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## THEORY OF PARAMETRIC ACOUSTIC ARRAYS

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Résumé. - Ce travail est une contribution à la théorie de l'antenne paramétrique formée par l'interaction nonlinéaire de deux ondes sonores dans un fluide thermovisqueux. La source des faisceaux primaires est un piston circulaire. L'attention est principalement portée sur l'analyse des propriétés à grande distance de la source de celle des composantes du champ engendré qui a une fréquence égale à la différence des fréquences des faisceaux primaires.

Les effets dus à l'interaction qui se produit dans les régions de divergence des ondes primaires sont comparés aux effets produits par l'interaction de deux ondes primaires considérées comme planes et collimées à faible distance de la source, et comme divergentes à grande distance de celle-ci.

Différents modèles sont passés en revue. La discussion de leurs domaines de validité fait l'objet d'une attention particulière. Les résultats théoriques et expérimentaux sont comparés.

Abstract. - The paper is a contribution to the theory of the parametric acoustic array formed by the nonlinear interaction of sound waves in a thermoviscous fluid. The source of the primary waves is a circular piston, and emphasis is put on an analysis of the farfield properties of the generated difference frequency sound field from two sound beams.

Effects from interaction in the divergence regions are studied and compared with the effects from a model composed of a plane collimated nearfield region of interaction, and a farfield divergence region.

Previous models are analysed with emphasis on a discussion of the range of validity. Theoretical results are compared with experimental observations.

1. Introduction. - There exists today an extensive literature concerned with the problem of mutual nonlinear interaction between two sound beams. The theory of parametric acoustic array has been treated by so many authors since the first paper was published by Westervelt in 1863, that there would seem by now to be little to add on this point. However, there still appear new results of interest.

Reviewing available experimental observations (see table I) we find that these are obtained under various conditions, and direct comparison between the results are therefore difficult. Often, for example, the observations are so near the sources that any comparison with the existing asymptotic theory is not justified. Further, theory valid within the interaction region, very near the source

Table I. - The experimental conditions of parametric arrays with circular primary sound source, reported in the literature.

Author	$f_a$ (kHz)	$f_-$ (kHz)	$a$ (cm)	Type of medium	$T$ (°C)	$L_A$ (m)	$R_{obs}$ (m)	$N_F$	$N_D$	$(\xi/\xi^W)_{3dB}$	$R_a/L_s$	Domain of $f_-$ (kHz)
Bennett et al. (1974)	21.1	5	2.92	Air	-	8.8	0.61	4.90	2.25	2.80	0.01	
Bjørnø et al. (1976)	910	40	1.0	Brackish water	15*	18.4 <sup>+</sup>	10	7.1	1.49	1.52	0.2	
Eller (1973)	1460	50	1.0	Fresh water	-	12.5	3.7	4.49	0.83	1.28	0.21	25-200
Esipov et al. (1975)	107.5	5	46	Sea water	-	130 <sup>+</sup>	80	1.2	0.25	1.32	0.04	1- 5
Hobaek et al. (1977)	4270	100	0.25	Fresh water	22	1.12	2	3.2	0.48	0.80	<0.04	24-500
Moffett et al. (1976)	270	50	5.1	Sea water	30	54	85	4.56	1.96	2.07	1.60	
Muir et al. (1972)	482	64	3.81	Fresh water	12.8*	76.5*	130	5.04	1.84	2.24	0.08	
Smith (1971)	3500	325	0.25	Fresh water	-	1.63	4.5	4.21	1.28	1.57	-	80-540

+ estimated quantities.

\* information obtained from other sources.

$L_s$  = shock formation distance.

of the carrier (primary) waves, can only cover a few of the observations. To illustrate some main features of the observations, we have in fig. 1.

