Assignment 2 (ML for TS) - MVA 2022/2023

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February 27, 2023

1 Introduction

Objective. The goal is to better understand the properties of AR and MA processes, and do signal denoising with sparse coding.

Warning and advice.

- Use code from the tutorials as well as from other sources. Do not code yourself well-known procedures (e.g. cross validation or k-means), use an existing implementation.
- The associated notebook contains some hints and several helper functions.
- Be concise. Answers are not expected to be longer than a few sentences (omitting calculations).

Instructions.

- Fill in your names and emails at the top of the document.
- Hand in your report (one per pair of students) by Monday 27$^{\text{th}}$ February 11:59 PM.
- Rename your report and notebook as follows:
  `FirstnameLastname1_FirstnameLastname1.pdf` and
  `FirstnameLastname2_FirstnameLastname2.ipynb`.
  For instance, `LaurentOudre_CharlesTruong.pdf`.
- Upload your report (PDF file) and notebook (IPYNB file) using this link: .

2 General questions

A time series $\{y_t\}_t$ is a single realisation of a random process $\{Y_t\}_t$ defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e. $y_t = Y_t(w)$ for a given $w \in \Omega$. In classical statistics, several independent realisations are often needed to obtain a “good” estimate (meaning consistent) of the parameters of the process. However, thanks to a stationarity hypothesis and a "short-memory" hypothesis, it is still possible to make “good” estimates. The following question illustrates this fact.

Question 1

An estimator $\hat{\theta}_n$ is consistent if it converges in probability when the number n of samples grows to ∞ to the true value $\theta \in \mathbb{R}$ of a parameter, i.e. $\hat{\theta}_n \xrightarrow{\mathcal{D}} \theta$.

- Recall the rate of convergence of the sample mean for i.i.d. random variables with finite variance.
- Let $\{Y_t\}_{t\geq 1}$ a wide-sense stationary process such that $\sum_k |\gamma(k)| < +\infty$. Show that the sample mean $\bar{Y}_n = (Y_1 + \cdots + Y_n)/n$ is consistent and enjoys the same rate of convergence as the i.i.d. case. (Hint: bound $\mathbb{E}[(\bar{Y}_n - \mu)^2]$ with the $\gamma(k)$ and recall that convergence in L_2 implies convergence in probability.)

Answer 1

- From Bienaymé-Tchebychev, the convergence rate is at least $\frac{1}{n}$:

$$\mathbb{P}(|S_n - \mathbb{E}[Y_1]| > \epsilon) \leq \frac{V(S_n)}{\epsilon^2} = \frac{V(Y_1)}{n\epsilon^2}$$

- $$\mathbb{E}[(\bar{Y}_n - \mu)^2] = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[(Y_i - \mu)(Y_j - \mu)] = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma(j-i)$$

  We do the change of variables $k = j - i$:

$$\mathbb{E}[(\bar{Y}_n - \mu)^2] = \frac{1}{n^2}\sum_{k=-(n-1)}^{(n-1)}\sum_{j=max(1,1+k)}^{min(n,n+k)}\gamma(k)$$

  Whether k is negative or positive there are $n - |k|$ terms in the sum:

$$\mathbb{E}[(\bar{Y}_n - \mu)^2] = \frac{1}{n}\sum_{k=-(n-1)}^{(n-1)}(1 - \frac{|k|}{n})\gamma(k) \leq \frac{1}{n}\sum_{k\in\mathbb{Z}}|\gamma(k)|$$

  Hence using Markov inequality :

$$\boxed{\mathbb{P}(|\bar{Y}_n - \mathbb{E}[Y_1]| > \epsilon) \leq \frac{\sum_{k\in\mathbb{Z}}|\gamma(k)|}{n\epsilon^2}}$$

3 AR and MA processes

Question 2 *Infinite order moving average MA(∞)*

Let $\{Y_t\}_{t\geq 0}$ be a random process defined by

$$Y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} \tag{1}$$

where $(\psi_k)_{k\geq 0} \subset \mathbb{R}$ ($\psi = 1$) are square summable, i.e. $\sum_k \psi_k^2 < \infty$ and $\{\varepsilon_t\}_t$ is a zero mean white noise of variance σ_ε^2. (Here, the infinite sum of random variables is the limit in L_2 of the partial sums.)

