Econ 206 - Homework 1 (1.4, 1.8, 1.9, 1.13)

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1.4) Prove that if $\succsim$ is a preference relation, then the relation $\succ$ is transitive and the relation $\sim$ is transitive.
Also show that if $x^1 \sim x^2 \succsim x^3$, then $x^1 \succsim x^3$.

Using the definition of "$\succ$" : $x^1 \succ x^2$ IFF $x^1 \succsim x^2$ and $x^2 \not\succsim x^1$ ($\star$)

•) to prove "$\succ$" is transitive we need to show that :

if $x^1 \succ x^2$ and $x^2 \succ x^3$ then $x^1 \succ x^3$

using ($\star$) it implies :

$x^1 \succ x^2 \iff x^1 \succsim x^2$ and $x^2 \not\succsim x^1$ ... (1)

$x^2 \succ x^3 \iff x^2 \succsim x^3$ and $x^3 \not\succsim x^2$ ... (2)

THUS we have $x^1 \succsim x^2$ and $x^2 \succsim x^3$, which by transitivity implies $x^1 \succsim x^3$ (3)

Proof by contradiction :

Assume $x^3 \succsim x^1$

from (1) $x^1 \succsim x^2$, which implies $x^3 \succsim x^2$ (by transitive property)

which contradicts (2)

so we have $x^3 \not\succsim x^1$ ... (4)

From (3) and (4) we have $x^1 \succsim x^3$ and $x^3 \not\succsim x^1 \iff x^1 \succ x^3$. ■

•) to prove "$\sim$" is transitive we have to show that

if $x^1 \sim x^2$ and $x^2 \sim x^3$ then $x^1 \sim x^3$

Using the definition of "$\sim$" ; $x^1 \sim x^2$ iff $x^1 \succsim x^2$ and $x^2 \succsim x^1$ ($\star\star$)

which implies :

$x^1 \sim x^2 \iff x^1 \succsim x^2$ and $x^2 \succsim x^1$ ... (5)

$x^2 \sim x^3 \iff x^2 \succsim x^3$ and $x^3 \succsim x^2$ ... (6)

thus, we have $x^1 \succsim x^2$ and $x^2 \succsim x^3$, which by transitivity implies $x^1 \succsim x^3$ (7)

and we have $x^3 \succsim x^2$ and $x^2 \succsim x^1$, which by transitivity implies $x^3 \succsim x^1$ (8)

From (7) $x^1 \succsim x^3$ and (8) $x^3 \succsim x^1 \iff x^1 \sim x^3$ ■

1.4*) o) Show that if $x^1 \sim x^2 \succsim x^3$ then $x^1 \succsim x^3$

*continued...

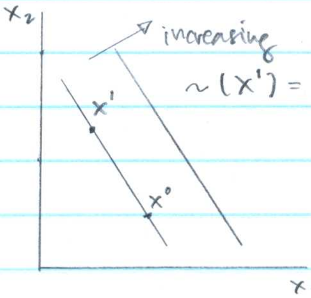
Let $x^1 \sim x^2$ and $x^2 \succsim x^3$

By definition of "$\sim$" $x^1 \sim x^2 \iff x^1 \succsim x^2$ and $x^2 \succsim x^1$

THUS we have $x^1 \succsim x^2$ and $x^2 \succsim x^3$, which by transitivity implies:

$$x^1 \succsim x^3 \blacksquare$$

1.8) Sketch a map of indifference sets that are parallel, negatively sloped straight lines, with preference increasing northeasterly. We know that preferences such as these satisfy axioms 1, 2, 3 and 4. Prove that they also satisfy axiom 5'. Prove that they do not satisfy axiom 5.



$\sim(x^1) = \sim(x^0)$ by definition of Axiom 5 and 5'

o) convex

IF $x^1 \succsim x^0$ THEN $tx^1 + (1-t)x^0 \succsim x^0 \quad \forall t \in [0,1]$

o) strictly convex

IF $x^1 \neq x^0$ AND $x^1 \succsim x^0$

THEN $tx^1 + (1-t)x^0 \succ x^0 \quad \forall t \in (0,1)$

From the picture above, whenever we pick any two points, it shows that

$$tx^1 + (1-t)x^0 \succsim x^0 \quad \forall t \in [0,1]$$

which implies that it satisfies Axiom 5 $\blacksquare$

Since $x^1 \sim x^0$

$$tx^1 + (1-t)x^0 \succsim x^1 \quad \forall t \in [0,1]$$

which means that $(tx^1 + (1-t)x^0)$ belongs to the indifference set of x^0

$$tx^1 + (1-t)x^0 \in \sim(x^0)$$

using the negation of definition

$$x^a \succ x^b \iff x^a \succsim x^b \text{ AND } x^b \not\succsim x^a$$

$$\neg(x^a \succ x^b) \iff \neg(x^a \succsim x^b \text{ AND } x^b \not\succsim x^a)$$

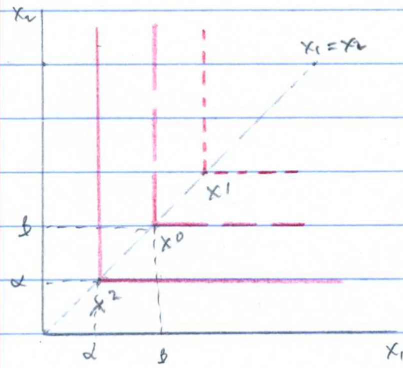
$$x^a \not\succ x^b \iff x^a \not\succsim x^b \text{ OR } x^b \succsim x^a$$

thus, if $x^b \succsim x^a$ it is sufficient to state $x^a \not\succ x^b$

since $x^b \sim x^a \iff x^b \succsim x^a$ and $x^a \succsim x^b$, then it is also sufficient to state $x^a \not\succ x^b$

Since $tx^1 + (1-t)x^0 \in \sim(x^0)$, we have $tx^1 + (1-t)x^0 \not\succ x^0$ (not strictly convex) $\blacksquare$

1.9) Sketch a map of indifference sets that are all parallel right angles that "kink" on the line $x_1 = x_2$. If preferences increases northeasterly, these preferences will satisfy axioms 1, 2, 3 and 4'. Prove that they also satisfy axiom 5'. Do they satisfy Axiom 4? Do they satisfy Axiom 5?



Axiom 4 : strict monotonicity
For all $x^0, x^1 \in \mathbb{R}^n_+$, if $x^0 \geq x^1$, then $x^0 \succsim x^1$, while if $x^0 \gg x^1$, then $x^0 \succ x^1$.
The graph shows preference increases northeasterly which means, since $x^0 \gg x^1$, then $x^0 \succ x^1$ ... (1)

The indifference sets are parallel right angles that "kink" on the line $x_1 = x_2$ and preference increases northeasterly means that if $x^0 \geq x^1$ implies $x^0 \succsim x^1$. (2)
Statements (1) and (2) fulfills the strict monotonicity Axiom.

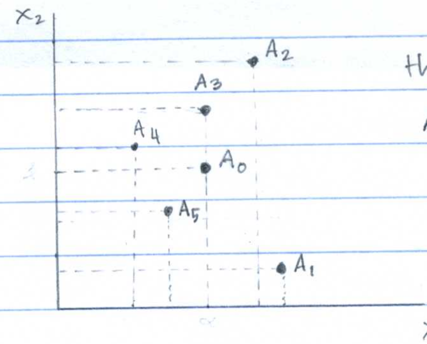
Axiom 5 : strict convexity. If $x^1 \neq x^0$ and $x^1 \succsim x^0$, then $tx^1 + (1-t)x^0 \succ x^0$ for all $t \in (0,1)$
From the sketch, pick point (α, α) and (α, β) where $\alpha < \beta$.
Since the point $(\alpha, t\alpha + (1-t)\beta)$ where $t \in (0,1)$ lies between (α, α) and (α, β) in the same indifference curve, we cannot have $(\alpha, t\alpha + (1-t)\beta)$ $\forall t \in (0,1) \succ (\alpha, \alpha)$. This implies that Axiom 5 (strict convexity) is not satisfied.