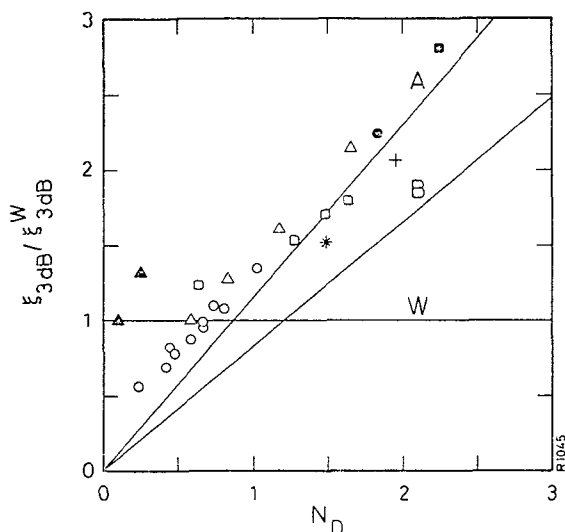


Fig. 1. - Observed half power angles normalized with respect to the half power angles of the Westervelt formula, plotted as a function of  $N_D$ . Curve A corresponds to the primary wave half power angle, curve B to the source density half power angle, and curve W to the Westervelt formula. Data are from (see Table I) :

- : Bennett et al. (1974)
- \* : Bjørnø et al. (1976)
- △ : Eller (1973)
- ▲ : Esipov et al. (1975)
- : Hobaek et al. (1977)
- ⊕ : Moffett et al. (1976)
- : Muir et al. (1972)
- : Smith (1971)

replotted a normalized half-power angle as a function of a divergence parameter  $N_D$  (defined below). The observations by the various authors were made at different distances from the sound sources. The half-power angle in the model treated by Westervelt is independent of  $N_D$ . In Fig. 1. it is given by the horizontal line marked W. The curves A and B represent respectively the half-power angle of the primary beam of highest frequency and of the source density. Some authors (see, for instance Berkta and Leahy (1974), Fenlon (1974), Moffett and Mellen (1976) have argued that the difference frequency directivity should be given by the source density directivity near the symmetric axis of the beams, if the primary beams are very much broader than the beamwidth in the model studied by Westervelt. Thus, for  $N_D \gg 1$ , the results should approach the curve B. The observations do not seem to support this. We have included the curve A in order to show the close relationship of the available obser-

ved results to the half power angles of the primary beams.

In the following we shall present numerical results from an asymptotic formula that seem to explain the main features of the observations for  $N_D \geq 0.6$ . If  $N_D < 0.6$  this asymptotic theory leads to results that approach curve W if the difference frequency,  $\omega_-$  is low enough as compared with the frequencies,  $\omega_a$  and  $\omega_b$  of the primary waves. A sufficient condition is that  $\frac{\omega_-}{\omega_a} \ll \frac{1}{8}$ . For higher values of this frequency ratio, the results are below curve W. However, the observations obtained by Hobaek and Vestrheim (1977) for  $N_D < 0.6$  can not be explained by this theory. They can, we believe, only be explained by the minimum beam-width effect first reported by the same authors at the Birmingham symposium (1971) as they probably are not in the far field region where the asymptotic theory is valid. This effect has until now not been explained theoretically. We have therefore analysed it a little closer, and have found that the recent theory by Novikov, Rudenko and Soluyan, with a slightly modification in the expansion technique, can be used to study an affect of this character.

Before we proceed with the theory we shall now summarize briefly some of our earlier results on the parametric acoustic array obtained at the University of Bergen. This will motivate the subsequent development.

In a paper from 1964 Lauvstad, Naze and Tjøtta analysed the case of far-field interaction between primary waves radiated from a circular piston, i.e., with Bessel directivity. In this model a screen was introduced at the Rayleigh distance from the piston in order to exclude the interaction in the Fresnel zones. Finite aperture effects were accounted for in this work, and in a following work on the plane-beam model (Naze and Tjøtta 1965). We shall return to these two works in the discussion later on, and add some comments on the range of validity of the theory.

It was observed by Hobaek (1967), in an experiment with sound beams of finite aperture, that the generated sound of difference frequency has a maximum much nearer the piston source than expected from the infinite plane-wave model (Zverev and Kalachev (1968) reported independently on similar observations using a square sound source).

