Seminar 12. Auctions and previous exam problems

November 14, 2022

The following review part is based on Jehle & Reny pp.428-432

First-price sealed-bid auction

There are N bidders in an auction, and any bidder i has a private value v_i for the auctioned good. Bidder i can submit any sealed bid b_i. We define the bid that can maximize the bidder's payoff (best response) as $b_i^* = \hat{b}(v_i)$. We assume

- Private value v_i is independent among the bidders; the CDF of v_i is $F(x)$
- Best response bidding function $\hat{b}(\cdot)$ is strictly increasing and identical for all bidders (we want to find a symmetric NE).

What is the BR $\hat{b}(v_i)$ for bidder i ?

If bidder i with $v_i = r$ submits a price higher than bidder 1, we must have:

$$\hat{b}(v_1) < \hat{b}(r) \iff v_1 < r, i \neq 1$$

Therefore bidder i can guess how likely he/she defeats bidder 1,

$$Pr\{\text{bidder } i \text{ defeats bidder 1}\} = Pr\{v_1 < r\} = F(r)$$

and similarly,

$$Pr\{\text{bidder } i \text{ defeats all the other N-1 bidders}\} = F^{(N-1)}(r)$$

Here is another interesting way to understand function $F^{(N-1)}(x)$:
Since "*bidder i defeats all the other N-1 bidders*" $\iff$ "*bidder i defeats the bidder with the second-highest private value*", if we denote the second-highest private value as θ, we can also write the probability above as

$$Pr\{\theta < r\} = F^{(N-1)}(r) \tag{1}$$

We find that $F^{(N-1)}(x)$ is actually the CDF of θ! We will use the conclusion later in equation 2.

The maximized expected utility function of bidder i is:

$$\begin{aligned} u(r) &= Pr\{\text{bidder } i \text{ defeats all the other N-1 bidders}\} \times (r - \hat{b}(r)) \\ &+ Pr\{\text{bidder } i \text{ lose the auction}\} \times 0 \\ &= F^{(N-1)}(r)(r - \hat{b}(r)) + (1 - F^{(N-1)}(r)) \times 0 \\ &= F^{(N-1)}(r)(r - \hat{b}(r)) \end{aligned}$$

(To calculate $\hat{b}(\cdot)$) Imagine the real private value of bidder i is actually v, not r, but the bidder reacted (by mistake) as if his/her private value was r, i.e. bid at $\hat{b}(r)$ instead of $\hat{b}(v)$. When the bidder realized the mistake, the price had already been submitted...

The expected utility function due to the mistake is (eq. 9.1, Jehle & Reny pp.430):

$$u(r, v) = F^{(N-1)}(r)(v - \hat{b}(r))$$

We know only when the wrong private value r is luckily the same as the real private value v, will the expected payoff be maximized (since $\hat{b}(v)$ instead of $\hat{b}(r)$ is the best response). That is to say, $r = v$ solves $\max_r u(r, v)$.

The FOC of $\max_r u(r, v)$ is (eq. 9.2, Jehle & Reny pp.430):

$$\frac{dF^{(N-1)}(r)(v - \hat{b}(r))}{dr} = (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - F^{(N-1)}(r)\hat{b}'(r) = 0$$

Since $r = v$ solves $\max_r u(r, v)$, it must solve the FOC above, i.e. (eq. 9.3, Jehle & Reny pp.430)

$$(N-1)F(v)^{N-2}f(v)(v - \hat{b}(v)) - F^{(N-1)}(v)\hat{b}'(v) = 0$$

Which gives,

$$(N-1)F(v)^{N-2}f(v)\hat{b}(v)) + F^{(N-1)}(v)\hat{b}'(v) = (N-1)F(v)^{N-2}vf(v)$$

(If we treat v as a variable:) $$\frac{dF^{(N-1)}(v)\hat{b}(v)}{dv} = (N-1)F(v)^{N-2}vf(v)$$

$$F^{(N-1)}(v)\hat{b}(v) = \int_0^v (N-1)F^{(N-2)}(x)xf(x)dx + C$$

(C is constant)