Axiom 5' : convexity : If $x^1 \succsim x^0$ then $tx^1 + (1-t)x^0 \succsim x^0$ for all $t \in [0,1]$
Let $x \sim y$, then we have $\min[x_1, x_2] = \min[y_1, y_2]$
we need to show that $\min[tx_1 + (1-t)y_1, tx_2 + (1-t)y_2] \geq \min(x_1, x_2) = \min[y_1, y_2]$ $\forall t \in [0,1]$.

$$\min[tx_1 + (1-t)y_1, tx_2 + (1-t)y_2] \geq \min[tx_1, tx_2] + \min[(1-t)y_1 + (1-t)y_2]$$
$$= \min[y_1, y_2] + t[\min(x_1, x_2) - \min(y_1, y_2)] = \min[y_1, y_2]$$

1.13) A consumer has lexicographic preferences over $x \in \mathbb{R}^2_+$ if the relation $\succsim$ satisfies $x^1 \succsim x^2$ whenever $x_1^1 > x_1^2$, or $x_1^1 = x_1^2$ and $x_2^1 \geq x_2^2$

a) Sketch an indifference map for these preferences.



the order of preference for this graph is

$A_1 > A_2 > A_3 > A_0 > A_5 > A_4$

b) Can these preferences be represented by a continuous utility function?

The Axiom of continuity requires there is no sudden preference reversals at all.

Let $x^1 = (20, 100)$ and $x^2 = (20, 50)$

from the definition, we will have $x^1 > x^2$

If we add $\varepsilon > 0$, no matter how small ε is to x_1^2,

we will have $x^1 = (20, 100)$ and $x^2 = (20 + \varepsilon, 50)$

since $x_1^2 > x_1^1$, this implies a reversal in prefrence, where

$x^2 > x^1$

2nd Homework

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1.20 Suppose preferences are represented by the Cobb-Douglass utility function $u(x_1,x_2) = Ax_1^{\alpha} x_2^{1-\alpha}$, $0<\alpha<1$, and $A>0$. Assuming an interior solution, solve for the Marshallian demand functions

↳ budget constraint : $p_1x_1 + p_2x_2 = y$

Lagrange: $\mathcal{L} = u(x_1,x_2) + \lambda(y - p_1x_1 - p_2x_2)$

$$= Ax_1^{\alpha} x_2^{1-\alpha} + \lambda(y - p_1x_1 - p_2x_2)$$

FOC: $\partial\mathcal{L}/\partial x_1 = A\alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0$ .... (1)

$\partial\mathcal{L}/\partial x_2 = A(1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0$ .... (2)

$\partial\mathcal{L}/\partial\lambda = y - p_1x_1 - p_2x_2 = 0$ .... (3)

From (1) and (2)

$$\lambda = \frac{\cancel{A}\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \frac{\cancel{A}(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2}$$

$$\Leftrightarrow p_2 \alpha \frac{x_2^{1-\alpha}}{x_2^{-\alpha}} = p_1(1-\alpha)\frac{x_1^{\alpha}}{x_1^{\alpha-1}} \Leftrightarrow \alpha p_2 x_2 = (1-\alpha)p_1x_1$$

$$\Leftrightarrow x_1 = \frac{\alpha}{1-\alpha}\frac{p_2}{p_1}x_2 \quad \text{or} \quad x_2 = \frac{1-\alpha}{\alpha}\frac{p_1}{p_2}x_1$$

Substitute to (3)

$$y = p_1x_1 + p_2x_2 \Leftrightarrow y = p_1x_1 + \frac{(1-\alpha)}{\alpha}p_1x_1 = \left[1 + \frac{(1-\alpha)}{\alpha}\right]p_1x_1 = \frac{1}{\alpha}p_1x_1$$

$$\Leftrightarrow x_1 = \frac{\alpha y}{p_1}$$

Thus $x_2 = \frac{1-\alpha}{\cancel{\alpha}}\frac{\cancel{p_1}}{p_2}\frac{y}{\cancel{p_1}}\cancel{\alpha} = \frac{(1-\alpha)y}{p_2}$

1.22 We can generalize further the result of the preceding exercize. Suppose that preferences are represented by the utility function $u(x)$. Assuming an interior solution, the consumer's demand function, $x(p,y)$ are determined implicitly by the conditions in (1.10). Now consider the utility function $f(u(x))$, where $f'>0$, and show that the FOC characterizing the solution to the consumer's problem in both cases can be reduced to the same set of equations. Conclude from this that the consumer's demand behavior is invariant to positive monotonic transforms of the utility function

↳ max $u(x)$ subject to $p\cdot x \leq y$

$\mathcal{L}_1(x,\lambda) = u(x) + \lambda_1(y - p\cdot x)$ with FOC $\partial\mathcal{L}_1/\partial x_i = \frac{\partial u(x^*)}{\partial x_i} - \lambda_i^* p_i = 0$

$\partial\mathcal{L}_1/\partial\lambda_1 = y - p\cdot x^* = 0$.

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↳ consider the utility function after positive monotonic transforms

$v(x) = f(u(x))$

max $v(x)$ subject to $p \cdot x \leq y$

$\mathcal{L}_2(x, \lambda) = v(x) + \lambda_2(y - px) = f(u(x)) + \lambda_2(y - px)$

and the FOC

$\partial \mathcal{L}_2 / \partial x_i = \frac{\partial v(x^*)}{\partial x_i} - \lambda_2^* p_i = 0 \Leftrightarrow f'(u(x^*)) \cdot \frac{\partial u(x^*)}{\partial x_i} - \lambda_2^* p_i = 0$

$\partial \mathcal{L}_2 / \partial \lambda_2 = y - px^* = 0$

↳ compare the MRS before and after transformation

$u(x)$ MRS_{ij} at $x^* = \frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j}$

$v(x) = f(u(x))$ MRS_{ij} at $x^* = \frac{\partial v(x)/\partial x_i}{\partial v(x)/\partial x_j} = \frac{f'(u)\, \partial u(x)/\partial x_i}{f'(u)\, \partial u(x)/\partial x_j}$

since $f' > 0$ we can cancel out $f'(u)/f'(u)$

$= \frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j}$ which is the same

1.27 A consumer of two goods faces positive income. His utility function is $u(x_1, x_2) = \max[ax_1, ax_2] + \min[x_1, x_2]$ where $a \in (0,1)$. Derive the Marshallian demand functions

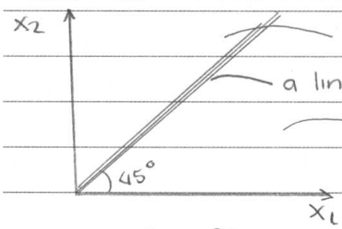
We will have three different situations

↳ if $x_1 > x_2$ $\quad u(x_1, x_2) = \max[ax_1, ax_2] + \min[x_1, x_2]$
$\quad = ax_1 + x_2$

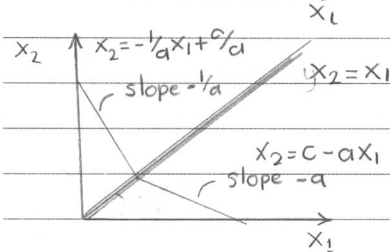
↳ if $x_1 < x_2$ $\quad u(x_1, x_2) = \max[ax_1, ax_2] + \min[x_1, x_2]$
$\quad = ax_2 + x_1$

↳ if $x_1 = x_2$ $\quad u(x_1, x_2) = ax_1 + x_1 = (a+1)x_1$ or $(a+1)x_2$

Draw the indifference curve to help us find the MRS



For $x_1 > x_2$
$u(x_1, x_2) = ax_1 + x_2 = c$
$\Leftrightarrow \quad x_2 = c - ax_1$
slope is $-a$



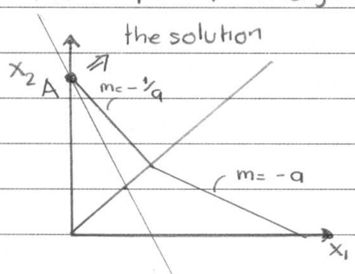
For $x_2 > x_1$
$u(x_1, x_2) = x_1 + ax_2 = c$
$\Leftrightarrow \quad x_2 = \frac{c}{a} - \frac{1}{a}x_1$
slope is $-\frac{1}{a}$