- Derive $\mathbb{E}(Y_t)$ and $\mathbb{E}(Y_t Y_{t-k})$. Is this process weakly stationary?
- Show that the power spectrum of $\{Y_t\}_t$ is $S(f) = \sigma_\varepsilon^2 |\phi(e^{-2\pi i f})|^2$ where $\phi(z) = \sum_j \psi_j z^j$. (Assume a sampling frequency of 1 Hz.)

The process $\{Y_t\}_t$ is a moving average of infinite order. Wold's theorem states that any weakly stationary process can be written as the sum of the deterministic process and a stochastic process which has the form (1).

Answer 2

- As we have an infinite sum, we first need to prove that we can invert the infinite sum and the expectation.

  Going back to the partial sums $S_N(t) = \sum_{k=0}^{N} \psi_k \epsilon_{t-k}$, the convergence in L_2 is defined as $\mathbb{E}[(S_N(t) - Y_t)^2] \to 0$. So we write :

  $$|\mathbb{E}[Y_t] - 0| = |\mathbb{E}[Y_t - S_N(t)]| \leq \sqrt{\mathbb{E}[(Y_t - S_N(t))^2]} \to 0$$

  The first equality comes from the fact that the noise has zero mean so the partial sums have also zero mean. The second inequality comes from the Cauchy-Schwarz inequality. Hence $\boxed{\mathbb{E}[Y_t] = 0}$.

  Moreover. If $x_n \to x$ and $x'_n \to x'$ we should write :

  $$x_n x'_n - x x' = (x_n - x)(x'_n - x') + x'(x_n - x) + x(x'_n - x')$$

  So if we take $x_N = S_N(t), x'_N = S_N(t-k)$ and $x = Y_t, x' = Y_{t-k}$ and then take the expectation:

  $$\mathbb{E}[S_N(t)S_N(t-k) - Y_t Y_{t-k}] = \mathbb{E}[(S_N(t) - Y_t)(S_N(t-k) - Y_{t-k})] + \mathbb{E}[Y_{t-k}(S_N(t) - Y_t)]$$
  $$+ \mathbb{E}[Y_t(S_N(t-k) - Y_{t-k})]$$

  Using C-S inequality on the three terms leads to :

  $$\mathbb{E}[Y_t Y_{t-k}] = \lim_N \mathbb{E}[S_N(t) S_N(t-k)]$$

$$\mathbb{E}[Y_t Y_{t-k}] = \lim_N \sum_{n=0}^{N}\sum_{m=0}^{N} \psi_n \psi_m \mathbb{E}[\epsilon_{t-n}\epsilon_{t-k-m}]$$

The expectation in the sum is zero if we don't have $t-n = t-k-m$ ie $m = n-k$ (only possible if $n \geq k$). So :

$$\boxed{\mathbb{E}[Y_t Y_{t-k}] = \sigma_\epsilon^2 \sum_{n=max(0,k)}^{\infty} \psi_n \psi_{n-k}}$$

Because the above quantities don't depend on t, the process is weakly stationary.

- $S(f) = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) e^{-2i\pi f\tau/f_s} = \sum_{\tau=-\infty}^{+\infty} \mathbb{E}[Y_t Y_{t+\tau}] e^{-2i\pi f\tau}$

$$= \sum_{\tau=-\infty}^{+\infty} \mathbb{E}[Y_t Y_{t-\tau}] e^{2i\pi f\tau}$$

$$= \sum_{\tau=-\infty}^{+\infty} \sigma_\epsilon^2 \sum_{n=max(0,\tau)}^{\infty} \psi_n \psi_{n-\tau} e^{2i\pi f\tau}$$

$$= \sigma_\epsilon^2 \sum_{n=0}^{\infty}\sum_{\tau=-\infty}^{n} \psi_n \psi_{n-\tau} e^{2i\pi f\tau}$$

$$= \sigma_\epsilon^2 \sum_{n=0}^{\infty}\sum_{\tau=-\infty}^{n} \psi_n (e^{2i\pi f})^n \psi_{n-\tau} (e^{2i\pi f})^{\tau-n}$$