Since a bidder with $v = 0$ must bid 0 to maximize the expected payoff, we know $\hat{b}(0) = 0$ in the NE.

$$\begin{aligned} F(0)^{N-1}\hat{b}(0) &= \int_0^0 (N-1)F^{(N-2)}(x)xf(x)dx + C \\ 0 &= 0 + C \end{aligned}$$

Thus,

$$\begin{aligned} F^{(N-1)}(v)\hat{b}(v) &= \int_0^v (N-1)F^{(N-2)}(x)xf(x)dx \\ \Rightarrow \quad \hat{b}(v) &= \frac{\int_0^v (N-1)F^{(N-2)}(x)xf(x)dx}{F^{(N-1)}(v)} \\ &= \frac{\int_0^v xdF^{(N-1)}(x)}{F^{(N-1)}(v)} \\ &= \frac{1}{F^{(N-1)}(v)}\int_0^v xdF^{(N-1)}(x) \qquad (eq.9.5) \\ &= \int_0^v xd\frac{F^{(N-1)}(x)}{F^{(N-1)}(v)} \end{aligned}$$

$\hat{b}(v)$ as the conditional expectation

Recall that in Seminar 11, we already know: for a random variable $\theta \in [a,b] \subset \mathbb{R}_1$ and $m \in (a,b)$

$$E(\theta|\theta \le m) = \int_a^m xdF(x|\theta \le m) = \int_a^m xd\frac{F(x)}{F(m)}$$

where $F(x) = Pr(\theta \le x)$ is the CDF of θ.

$\hat{b}(v)$ is actually an expectation of some variable θ conditional on $\theta \le v$, and the variable θ's CDF is $Pr(\theta \le x) = F^{(N-1)}(x)$. In equation 1 we already showed θ is actually the second highest private value.

In conclusion, the the BR of any bidder i with private value v, is the expectation of the second highest private value conditional on v is higher than the second highest private value (i.e. bidder i wins):

$$\hat{b}(v) = E(\theta|\theta \le v) \tag{2}$$

1 Jehle & Reny pp.484, exercise 9.2

Show in two ways that the symmetric equilibrium bidding strategy of a first-price auction with N symmetric bidders each with values distributed according to F, can be written as

$$\hat{b}(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx$$

For the first way, use our solution from the text and apply integration by parts. For the second way, use the fact that $F^{N-1}(r)(v-\hat{b}(r))$ is maximised in r when $r = v$ and then apply the envelope theorem to conclude that $d(F^{N-1}(v)(v-\hat{b}(v))/dv = F^{N-1}(v)$; now integrate both sides from 0 to v.

Method 1: Integration by parts

Integration by parts:

$$\int_a^b u(x)\cdot v'(x)dx = [u(x)\cdot v(x)]_a^b - \int_a^b u'(x)\cdot v(x)dx$$

We start from the euqation above equation (9.5) on Jehle & Reny pp.431, let's call it equation 3,

$$\hat{b}(v) = \frac{(N-1)}{F^{(N-1)}(v)}\int_0^v xf(x)F^{(N-2)}(x)dx \tag{3}$$

Denote,

$$\begin{cases} u(x) = x \\ v(x) = \frac{F^{(N-1)}(x)}{(N-1)} \end{cases} \Rightarrow \begin{cases} u'(x) = 1 \\ v'(x) = F^{(N-2)}(x)f(x) \end{cases}$$

Equation 3 can be rewritten as,

$$\begin{aligned}
\hat{b}(v) &= \frac{(N-1)}{F^{(N-1)}(v)}\int_0^v xf(x)F^{(N-2)}(x)dx \\
&= \frac{(N-1)}{F^{(N-1)}(v)}\int_0^v u(x)\cdot v'(x)dx \\
\text{(Integration by parts:)} \quad &= \frac{(N-1)}{F^{(N-1)}(v)}\left\{[u(x)\cdot v(x)]_0^v - \int_0^v u'(x)\cdot v(x)dx\right\} \\
&= \frac{(N-1)}{F^{(N-1)}(v)}\left\{\left[x\cdot\frac{F^{(N-1)}(x)}{(N-1)}\right]_0^v - \int_0^v 1\cdot\frac{F^{(N-1)}(x)}{(N-1)}dx\right\} \\
&= \frac{(N-1)}{F^{(N-1)}(v)}\left\{v\cdot\frac{F^{(N-1)}(v)}{(N-1)} - \frac{1}{N-1}\int_0^v F^{(N-1)}(x)dx\right\} \\
&= v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{(N-1)} dx
\end{aligned}$$

