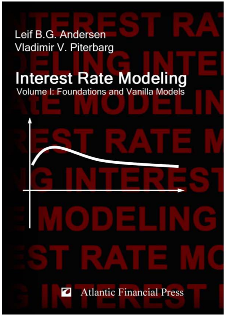
Introduction to Interest Rate Modelling and Derivative Pricing

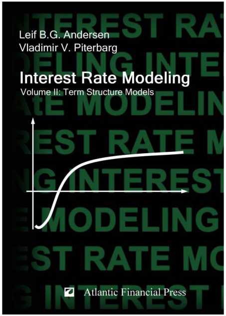
Zhengqi Wang

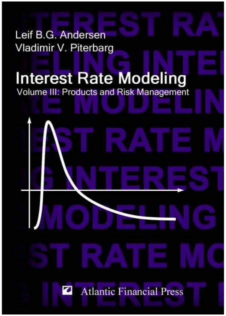
August 2, 2024

Table of Contents

“Embark on a journey through the intricate world of interest rates with Andersen and Piterbarg’s comprehensive guide.”







***Leif B.G. Andersen** is the Managing Director and Global Head of the Quantitative Strategies Group at Bank of America Merrill Lynch, recognized for his extensive work in financial modeling, derivatives, and risk management. **Vladimir V. Piterbarg**, Head of Quantitative Research and Quantitative Trading at Rokos Capital Management and former Global Head of Quantitative Analytics at Barclays Capital, is renowned for his contributions to interest rate modeling and quantitative finance.*

Content of Volum 1

Foundations and Fundamental Concepts of Interest Rate Models

1. A simple Interest Rate Model (IRM) with only parallel shifts of the curve.
2. Risk premium and risk neutralization.
3. From yield curve models to short rate models and back.
4. Estimation and calibration of interest rate model parameters.

Content of Volum 2

Advanced Term Structure Models and Their Applications

1. The general Heath-Jarrow-Morton (HJM, 1992) framework.
2. The Markov Property and the Cheyette (1992) model family.
3. Volatility, skew and kurtosis in Cheyette models.
4. The Multi-Factor Cheyette (MFC) model.

A Simple Interest Rate Model

The simplest dynamic IRM: Ho-Lee (1986) under a Heath-Jarrow-Morton (1992) approach.

IRM = Model of the Whole Yield Curve

1. Black-Scholes and extensions (local volatility: Dupire and stochastic volatility)
    - Given today's underlying asset price.
    - Model its future arbitrage-free evolution and value options and other contingent claims.
    - Applies when we model one interest rate (e.g. to price one swaption) but not when we model all interest rates.
2. In the context of multiple interest rates, what is the underlying asset?
    - Clean theoretical grounds for interest rate modelling introduced by Heath-Jarrow-Morton.
    - Underlying "asset" = whole curve = collection of all rates of all maturities at a given time.
    - Today's curve is given and we model its future evolution.
    - In order to price options and exotics, and estimate exposures.

Discount Factors (DF)

- For simplicity, we neglect spreads and credit and consider a single YC.
- Yield Curve (YC) at time `t = rates` of all maturities T at time `t = all discount factors of all maturities T at time t`, that is: `DF(t,T)`.
- `DF(t,T)` = price at time t of 1 monetary unit paid at maturity T
- For a fixed maturity T, `DF(t,T)` is the price series of a tradable asset = zero-coupon bond of maturity T = delivers 1 at T.
- At a time t, if we know all `DF(t,T)`, we also know all the forward Libors and par swap rates:

$$L(t, T_1, T_2) = \frac{DF(t, T_1) - DF(t, T_2)}{(T_2 - T_1)DF(t, T_2)}, FSR(t, T_1, T_2) = \frac{DF(t, T_1) - DF(t, T_2)}{\sum_i (T_i^e - T_i^s) DF(t, T_i^e)}$$

These are simplistic formulas, only correct in absence of spreads!

