

A Vector Algebra Formulation of Kinematics of Wheeled Mobile Robots

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Abstract

This document presents a straightforward yet general approach for solving problems in the kinematics of wheeled mobile robots. There are two problems. Control algorithms must translate the desired linear and angular velocity of the vehicle into the velocities and steer angles of the wheels. Conversely, estimation systems must do the reverse - compute the linear and angular velocity given the steer angles and wheel rotation rates.

It turns out that the problem is very elegantly solved in the general case by appealing to the physics of rigid body motion, especially the instantaneous center of rotation, and the mathematics of moving coordinate systems. The presented approach is general enough to handle any number of driven and steered wheels in any configuration, and each wheel may be offset from its steering pivot point. Flat terrain is assumed in the examples but the underlying technique makes no such assumption. The solution also does not assume that all wheels are at the same elevation - they may articulate in arbitrary ways. While the examples assume that the booms connecting wheels to the body are fixed, the general formulation can easily incorporate knowledge of the steering rates.

The formulation elegantly avoids the solution of nonlinear simultaneous equations by working, at times, in terms of the velocities of the wheel pivot points. In this way, the wheel steer angles are not expressed in terms of themselves in the actuated inverse solution.

The approach is applied to several examples including differential steer, Ackerman steer, a generalized bicycle model, and the difficult case of 4 steered and driven wheels whose steer axes are offset from the wheel contact points.

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1. Introduction

For control purposes, the kinematics of wheeled mobile robots (WMRs) that we care about are the *rate* kinematics. Of basic interest are two questions:

- Forward Kinematics: How do measured motions of the wheels translate into equivalent motions of the robot.
- Inverse Kinematics: How do desired motions of the robot translate into desired motions of the wheels.

1.1 Conventions

Wheels normally have up to two degrees of freedom (steer and drive) with respect to the vehicle to which they are attached. The relationship between wheel angular velocity and the linear velocity of the contact point is:

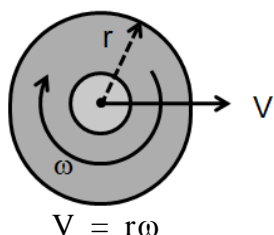


Figure 1: Wheel Linear and Angular Velocity: Assuming a point contact at the precise bottom of the wheel, and a known radius, the linear and angular velocity of a wheel are related as shown.

To accommodate modeling of passive castors, we allow the steer axis to be potentially different from the the contact point. Define the following frames of reference:

- w: world, fixed to the environment
- v: vehicle, fixed to some point on the vehicle whose motion is of interest.
- s: steer, positioned at the hip/steer joint. Moves with the boom to the wheel.
- c: contact point, moves with the contact point. Has the orientation of wheel in the plane.

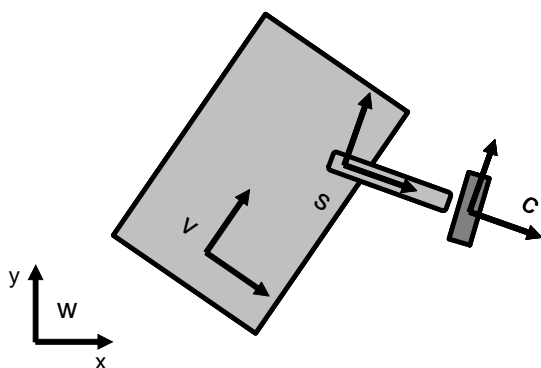


Figure 2: Frames for WMR Kinematics. The four frames necessary for the relation of wheel rotation rates and to vehicle speed and angular velocity

With the exception of rotations of wheels on their axles and around their steering axes, we will regard all vehicles in this section to be rigid bodies. Furthermore, while the wheel contact point is a point that moves on both the wheel and the floor, it is fixed with respect to the wheel frame and it can be treated as a fixed point in the wheel frame.

1.2 Vector Quantities

Vecor quantities require, in general, three objects for their precise specification. The notation that will be used consistently throughout the document can be visualized as follows:

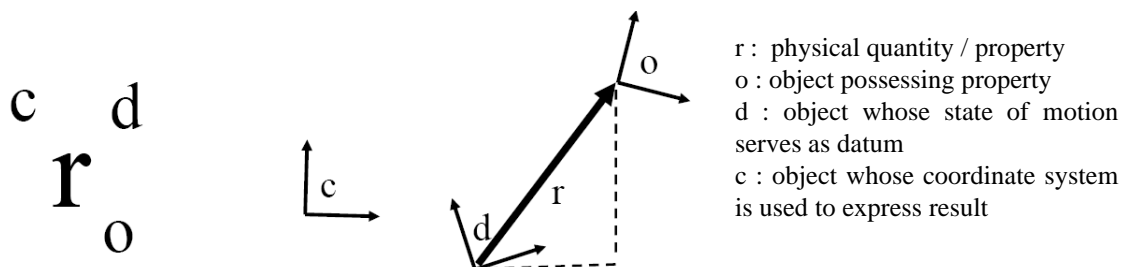


Figure 3: Notational Conventions. Letters denoting physical quantities may be adorned by designators for as many as three objects. The right subscript identifies the object to which the quantity is attributed. The right superscript identifies the object whose state of motion is used as datum. The left superscript will identify the object providing the coordinate system in which to express the quantity

When the coordinate system object is unspecified, it will be the same object as the datum and vice-versa. We will use the following conventions for specifying relationships:

- \hat{r}_a denotes the vector r property of a expressed in coordinate system independent form.
- \underline{r}_a denotes the vector r property of a expressed in the default coordinate system associated with object a . Thus $\underline{r}_a = \overset{a}{r}_a$.
- \hat{r}_a^b denotes the vector r property of a relative to b in coordinate system independent form.
- \underline{r}_a^b denotes the vector r property of a relative to b expressed in the default coordinate system associated with object b . Thus $\underline{r}_a^b = \overset{b}{r}_a^b$.
- $\overset{c}{r}_a^b$ denotes the vector r property of a relative to b expressed in the default coordinate system associated with object c .