Since $a \in (0,1)$ $\quad \frac{1}{a} > a$

It can be seen from the picture that we will have 5 possible solutions depend on the slope of budget functions $\quad y = p_1x_1 + p_2x_2 \Leftrightarrow x_2 = -\frac{p_1}{p_2}x_1 + \frac{y}{p_2}$

↳ First case

the slope of budget function $< -\frac{1}{a} \Leftrightarrow -\frac{p_1}{p_2} < -\frac{1}{a} \Leftrightarrow \frac{p_1}{p_2} > \frac{1}{a}$



If $\frac{p_1}{p_2} > \frac{1}{a}$ this consumer will only consume x_2 to maximize his/her utility

$x_1 = 0$ and $\quad p_1 x_1 + p_2 x_2 = y$

$\Leftrightarrow \quad 0 + p_2 x_2 = y \Leftrightarrow x_2 = \frac{y}{p_2}$

budget $\quad x_2 = -\frac{p_1}{p_2}x_1 + \frac{y}{p_2}$

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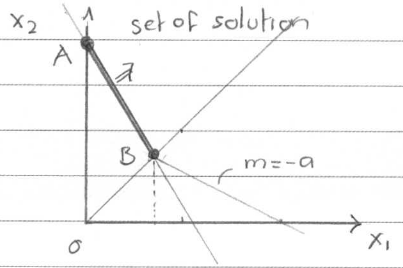
↳ 2^{nd} case

the slope of the budget function $= -1/a \Leftrightarrow -P_1/P_2 = -1/a \Leftrightarrow P_1/P_2 = 1/a$



If $P_1/P_2 = 1/a$ the consumption bundle is the AB line

point B is the intersection between $x_2 = x_1$ and

$x_2 = -P_1/P_2\, x_1 + y/P_2$ so

$\Leftrightarrow x_1 = -P_1/P_2\, x_1 + y/P_2 \Leftrightarrow (1 + P_1/P_2)\, x_1 = y/P_2$

$\Leftrightarrow x_1 = y/(P_1+P_2)$. point $A = (0, y/P_2)$

point $B = (y/(P_1+P_2), y/(P_1+P_2))$

budget $x_2 = -P_1/P_2\, x_1 + y/P_2$

Thus $x_1 \in [0, y/(P_1+P_2)]$ and $x_2 = -P_1/P_2\, x_1 + y/P_2$

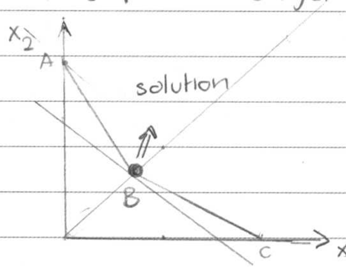
or we can express this by using a linear combination of point A and B

$x(P,y) = \alpha \vec{A} + (1-\alpha)\vec{B}$ with $\alpha \in [0,1]$

↳ 3^{rd} case

the slope of the budget function is between $-1/a$ and $-a \Leftrightarrow -1/a < -P_1/P_2 < -a$

$\Leftrightarrow 1/a > P_1/P_2 > a$



In this case, the consumer will spend his/her money to buy x_1 and x_2 in the same amount

$x_1 = x_2$ thus $p_1 x_1 + p_2 x_2 = y$

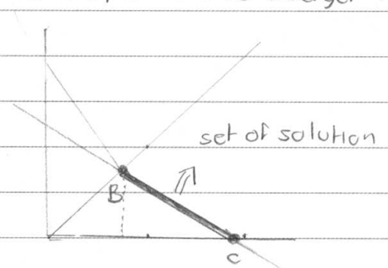
$\Leftrightarrow p_1 x_1 + p_2 x_1 = y$

budget $x_2 = -P_1/P_2\, x_1 + y/P_2$

$\Leftrightarrow x_1 = y/(P_1+P_2) = x_2$

↳ 4^{th} case

the slope of the budget function is $-a \Leftrightarrow -P_1/P_2 = -a \Leftrightarrow P_1/P_2 = a$



This case is similar with 2^{nd} case, but the set of solution is the BC line

point B is $(y/(P_1+P_2), y/(P_1+P_2))$

and point C is $(y/P_1, 0)$

budget $x_2 = -P_1/P_2\, x_1 + y/P_2$

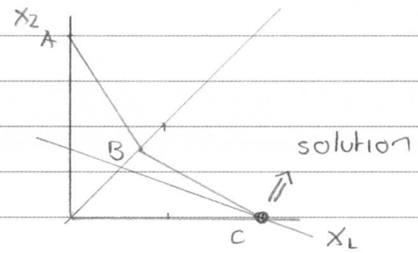
Thus the marshallian demand is

$x_1 \in [y/(P_1+P_2), y/P_1]$ and $x_2 = -P_1/P_2\, x_1 + y/P_2$

or in term of x_2, $x_2 \in [0, y/(P_1+P_2)]$ and $x_1 = -P_2/P_1\, x_2 + y/P_1$

↳ 5^{th} case

the slope of the budget function $> -a \Leftrightarrow -p_1/p_2 > -a \Leftrightarrow p_1/p_2 < a$



In this case the consumer will only purchase X_1 thus $X_2 = 0$

$p_1 x_1 + p_2 x_2 = y$

$\Leftrightarrow p_1 x_1 = y \Leftrightarrow x_1 = y/p_1$

budget $x_2 = -p_1/p_2 \, x_1 + y/p_2$

As a summery, our marshallian demand for $x_1(p,y)$ and $x_2(p,y)$

↳ IF $p_1/p_2 > 1/a$

$x_1(p,y) = 0 \quad , \quad x_2(p,y) = y/p_2$

↳ IF $p_1/p_2 = 1/a$

$x_1 \in [0, y/(p_1+p_2)]$ and $x_2(p,y) = -p_1/p_2 \, x_1 + y/p_2$

↳ IF $1/a > p_1/p_2 > a$

$x_1(p,y) = x_2(p,y) = y/(p_1+p_2)$

↳ IF $p_1/p_2 = a$

$x_1 \in [y/(p_1+p_2), y/p_1]$ and $x_2(p,y) = -p_1/p_2 \, x_1 + y/p_2$

↳ IF $p_1/p_2 < a$

$x_1(p,y) = y/p_1 \quad x_2(p,y) = 0$

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1.28 An infinetely lived agent owns 1 unit of a commodity that she consumes over her lifetime. The commodity is perect storable and she will receive no more than she has now. Consumption of the commodity in period t is denoted X_t and her lifetime utility function is given by

$$u(x_0, x_1, x_2, \dots) = \sum_{t=0}^{\infty} \beta^t \ln(x_t), \text{ where } 0<\beta<1$$

Calculate her optimal level of consumption in each period!

↳ $u(x_0, x_1, x_2, \dots) = \beta^0 \ln(x_0) + \beta^1 \ln(x_1) + \dots \beta^t \ln(x_t)$

such that $\sum_{t=0}^{\infty} x_t = 1$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(x_t) + \lambda\left(\sum_{t=0}^{\infty} x_t - 1\right)$$

FOC $\partial\mathcal{L}/\partial x_t = \beta^t/x_t - \lambda = 0$, $t = 0, 1, \dots$ .... (1)

$\partial\mathcal{L}/\partial\lambda = \sum_{t=0}^{\infty} x_t - 1 = 0$ ... (2)

from (1) it is implied that $x_t = \beta^t/\lambda$

substitute to (2)

$$\sum_{t=0}^{\infty} x_t = 1 \Leftrightarrow \sum_{t=0}^{\infty} \frac{\beta^t}{\lambda} = 1 \Leftrightarrow \frac{1}{\lambda}\sum_{t=0}^{\infty} \beta^t = 1$$

since $0<\beta<1$, our geometric series converges $\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$

Thus $\frac{1}{\lambda} \cdot \sum_{t=0}^{\infty} \beta^t = 1 \Leftrightarrow \frac{1}{\lambda} \cdot \frac{1}{1-\beta} = 1 \Leftrightarrow \lambda = 1/1-\beta$

↳ from (1) $\frac{\beta^t}{x_t} = \lambda \Leftrightarrow \frac{\beta^t}{x_t} = \frac{1}{1-\beta} \Leftrightarrow x_t^* = \beta^t(1-\beta)$

Advanced Micro H.W3

#1.29

This is about Price Indifference Maps.