Theoretical results explaining these observations were presented at the Birmingham Symposium in 1971 (Faltinsen and Tjøtta, Hobaek and Vestrheim). It is of interest to note that this maximum - or more precisely the position of maximum in the product  $R \cdot p_{\perp}$  of distance from the source and the pressure amplitude - is strongly related to the observed beamwidth minimum already mentioned above. This will be discussed later on in the present paper. Further, increasing intensity has significant effect on this maximum and the beam-width, as reported by Vestrheim and Hobaek at the same symposium. The paper by Faltinsen and Tjøtta also contains an analysis of the interaction problem on the basis of Burgers' equations. The exact solution of this equations is expanded in a series in the parameters  $\frac{M_a}{S_a}$  and  $\frac{M_b}{S_b}$ , where  $M_a$  and  $M_b$  denote the Mach number,  $S_a$  and  $S_b$  the Stokes numbers of the two interacting waves. The series converges for all finite values of the expansion parameters. The three first terms in the expansion are given explicitly, and is used to estimate the range of validity of the results that one obtained by the usual quasi-linear approximation. The third order in this expansion contains terms that can be interpreted as shown in Table II.

Table II. - Interpretation of third order terms.

Term	Interaction
$\cos \omega_a \tau$	$\omega_a \circ 2\omega_a$
$\cos \omega_b \tau$	$\omega_a \circ (\omega_a - \omega_b)$ and $\omega_a \circ (\omega_a + \omega_b)$
$\cos 3\omega_a \tau$	$\omega_a \circ 2\omega_a$
$\cos\{(2\omega_a - \omega_b)\tau\}$	$\omega_a \circ (\omega_a - \omega_b)$ and $\omega_b \circ 2\omega_a$
$\cos\{(2\omega_a + \omega_b)\tau\}$	$\omega_a \circ (\omega_a + \omega_b)$ and $\omega_b \circ 2\omega_a$

A similar table is obtained by interchanging  $\omega_a$  and  $\omega_b$ . Here  $\tau = t - x/c_0$  is the retarded time for propagation in x-direction, and  $\omega$  and  $\Omega$  denote the nonlinear interaction between waves with frequencies  $\omega$  and  $\Omega$  in a first order (quasi-linear) approximation.

Finally, we also refer to a report (Hobaek 1970) in which a discussion is given of the linear field

from a circular piston source.

## 2. - The Westervelt model, validity of the asymptotic formula.

The Westervelt model (1963) represents the difference-frequency pressure that is generated by two narrow sound beams, assuming a model of a continuous line array of simple sources and phase to produce radiation in the endfire direction. Its asymptotic solution is based on the Born approximation of the Green's function for the wave equation, viz.

$$G(\underline{R}, \underline{r}) = \frac{e^{i\mathcal{L}A} |\underline{R} - \underline{r}|}{|\underline{R} - \underline{r}|} \approx \frac{1}{R} e^{i\mathcal{L}A(R-r \cos \psi)} \quad (1)$$

where  $\psi$  is the angle between the directions to the observer position  $\underline{R} = \{R, \xi, 0\}$  and the position  $\underline{r} = (r, \theta, \varphi)$  of a source point in the integrals giving the solution. (The  $\varphi$  dependence drops out in the line array model, and  $\cos \psi \rightarrow \cos \theta$ ).

We find that this is a valid approximation if

$$R \gg \sup \left\{ \frac{1}{2} k_- L^2 \sin^2 \frac{\xi}{2}, L \right\} \quad (2)$$

which is a sufficient condition for the phase correction to be small compared to unity within the integration limits, and it also allows us to put  $R$  for  $r$  in the dominator. Here  $k_- = \omega_-/c$ ,  $L$  is the length of the interaction region, for example,  $L = L_A$ , where  $L_A = 1/(\alpha_a + \alpha_b - \alpha_-)$ ,  $\alpha_a, \alpha_b$  are absorption coefficients for the primary waves and  $\alpha_-$  that of difference frequency wave.

Remark: This may explain the results of Berktaay and Shooter (1973), Fig. 2 and Table I in their paper, that the nearfield-farfield transition for the type of array considered depends on the range (in array length), but is independent of the wavelength. For the parameter values used,  $L_A > \frac{1}{2} k_- L_A^2 \sin^2 \frac{\xi}{2}$ , and the condition above becomes  $R \gg L_A$ .