Typically, r will be a position or displacement vector, or a velocity v . These conventions are just complicated enough to capture some of the subtly defined quantities that will be used later. Wheeled robot kinematics can often be most easily expressed in body coordinates. For example, a wheel encoder measures the velocity of the wheel relative to the earth $\overset{e}{v}_w$ and we will find it convenient to express this quantity in body coordinates $\overset{b}{v}_w^e$.

1.3 Curvature, Heading Rate, and Velocity

Consider any vehicle moving in the plane. In order to describe its motion, it is necessary to choose a reference point - the origin of the vehicle frame. This reference point is a particle moving along a path in space.

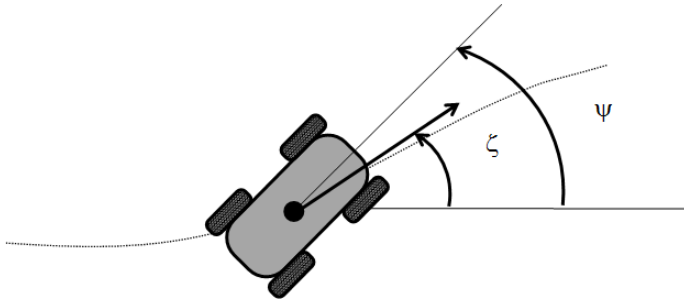


Figure 4: Distinguishing Heading from Yaw . Yaw is where a vehicle is pointing whereas heading is the direction of the velocity vector.

The *heading* ζ is the angle formed by the path tangent with the some specified datum direction fixed to the earth. Conversely, the *yaw* ψ of the vehicle body, is the angle formed by the forward looking axis of the vehicle body with an earth fixed datum. These two angles may be related or completely independent of each other. Some vehicles must go where they are pointed and some need not. If we allow lateral wheel slip to occur, then these two angles are never the same.

Curvature κ is a property of the 2D path followed by the vehicle reference point - regardless of the orientation of the vehicle during the motion:

$$\kappa = \frac{d\zeta}{ds} \quad (1)$$

The *radius of curvature* is defined as the reciprocal of curvature:

$$R = 1/\kappa \quad (2)$$

The time derivative $\dot{\zeta}$ is precisely the rate of change of heading (the direction of the velocity vector). This can be equal to the angular velocity and it may have no relationship to angular velocity. Some vehicles can move in any direction (linear velocity) and while orienting themselves arbitrarily (angular velocity)¹.

The rotation rate of the tangent vector to the path can be obtained from the chain rule of differentiation using the speed V as follows:

$$\dot{\zeta} = \frac{d\zeta ds}{ds dt} = \kappa V \quad (3)$$

1.4 The Coriolis Law

Consider two frames of reference which are rotating with respect to each one another at some *instantaneous* angular velocity ω , (of the second wrt the first). The Coriolis Law concerns the fact that two observers in relative rotation will disagree on the derivative of *any vector*. That they disagree is easy to see if we imagine that the vector is fixed with respect to one observer. In which case, the other will see a moving vector due to the relative rotation of the two frames.

It can be convenient to imagine that the two frames have coincident origins but this is not necessary

1. Some in robotics define a *holonomic* robot to be one which can do this. As the term is used in dynamics, the wheels of such vehicles are often still subject to nonholonomic constraints. Still other vehicles are truly holonomic - their wheels are not restricted from moving sideways.

in the case of free vectors like displacement, velocity, force etc. We will call the first frame *fixed* and the second *moving* - though this is completely arbitrary. We will indicate quantities measured in the first frame by an f subscript and those measured in the second by an m subscript.

It is a tedious but straightforward exercise to show that:

Law of Coriolis

$$\left(\frac{d\vec{v}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{v}}{dt}\right)_{\text{moving}} + \vec{\omega} \times \vec{v} \quad (4)$$

\vec{v} is any vector (position, velocity, acceleration, force)

$\vec{\omega}$ is ang velocity of moving frame wrt fixed one

One must be very careful when using this equation to keep the reference frames straight. The equation is used to relate time derivatives of the *same vector* computed by two observers in relative rotational motion. The vector is the same in all its instances in the equation - only the derivatives are different.

A simple example of the use of the formula is to consider a vector of constant magnitude which is rotating relative to a “fixed” observer. If we imagine a moving observer which is moving with the vector, then:

$$\left(\frac{d\vec{v}}{dt}\right)_{\text{moving}} = 0 \quad (5)$$

and the law reduces to:

$$\left(\frac{d\vec{v}}{dt}\right)_{\text{fixed}} = \vec{\omega} \times \vec{v} \quad (6)$$

Again, \vec{v} is an arbitrary vector quantity. If it represents the position vector from an instantaneous center of rotation¹ to any particle, then the law provides the velocity of the particle with respect to the fixed observer:

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} = \vec{\omega} \times \vec{r} = \vec{v}_{\text{fixed}} \quad (7)$$

1. We choose to use this point because it is fixed with respect to the fixed observer. The time derivative will then represent the velocity of the particle wrt the fixed observer.