Instantaneous Forward Rates (IFR)

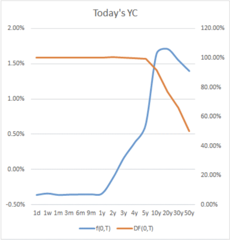
- Another convenient representation of `YC(t)` is the collection of Instantaneous Forward Rates (IFR) of all maturities T at time t.
- `f(t,T) = seen` at time t, par rate for a short term forward loan maturity `T = short term forward Libor maturity`.
- Forward rates are deduced from discount factors and vice-versa:

$$f(t,T) \equiv \lim_{\epsilon \to 0} L(t,T,T+\epsilon) = -\frac{\partial \log DF(t,T)}{\partial T} \iff DF(t,T) = \exp\left[-\int_t^T f(t,u)\,du\right] \quad (1)$$

- Note rates are not tradable assets, just a convenient "view" over bonds prices / discount factors.
- One particular IFR: forward rate maturity t at time t = short rate at time t. $r_t \equiv f(t,t)$

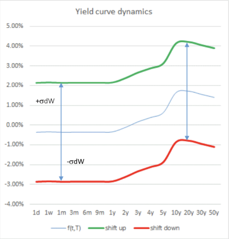
Today's Yield Curve

- Today's Yield Curve `YC(0)`.
  - Collection of all discount factors `DF(0,T)`.
  - Or equivalently all forward rates `f(0,T)`.
- YC construction is a sophisticated process, out of scope for this presentation.
- In practice, `YC(0)` is constructed out of linear instruments: swaps, coupon bonds.
- Here we assume todays `YC`, `DF(0,T)` and `f(0,T)` is known.



Yield Curve Model

- Today's `YC(0)` is given, IRM specifies how YC evolves from here.
- Simplest model = parallel shifts = flat deformations.
- Modeled with a Brownian Motion under the historical probability P: $df(t, T) = \sigma dW$.
- (Constant) sigma = (annual) volatility of (all) rates.
- Then all rates are normally distributed, with (same) (annual) variance σ^2, and 100% correlation between rates of different maturities.
- This model is far too simplistic for practical use.



Convexity Arbitrage

- Dynamics of (tradable) bonds: Ito's lemma: 2nd order expansion of $DF(t,T) = \exp\left[-\int_t^T f(t,u)\,du\right]$.
  Proof:

$$\frac{dDF(t,T)}{DF(t,T)} = d\log DF(t,T) + \frac{[d\log DF(t,T)]^2}{2}$$

DF(t, T) = exp $\left[-\int_t^T f(t,u)\,du\right] \Rightarrow d\log DF(t,T) = -d\int_t^T f(t,u)\,du = r_t dt - \sigma(T-t)dW$

$$[d\log DF(t,T)]^2 = \sigma^2(T-t)^2\,dt$$

$$\boxed{\underbrace{\frac{dDF(t,T)}{DF(t,T)}}_{\text{bond return}} = \underbrace{r_t dt}_{\text{earns risk free rate}} - \underbrace{(T-t)\sigma dW}_{\text{volatility}\propto\text{duration}} + \underbrace{\frac{(T-t)^2\sigma^2}{2}dt}_{\text{convexity}\propto\text{duration}^2}}$$

- Arbitrage = take advantage of convexity
  - Buy long term bond, for example 10y.
  - Hedge with short term bond, for example 1y.
  - Match durations, sell ˜10 1y bonds for each 10y bond.
  - End up with positive convexity and no risk.

- Formally: buy T2 bonds, sell T1 bonds (T2 $\geq$ T1).
    - Buy $DF(t, T_1)(T_1 - t)$ bonds T2.
    - Sell $DF(t, T_2)(T_2 - t)$ bonds T1.
    - Value at t: $\pi_t = DF(t, T_1)DF(t, T_2)(T_1 - T_2) \leq 0$
- Change in value:

$$d\pi_t = \underbrace{\left[r_t \pi_t - (T_1 - t)(T_2 - t)\frac{\sigma^2}{2}\pi_t\right]}_{\substack{> r_t \pi_t \\ < 0}} dt + \underbrace{0dW}_{\text{no risk}}$$

- We see that this portfolio has no risk and earns more than risk free rate.
- Means parallel shift dynamics is impossible. Fixed income hedge funds and trading desks would execute and exhaust the arbitrage.
- If random YC deformations are really parallel:
    - There must be a simultaneous steepening.
    - Causing long term bonds to decrease in value.
    - And neutralize the arbitrage.