Attempts have been made to improve the result based on the Born approximation. Thus Berktaay (1971, 1972) has included one more term in the expansion of  $|\underline{R} - \underline{r}|$  in the exponential function, i.e., he puts

$$G \approx \frac{1}{R} e^{ik_-(R-r \cos \psi + \frac{r^2}{2R} \sin^2 \psi)} \quad (3)$$

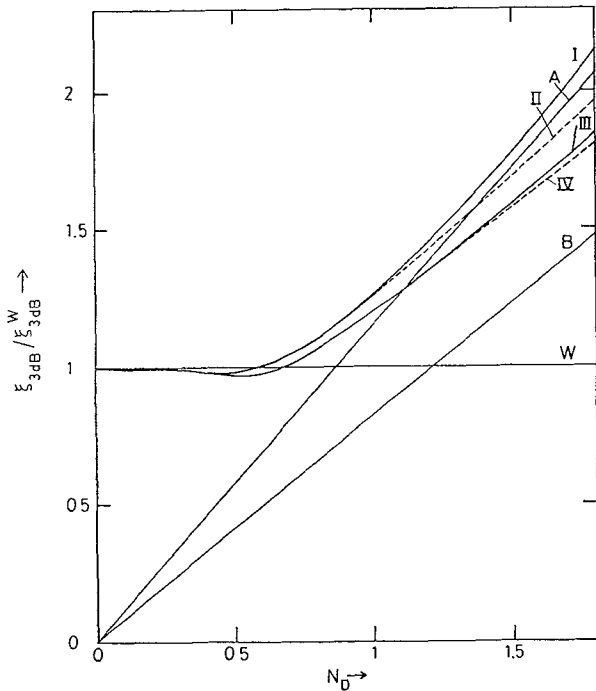


Fig. 2. - Computed half power angles normalized with respect to the half power angles of the Westervelt formula, plotted as a function of  $N_D$ . Curve A, B and W are the same as in Fig. 1. Curve I to IV are subject to the following approximations in the expansion,

- I :  $J_0 = 1$  and  $J_n = 0$ ,  $n \neq 0$ .
- II : Keeping the first term only ( $J_0$ )
- III : Keeping the two first terms ( $J_0$  and  $J_1$ )
- IV : Keeping the three first terms ( $J_0$ ,  $J_1$  and  $J_2$ ) and adding contribution from the near field region.

In this case we find the following sufficient condition for the phase approximation, instead of (2) :

$$R \gg \sup \left\{ \frac{1}{2} L \sin \xi (k_- L \sin \xi)^{\frac{1}{3}}, L \sin \xi \left( \frac{1}{2} k_- L \sin \xi \right)^{\frac{1}{2}}, L \right\} \quad (4)$$

For small angles such that  $\frac{1}{2} k_- L^2 \sin^2 \xi < L$ , i.e., for  $\xi < \xi_{3dB} = (2/k_- L_A)^{\frac{1}{2}}$  (if we put  $L \approx L_A$ ) both conditions, (2) and (4) yield  $R \gg L_A$ , and the two solutions have the same range of validity. We also find that the error introduced by the approximation

$$\frac{1}{|R-r|} \approx \frac{1}{R} \text{ in the dominator may be of the same}$$

order of magnitude as the correction obtained by Berktaý. [ This point has recently been studied in details by böe (1976)]. Nevertheless, including higher order terms in the exponential, and in the denominator as well, the result may be improved

numerically and qualitatively.

Our results for a line array have been generalized to the case of an array with finite width by Böe (1976). Considering interaction within a cylindrical volume of radius  $a$  and length  $L$ , he obtains

$$R \gg \sup \left\{ \frac{1}{2} k_- (L \sin \xi + a \cos \xi)^2, (L^2 + a^2)^{\frac{1}{2}} \right\} \quad (5)$$

and

$$R \gg \sup \left\{ (L \sin \xi + a \cos \xi) \left( \frac{1}{2} k_- (L \sin \xi + a \cos \xi) \right)^{\frac{1}{3}}, (L \sin \xi + a \cos \xi) \left( \frac{1}{2} k_- (L \cos \xi + a \sin \xi) \right)^{\frac{1}{2}}, (L^2 + a^2)^{\frac{1}{2}} \right\} \quad (6)$$

which generalize eqs. (2) and (4), respectively.

### 3. - Primary sound field.

A mathematical model describing adequately the field from a physically realizable parametric acoustic array requires the use of a more realistic representation of the source region, i.e., of the primary wave field in the zone of interaction. We shall here restrict our discussion to primary waves generated by a baffled circular piston, although many of the overall features are common to source fields regardless the form of the sound sources. The field from a circular piston is, of course, discussed in most textbooks on acoustics, but nevertheless details concerning the field near the piston are not readily found. A report (Hobaek 1970) was therefore worked out in order to display the more salient features of the piston field. A paper by Zemanek (1971) on this subject appeared about the same time. We shall now summarize a few facts from our knowledge about the primary sound field, to be used later on.