1.5 Nonholonomic Constraints

Wheeled vehicles are almost always subject to mobility constraints that are known as *nonholonomic*. Consider a wheeled vehicle rolling without slipping in the plane:

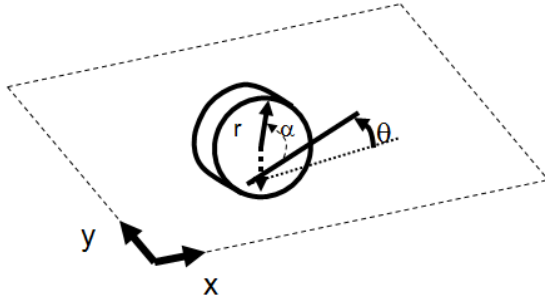


Figure 5: Nonolonomic Motion The constraint expressing that the wheel rolls without slipping cannot be integrated.

The constraint of “rolling without slipping” means that \dot{x} and \dot{y} must be consistent with the direction of rolling - they are not independent. The vector $\begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}^T$ must have a vanishing dot product with the *disallowed direction* $\begin{bmatrix} \sin\theta & -\cos\theta \end{bmatrix}^T$:

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \quad (8)$$

If we define the wheel configuration vector:

$$\underline{q} = \begin{bmatrix} x & y & \theta & \alpha \end{bmatrix}^T \quad (9)$$

and the weight vector:

$$\underline{w}(\underline{q}) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \end{bmatrix} \quad (10)$$

The constraint is of the form:

$$\underline{w}(\underline{q})\dot{\underline{q}} = 0 \quad (11)$$

Simply put, this constraint is nonholonomic because it cannot be re-written in the form:

$$h(\underline{q}) = 0 \quad (12)$$

Naturally, the process to remove $\dot{\underline{q}}$ in favor of \underline{q} would be integration - but the expression cannot be integrated to generate the above form. The integral would be:

$$\int_0^t \underline{w}(\underline{q})\dot{\underline{q}} dt = \int_0^t (\dot{x} \sin\theta(t) - \dot{y} \cos\theta(t)) dt \quad (13)$$

And the integrals of sine functions of anything more complicated than a quadratic polynomial in time have no closed form solution. In plain terms, because wheels cannot move sideways, vehicles often end up restricted to motions that do not move sideways.

2. Kinematic Steering Models

For wheeled vehicles, the transformation from the steer angles and rotation rates of the wheels onto path curvatures and linear and angular velocities can be very complicated. There can be more degrees of freedom of steer and/or drive than are necessary, creating an *overdetermined* configuration. Often, the available control inputs also map nonlinearly onto the variables of interest. Nonetheless, the velocity kinematics of wheeled mobile robots is a much more straightforward topic than the kinematics of manipulators.

2.1 Instantaneous Center of Rotation

Suppose a particle P is executing a pure rotation about a point in the plane. Clearly, since its trajectory is a circle, its position vector \vec{r}_p is given by:

$$\vec{r}_p = r(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}) \quad (14)$$

where θ is the angle that $\vec{r}_p(t)$ make with $\vec{r}_p(0)$. Since r is fixed, the particle velocity is:

$$\vec{v}_p = r\omega(-\sin(\theta)\hat{i} + \cos(\theta)\hat{j}) \quad (15)$$

... which is orthogonal to \vec{r}_p

It is easy to show that this is equivalent to:

$$\vec{v}_p = \vec{\omega} \times \vec{r}_p \quad (16)$$

And note especially that the magnitudes are related by:

$$v_p = r_p\omega \quad (17)$$

Now consider a rigid body executing a general motion in the plane.

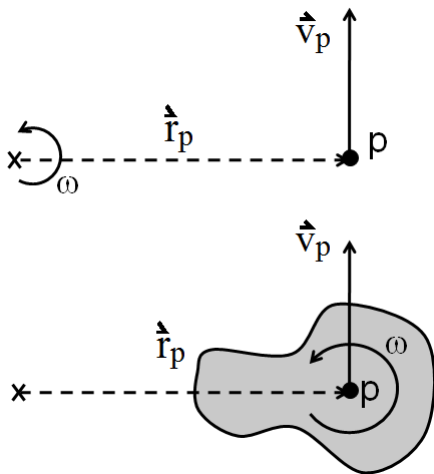


Figure 6: Pure Rotation and Instantaneous Center of Rotation . All planar motions can be described as a pure rotation about an instantaneous center of rotation. Top: A particle undergoing pure rotation. Bottom: Every particle on this body is in pure rotation about the ICR

A basic theorem of mechanics shows that all rigid body motions in the plane can be considered to be a rotation about some point - the instantaneous center of rotation (ICR). This is easy to see by

noting that the most general motion in the plane consists of a linear velocity in the plane and a rotation about the normal to the plane. Define the ratio r that relates the linear and angular velocity with respect to some world frame thus:

$$r = v_p^w / \omega \quad v_p^w = r\omega \quad (18)$$

The above equation is the equation describing the motion of the point on the object if it were rotating about a point positioned r units along the normal to the instantaneous velocity vector. In vector terms this is:

$$\dot{v}_p^{icr} = \vec{\omega} \times \dot{r}_p^{icr} \quad (19)$$

In such a case, r is called the *radius of curvature* because the curvature of its trajectory is $\kappa = 1/r$. That is true for one point. Consider a neighboring point q . The position vector to q can be written as:

$$\dot{r}_q^{icr} = \dot{r}_p^{icr} + \dot{r}_q^p \quad (20)$$

Taking time derivatives in a frame fixed to the world:

$$\left. \frac{d\dot{r}_q^{icr}}{dt} \right|_w = \left. \frac{d\dot{r}_p^{icr}}{dt} \right|_w + \left. \frac{d\dot{r}_q^p}{dt} \right|_w \quad (21)$$