Arbitrage Free Dynamics

- Remove arbitrage with average steepening.
  - Modelled by tenor dependent drift:

$$df(t,T) = \sigma dW + \mu(T-t)dt$$

- Computation of the drift
  - Updated dynamics of bonds:

$$\frac{dDF(t,T)}{DF(t,T)} = r_t dt - (T-t)\sigma dW + (T-t)^2 \frac{\sigma^2}{2} dt - \left[\int_t^T \mu(T-u)du\right] dt$$

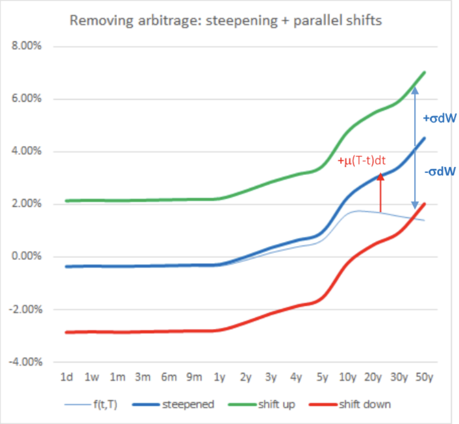
  - Updated dynamics of arbitrage portfolio:

$$\frac{d\pi_t}{dt} = r_t \pi_t + DF(t,T_1)DF(t,T_2)(T_1-t)(T_2-t)$$
$$\times \left((T_2 - T_1)\frac{\sigma^2}{2} - \left[\frac{\int_t^{T_2} \mu(T_2-u)\,du}{T_2-t} - \frac{\int_t^{T_1} \mu(T_1-u)\,du}{T_1-t} \right] \right)$$

  - To prevent arbitrage:

$$\frac{d\pi_t}{dt} = r_t\pi_t \iff \frac{\frac{\int_t^{T_2} \mu(T_2-u)\,du}{T_2-t} - \frac{\int_t^{T_1} \mu(T_1-u)\,du}{T_1-t}}{T_2-T_1} = \frac{\sigma^2}{2} \quad (\forall T_2 > T_1)$$

hence $\mu(T-t) = \sigma^2(T-t) + c(t)$


Removing arbitrage: steepening + parallel shifts
8.00%
6.00%
4.00%
2.00%
0.00%
-2.00%
-4.00%
1d
1w
1m
3m
6m
9m
1y
2y
3y
4y
5y
10y
20y
30y
50y
+σdW
+μ(T-t)dt
-σdW
f(t,T)
steepened
shift up
shift down

- Under the historical probability, in the simple flat deformation model, changes in forward rates must satisfy:

$df(t,T) = \sigma dW + \left[\sigma^2(T-t) + c(t)\right] dt$ define $\eta_t \equiv -\frac{c(t)}{\sigma}$

and get df(t, T) = $\left[\underbrace{\sigma^2(T-t)}_{\text{convexity arbitrage adjustment}} - \underbrace{\eta_t \sigma}_{\text{risk premium}}\right] dt + \underbrace{\sigma dW}_{\text{random parallel shifts}}$

  - The quantity η is called risk premium, it could depend on time t and even have stochastic dynamics. But it must be the same for all rates of all maturities T.

- The bond dynamics is: $\frac{dDF(t,T)}{DF(t,T)} = [r_t + \eta_t \sigma (T-t)]\, dt - \sigma (T-t) dW$
  - Hence the risk premium is excess return per unit of (bond) volatility.
  - And we reiterate that risk premium must be the same for all bonds.

- To find the short rate dynamics:
  - First integrate forward rate: $f(t,T) = f(0,T) + \sigma^2 t \left(T - \frac{t}{2}\right) - \sigma \int_0^t \eta_s\, ds + \sigma W_t$
  - Find integrated form for r: $r_t = f(t,t) = f(0,t) + \frac{\sigma^2 t^2}{2} - \sigma \int_0^t \eta_s\, ds + \sigma W_t$
  - Differentiate:

$$dr_t = \left[\underbrace{\frac{\partial f(0,t)}{\partial t}}_{\text{follow forwards}} \underbrace{+\sigma^2 t}_{\text{convexity adjustment}} \underbrace{-\sigma \eta_t}_{\text{risk premium}}\right] dt + \sigma dW$$