Let  $\Phi$  denote the velocity potential, or another variable describing the field. At a sufficient distance  $r$  from the piston source, we now put

$$\Phi(r, \theta, \varphi) = \psi(r) D(\theta, \varphi) \quad (7)$$

where spherical coordinates  $(r, \theta, \varphi)$  are used. The angle  $\theta$  is measured from the symmetry axis,  $\varphi$  is the azimuthal angle.

We use this to define "the far field," where the field is separable in angular and radial coordinates. At shorter distances, where the field is not separable in this way, it is accordingly denoted as "the near field".

A similar definition applies to fields radiated from parametric sources as well.

The Rayleigh distance  $R_0 = \frac{S}{\lambda}$ , where  $S$  is the source surface area and  $\lambda$  the wavelength, may serve as a convenient measure of the limit between the near field and the far field.

The Fraunhofer approximation of the far field, with the "Bessel directivity",

$$p(r, \theta) = -ip' \frac{R_0}{r} e^{i\chi r} \frac{2J_1(ka \sin \theta)}{ka \sin \theta} \quad (8)$$

is only valid for  $r \gg R_0 = \frac{ka^2}{2}$ , if a phase error much less than 1 in the Green's function is required. Also  $r \gg a$  is assumed in this approximation. Here  $\chi = k + i\alpha$ ,  $k = \omega/c_0$ ,  $a$  is the piston radius and  $J_1$  is the Bessel-function of order 1. A time variation  $e^{-i\omega t}$  is assumed.

An improved approximation is sometimes obtained if we keep higher order terms in the expansion of the Green's function. For instance, keeping square terms, we get the following sufficient condition for the phase approximation, instead of  $r \gg R_0$  as in the Born approximation :

$$r \gg \frac{a}{2} (ka)^{\frac{1}{3}} \quad (9)$$

Due to the cube root this may improve considerably the range of validity of the phase approximation, when  $ka \gg 1$ . This is valid on and near the symmetry axis. Generalized results valid far off the axis are obtained by putting  $L=0$  in eqs. (5) and (6).

The near field is highly fluctuating. However, the large amplitude of the fluctuation on the axis is due to symmetry. Outside a small region surrounding the axis, "the paraxial region", the fluctuations are much smaller. In fact, for  $r \leq R_0/2\pi$  the sound source may be considered as a cylindrical beam of plane waves, with radius approximately equal to  $a$ . The pressure amplitude is nearly constant across the beam, but it has a ridge near the beam edge where it is about 20% higher than elsewhere in the beam.

The phase angle of the field within the beam

also behaves nearly as for a plane wave, except in the paraxial region.

Thus, the near field may roughly be described as a slowly diffusing beam of plane waves out to about  $R_0/2\pi$ , followed by a transition region where the field changes from plane waves to spherically spreading waves. In this transition region, extending somewhat beyond  $R_0$ , the beam structure dissolves while the paraxial region grows in lateral extension with the distance and dominates the field for  $r > R_0/2\pi$ . For certain purposes the Bessel directivity adequately describes the field down to a distance a little above  $R_0/2\pi$  (see, for example, Zemanek (1971)). However, this does not allow for a general application of the far field solution within the transition regions.

Finally, one important point to consider is that there is a  $\pi/2$  phase shift between the plane wave region of the beam and the spherically diverging wave of the main lobe (a reference to the same propagation distance is tacitly assumed). This phase shift brings about a fundamental difference between the resulting difference-frequency field of the parametric array, and the field of the sum frequency as well as the 2<sup>nd</sup> harmonics of each of the primaries. This occurs because the source function,  $Q_{ab}$  in the quasi linear differential equation (see eq.(10)) giving the difference frequency pressure contains  $p_a p_b^*$  (asterisk denotes complex conjugate) such that a constant shift in the phase in both of the primary waves will be cancelled out in the source function. On the other hand the source function of the sum frequency contains  $p_a p_b$ , which implies that the sources in the spreading region will be in exactly opposite phase to the sources in the plane wave region (in addition the sources are phased like the generated wave, of course). This also applies for the 2<sup>nd</sup> harmonics. Thus, at a certain distance from the primary source there should exist a domain where the contribution from the spreading primary waves cancels the contribution from the plane wave region. Here the generated field should vanish provided the phase shift in the primary field is abrupt, but since the phase shift takes place gradually over a transition region, complete cancellation is not to be expected. Instead, the field should experience a minimum in this domain. This is probably the physical interpretation of the minimum which was observed in the 2<sup>nd</sup> harmonic field by Gould, Smith, Williams, Ryan (1966) and obtained