But the last derivative can be rewritten in terms of a derivative taken in the body frame:

$$\left. \frac{d\dot{r}_q^p}{dt} \right|_w = \left. \frac{d\dot{r}_q^p}{dt} \right|_b + \vec{\omega} \times \dot{r}_q^p = \vec{\omega} \times \dot{r}_q^p \quad (22)$$

Hence the derivative is:

$$\dot{v}_q^{icr} = \dot{v}_p^{icr} + \vec{\omega} \times \dot{r}_q^p \quad (23)$$

Substituting for the velocity of p relative to its ICR:

$$\dot{v}_q^{icr} = \vec{\omega} \times \dot{r}_p^{icr} + \vec{\omega} \times \dot{r}_q^p = \vec{\omega} \times (\dot{r}_p^{icr} + \dot{r}_q^p) = \vec{\omega} \times \dot{r}_q^{icr} \quad (24)$$

Remarkably, the motion of an arbitrary other point q on the body is also a pure rotation about the ICR. We have shown that **all** particles on a rigid body move in a manner that can be instantaneously described as a pure rotation about a point called the ICR.

2.2 Jeantaud Diagrams

Note that fixing just the directions of the linear velocity of two points will fix the position of the ICR. In kinematic steering, we typically try to reduce or eliminate wheel skid (and the associated energy losses) by actuating all wheels in order to be consistent with a single instantaneous center of rotation. If all wheels are consistent, any two of them, and one velocity can be used to predict

the motion.

Consider 4 wheel steer in the following *Jeantaud Diagram*:

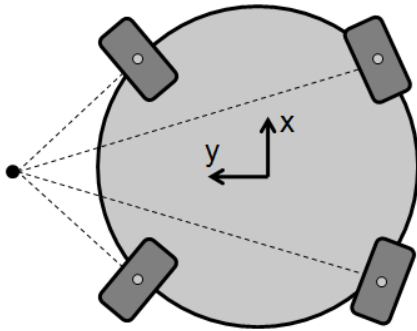


Figure 7: Jeantaud Diagram. When the wheels are set to a steering configuration that is consistent with rigid body motion, the wheels do not slip. In this configuration their axles all point to the same instantaneous center of rotation.

This vehicle can *crab steer* - move in any direction - without changing heading. Equivalently, it can be pointed in any direction while moving in any other. If there are no limits on steer angles, such schemes are extremely maneuverable - they can even spin about any point within the vehicle envelope. However, if there are limits on steer angles, there is a forbidden region where the instantaneous center of rotation (ICR) cannot lie.

3. Forward Rate Kinematics (Actuated Inverse Solution)

Consider again Figure 2. Given the linear $\dot{\mathbf{v}}_v^w$ and angular $\dot{\omega}_v^w$ velocity of the vehicle frame with respect to the world frame, the linear velocity of the wheel contact point can be computed. Clearly:

$$\dot{\mathbf{r}}_c^w = \dot{\mathbf{r}}_v^w + \dot{\mathbf{r}}_s^v + \dot{\mathbf{r}}_c^s \quad (25)$$

3.1 Offset Wheel Case

Lets differentiate this in the world frame.

$$\begin{aligned} \dot{\mathbf{v}}_c^w &= \dot{\mathbf{v}}_v^w + \left. \frac{d}{dt}(\dot{\mathbf{r}}_s^v) \right|_w + \left. \frac{d}{dt}(\dot{\mathbf{r}}_c^s) \right|_w \\ \dot{\mathbf{v}}_c^w &= \dot{\mathbf{v}}_v^w + \cancel{\dot{\mathbf{y}}_s^v} + \dot{\omega}_v^w \times \dot{\mathbf{r}}_s^v + \dot{\omega}_s^w \times \dot{\mathbf{r}}_c^s \end{aligned} \quad (26)$$

$$\dot{\mathbf{v}}_c^w = \dot{\mathbf{v}}_v^w + \dot{\omega}_v^w \times \dot{\mathbf{r}}_s^v + (\dot{\omega}_v^w + \dot{\omega}_s^w) \times \dot{\mathbf{r}}_c^s$$

This result is important enough to be called the *wheel equation*.

We also need to know rate of rotation of the steer frame with respect to the vehicle $\dot{\omega}_s^v$. This is measureable or otherwise known. The other vectors $\dot{\mathbf{r}}_s^v$ and $\dot{\mathbf{r}}_c^s$ are known vehicle dimensions. This is the solution for the general (planar) case. In the 3D case, if wheels are articulating out of the plane, the cross products, when interpreted in 3D, produce the correct wheel velocities. For example, if the vehicle frame is above the wheel frames, the cross product correctly produces the planar projection of the wheel position vector.

To use equation (26), it must be expressed in a particular coordinate system. If all vectors are expressed in *vehicle*¹ coordinates, the position vector $\dot{\mathbf{r}}_s^v$ is constant:

$$\dot{\mathbf{v}}_c^w = \dot{\mathbf{v}}_v^w + \dot{\omega}_v^w \hat{\mathbf{k}} \times \dot{\mathbf{r}}_s^v + \dot{\omega}_c^w \hat{\mathbf{k}} \times \mathbf{R}_s^v \dot{\mathbf{r}}_c^s \quad (27)$$

and its now clear that we also need the steer/caster angle² to get \mathbf{R}_s^v . This result is of the form

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \mathbf{H} \begin{bmatrix} \dot{v}_x & \dot{v}_y & \dot{\omega} & \dot{\theta} \end{bmatrix}^T \quad (28)$$

where $\dot{\theta}$ is the steer angle rate. Moreover, the relationship is actually linear in the vehicle velocities.