By changing index $k = n-\tau$ in the second sum :

$$= \sigma_\epsilon^2 \sum_{n=0}^{\infty} \psi_n (e^{2i\pi f})^n \sum_{k=0}^{+\infty} \psi_k (e^{2i\pi f})^{-k}$$

$$= \sigma_\epsilon^2 |\sum_{n=0}^{\infty} \psi_n (e^{2i\pi f})^n|^2$$

So $\boxed{S(f) = \sigma_\epsilon^2 |\phi(e^{-2i\pi f})|^2}$

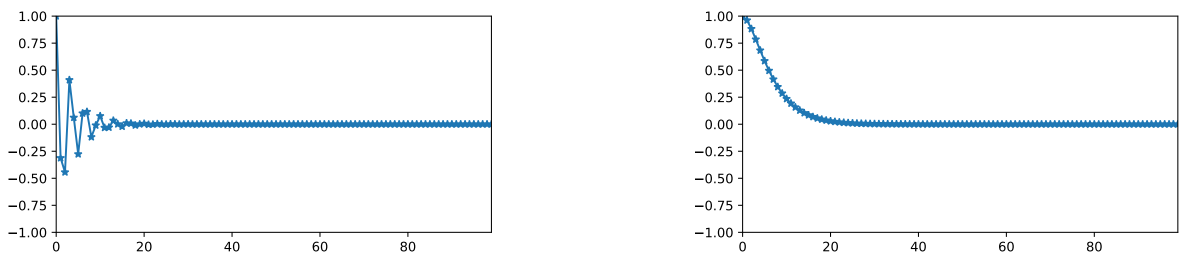
Question 3 *AR(2) process*

Let $\{Y_t\}_{t\geq 1}$ be an AR(2) process, i.e.

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \tag{2}$$

with $\phi_1, \phi_2 \in \mathbb{R}$. The associated characteristic polynomial is $\phi(z) := 1 - \phi_1 z - \phi_2 z^2$. Assume that ϕ has two distinct roots (possibly complex) r_1 and r_2 such that $|r_i| > 1$. Properties on the roots of this polynomial drive the behaviour of this process.

- Express the autocovariance coefficients $\gamma(\tau)$ using the roots r_1 and r_2.
- Figure 1 shows the correlograms of two different AR(2) processes. Can you tell which one has complex roots and which one has real roots?
- Express the power spectrum $S(f)$ (assume the sampling frequency is 1 Hz) using $\phi(\cdot)$.
- Choose ϕ_1 and ϕ_2 such that the characteristic polynomial has two complex conjugate roots of norm $r = 1.05$ and phase $\theta = 2\pi/6$. Simulate the process $\{Y_t\}_t$ (with $n = 2000$) and display the signal and the periodogram (use a smooth estimator) on Figure 2. What do you observe?



Correlogram of the first AR(2) — Correlogram of the second AR(2)

Figure 1: Two AR(2) processes

Answer 3

- Here we are assuming that the roots are outside the unit circle, which means that the process is weakly stationary. This implies also that the mean is zero. We have :

$$\gamma(\tau) = \mathbb{E}[Y_t Y_{t+\tau}] = \mathbb{E}[Y_t(\phi_1 Y_{t+\tau-1} + \phi_2 Y_{t+\tau-2} + \epsilon_{t+\tau})]$$

- For $\tau = 0$, $\gamma(0) = \phi_1\gamma(-1) + \phi_2\gamma(-2) + \sigma^2$
- For $\tau \geq 1$, $\gamma(\tau) - \phi_1\gamma(\tau-1) - \phi_2\gamma(\tau-2) = 0$

  Hence the sequence can be solved with the roots of the characteristic polynomial $z^2\phi(z^{-1})$, first for $\tau \geq 0$ and then for $\tau < 0$ by symmetrization :

$$\gamma(\tau) = a_1(\frac{1}{r_1})^{|\tau|} + a_2(\frac{1}{r_2})^{|\tau|}$$

  We then need two initial conditions to find the parameters a_1 and a_2.

- The first correlogram shows oscillations, so the roots are complex (they combine in an amortized cosinus), while the second is just an exponential decrease so the roots are real.

- Let's define the operator $L : Y_t \to Y_{t-1}$. Using the recurrence formula above we have :

$$\phi(L)Y_t = \epsilon_t$$

  Then because $\phi(z) = (z - r_1)(z - r_2) = r_1 r_2 (1 - \frac{z}{r_1})(1 - \frac{z}{r_2})$, we have that :

$$\phi^{-1}(z) = \frac{1}{r_1 r_2 (1 - \frac{z}{r_1})(1 - \frac{z}{r_2})} = \frac{1}{r_1 r_2}(\sum_{i=0}^{+\infty}(\frac{z}{r_1})^i)(\sum_{j=0}^{+\infty}(\frac{z}{r_2})^i) \text{ for z < min}(r_1, r_2)$$

