# 1MAT11 - Linear Algebra

*Lecture Notes*

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# Contents

# 1 Vector Space

## 1.1 Definition

Let $V$ be a non-empty set, and let $\mathbb{K}$ be a field (like $\mathbb{R}$).

We say that $V$ is a **vector space over** $\mathbb{K}$ if:

- There is a binary operation **addition**: $+ : V \times V \to V$ such that $u + v \in V$ for all $u, v \in V$.
- There is a **scalar multiplication**: $\cdot : \mathbb{K} \times V \to V$ such that $a \cdot u \in V$ for all $a \in \mathbb{K}$ and $u \in V$.
- Eight axioms hold for all $u, v, w \in V$ and $a, b \in \mathbb{K}$:
  1. **Associativity**:
  $$u + (v + w) = (u + v) + w \tag{1}$$
  2. **Commutativity**:
  $$u + v = v + u \tag{2}$$
  3. **Existence of zero vector**:
  $$u + 0 = u \tag{3}$$
  4. **Existence of additive inverse**:
  $$u + (-u) = 0 \tag{4}$$
  5. **Distributivity over vector addition**:
  $$a(u + v) = au + av \tag{5}$$
  6. **Distributivity over scalar addition**:
  $$(a + b)u = au + bu \tag{6}$$
  7. **Associativity of scalar multiplication**:
  $$a(bu) = (ab)u \tag{7}$$
  8. **Multiplicative identity**:
  $$1u = u \tag{8}$$

## 1.2 What *is* a vector space, really?

Once we formalize the idea of vector spaces as sets of *objects that we can add and scale*, the concept applies **not just to arrows in space**, but to many other things: functions, polynomials, matrices, sequences, etc.

All those are **vector spaces** as long as they satisfy the axioms above.

Then, the cool part about vector spaces is that the **abstraction** lets us transfer methods and results from one domain to another. In other words, if we can prove something about vector spaces in general, we can apply it to all the specific cases we care about.

# 2 Vector Subspace

## 2.1 Definition

Let $V$ be a vector space over a field $\mathbb{K}$. A subset $W \subseteq V$ is a **vector subspace** of $V$ if $W$ itself is a vector space under the same operations of vector addition and scalar multiplication defined in $V$.

To be a subspace, a subset $W$ must satisfy the following three conditions:

1. **Non-emptiness**:
$$0 \in W \tag{9}$$

2. **Closure under addition**:
$$u, v \in W \implies u + v \in W \tag{10}$$

3. **Closure under scalar multiplication**:
$$u \in W, a \in \mathbb{K} \implies au \in W \tag{11}$$