1. The notation $\dot{\mathbf{v}}_a^b$ means the velocity of frame a with respect to frame b (i.e $\dot{\mathbf{v}}_a^b$ expressed in the coordinates frame c. Why such a confused notion? Contact point velocities are of the wheel (a) with respect to the world (b) expressed in vehicle coordinates (c).

2. It is not unusual to find that expressing a vector equation in some coordinate system leads to a need to know more things. That's what the gyros are for in inertial guidance.

3.2 Wheel Control

The above results have computed the two components of the velocity of each wheel contact with respect to the world frame expressed in body coordinates. For any wheel, the velocity vector is oriented with respect to the body at an angle:

$$\beta_k = \text{atan2}(v_{ky}, v_{kx}) \quad (29)$$

If the wheel is steered to this angle, it will roll without slipping.

The individual speeds to be commanded of the wheels are given by the magnitudes of the velocities:

$$v_k = \sqrt{v_{kx}^2 + v_{ky}^2} = \sqrt{\underline{v}_k^T \underline{v}_k} = \sqrt{\underline{V}^T \underline{H}_k^T \underline{H}_k \underline{V}} \quad (30)$$

These can be converted to angular velocities of the wheels (about their axes) using:

$$\omega_k = v_k / r_k \quad (31)$$

where r_k is the wheel radius. Once the sign of the wheel velocity is known, the wheel steer angle is given by:

$$\theta_k = \begin{cases} \beta_k & \text{if } v_k \geq 0 \\ (\beta_k \oplus 2\pi) & \text{if } (v_k < 0) \end{cases} \quad (32)$$

where the \oplus operator indicates angle addition including a return of the result, if necessary, to the range $-\pi < \theta \leq \pi$. Also, the signs of the wheel angular velocities may need to be flipped depending on the conventions in effect for the positive sense.

3.3 Multiple Offset Wheels

The matrix \underline{H} will be called the *forward velocity Jacobian*. We can write equation (28) for each wheel and easily compute the velocity of each wheel contact point. Stacking all equations together leads to:

$$\underline{v}_c = \underline{v}_s + \underline{H}_c(\underline{\theta}) \underline{V}_c = \underline{H}_s \underline{V}_s + \underline{H}_c(\underline{\theta}) \underline{V}_c \quad (33)$$

where \underline{v}_s is the vector of steer axis linear velocities with respect to the world and the expression $\underline{H}_c(\underline{\theta}) \underline{V}_c$ captures the effect of the steer angle rates (the term $(\underline{\omega}_v + \underline{\omega}_s) \times \underline{r}_c$) on the wheel velocities. $\underline{\theta}$ is the vector of steer angles.

If the intent is to control the wheels to achieve a desired vehicle motion \underline{V}_s , then the desired steer angles must be determined. In this case, the problem can be solved by first computing the steer axis velocities with the first half of equation (33)

$$\underline{v}_s = \underline{H}_s \underline{V}_s \quad (34)$$

The consistent steer angles $\underline{\theta}$ can then be computed from these velocities as described in equation

(29). Once these angles are known the desired wheel velocities can be computed with:

$$\underline{v}_c = \underline{v}_s + H_c(\underline{\theta})\underline{V}_c \quad (35)$$

It will be necessary to know actual steer rates in \underline{V}_c . This is not a strong assumption in a control context since the steering angles will need to be controlled anyway and at least the feedback can be differentiated to produce the needed rates.

3.4 No Wheel Offsets Case

When the steer axis is coincident with the contact point, we can align frames c and s. Then and we have $\dot{\underline{r}}_c^s = 0$ and the solution reduces to:

$$\boxed{\dot{\underline{v}}_c^w = \dot{\underline{v}}_v^w + \omega \times \underline{r}_c^v} \quad (36)$$

Expressed in body coordinates, this is:

$$\boxed{\underline{v}_c^w = \underline{v}_v^w + \omega_v^w \hat{k} \times \underline{r}_c^v} \quad (37)$$

This result is of the form:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = H \begin{bmatrix} V_x \\ V_y \\ \omega \end{bmatrix} \quad (38)$$

3.5 Multiple Wheels with No Offsets

When the wheels are not offset from their steer axes, the above result reduces to:

$$\underline{v}_c = \underline{v}_s = H_s \underline{V}_s \quad (39)$$

4. Inverse Rate Kinematics (Sensed Forward Solution)

Now for the opposite problem. Suppose we have measurements of wheel angular velocities ω_k and steer angles. We want to find the associated vehicle linear and angular velocity. Once again, the forward solution for the steer axis velocities for any wheel is:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = H \begin{bmatrix} V_x & V_y & \omega \end{bmatrix}^T \quad \underline{v}_s = H_s \underline{V}_s \quad (40)$$

Which is in the form of an observer. The unknowns are $\begin{bmatrix} V_x & V_y & \omega \end{bmatrix}^T$ in this case.

4.1 Wheel Sensing

Typically, the wheel angular velocity and steer angles are sensed more or less directly. Under this assumption, the linear velocities of the wheels are derived by inverting equation (31):

$$v_k = r_k \omega_k \quad (41)$$

The wheel velocity components in the body frame are then:

$$v_{kx} = v_k \cos(\theta_k) \quad v_{ky} = v_k \sin(\theta_k)$$

4.2 Multiple Wheels with No Offset

Usually, there are three unknowns and only two constraints are provided per wheel. Clearly, the linear velocity of one point on a rigid body does not constrain all 3 dof of planar motion.