  That way we can invert the equality and get :

$$Y_t = \phi^{-1}(L)\phi(L)Y_t = \phi^{-1}(L)\epsilon_t = \frac{1}{r_1 r_2}(\sum_{i=0}^{+\infty}\frac{L^i}{(r_1)^i})(\sum_{j=0}^{+\infty}\frac{L^j}{(r_2)^j})\epsilon_t = \frac{1}{r_1 r_2}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\epsilon_{t-i-j}}{(r_1)^i (r_2)^j}$$

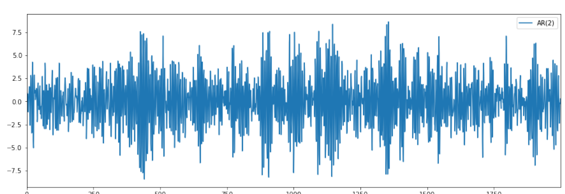
  Using the formula for the power spectrum for an MA(∞) process, we have ;

$$S(f) = \sigma^2 |\frac{1}{r_1 r_2}(\sum_{k=0}^{+\infty}(\frac{e^{2i\pi f}}{r_1})^k)(\sum_{j=0}^{+\infty}(\frac{e^{2i\pi f}}{r_2})^j)|^2 = \frac{\sigma^2}{|\phi(e^{2i\pi f})|^2}$$

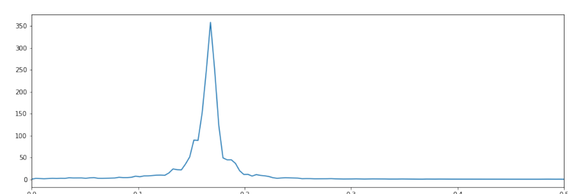
- $\phi = 1 - \phi_1 X - \phi_2 X^2$ has the same roots as $X^2 + \frac{\phi_1}{\phi_2}X - \frac{1}{\phi_2}$.

  Using the roots-coefficients relations, we have that :

$$\begin{cases} -\frac{\phi_1}{\phi_2} = r_1 + r_2 = 2x1.05\cos(\frac{2\pi}{6}) \\ -\frac{1}{\phi_2} = r_1 r_2 = 1.05^2 \end{cases} \Leftrightarrow \begin{cases} \phi_1 = \frac{2.1}{1{,}05^2}\cos(\frac{2\pi}{6}) \\ \phi_2 = -1.05^{-2} \end{cases}$$



Signal



Periodogram

Figure 2: AR(2) process

- We can observe the stationary of the process as well as a non negligeable variance. Furthermore we can observe a frequence at about 0.16 of the sampling frequency.

4 Sparse coding

The modulated discrete cosine transform (MDCT) is a signal transformation often used in sound processing applications (for instance to encode a MP3 file). A MDCT atom $\phi_{L,k}$ is defined for a length 2L and a frequency localisation k ($k = 0, \ldots, L-1$) by

$$\forall u = 0, \ldots, 2L-1, \quad \phi_{L,k}[u] = w_L[u]\sqrt{\frac{2}{L}}\cos[\frac{\pi}{L}\left(u + \frac{L+1}{2}\right)(k + \frac{1}{2})] \tag{3}$$

where w_L is a modulating window given by

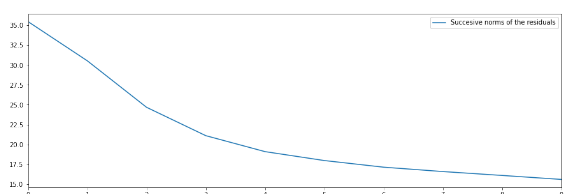
$$w_L[u] = \sin\left[\frac{\pi}{2L}\left(u + \frac{1}{2}\right)\right]. \tag{4}$$

Question 4 *Sparse coding with OMP*

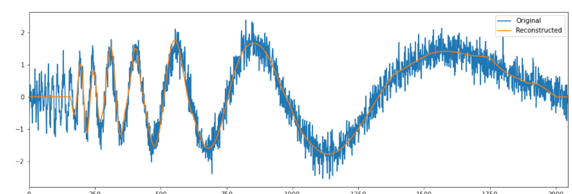
For the signal provided in the notebook, learn a sparse representation with MDCT atoms. The dictionary is defined as the concatenation of all shifted MDCDT atoms for scales L in $[32, 64, 128, 256, 512, 1024]$.

- For the sparse coding, implement the Orthogonal Matching Pursuit (OMP). (Use convolutions to compute the correlations coefficients.)
- Display the norm of the successive residuals and the reconstructed signal with 10 atoms.

Answer 4



Norms of the successive residuals



Reconstruction with 10 atoms

Figure 3: Question 4