Method 2: Envelop theorem

Envelop theorem (unconstraint case):

$f(x;a)$ is a function of x with a as paratmeter. Given any a, let x^* be the solution maximizing or minimizing object function $f(x,a)$, i.e. $\max_x f(x,a) = f(x^*,a)$, then

$$\frac{df(x^*,a)}{da} = \frac{f(x,a)}{da}\Big|_{x=x^*}$$

We start from euqation (9.1) on Jehle & Reny pp.430 (the one called as *"expected utility function due to the mistake"* in the review part),

$$u(r,v) = F^{(N-1)}(r)(v - \hat{b}(r))$$

We know $r^* = v$ maximizes $u(r,v)$, i.e.

$$\max_r u(r,v) = u(r^*,v) = F^{(N-1)}(v)(v - \hat{b}(v)) \tag{4}$$

On the other hand, by envelop theorem,

$$\begin{aligned}\frac{du(r^*,v)}{dv} &= \frac{f(r,v)}{dv}\Big|_{r=v} \\ &= F^{(N-1)}(r)\Big|_{r=v} \\ &= F^{(N-1)}(v)\end{aligned}$$

we find another way to express $u(r^*,v)$:

$$u(r^*,v) = \int_0^v F^{(N-1)}(x)dx + C \tag{5}$$

($C = 0$ since $u(v=0) = 0 = 0 + C$)
Therefore, by equation 4 and equation 5,

$$\begin{aligned}F^{(N-1)}(v)(v - \hat{b}(v)) &= \int_0^v F^{(N-1)}(x)dx \\ \hat{b}(v) &= v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{(N-1)} dx\end{aligned}$$

2 Jehle & Reny pp.484, exercise 9.1 - Show that the bidding strategy in (9.5) is strictly increasing.

By exercise 9.1 we know,

$$\hat{b}(v) = v - \frac{\int_0^v F^{N-1}(x)dx}{F^{N-1}(v)}.$$

Then:

$$\begin{aligned}
\frac{d}{dv}\left(\hat{b}(v)\right) &= \frac{d}{dv}\left(v - \frac{\int_0^v F^{N-1}(x)dx}{F^{N-1}(v)}\right) \\
&= 1 - \frac{\frac{\int_0^v F^{N-1}(x)dx}{dv} \cdot [F^{N-1}(v)] - \left[\int_0^v [F^{N-1}(x)]dx\right] \cdot \frac{F^{N-1}(v)}{dv}}{\left[F^{N-1}(v)\right]^2} \\
&= 1 - \frac{F^{N-1}(v) \cdot [F^{N-1}(v)] - \left[\int_0^v [F^{N-1}(x)]dx\right] \cdot [(N-1)F^{N-2}(v)f(v)]}{F^{2N-2}(v)} \\
&= 1 - 1 + \frac{\left[\int_0^v [F^{N-1}(x)]dx\right] \cdot [(N-1)f(v)]}{F^N(v)} \\
&= \frac{\left[\int_0^v [F^{N-1}(x)]dx\right] \cdot [(N-1)f(v)]}{F^N(v)}
\end{aligned}$$

If we assume $F(x)$ is strictly increasing and $N > 1$, then $\hat{b}(v) > 0$.

3 Jehle & Reny pp.485, exercise 9.3

This exercise will guide you through the proof that the bidding function in (9.5) is in fact a symmetric equilibrium of the first-price auction.