analytically by Ingenito and Williams (1971).

Likewise there is a phase shift of  $\pi$  between the field of two adjacent lobes. Therefore, if in the interacting wave field two adjacent lobes of one field overlap a common lobe in the other, the source function of the difference frequency will be of opposite phase in the two lobes. The effect of this is to reduce the contribution to the difference frequency field from the higher order lobes when interactions in the far field of the primary waves are important.

#### 4. - Scaling parameters.

Returning to the parametric array problem it should be evident that the nature of the array will be strongly dependent on the length of the interaction region (array length) compared with the near field length of the primary waves. Thus, if  $L_A \ll R_a$  interaction takes place within the plane collimated wave region of the primary fields, whereas interaction within the spherically diverging wave region will be important if  $L_A \gg R_a$ . Here  $R_a$  refers to the Rayleigh distance of the primary wave of highest frequency. (The Rayleigh distance of the other primary wave will be denoted  $R_b$ ). That there is a significant difference between these two primary fields is easily revealed by considering contributions to the generated sound on the axis, from source points lying on an equi-phase surface. The phase difference between such contributions vanishes as  $R \rightarrow \infty$  for the collimated plane primary wave, but reaches a finite value when the primary wavefronts are curved. Thus, destructive interference may occur if the spreading angle of the diverging primary wave is large enough.

It has proved very convenient to use a set of parameters to characterize the state of the primary field in relation to the array length and the difference frequency wave length. Two independent parameters are sufficient for this purpose if the primary field is axially symmetric and nonlinear attenuation is negligible. Vestrheim (1973) introduced two basic parameters as follows :

$$N_f = \omega_- / \omega_a, \quad N_F = \sqrt{L_A / 2R_a}.$$

Any array specified by frequencies, primary source radius, sound velocity and attenuation coefficient of the medium, will be assigned definite

$N_f, N_F$  values accordingly, and is thus represented by a point in the  $N_f$ - $N_F$ -plane. The graphical representation of the  $N_f$ - $N_F$ -plane will be referred to as "the parameter diagram".

The basic parameters may be combined in different ways, and two important derived parameters are  $N_D = \sqrt{(N_f)} \cdot N_F$  and  $N_A = \sqrt{(N_f)} / N_F$ .  $N_D$  gives a measure of the (maximum) destructive phase-shift within the array, due to curved wave fronts, while  $N_A$  may be interpreted as a measure of the ratio of the difference frequency Rayleigh distance to the array length :

$$N_A = \sqrt{2R_- / L_A}, \quad \text{where } R_- = \pi a^2 / \lambda_- . \quad N_A \text{ is also}$$

proportional to the ratio of the Westervelt beamwidth to the beamwidth of difference frequency sound radiated from a piston of diameter  $2a$ .

The parameter diagram can be divided into three regions, all with characteristic properties of the parametric arrays, by drawing the curves where  $N_A = 1$  and  $N_D = 1$  respectively. We label the regions according to Vestrheim (1973). Typical of region I is the plane collimated primary wave array, when  $N_A \gg 1$ . Typical of region II is the Westervelt array, and of region III, the diverging primary wave array, when  $N_D \gg 1$ .

#### 5. - Diverging primary waves.

We shall now consider models where the primary waves are divergent, in an attempt to approximate interaction in the far field. Thus, when related to real parametric arrays, such models are of interest when  $L_A > R_a$ , i.e. in the domains II and III in the parameter diagram. The model introduced by Luvstad et al. (1964) will form the basis of our investigation.