This is the locus of prices s.t

$\{ (P_1, P_2) \mid V(P_1, P_2, y) = V \}$ for $V \in \mathbb{R}$

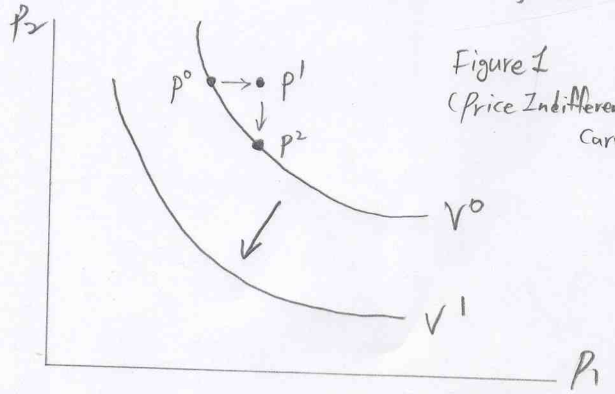

Figure 1 (Price Indifference Curve)

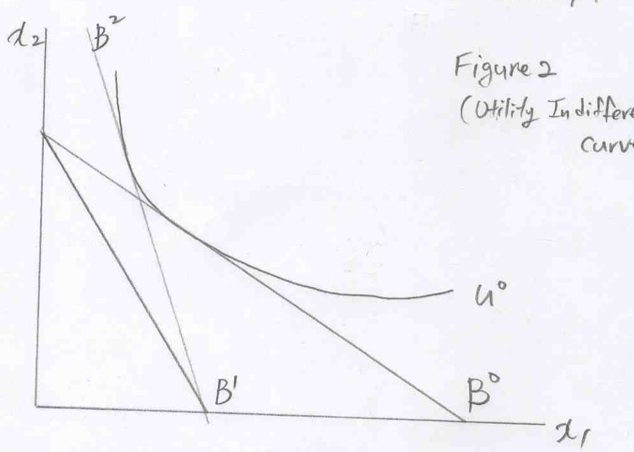

Figure 2 (Utility Indifference curve)

① The slope is negative
② the direction is towards the origin
③ The curve is convex.

For ①

Consider a movement p^0 to p' in Figure 1.

This corresponds a movement B^0 to B' in Figure 2.

However, in B' it can not reach same utility level.

So, we need to decrease the price of good 2.

Then, as $P' \to P^2$ in Figure 1,

$\Rightarrow B' \to B^2$ in Figure 2

Thus, it is negative slope.

For ②

Consider again moving P^0 to P'.

But Utility is lower than before in Figure 2.

Hence, we can intuitively know that the price indifference curves increase towards the origine.

For ③

To understand the convex shape, we need to check that indirect utility is quasiconvex in prices and income.

Thus, the set $\{ (P_1, P_2, y) \mid V(P_1, P_2, y) \leq V \}$ must be convex.

Consider any price, income combinations (p', y) and (p^2, y) ...

By quasiconvexity it follows that

$\{ (P_1, P_2) \mid V(P_1, P_2, y) \leq V \}$

is convex for any V and for any y.

#1.30

$V(p,y) = y(p_1^r + p_2^r)^{-\frac{1}{r}}$

Case 1 : $r \in (0,1)$

If $r \in (0,1)$, then the function $p_1^r + p_2^r$ is concave then $[p_1^r + p_2^r]^{\frac{1}{r}}$ is also concave

Therefore, the indirect utility

$$V(p,y) = \frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}}$$

is convex for any $y>0$.

By definition of convex function, we have that the set of points on and above the graph of the function are a convex set. The set is

$$\left\{ (p_1, p_2, v) \,\middle|\, \frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}} \le v \right\}$$

for any $y>0$

The convexity of this set implies convexity of

$$\left\{ (p_1, p_2) \,\middle|\, \frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}} \le v \right\}$$ for any $y>0$ any $v \in \mathbb{R}$

Since the set is convex for any $y>0$,

$$\left\{ (p_1, p_2, y) \,\middle|\, \frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}} \le v \right\}$$ for any $v \in \mathbb{R}$

this is the definition of a quasiconvex function.

Case 2) : $r<0$

If $r<0$, $p_1^r + p_2^r$ is convex, then

$\frac{1}{[p_1^r + p_2^r]^{\frac{1}{r}}}$ is also convex.

So, the set $\left\{ (p_1, p_2, v) \,\middle|\, \frac{1}{[p_1^r + p_2^r]^{\frac{1}{r}}} \le v \right\}$

is convex for any v

for any $y>0$, the function $\frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}}$ is still convex

thus $\left\{ (p_1, p_2, v) \,\middle|\, \frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}} \le v \right\}$ for any $y>0$

it follows that

$$\left\{ (p_1, p_2, y) \,\middle|\, \frac{y}{[p_1^r + p_2^r]^{\frac{1}{r}}} \le v \right\}$$

is convex for any v

this is the definition of quasiconvex function.

#1.33

proof by contradiction.

Assume that $e(p,u)$ is bounded above in u.

Then, there exists an $M \in \mathbb{R}$ s.t

$$e(p,u) \le M. \text{ for every } u \in \mathbb{R}$$

Let u^* be the utility level s.t

$$e(p,u^*) = M$$

Then, consider $u' > u^*$

Since $e(p,u)$ is strictly increasing in u,

$$e(p,u') > e(p,u^*) = M$$

contradiction!

Therefore, the expenditure function is unbounded above in u. $\blacksquare$

1.36

From the definition,

a) $e(p,u) \equiv \min_{x \in \mathbb{R}^n_+} p \cdot x \quad st \quad u(x) \geq u$

$e(p,u) = p \cdot x^h(p,u)$

If $p^\circ \gg 0$ and $x^\circ = x^h(p^\circ, u^\circ)$,

then $e(p,u^\circ) \leq p \cdot x^\circ$ for all $p \gg 0$

with equality when $p = p^\circ$

b) when $p = p^\circ$, then $e(p^\circ, u^\circ) = p^\circ x^\circ$

x° is the cheapest bundle to achieve u°

Let $f(p) = e(p,u) - p \cdot x$

non positive function

at $p = p^\circ$, $e(p^\circ, u) = p^\circ \cdot x$

c) $f(p^\circ) = 0 \quad \rightarrow$ maximum value.

d) $\dfrac{\partial f(p_0)}{\partial p} = \dfrac{\partial e(p,u)}{\partial p} - x_i = 0$

$$\frac{\partial e(p,u)}{\partial p} = x^*$$

$\blacksquare$

Additional proof for #1.36 (A)

- Using the definition e, Show that

If $p^\circ \gg 0$ and $x^\circ = x^h(p^\circ, u^\circ)$

then $e(p,u^\circ) \leq p \cdot x^\circ$, $\forall p \gg 0$

, equality $p = p^\circ$

Using contradiction,

by definition,

$e(p,u) = \min p \cdot x \quad s.t \quad u(x) = u$

let's assume

$e(p^*, u^\circ) > p^* \cdot x^\circ$

lets x^* is optimal at U°

which means at p^*, $p^* x^*$ is preferred that $p^* x^\circ$

it must be that the budget $p^* x^* < p^* x^\circ$

, otherwise he will choose x°

since they are in the same utility fn U°

$p^* x^* < p^* x^\circ$

Since x^* is the solution (minimum at U°)

then, $p^* x^* = e(p^*, u^\circ)$

$e(p^*, u^\circ) < p^* x^\circ$ contradiction!

Econ 206 Homework # 4

Jehle / Reny 1.40, 1.44, 1.63, 1.65

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1.40. Prove that Hicksian demands are homogenous of degree zero in prices.

↳ we want to show that for any $t > 0$,

$$x^h(tp, u) = x^h(p, u)$$

By Shephard's lemma, the first partial derivative of the expenditure function (homogeneous of degree one in p) with respect to p is the Hicksian demand.