(a) Another way to write the derivative

Recall from (9.2) that

$$\frac{du(r,v)}{dr} = (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - F^{N-1}(r)\hat{b}'(r).$$

Using (9.3), show that

$$\begin{aligned}
\frac{du(r,v)}{dr} &= (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - (N-1)F^{N-2}(r)f(r)(r - \hat{b}'(r)) \\
&= (N-1)F^{N-2}(r)f(r)(v - r)
\end{aligned}$$

Equation 9.3 on Jehle & Reny pp.430 is,

$$(N-1)F(v)^{N-2}f(v)\hat{b}(v)) + F^{(N-1)}(v)\hat{b}'(v) = (N-1)F(v)^{N-2}vf(v)$$

Equation 9.3 holds for any v, since v is the variable. It will also hold if we take $v = r$, i.e.

$$(N-1)F^{N-2}(r)f(r)\hat{b}(r)) + F^{(N-1)}(r)\hat{b}'(r) = (N-1)F^{N-2}(r)rf(r)$$

which yields,

$$\begin{aligned}
F^{(N-1)}(r)\hat{b}'(r) &= (N-1)F^{N-2}(r)rf(r) - (N-1)F^{N-2}(r)f(r)\hat{b}(r)) \\
&= (N-1)F^{N-2}(r)f(r)\left[r - \hat{b}(r)\right]
\end{aligned}$$

Substituting the result above into equation 9.2 yields,

$$\begin{aligned}\frac{du(r,v)}{dr} &= (N-1)F^{N-2}(r)f(r)(v-\hat{b}(r)) - F^{N-1}(r)\hat{b}'(r)\\ &= (N-1)F^{N-2}(r)f(r)(v-\hat{b}(r)) - (N-1)F^{N-2}(r)f(r)\left[r-\hat{b}(r)\right]\\ &= (N-1)F^{N-2}(r)f(r)(v-r)\end{aligned}$$

(b) Minimum or maximum?

Use the result in part (a) to conclude that $du(r,v)/dr$ is positive when $r<v$ and negative when $r>v$, so that $u(r,v)$ is maximised when $r=v$.

$$\frac{du(r,v)}{dr} = (N-1)F^{N-2}(r)f(r)(v-r)$$

Obviously, if we assume $F(x)$ is strictly increasing and $N>1$, $\frac{du(r,v)}{dr}>0$ when $r<v$ and $\frac{du(r,v)}{dr}<0$ when $r>v$. Therefore $r=v$ maximizes $u(r,v)$.

4 Problems 2 of the exam in ECON4240, Spring 2005

Consider a strategic situation between an employer (E) and a worker (W). E can either accept (A) or reject (R) W. W can either become skilled (S) through education, or remain unskilled (U). W can be of two types; either he is inherently high ability (H) or he is inherently low ability (L). The players' payoffs depending on their actions and W's type is shown below.

H

	S	U
A	2, 3	-1, 2
R	0, 1	0, 0

L

	S	U
A	-1,-1	-3,-2
R	0, -3	0, 0

a) Rationalizable strategies & NE

For each of these games, determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy Nash equilibria.

In Game H: Only A and S are rationalizable. (A,S) is the unique NE.
In Game L. Only R and U are rationalizable. (R,U) is the unique NE.

b) Bayesian normal form

Assume next that only W knows his own type, while player E thinks that the two types of W are equally likely. Model this situation in an ex ante perspective by specifying the Bayesian normal form.

Denote the Worker's contingent choices are: If Game H, S and U; If Game L, S' and U'. The Baysian normal form is:

	SS'	SU'	US'	UU'
A	½, 1	-½, ½	-1, ½	-2, 0
R	0, -1	0, ½	0,-3/2	0, 0

c) Rationalizable strategy & BNE

For the Bayesian normal form found in part (b), determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy and/or mixedstrategy Nash equilibria.

	SS' p	SU' 1-p	US'	UU'
A q	½, 1	-½, ½	-1, ½	-2, 0
R 1-q	0, -1	0, ½	0,-3/2	0, 0

(1) Rationalizable strategy

For the Employer, both A and R are rationalizable.