Therefore, we need at least two sets of wheel (steer, speed) measurements in general. However, the problem then becomes overdetermined and a best fit solution is called for which tolerates the inevitable inconsistency in the measurements.

$$\underline{V}_s = [H_s^T H_s]^{-1} H_s^T \underline{v}_s \quad (42)$$

This solution produces the linear and angular velocities which are most consistent with the measurements even if the measurements do not agree.

4.3 Multiple Offset Wheels

When the wheel contact points are offset from the steering axes (s frame origins), then the steer axis velocities can be determined by solving equation (35)

$$\underline{v}_s = \underline{v}_c - H_c(\theta) \underline{V}_c \quad (43)$$

and these velocities can be substituted above in equation (42).

5. Examples

5.1 Differential Steer

The case of differential steer is pretty easy:

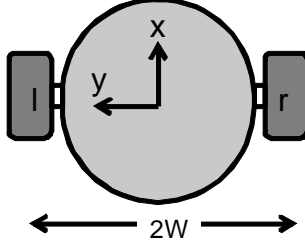


Figure 8: Differential Steer In this very simple configuration, there are two fixed wheels that are unable to steer.

5.1.1 Forward Kinematics

Let the two wheel frames be called l and r for left and right. Let the *forward* velocity be V and the angular velocity is ω . The dimensions are:

$$\mathbf{r}_r^v = \begin{bmatrix} 0 & W \end{bmatrix}^T \quad \mathbf{r}_l^v = \begin{bmatrix} 0 & -W \end{bmatrix}^T \quad (44)$$

Written in the body frame, equation (38) for each wheel reduces to two scalar equations for the velocities in the x direction:

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} V + \omega W \\ V - \omega W \end{bmatrix} = \begin{bmatrix} 1 & W \\ 1 & -W \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix} \quad (45)$$

where V is the velocity of the robot center in the x direction. The equation for the wheel velocities in the *sideways* direction vanishes (when expressed in body coordinates), if we assume that the wheels cannot move sideways. Essentially, the nonholonomic constraint is substituted into the kinematics to eliminate one unknown. We are therefore able to solve for the (normally 3 dof) motion using two measurements.

5.1.2 Inverse Kinematics

In body coordinates, the constraints for two wheels generate two scalar equations. equation (45) is easy to invert:

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} 1 & W \\ 1 & -W \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix} \quad \begin{bmatrix} V \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} v_r \\ v_l \end{bmatrix} \quad (46)$$

This case was a special one since the equations were not overdetermined.

5.2 Ackerman Steer

This is the configuration of the ordinary automobile:

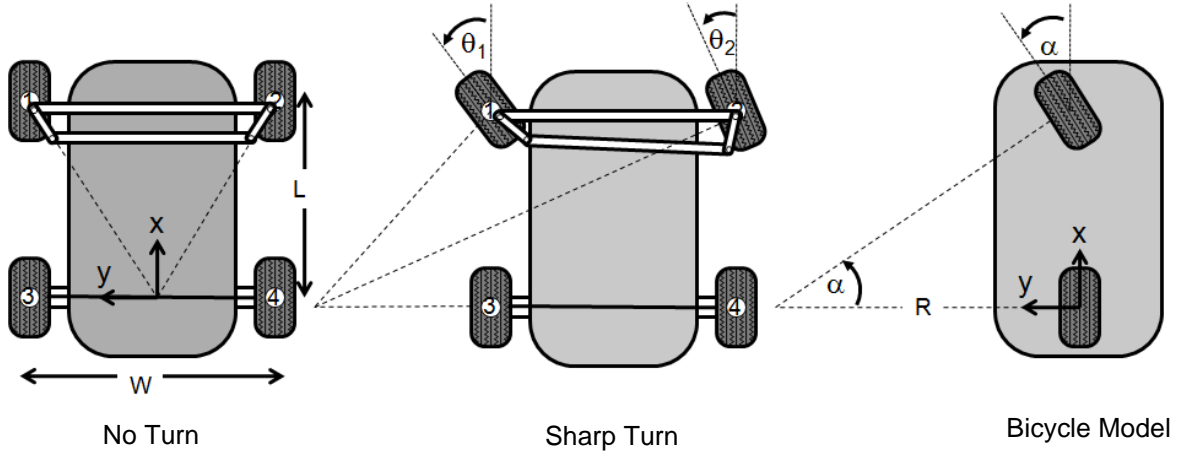


Figure 9: Ackerman Steer. The two front wheels are connected by a mechanism that causes the two front wheels to turn through increasingly different angles as the steering wheel turns. In this way, wheel slip is minimized.

We can model this vehicle by just two wheels as described in figure 9. This is called the *bicycle model*.

5.2.1 Forward Kinematics

In this case, it is convenient to place the vehicle frame at the center of the rear axle. Let the forward velocity of the vehicle frame be denoted V and the angular velocity is ω . The position vector to the front wheel is:

$$\mathbf{r}_f^v = \begin{bmatrix} L & 0 \end{bmatrix}^T \quad (47)$$

Written in the body frame, equation (36) for this wheel reduces to:

$$\begin{aligned} \mathbf{v}_{c1}^v &= \mathbf{v}_v^w + \omega_v^w \hat{\mathbf{k}} \times \mathbf{r}_c^v \\ \mathbf{v}_{f1}^w &= \mathbf{v}_v^w + \omega_v^w \hat{\mathbf{k}} \times (L\hat{\mathbf{i}} + 0\hat{\mathbf{j}}) \Rightarrow \mathbf{v}_1 = \begin{bmatrix} V & \omega L \end{bmatrix}^T \end{aligned} \quad (48)$$

If we define $\mathbf{V} = \begin{bmatrix} V & \omega \end{bmatrix}^T$ this is of the form $\mathbf{v}_1^w = \mathbf{H}_k \mathbf{V}$:

$$\begin{bmatrix} v_{fx} \\ v_{fy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix} \quad (49)$$

The velocity vector of the front wheel is oriented with respect to the body at an angle:

$$\alpha = \text{atan2}(\omega L, V) \quad (50)$$

or:

$$\tan(\alpha) = \frac{\omega L}{V} = \kappa L = \frac{L}{R} \quad (51)$$

where R is the instantaneous radius of curvature. This result relating curvature to steer angle could have been derived by geometry from the figure. An equivalent formula for curvature can be generated for any two wheels of a vehicle that are not slipping.