We also know that the derivative's degree of homogeneity is $k-1$.

Then Hicksian demand functions must be homogeneous of degree 0 $(1-1)$ in prices.

1.44. $x^0 = x(p^0, y^0)$

$x^s(p, x^0) = x(p, p \cdot x^0)$

$u^0 = u(x^0)$

$$\frac{\partial x^s(p^0, x^0)}{\partial p_j} = \frac{\partial x(p^0, y^0)}{\partial y} \cdot \frac{\partial y}{\partial p_j} + \frac{\partial x(p^0, y^0)}{\partial p_j}$$

$$= \underbrace{\frac{\partial x^h(p^0; u^0)}{\partial y}}_{=0} \cdot \frac{\partial y}{\partial p_j} + \frac{\partial x^h(p^0, u^0)}{\partial p_j}$$

$$= \frac{\partial x^h(p^0, u^0)}{\partial p_j}$$

1.63. a). $\dfrac{\partial\left(\frac{x_i}{x_j}\right)}{\partial y} = 0$

$y = \dfrac{u}{v}$

$y' = \dfrac{u'v - v'u}{v^2} = \dfrac{1}{v} u' - \dfrac{u}{v^2} v'$

$$\frac{\partial\left(\frac{x_i(p_i, y)}{x_j(p_j, y)}\right)}{\partial y} = \frac{1}{x_j} \cdot \frac{\partial x_i}{\partial y} - \frac{x_i}{x_j^2} \cdot \frac{\partial x_j}{\partial y} = 0$$

$$\frac{1}{\cancel{x_j}} \frac{\partial x_i}{\partial y} = \frac{x_i}{x_j^{\cancel{2}}} \frac{\partial x_j}{\partial y}$$

$$\frac{1}{x_i} \frac{\partial x_i}{\partial y} = \frac{1}{x_j} \cdot \frac{\partial x_j}{\partial y}$$

$$\underbrace{\frac{y}{x_i} \frac{\partial x_i}{\partial y}}_{n_i} = \underbrace{\frac{y}{x_j} \frac{\partial x_j}{\partial y}}_{n_j}$$

$$n_i = n_j$$

b) $\dfrac{\partial x_i}{\partial y} \cdot \dfrac{y}{x_i} = \dfrac{\partial x_j}{\partial y} \cdot \dfrac{y}{x_j} = c \quad \forall i j$

we know that $\sum x_i \cdot p_i = y = x_1 p_1 + x_2 p_2 + \dots x_n p_n$

total derivation w.r.t y

$$\frac{\partial y}{\partial y} = 1 = p_1 \cdot \frac{\partial x_1}{\partial y} + p_2 \frac{\partial x_2}{\partial y} + \dots + p_n \cdot \frac{\partial x_n}{\partial y}$$

$$1 = p_1 \cdot \frac{x_1}{y} \boxed{\frac{y}{x_1} \cdot \frac{\partial x_1}{\partial y}} + p_2 \cdot \frac{x_2}{y} \boxed{\frac{y}{x_2} \frac{\partial x_2}{\partial y}} + \dots + p_n \frac{x_n}{y}$$

$$+ p_n \frac{x_n}{y} \cdot \boxed{\frac{y}{x_n} \cdot \frac{\partial x_n}{\partial y}}$$

$$1 = p_1 \cdot \frac{x_1}{y} \cdot c + p_2 \frac{x_2}{y} \cdot c + \dots + p_n \frac{x_n}{y} \cdot c$$

$$1 = \frac{c}{y} \sum p_i x_i = \frac{c}{\cancel{y}} \cdot \cancel{y}$$

$\therefore$ thus, $c = 1 \implies$ (a) and (b) ...

1.63 c) Homothetic utility function is a utility function u that satisfies $u(x) \geq u(y) \iff u(kx) \geq u(ky)$ for all k

$\Rightarrow$ $v(p, y) = \max u(x)$ s.t $p \cdot x \leq y$

by homothetic

$V(p, y) = y \cdot \max \frac{1}{y} u(x)$ s.t $p \cdot \frac{x}{y} \leq 1$

$V(p, y) = y \cdot \max u\left(\frac{x}{y}\right)$ s.t $p \cdot \frac{x}{y} \leq 1$

$\Rightarrow$ $V(p, y) = y \cdot \max u(x')$ s.t $p \cdot x' \leq 1$

$\Rightarrow$ $V(p, y) = y \cdot V(p, 1)$

$\therefore \frac{\partial V(p, y)}{\partial y} = V(p, 1)$

the marginal utility function of income is independent of income and depends on price (false)

1.65. A consumer with income y^0, faces prices p^0 and enjoys utility $u^0 = v(p^0, y^0)$. Cost of living index (I) is denoted by:

$$I(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)}$$

a) $e(p^1, u^0) = e\left(\frac{p^1}{p^0} \cdot p^0, u^0\right) = \frac{p^1}{p^0} e(p^0, u^0)$

if $p^1 > p_0$, then $p^1 x^0 > p^0 x^0$

thus, $e(p^1, u^0) > e(p^0, u^0)$

$$I(p^0, p^1, u^0) = \frac{p_1}{p_0} > 1$$

similarly for $I(p^0, p^1, u^0) < 1$, where $p^1 < p^0$.

b) from (a), we have

$$\frac{y^1}{y^0} > I = \frac{p^1}{p^0}$$

$$\frac{y^1}{y^0} = \frac{p^1 x^1}{p^0 x^0} > \frac{p^1}{p^0} \implies x^1 > x^0$$

thus $u(x^1) > u(x^0) \rightarrow$ by strict monotonicity

similarly for $\frac{y_1}{y_0} < I$, where $p_1 < p_0$.

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2.1 Show that budget balancedness and homogeneity of $x(p,y)$ are unrelated conditions in the sense that neither implies the other

by homogeneity of $x(p,y)$

$u(x) = u(x(p,y)) = u(x(tp,ty))$

because budget (p,y) and (tp,ty) are the same s.t. $p.x \leq y$

$x(p,y)$ and $x(tp,ty)$ are feasible thus $x(p,y) = x(tp,ty)$

the demand for every good $x_i(p,y)$ is homogeneous of degree zero regardless of budget balancedness

↳ by theorem 2.5

Budget Balancedness AND Symmetry $\Rightarrow$ Homogeneity

which means Homogeneity does not necessarily imply budget balancedness

or budget balancedness ALONE does not imply homogeneity

from page 81

by budget balancedness

$tp \cdot x(tp,ty) = ty \Leftrightarrow p\,x(tp,ty) = y$

and by theorem 1.17 $\sum_{j=1}^{n} p_j \dfrac{\partial x_j(p,y)}{\partial p_i} = -x_i(p,y)$ and $\sum_{j=1}^{n} p_j \dfrac{\partial x_j(p,y)}{\partial y} = 1$

but to prove $f_i'(t) = 0$ with $f_i(t) = x_i(tp,ty)$

we need another assumption that $S(p,y)$ is symmetric

2.2 Suppose that $x(p,y) \in \mathbb{R}^n_+$ satisfies budget balancedness and homogeneity on $\mathbb{R}^{n+1}_{++}$. Show that for all $(p,y) \in \mathbb{R}^{n+1}_{++}$, $s(p,y) \cdot p = 0$ where $s(p,y)$ denotes the Slutsky matrix associated with $x(p,y)$

↳ Recall from the proof of Theorem 1.17
when budget balancedness holds, we may differentiate the budget equation with respect to prices and income to obtain for $i = 1, \dots n$

$$\sum p_j \frac{\partial x_j(p,y)}{\partial p_j} = -x_j(p,y) \qquad \text{and} \qquad \sum p_j \frac{\partial x_j(p,y)}{\partial y} = 1$$

↳ homogeneity
With fixed p and y, $x_i(tp,ty) = x_i(p,y) \quad \forall t$
then if $x_i(tp,ty) = f_i(t)$ = constant or $f_i'(t) = 0$