For the Worker, US' is dominated by SS'; UU' is dominated by SU'. The rationalizable strategies are SS' and SU'.

(2) Pure-strategy NE

$$(A, SS'), (R, SU')$$

(3) Mixed-strategy NE

Since US' and UU' are dominated, the Employer believes the Worker will choose them with probability (0,0).

The Employer believes the Worker chooses SS' and SU' with probability $(p, 1-p)$; The Worker believe the Employer chooses A and R with probability $(q, 1-q)$.

$$E(U_A^E) = p \times 0.5 + (1-p) \times (-0.5) = p - 0.5$$
$$E(U_R^E) = p \times 0 + (1-p) \times 0 = 0$$
$$E(U_A^E) = E(U_R^E) \Rightarrow p = 0.5$$

$$E(U_{SS'}^W) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$E(U_{SU'}^W) = q \times 0.5 + (1-q) \times 0.5 = 0.5$$
$$E(U_{SS'}^W) = E(U_{SU'}^W) \Rightarrow 2q - 1 = 0.5 \Rightarrow \frac{3}{4}$$

The Mixed-strategy NE is: the Employer chooses A and R with probability $(\frac{3}{4}, \frac{1}{4})$; Worker chooses SS' and SU' with probability $(0.5, 0.5)$, i.e.

$$\{(\frac{3}{4}, \frac{1}{4}), (0.5, 0.5, 0, 0)\}$$

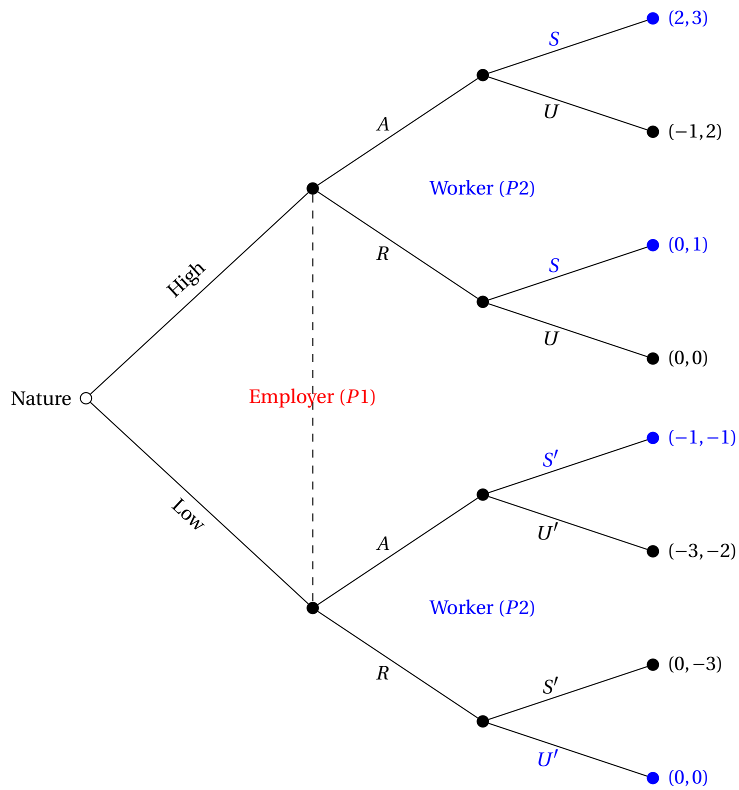
5 Problems 3 of the exam in ECON4240, Spring 2005

Problem 3 (20 %) Consider again the strategic situation between an employer (E) and a worker (W) described in Problem 2. Assume (as in parts b and c) of Problem 2) that only W knows his own type, while player E thinks that the two types of W are equally likely.

a) (Screening)

Assume now that E acts before W, and that E's choice of A or R can be observed by W before he makes his choice of S or U. Show that there is a unique subgame perfect Nash equilibrium.