The basic equation governing  $p_-$  can be written on the form

$$\nabla^2 p_- + k_-^2 p_- = - \frac{\Lambda + 2}{\rho_0 c_0^2} \frac{\partial^2}{\partial t^2} (p_a p_b^*) \equiv Q_{ab} \quad (10)$$

The solution of this is expressed by the Green's function, eq. (1). Substituting

$$p_n(r, \theta) = \dots i p_n' \frac{R_n}{r} D_r(\theta) e^{i(\mathcal{K}_n r - \omega_n t)}, \quad n = a, b$$

for the primary waves, we obtain from eq. (10)

$$p_-(R) = - \frac{(\Lambda+2) p_a' p_b'}{8\pi \rho_0 c_0^2} k_-^2 R_a R_b \frac{e^{i\mathcal{K}_- R}}{R} \int_L^\infty \int_0^{\pi/2} D_a(\theta) D_b(\theta) \sin\theta d\theta dr \quad (11)$$

$$\times \int_0^{2\pi} e^{i\mathcal{K}_- [r - (R-r)/R] - r/L_A} d\varphi$$

which is equivalent to eq.(5.1) in Lauvstad et al. (1964). Here  $D_a$  and  $D_b$  are the Bessel directivity of the two waves, and  $\Lambda = (d^2 p/d\rho^2)_{\rho_0 c_0^2}^{-2}$ . By putting the lower limit  $L$  equal to  $R_a$ , only interactions in the far fields of the primary waves are included.

The integration over  $r$  is elementary. Further, by expanding in a Fourier-Bessel series, we are left with a simple numerical integration to perform. This also makes a discussion of contributions from different parts relatively simple. The analytical results and further discussions will be published elsewhere. In Fig. 2 we present some numerical results. The near-field contribution is obtained from the formula of Naze and Tjøtta (1965). In Fig. 2, as in Fig. 1, the normalized half power angle is plotted as a function of the divergence number  $N_D$ . The curves are computed for parameters corresponding to the experiment of Hobaek and Vestrheim (1977),

keeping the field number  $N_F$  approximately constant  $\approx 3.1$ , while changing  $k_-/k_a$ . The different curves I to IV correspond to increasing degree of accuracy, from putting  $J_0 = 1$ ,  $J_n = 0$ ,  $n \neq 0$  (I), to keeping the three first terms in the series as well as including the contribution from the near field region (IV). The near field contribution is almost negligible at  $\xi_{3dB}$ , and its effect is actually not visible in curve IV.

The increase in directivity over the Westervelt directivity at values of  $N_D$  about 0.5, also noticed by Berktaay and Leahy (1974), is significant. It becomes larger when  $k_-/k_a$  increases. For instance,

if  $k_-/k_a = 0.2$  we find  $\xi_{3dB}/\xi_{3dB}^W = 0.87$  at  $N_D = 0.56$ .

The asymptotic theory seems to explain the main features of the observations, except the ones by Hobaek and Vestrheim (1977) at  $N_D < 0.6$ . However, there is reason to believe that these measurements are too close to the interaction region to justify comparison with an asymptotic theory. We have computed the pressure on the axis as a function of distance by using an approach suggested by Muir and Willette (1972) for primary beams with Bessel directivity. If we plot  $R \cdot p_-$  versus  $R$  for different values of  $N_D$  we obtain curves similar to the observations presented in Fig. 5 in the paper by Hobaek and Vestrheim (1977), if  $N_D > 0.6$ . When  $N_D \leq 0.6$ , however, these curves show pronounced maxima at  $R = 1 - 3 L_A$ . This effect becomes more pronounced as  $N_D$  decreases. Since it occurs for the same range of  $N_D$  values, as the observed directivity is stronger than predicted asymptotically, we believe that these two phenomena are connected. In the next section we demonstrate a theoretical prediction of the minimum beam-width effect in the near field of a collinear parametric array. Also in this case a close relation to a maximum of  $R \cdot p_-$  is found. Thus, there is strong evidence that a minimum beam-width effect exists in parametric arrays with diverging primary waves, when  $N_D < 0.6$ .

## 6. - Collinear primary sound beams.

We will now introduce a different method which is applicable when the primary waves are approximated by a beam of plane waves. This model is suitable when considering interaction in the near field.