↳ make a total derivative of $f_i(t) = x_i(tp,ty)$

$$f_i'(t) = \sum_{j=1}^{n} \frac{\partial x_i(tp,ty)}{\partial p_j} \cdot p_j + \frac{\partial x_i(tp,ty)}{\partial y} \cdot y = 0$$

since $ty = tp \cdot x(tp,ty)$, and by dividing by $t > 0$

$$y = \sum_{j=1}^{n} p_j x_j(tp,ty)$$

thus
$$f_i'(t) = \sum_{j=1}^{n} \left[\frac{\partial x_i(tp,ty)}{\partial p_i} p_j + \frac{\partial x_i(tp,ty)}{\partial y} \cdot x_j(tp,ty) \cdot p_j \right]$$

$$0 = \sum_{j=1}^{n} p_j \cdot \left[\frac{\partial x_i(tp,ty)}{\partial p_j} + \frac{\partial x_j(tp,ty)}{\partial y} x_j(tp,ty) \right]$$

If we use
$$S(p,y) = \frac{\partial x_i(tp,ty)}{\partial p_j} + \frac{\partial x_i(tp,ty)}{\partial y} x_j(tp,ty)$$

that implies
$$0 = \sum_{j=1}^{n} p_j \cdot S(p_j, y)$$

$$0 = S(p,y) \cdot p$$

2.4 Suppose that the function $e(p,u) \in \mathbb{R}_{++}$ not necessarily an expenditure function and $x(p,y) \in \mathbb{R}^n_+$, not necessarily a demand function, satisfy the system of partial differential equations given in Section 2.2. Show the following:

(a) If $x(p,y)$ satisfies budget balancedness, then $e(p,u)$ must be homogeneous of degree one in p.

(b) If $e(p,u)$ is homogeneous of degree one in p and for each p, it assumes every nonnegative value as u varies, then $x(p,y)$ must be homogeneous of degree zero in (p,y)

a) ↳ according to (P.83) $\dfrac{\partial e(p,u)}{\partial p_i} = x_i(p, e(p,u))$ for $i = 1 \dots n$ ------ (1)

↳ If $x(p,y)$ satisfies the budget balancedness condition

$$\sum_{i=1}^{n} p_i x_i(p,y) = y \quad \text{for any } y$$

consider $y = e(p,u)$ and we get

$$\sum p_i x_i(p, e(p,u)) = e(p,u)$$

substitute (1) $\displaystyle \sum p_i \left[\frac{\partial e(p,u)}{\partial p_i}\right] = e(p,u) = 1 \cdot e(p,u)$ ---- (2)

↳ by theorem A.27 (Euler's Theorem)

$$\sum x_i \left[\frac{\partial f(x)}{\partial x_i}\right] = k f(x) \Leftrightarrow f(x) \text{ is homogeneous of degree } k \text{ in } x$$

then (2) implies that $e(p,u)$ is homogeneous of degree one in p

b). ↳ If $e(p,u)$ is homogeneous of degree one in p and for each p.

$e(tp,u) = t \cdot e(p,u)$ for any $a > 0$ according to Theorem A.26

↳ Differentiate left hand side (LHS)

$$\frac{\partial e(tp,u)}{\partial p} = \frac{\partial e(tp,u)}{\partial p} \frac{\partial (tp)}{\partial p} = \frac{\partial e(tp,u)}{\partial p} \cdot t$$

↳ Differentiate right hand side (RHS)

$$\frac{\partial [t \cdot e(p,u)]}{\partial p} = t \cdot \frac{\partial e(p,u)}{\partial p}$$

↳ Use P.83 and bring LHS and RHS together

$$\frac{\partial e(tp,u)}{\partial p} \cdot t = \frac{\partial e(p,u)}{\partial p} \cdot t \Leftrightarrow x(tp,y) \cdot \cancel{t} = x(p,y) \cdot \cancel{t}$$

which implies $x(tp,y) = x(p,y) \quad \forall t$

$x(p,y)$ is homogeneous of degree zero

2.6 A consumer has expenditure function $e(p_1, p_2, u) = up_1p_2/(p_1+p_2)$. Find a direct utility function $u(x_1, x_2)$ that rationalizes this person's demand behavior.

$$e(p_1,p_2,u) = \frac{u\, p_1 \cdot p_2}{(p_1+p_2)} \Leftrightarrow e(p_1,p_2,v(p_1,p_2,y)) = \frac{v(p_1,p_2,y)\cdot p_1 \cdot p_2}{(p_1+p_2)} = y$$

$$\Leftrightarrow v(p_1,p_2,y) = \frac{y\cdot(p_1+p_2)}{p_1\cdot p_2}$$

Let assume that $y = 1$ where $p_1 \cdot x_1 + p_2 x_2 = 1$

$$\Leftrightarrow v(p_1,p_2,1) = \frac{(p_1+p_2)}{p_1\cdot p_2}$$

$$\hookrightarrow u(x_1,x_2) = \min_{\{p\}} v(p_1,p_2,1) \quad \text{s.t } p_1x_1 + p_2x_2 = 1$$

$$= \min_{\{p\}} \frac{p_1+p_2}{p_1p_2} \quad \text{s.t } p_1x_1+p_2x_2 = 1$$

$\hookrightarrow$ Lagrange: $\mathcal{L} = \frac{p_1+p_2}{p_1p_2} + \lambda(1 - p_1x_1 - p_2x_2)$

$\hookrightarrow$ FOC

$$\frac{\partial \mathcal{L}}{\partial p_1} = \frac{(p_1\cdot p_2) - (p_1+p_2)p_2}{(p_1 p_2)^2} - \lambda x_1 = 0 \quad \text{.... (1)}$$

$$\frac{\partial \mathcal{L}}{\partial p_2} = \frac{(p_1\cdot p_2) - (p_1+p_2)p_1}{(p_1p_2)^2} - \lambda x_2 = 0 \quad \text{.... (2)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - p_1x_1 - p_2x_2 = 0 \quad \text{.... (3)}$$

$\hookrightarrow$ substitute (1) and (2)

$$\frac{x_1}{x_2} = \frac{\frac{p_1p_2 - (p_1+p_2)p_2}{\cancel{(p_1p_2)^2}}}{\frac{p_1p_2 - (p_1+p_2)p_1}{\cancel{(p_1p_2)^2}}} = \frac{\cancel{p_1p_2} - \cancel{p_1p_2} - p_2^2}{\cancel{p_1p_2} - \cancel{p_1p_2} - p_1^2} = \frac{-p_2^2}{-p_1^2} = \frac{p_2^2}{p_1^2} \Leftrightarrow \frac{x_1}{x_2} = \frac{p_2^2}{p_1^2} \Leftrightarrow \frac{p_2}{p_1} = \sqrt{\frac{x_1}{x_2}}$$

$\hookrightarrow$ substitute to (3)

$$1 - p_1x_1 - p_2x_2 = 0 \Leftrightarrow 1 = p_1x_1 + p_1\cdot\sqrt{\frac{x_1}{x_2}}\cdot x_2 \Leftrightarrow 1 = p_1\left(x_1 + \sqrt{x_1x_2}\right)$$

we have $p_1 = \frac{1}{x_1+\sqrt{x_1x_2}}$ and $p_2 = \frac{1}{x_2+\sqrt{x_1x_2}}$

$$\hookrightarrow u(x_1,x_2) = v(p_1,p_2,1) = \frac{(p_1+p_2)}{p_1\cdot p_2} = \frac{\frac{1}{x_1+\sqrt{x_1x_2}} + \frac{1}{x_2+\sqrt{x_1x_2}}}{\frac{1}{(x_1+\sqrt{x_1x_2})(x_2+\sqrt{x_1x_2})}} = \frac{\frac{(x_2+\sqrt{x_1x_2}) + (x_1+\sqrt{x_1x_2})}{\cancel{(x_1+\sqrt{x_1x_2})(x_2+\sqrt{x_1x_2})}}}{\frac{1}{\cancel{(x_1+\sqrt{x_1x_2})(x_2+\sqrt{x_1x_2})}}}$$

$$u(x_1,x_2) = x_1+x_2+2\sqrt{x_1x_2}$$

Advanced Microeconomics H·W6

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2.8

(a)

	x^0	x^1
p^0	10	6
p^1	22	14

$p^0 \cdot x^1 = 6 \leq 10 = p^0 \cdot x^0$

$\Rightarrow p^1 \cdot x^0 = 22 > 14 = p^1 \cdot x^1$

WARP is satisfied.