Extensive Form:



The strategy of the $Worker$:

- If High:
  - when E chooses A, W chooses S
  - when E chooses R, W chooses S
- If Low:
  - when E chooses A, W chooses S'
  - when E chooses R, W chooses U'

For the $Employer$:

$$E(U_A^E) = \frac{1}{2} \times 2 + \frac{1}{2} \times (-1) = 0.5$$
$$E(U_R^E) = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0$$
$$\Rightarrow E(U_A^E) > E(U_R^E)$$

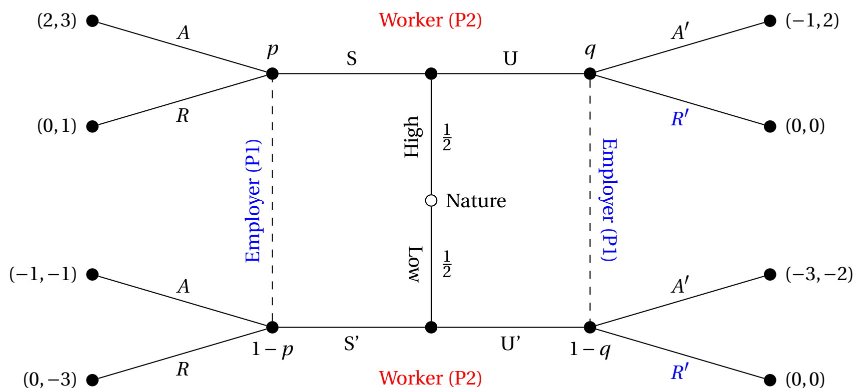
The strategy of the $Employer$ is to choose A.

Therefore there is a SPNE:
{(For High ability Worker:) S after A, S after R. (For Low ability Worker:) S' after A, U' after R ; (For Employer) A}

b) (Signaling)

Assume now that W acts before E, and that W's choice of S or U can be observed by E before she makes her choice of A or R. Show that there is a unique perfect Bayesian equilibrium.

Extensive Form:



The Worker has 4 possible strategies: SS', UU', SU', US'.

Denote the updated belief of the Employer :

- $Pr(\text{Strong}|P/P') = p$
- $Pr(\text{Weak}|U/U') = q$

(1) When the Employer believes the Worker chooses SS'
Then $p = Pr(\text{High}) = 0.5$,

$$E[U_A^E] = 0.5 \times 2 + 0.5 \times (-1) = 0.5$$
$$E[U_R^E] = 0.5 \times 0 + 0.5 \times 0 = 0$$
$$E[U_A^E] > E[U_R^E]$$

The BR of the Employer is to choose A.

If the Worker surprisingly chooses U/U', we can see that R' dominates A' for the Employer. Therefore, the Low ablility Worker will deviate from S' to U' to have a higher payoff (U' and R' leads to $(0,0)$)

$\Rightarrow SS'$ is not part of a PBE.

(2) When the Employer believes the Worker chooses UU'
Since R' dominates A', the Employer will always choose R'. For a High ability Worker, deviating from U to S will always lead to higher payoff (either 3 or 1), no matter how the Employer reacts.

$\Rightarrow UU'$ is not part of a PBE.

(3) When the Employer believes the Worker chooses SU'

Then $p = 1, q = 0$,

The BR of the Employer is to choose A after S/S' (payoff is be $(2,3)$) and R' after U/U'(payoff is be $(0,0)$).

If the High ability Worker eviates to U, the payoff is $(0,0)$, lower than $(2,3)$; If the Low ability Worker eviates to S, the payoff is $(-1,-1)$, lower than $(0,0)$;

There is no incentive for the Worker to deviate. $\Rightarrow SU'$ is part of a PBE.

(4) When the Employer believes the Worker chooses US'

Then $p = 0, q = 1$,

The BR of the Employer is to choose R after S/S' and R' after U/U'.

For a High ability Worker, deviating from U to S will increase the payoff from $(0,0)$ to $(0,1)$.

For a Low ability Worker, deviating from S' to U' will increase the payoff from $(0,-3)$ to $(0,0)$.

$\Rightarrow US'$ is not part of a PBE.

In conclusion, there is only one PBE:

$$\{SU', AR', p = 1, q = 0\}$$