\alpha_{t-2}}\, x_{t-3} + \sqrt{1-\alpha_t\alpha_{t-1}\alpha_{t-2}}\, \epsilon_{t-3}$$

$$\vdots$$

$$= \sqrt{\alpha_t\alpha_{t-1}\cdots\alpha_1}\, x_0 + \sqrt{1-\alpha_t\alpha_{t-1}\cdots\alpha_1}\, \epsilon$$

$\bar{\alpha}_t = \prod_{i=1}^{t} \alpha_i$

$$= \sqrt{\bar{\alpha}_t}\, x_0 + \sqrt{1-\bar{\alpha}_t}\, \epsilon$$

$$** \quad \therefore\ x_t = \sqrt{\bar{\alpha}_t}\, x_0 + \sqrt{1-\bar{\alpha}_t}\, \epsilon$$

→ we can directly sample x_t at any time step t