(b)

	x^0	x^1
p^0	40	32
p^1	55	44

$p^0 \cdot x^1 = 32 \leq 40 = p^0 \cdot x^0$

$\Rightarrow p^1 \cdot x^0 = 55 > 44 = p^1 \cdot x^1$

WARP is satisfied.

(c)

	x^0	x^1
p^0	5	5
p^1	8	6

$p^0 \cdot x^1 = 5 \leq 5 = p^0 \cdot x^0$

$\Rightarrow p^1 \cdot x^0 = 8 > 6 = p^1 \cdot x^1$

WARP is satisfied.

(d)

	x^0	x^1
p^0	100	60
p^1	110	74

$p^0 \cdot x^1 = 60 \leq 100 = p^0 \cdot x^0$

$p^1 \cdot x^0 = 110 > 74 = p^1 \cdot x^1$

WARP is satisfied

#2.9

(a)

Homogeneity of degree zero gives

$$-P_2 \cdot \frac{dX_2}{dP_2} = P_1 \cdot \frac{dX_2}{dP_1} + y \cdot \frac{dX_2}{dy}$$

Budget Balancedness gives

$$P_1 X_1 + P_2 X_2 = y$$

So, differentiating with respect to P_2 gives

$$P_1 \cdot \frac{dX_1}{dP_2} = -X_2 - P_2 \cdot \frac{dX_2}{dP_2}$$

$$= -X_2 + P_1 \frac{dX_2}{dP_1} + y \cdot \frac{dX_2}{dy} \quad (*)$$

Now, differentiate Budget balancedness wrt y and multiply through by X_2 to get

$$P_1 \cdot X_2 \cdot \frac{dX_1}{dy} = X_2 - P_2 X_2 \frac{dX_2}{dy} \quad (**)$$

Add (*) and (**), cancel $P_1 > 0$ and use (BB) again to get

$$\frac{dX_1}{dP_2} + X_2 \frac{dX_1}{dy} = \frac{dX_2}{dP_1} + X_1 \frac{dX_2}{dy}$$

This is required condition.

(b)

Budget balancedness and WARP imply $X(P,y)$ is homogeneous of degree zero in (P,y) <P.88>

So, by part (a), with two goods, BB and WARP imply that the slutsky matrix associated with $X(P,y)$ is symmetric.

WARP and BB imply that slutsky matrix is negative semidefinite <p.88-90>

So, two goods, BB and WARP imply symmetry and negative semidefinite.

By theorem 2.6 <P.83>

$X(p,y)$ is generated by Utility maximizing behaviour.

This means that

R has no intransitive cycles.

2.10

WARP

$$p^0 \cdot x^1 \le p^0 \cdot x^0 \Rightarrow p^1 \cdot x^0 > p^1 \cdot x^1$$

$$p^0 \cdot x^0 = 42, \quad p^1 \cdot x^1 = 36, \quad p^2 \cdot x^2 = 50$$

(a) ⓘ Compare x^0, x^1:

$p^0 \cdot x^1 = 48$, $p^1 \cdot x^0 = 33$.

$p^0 \cdot x^1 = 48 \nleq p^0 \cdot x^0 = 42$

$p^1 \cdot x^0 = 33 \le 36 = p^1 \cdot x^1 \Rightarrow p^0 x^1 = 48 > 42 = p^0 \cdot x^0$ (WARP satisfied)

Thus, x^1 is revealed preferred to x^0, that is $x^1 \succ^R x^0$

ⓘⓘ compare x^1, x^2:

$p^1 \cdot x^2 = 39$, $p^2 \cdot x^1 = 48$

$p^1 \cdot x^2 = 39 \nleq 36 = p^1 \cdot x^1$

$p^2 \cdot x^1 = 48 \le 50 = p^2 \cdot x^2 \Rightarrow p^1 \cdot x^2 = 39 > 36 = p^1 \cdot x^1$ (WARP satisfied)

Thus, x^2 is revealed preferred to x^1, that is $x^2 \succ^R x^1$.

ⓘⓘⓘ Compare x^2, x^0

$p^2 \cdot x^0 = 52$, $p^0 \cdot x^2 = 40$.

$p^2 \cdot x^0 = 52 \nleq p^2 \cdot x^2 = 40$

$p^0 \cdot x^2 = 40 < p^0 \cdot x^0 = 42 \Rightarrow p^2 \cdot x^0 = 52 > p^2 \cdot x^2 = 40$ (WARP satisfied)

Thus, x^0 is revealed preferred to x^2, that is $x^0 \succ^R x^2$

(b) from (a)

We have that $x^0 \succ^R x^2$, $x^2 \succ^R x^1$.

Transitivity implies that $x^0 \succ^R x^1$. However from ⓘ $x^1 \succ^R x^0$

It is impossible being true at the same time.

Thus, transitivity is not satisfied.

#2.11 (a) Suppose that a choice fn $X(p,y) \in \mathbb{R}^n_+$ is homogeneous of degree zero in (p,y)
Show that WARP is satisfied $\forall (p,y)$ iff it is satisfied on $\{(p,1) \mid p \in \mathbb{R}^n_{++}\}$.

($\Rightarrow$) WARP

$$p^0 x^1 \leq p^0 x^0 \Rightarrow p^1 x^0 > p^1 x^1$$

multiply $\frac{1}{y}$ to both sides, then

$$\frac{1}{y} p^0 \cdot x^1 \leq \frac{1}{y} p^0 \cdot x^0 \Rightarrow \frac{1}{y} p^1 \cdot x^0 > \frac{1}{y} p^1 \cdot x^1$$

$$\frac{p^0}{y} \cdot X(p^1, y) \leq \frac{p^0}{y} X(p^0, y) \Rightarrow \frac{p^1}{y} \cdot X(p^0, y) > \frac{p^1}{y} X(p^1, y)$$

By the property of homogeneous of degree zero in (p,y)

$$\frac{p^0}{y} \cdot X\left(\frac{p^1}{y}, 1\right) \leq \frac{p^0}{y} \cdot X\left(\frac{p^0}{y}, 1\right) \Rightarrow \frac{p^1}{y} \cdot X\left(\frac{p^0}{y}, 1\right) > \frac{p^1}{y} X\left(\frac{p^1}{y}, 1\right)$$

WARP is satisfied. $\blacksquare$

($\Leftarrow$)

$$\frac{p^0}{y} \cdot X\left(\frac{p^1}{y}, 1\right) \leq \frac{p^0}{y} \cdot X\left(\frac{p^0}{y}, 1\right) \Rightarrow \frac{p_1}{y} \cdot X\left(\frac{p^0}{y}, 1\right) > \frac{p^1}{y} X\left(\frac{p^1}{y}, 1\right)$$

multiply y to both sides, then

$$p^0 \cdot X\left(\frac{p^1}{y}, 1\right) \leq p^0 \cdot X\left(\frac{p^0}{y}, 1\right) \Rightarrow p_1 \cdot X\left(\frac{p^0}{y}, 1\right) > p_1 \cdot X\left(\frac{p^1}{y}, 1\right)$$

By the homogeneous of degree zero in (p,y) for X.

$$p^0 \cdot X(p^1, y) \leq p^0 \cdot X(p^0, y) \Rightarrow p^1 \cdot X(p^0, y) > p^1 \cdot X(p^1, y)$$

WARP is satisfied $\blacksquare$

Econ 206 - Hw. 7
(2.18, 2.30, 2.33, 2.34)

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2.18. Consider the following 'Independence Axiom' on a consumer's preferences, $\succsim$, over gambles: If $(p_1 \circ a_1, \dots, p_n \circ a_n) \sim (q_1 \circ a_1, \dots, q_n \circ a_n)$ then for every $\alpha \in [0,1]$, and every simple gamble $(r_1 \circ a_1, \dots, r_n \circ a_n)$

$$((\alpha p_1 + (1-\alpha) r_1) \circ a_1, \dots, (\alpha p_n + (1-\alpha) r_n) \circ a_n)$$
$$\sim ((\alpha q_1 + (1-\alpha) r_1) \circ a_1, \dots, (\alpha q_n + (1-\alpha) r_n) \circ a_n) \quad (*)$$