5.2.2 Inverse Kinematics

For this configuration, equation 49 became a diagonal matrix, decoupling linear and angular velocity, due to the choice of the rear wheel for the vehicle reference point. This makes the inverse kinematics particularly trivial.

$$\begin{bmatrix} V \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix}^{-1} \begin{bmatrix} v_{fx} \\ v_{fy} \end{bmatrix} = \begin{bmatrix} v_{fx} \\ v_{fy}/L \end{bmatrix} \quad (52)$$

5.3 Generalized Bicycle Model

It should be clear from the discussion of the ICR that any wheeled vehicle whose wheels roll without slipping moves in a manner that can be described by just two wheels with the same ICR as the original vehicle. Consider the following generalized bicycle model:

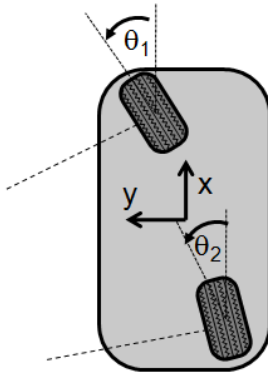


Figure 10: Generalized Bicycle Model. Any vehicle whose wheels roll without slipping can be modelled by just two wheels

5.3.1 Forward Kinematics

The model is simpler if the body frame is placed on one of the wheels but this is not always possible so the wheel positions are defined more generally. The wheel positions are:

$$\mathbf{r}_1^v = \begin{bmatrix} x_1 & y_1 \end{bmatrix}^T \quad \mathbf{r}_2^v = \begin{bmatrix} x_2 & y_2 \end{bmatrix}^T \quad (53)$$

The wheel velocities are by equation (37):

$$\begin{aligned}\underline{v}_i^w &= \underline{v}_v^w + \omega_v^w \hat{k} \times \underline{r}_i^v \\ \underline{v}_i^w &= \underline{v}_v^w + \omega_v^w \hat{k} \times (x_i \hat{i} + y_i \hat{j}) \Rightarrow v_i = \begin{bmatrix} V_x - \omega y_i & V_y + \omega x_i \end{bmatrix}^T\end{aligned}\quad (54)$$

Which is:

$$\begin{aligned}\underline{v}_s &= H_s \underline{V}_s \\ \begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{2x} \\ v_{2y} \end{bmatrix} &= \begin{bmatrix} V_x - \omega y_1 \\ V_y + \omega x_1 \\ V_x - \omega y_2 \\ V_y + \omega x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ \omega \end{bmatrix}\end{aligned}\quad (55)$$

5.3.2 Inverse Kinematics

This case requires the general solution for nonoffset wheels:

$$\underline{V}_s = [H_s^T H_s]^{-1} H_s^T \underline{v}_s \quad (56)$$

5.4 Four Wheel Steer

This is the case of 4 independently steerable wheels. Subject to any limits on steer angles, this vehicle configuration is very maneuverable. It can turn in place and it can drive in any direction while facing another direction:

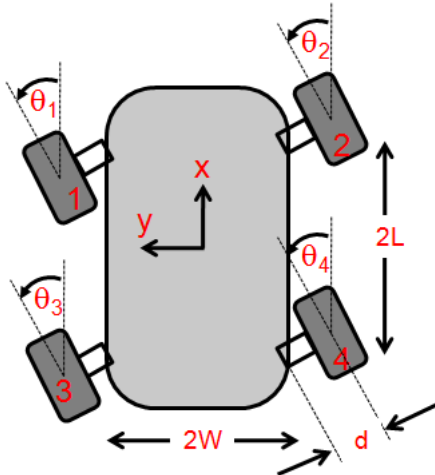


Figure 11: 4-Wheel Steer. All four wheels are steerable on this vehicle. As a result, it can turn in place or drive in any direction regardless of where it is pointing. This figure suggests driving slightly left while pointing forward. The wheel contact points are offset by a distance d from the steer centers

A related configuration is *double Ackerman*. In this case, there is an Ackerman mechanism at both the front and the rear and the vehicle can turn through smaller radii than single Ackerman as a result.

5.4.1 Forward Kinematics

For the four wheel steer configuration. Let the wheel frames be identified by numbers as shown. The centers of their contact points are assumed to be in the center of their vertical projections. Let the *forward* velocity of the vehicle frame be denoted V and the angular velocity is ω .