Let  $x$  denote the distance along the symmetry axis of the primary wave beam and  $\sigma$  the lateral distance from the axis. Then

$$\left. \begin{aligned} p_-(x, \sigma) &= p_-(x, \sigma) e^{i(k_x x + k_\sigma \sigma - \omega_- t)} \\ p_n(x, \sigma) &= p_n(\sigma) e^{i(k_n x - \omega_n t)}, \quad n = a, b \end{aligned} \right\} \quad (12)$$

$k_x$  is the component of  $k_-$  in the  $x$ -direction,  $k_\sigma$  in the lateral direction.



If we assume a small  $k_\sigma$  and a slow variation of  $p_-$  with  $x$  over a wavelength, eq. (10) becomes

$$\frac{\partial p'_-(x, \sigma)}{\partial x} - \frac{i}{2k_-} \nabla_\perp^2 p'_-(x, \sigma) = -\frac{i(\Lambda+2)k_-}{4\rho_0 c_0^2} p'_-(\sigma) e^{-\frac{x}{L_A}}, \quad (13)$$

where  $\nabla_\perp^2 = \frac{\partial^2}{\partial \sigma^2} + \frac{1}{\sigma} \frac{\partial}{\partial \sigma}$ . This is the equation used by Novikov et al. (1976). They derived it from the equation of Zabolotskaya and Khokhlov (1969). It is of first order in  $x$  and can be solved by using a suitable integral transform to eliminate the  $\sigma$ -dependence. If we assume the primary waves to have constant non-zero amplitudes for  $\sigma < a$  and zero amplitudes for  $\sigma > a$ , which is the model used for the nearfield interaction in Faltinsen and Tjøtta (1971) and Hobaek and Vestrheim (1971), the solution becomes cumbersome and is hardly tractable analytically. However, following Novikov et al. by assuming a Gaussian distribution, we obtain an integral solution for  $p'_-(x, a)$ . This can be explained in a power series and solved numerically.

Curves computed of the lateral amplitude distribution at different values of  $x$  display the building up of the difference frequency sound field, as was also shown by Novikov et al. In their work, however, the primary wave absorption was neglected. When absorption is included this building up of a sound field will reach a maximum at some distance and then decay. This effect is shown in Fig. 3, by tracing the lateral distances to points where the pressure is 3dB below the pressure at the axis. The parameter is  $L_d/L_A$ , which is closely related to  $N_A$ , and the ordinate is  $(\sigma/d)/(x/L_A) = \frac{L_A}{R} \text{tg} \xi$ , where  $\xi$  as before is the angle from the  $x$ -axis.  $L_d = \frac{1}{2} k_- d^2$  and  $d$  is a characteristic radius of the primary source. A remarkable feature is the beamwidth minimum which becomes more pronounced and moves inwards as  $L_d/L_A$  decreases. This is consistent with measurements by Vestrheim and Hobaek (1971) and Vestrheim (1973). Further, on the axis, computations agree with earlier results by Faltinsen and Tjøtta (1971) and Hobaek and Vestrheim (1971). It is found that maximum of  $x \cdot p_-$  is closely related to the beamwidth minimum shown in Fig. 3. A more exten-

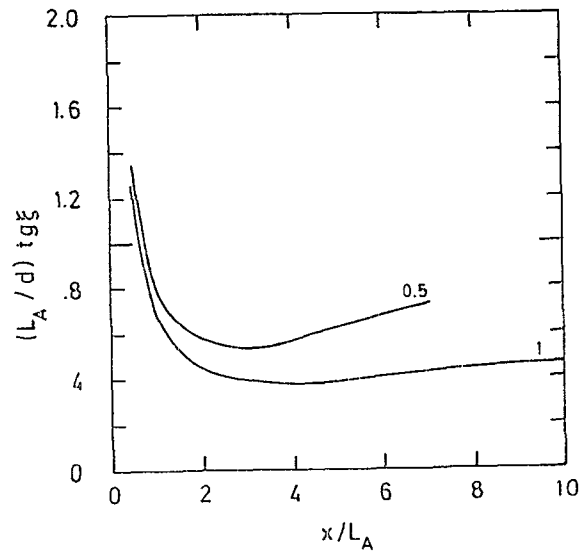


Fig. 3. - Variation of half power angle with distance, scaled in array lengths, as computed numerically. The parameter is  $L_d/L_A$ .

sive discussion of this topic will be published elsewhere.

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