Markov Property

- From the integrated equation on forward rates:

$$f(t,T) = \underbrace{f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right)}_{\text{deterministic depends on } T} - \underbrace{\sigma \int_0^t \eta_u\, du + \sigma W_t}_{\text{random does not depend on } T \text{ common to all forwards}}$$

- It follows that $f(t,T_2) - f(t,T_1) = f(0,T_2) - f(0,T_1) + \sigma^2 t(T_2 - T_1)$ and $f(t,T_2) = f(t,T_1) + f(0,T_2) - f(0,T_1) + \sigma^2 t(T_2 - T_1)$.
- All forward rates are deterministic functions of one another – and that function does not depend on risk premium.
  - This means that the dynamics of the entire YC can be reduced to the dynamics of one arbitrary forward rate of some maturity T^*.

$$df(t,T^*) = \left[\sigma^2(T^* - t) - \eta_t \sigma\right] dt + \sigma dW$$

  - And all other rates at time t are found as a function of `f(t,` T^*`)` with the reconstruction formula:

$$f(t,T) = f(t,T^*) + f(0,T) - f(0,T^*) + \sigma^2 t(T - T^*)$$

  - And reconstruct the whole future YC as a function of the short rate alone:

$$f(t,T) = r_t + \underbrace{f(0,T) - f(0,t)}_{\text{today's slope}} + \underbrace{\sigma^2 t(T-t)}_{\text{convexity adjustment}}$$

From Curve to Short Rate Model and Back

- Change of variable: $X_t \equiv r_t - f(0,t)$, where:
  - X: "random factor", distance of realized short rate from forward.
  - Then we have the factor dynamics: $dX_t = \left[\sigma^2 t - \sigma \eta_t\right] dt + \sigma dW$.
  - And reconstruction:

$$f(t,T) = \underbrace{f(0,T)}_{\text{today's forward}} + \underbrace{X_t}_{\text{common factor to all rates}} + \underbrace{\sigma^2 t(T-t)}_{\text{convexity adjustment}}$$

- Crucial property for an efficient implementation
  - No need simulate the whole YC.
  - Only simulate factor X.
  - And reconstruct the whole future YC as a known function of X.
- Equivalence between short rate model and YC model: Short rate at time t encapsulates the whole curve at time t.
- Markov property not satisfied for general YC models.
  - Satisfied in the simplistic case with only parallel shifts.

reconstruct future YC

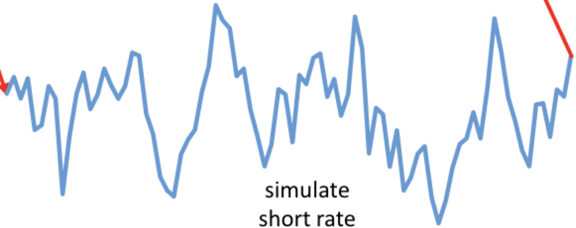
$$f(0,t)$$

$$f(t,T) = r_t + f(0,T) - f(0,t) + \sigma^2 t(T-t)$$

$$dr_t = \left[\frac{\partial f(0,t)}{\partial t} + \sigma^2 t - \sigma\eta_t\right]dt + \sigma dW$$

today's short rate

$$r_0 = f(0,0)$$



future short rate

$$r_{T*}$$

simulate short rate

Simple Parallel IRM: Take Away

- With parallel shifts, the dynamics or rates under the historical probability must be:

$$df(t,T) = \underbrace{\sigma^2 (T-t)\, dt}_{\text{deterministic steepening}} - \underbrace{\eta_t \sigma\, dt}_{\text{Risk premium}} + \underbrace{\sigma dW}_{\text{random parallel shift}}$$

- The risk premium must be the same for all rates, independently of their maturity T.
- We have one risk premium by factor (one in our model), but same for all assets (bonds).
- It is the unicity of the risk premium under the historical probability that makes the dynamics arbitrage-free.
- The induced dynamics on bonds is:

$$\frac{dDF(t,T)}{DF(t,T)} = \underbrace{r_t dt}_{\substack{\text{earns short rate as}\\ \text{maturity approaches}}} + \underbrace{\eta_t \sigma (T-t) dt}_{\text{risk premium}} - \underbrace{\sigma (T-t) dW}_{\text{volatility} \propto \text{duration}}$$

- That model satisfies the Markov property: all rates are deterministic functions of one another.

The End Thank You!

Questions?