The steer center position vectors are:

$$\begin{aligned} \mathbf{r}_{sk}^v &= \begin{bmatrix} x_k & y_k \end{bmatrix}^T & \mathbf{r}_{s1}^v &= \begin{bmatrix} L & W \end{bmatrix}^T & \mathbf{r}_{s2}^v &= \begin{bmatrix} L & -W \end{bmatrix}^T \\ \mathbf{r}_{s3}^v &= \begin{bmatrix} -L & W \end{bmatrix}^T & \mathbf{r}_{c4}^v &= \begin{bmatrix} -L & -W \end{bmatrix}^T \end{aligned} \quad (57)$$

The contact point offsets in the body frame depend on the steer angles. They are:

$$\begin{aligned} \mathbf{r}_{ck}^{sk} &= \begin{bmatrix} a_k & b_k \end{bmatrix}^T & \mathbf{r}_{c1}^{s1} &= d \begin{bmatrix} -s\theta_1 & -c\theta_1 \end{bmatrix}^T & \mathbf{r}_{c2}^{s2} &= d \begin{bmatrix} s\theta_2 & c\theta_2 \end{bmatrix}^T \\ \mathbf{r}_{c3}^{s3} &= d \begin{bmatrix} -s\theta_3 & -c\theta_3 \end{bmatrix}^T & \mathbf{r}_{c4}^{s4} &= d \begin{bmatrix} s\theta_4 & c\theta_4 \end{bmatrix}^T \end{aligned} \quad (58)$$

Written in the body frame, equation (26) for each wheel reduces to eight scalar equations of the form:

$$\begin{aligned} \dot{\mathbf{v}}_c^w &= \dot{\mathbf{v}}_v^w + \dot{\omega}_v^w \times \mathbf{r}_s^v + (\dot{\omega}_v^w + \dot{\omega}_s^v) \times \mathbf{r}_c^s \\ \mathbf{v}_{i1}^w &= \mathbf{v}_v^w + \omega_v^w \hat{k} \times (x_k \hat{i} + y_k \hat{j}) + (\omega_v^w + \omega_s^v) \hat{k} \times (a_k \hat{i} + b_k \hat{j}) \\ \mathbf{v}_i^w &= \begin{bmatrix} V_x - \omega y_k - (\omega + \dot{\theta}_i) b_k & V_y + \omega x_k + (\omega + \dot{\theta}_k) a_k \end{bmatrix}^T \end{aligned} \quad (59)$$

The problem is much simplified by splitting the calculation into two parts. If we define the two state vectors:

$$\mathbf{V}_s = \begin{bmatrix} V_x & V_y & \omega \end{bmatrix}^T \quad \mathbf{V}_c = \begin{bmatrix} \omega & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 \end{bmatrix}^T \quad (60)$$

These are all of the form $\underline{v}_k^w = H_{sk}\underline{V}_s + H_{ck}(\underline{\theta})\underline{V}_c$ where $\underline{\theta}$ is the vector of the four steer angles:

$$\begin{aligned} \underline{v}_i^w &= \begin{bmatrix} V_x - \omega y_i - (\omega + \dot{\theta}_i)b_i & V_y + \omega x_i + (\omega + \dot{\theta}_i)a_i \end{bmatrix}^T \\ \begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{2x} \\ v_{2y} \\ v_{3x} \\ v_{3y} \\ v_{4x} \\ v_{4y} \end{bmatrix} &= H_{s1}\underline{V}_s + H_{c1}\underline{V}_c = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ 1 & 0 & -y_3 \\ 0 & 1 & x_3 \\ 1 & 0 & -y_4 \\ 0 & 1 & x_4 \end{bmatrix} \underline{V}_s + \begin{bmatrix} -b_1 & -b_1 & 0 & 0 & 0 \\ a_1 & a_1 & 0 & 0 & 0 \\ -b_2 & 0 & -b_2 & 0 & 0 \\ a_2 & 0 & a_2 & 0 & 0 \\ -b_3 & 0 & 0 & -b_3 & 0 \\ a_3 & 0 & 0 & a_3 & 0 \\ -b_4 & 0 & 0 & 0 & -b_4 \\ a_4 & 0 & 0 & 0 & a_4 \end{bmatrix} \underline{V}_c \end{aligned} \quad (61)$$

Stacking all equations together leads to:

$$\underline{v} = \underline{v}_s + H_c \underline{V}_c = H_s \underline{V}_s + H_c \underline{V}_c \quad (62)$$

where:

$$\begin{aligned} \underline{V}_s &= \begin{bmatrix} V_x & V_y & \omega \end{bmatrix}^T \quad \underline{V}_c = \begin{bmatrix} \omega & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 \end{bmatrix}^T \\ \underline{v} &= \begin{bmatrix} v_{1x} & v_{1y} & v_{2x} & v_{2y} & v_{3x} & v_{3y} & v_{4x} & v_{4y} \end{bmatrix}^T \\ H_s &= \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ 1 & 0 & -y_3 \\ 0 & 1 & x_3 \\ 1 & 0 & -y_4 \\ 0 & 1 & x_4 \end{bmatrix} \quad H_c = \begin{bmatrix} -b_1 & -b_1 & 0 & 0 & 0 \\ a_1 & a_1 & 0 & 0 & 0 \\ -b_2 & 0 & -b_2 & 0 & 0 \\ a_2 & 0 & a_2 & 0 & 0 \\ -b_3 & 0 & 0 & -b_3 & 0 \\ a_3 & 0 & 0 & a_3 & 0 \\ -b_4 & 0 & 0 & 0 & -b_4 \\ a_4 & 0 & 0 & 0 & a_4 \end{bmatrix} \end{aligned} \quad (63)$$

The steer angles and steer rates $\underline{\theta}$ would be known in an estimation or control context, so the above solution computes the velocities of the wheels based on this information.

5.4.2 Inverse Kinematics

The inverse solution follows the general case described in Section 4.3. First, the steer axis velocities are computed with:

$$\underline{v}_s = \underline{v}_c - H_c(\theta)\underline{V}_c \quad (64)$$

Then, the vehicle velocity is computed with:

$$\underline{V}_s = [H_s^T H_s]^{-1} H_s^T \underline{v}_s \quad (65)$$

This solution produces the linear and angular velocities which are most consistent with the measurements even if the measurements do not agree.