↳ Suppose $x = (p_1 \circ a_1, \dots, p_n \circ a_n)$;
$y = (q_1 \circ a_1, \dots, q_n \circ a_n)$;
$z = (r_1 \circ a_1, \dots, r_n \circ a_n)$;
where $x \sim y$

we can rewrite $((\alpha p_1 + (1-\alpha) r_1) \circ a_1, \dots, (\alpha p_n + (1-\alpha) r_n) \circ a_n)$

into $((\alpha p_1 \circ a_1) + (1-\alpha) r_1 \circ a_1, \dots, (\alpha p_n \circ a_n) + (1-\alpha) r_n \circ a_n)$

$= \alpha ((p_1 \circ a_1), \dots ; (p_n \circ a_n)) + 1-\alpha ((r_1 \circ a_1) \dots ; (r_n \circ a_n))$

$= \alpha x + (1-\alpha) \cdot z \quad \dots (**)$

we can also rewrite $((\alpha q_1 + (1-\alpha) r_1) \circ a_1, \dots, (\alpha q_n + (1-\alpha) r_n) \circ a_n)$

into $\alpha y + (1-\alpha) z \quad (***)$

By $(*)$ and $(**)$, $(***)$, we have the relationship:

$$\alpha x + (1-\alpha) z \sim \alpha y + (1-a) z \quad \forall \alpha \in [0,1]$$

which imply

$$x \sim y \quad \blacksquare$$

2.30 If a VNM utility function displays constant absolute risk aversion, so that $R_a(w) = \alpha$ for all w, what functional form must it have?

$$R_a(w) = \alpha = \frac{-u''(w)}{u'(w)} \quad \text{for all } w$$

$$\int -\frac{u''(w)}{u'(w)}\, dw = \int \alpha\, dw$$

$$\int \frac{-1}{u'(w)}\, du'(w) = \int \alpha\, dw$$

$$\Rightarrow \ln u'(w) = -\alpha w + c_1$$

$$u'(w) = e^{-\alpha w + c_1}$$

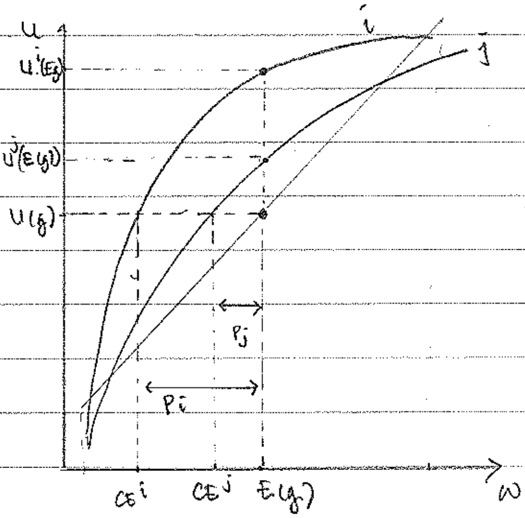
$$\int u'(w)\, dw = \int e^{-\alpha w + c_1}\, dw$$

$$u(w) = -\frac{1}{\alpha} e^{-\alpha w + c_1} + c_2$$

2.33. Let S^i be the set of all probabilities of winning such that individual i will accept a gamble of winning or losing a small amount of wealth, h. Show that for any two individuals i and j, where $R_a^i(w) > R_a^j(w)$, it must be that $S^i \subset S^j$. Conclude that the more risk averse the individual, the smaller the set of gambles he will accept.

$$R_a^i(w) > R_a^j(w) \implies CE^i < CE^j$$

Graphically ...



To make individual i join gamble g, we need to compensate the individual with P_i, while for individual j, we just need to compensate P_j. With $P_i > P_j$ this means individual i will have smaller set of gamble compared to individual j.

2.34 An infinitely lived agent must choose her lifetime consumption plan. Let x_t denote consumption spending in period t, y_t denote income expected in period t, and $r > 0$, the market rate of interest at which the agent can freely borrow or lend. The agent's intertemporal utility function takes the additive separable form.

$$u^*(x_0, x_1, x_2, \ldots) = \sum_{t=0}^{\infty} \beta^t u(x_t)$$

where $u(x)$ is increasing and strictly concave, and $0 < \beta < 1$. The intertemporal budget constraint requires that the present value of income:

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t x_t \le \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t .$$

(a) interpretation of β:

β denotes the <u>discount rate</u> since u^* presents the current level of utility.

(b) FOC for optimal choice of consumption in period t ..

$$L = u(x_0, \ldots) - \lambda\left(\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t - \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t x_t\right)$$

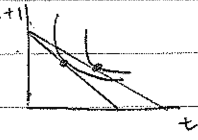
$$\frac{\partial L}{\partial x_t} = \beta^t u'(x_t) + \lambda\left(\frac{1}{1+r}\right)^t = 0$$

thus, $(\beta(1+r))^t u'(x_t) = \lambda$

(c) Assuming that consumption in all other periods remains constant, sketch an indifference curve showing the intertemporal trade-off between x_t and x_{t+1} alone. Carefully justify the slope and curvature depicted.

By FOC in (b) ... $[\beta(1+r)]^t u'(x_t) = \lambda$; $[\beta(1+r)]^{t+1} u'(x_{t+1}) = \lambda$

$$\frac{u'(x_{t+1})}{u'(x_t)} = \frac{\lambda / (\beta(1+r))^{t+1}}{\lambda / (\beta(1+r))^t} = \frac{1}{\beta(1+r)}$$



2-34 (d) How does the consumption in period t vary with the market interest rate (r)

From (b) we have $u'(x_t) = \dfrac{\alpha}{[\beta(1+r)]^t}$

if $r\uparrow$, then $u'(x_t)$ will decrease, which imply $x_t \uparrow$

(e) Show that lifetime utility will always increase with an income increase in any period.

assume : $\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t x_t = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \cdot y_t$

suppose $y_t \uparrow$, then $x_t \uparrow$ which will increase $u(x_t)$

(f) If $\beta = 1/(1+r)$, what is the consumption plan for the agent?

from (d), we know that

$$\frac{u'(x_{t+1})}{u'(x_t)} = \frac{1}{\beta(1+r)} = \frac{1}{\left(\frac{1}{1+r}\right)\cdot(1+r)}$$

$$u'(x_{t+1}) = u'(x_t)$$

$$\Longrightarrow x_{t+1} = x_t$$

$\therefore$ the agent will consume constant amount of good.

(g) if $\beta > \frac{1}{1+r}$; let $\beta = \frac{2}{1+r}$

$$\frac{u'(x_{t+1})}{u'(x_t)} = \frac{1}{\frac{2}{1+r}(1+r)} = \frac{1}{2} \;;\; 2u'(x_{t+1}) = u'(x_t)$$

$\therefore$ The agent will consume less in the future, meanwhile if $\beta < \frac{1}{1+r}$, then agent will consume more in the future.

3.6 Let $f(x_1,x_2)$ be nondecreasing and homogeneous of degree one. a) Show that the isoquants of f are radially parallel, with equal slope at all points along any given ray from the origin. b) Use this to demonstrate that the marginal rate of technical substitution depends only on input proportions. c) Further, show that MP_1 is nondecreasing and MP_2 is nonincreasing in input proportions, $R \equiv x_2/x_1$. d) Show that the same is true when the production function is homothetic.

↳ since $f(x_1,x_2)$ is non decreasing

given $x_1' > x_1$ or $x_2' > x_2$ $\quad f(x_1',x_2') \geq f(x_1,x_2)$

$f(x_1,x_2)$ is homogeneous degree one

then $f(kx_1,kx_2) = k f(x_1,x_2)$ and $f_1(kx_1,kx_2) = f_1(x_1,x_2)$
$f_2(kx_1,kx_2) = f_2(x_1,x_2)$

$\partial f/\partial x_1$, $\partial f/\partial x_2$ homogeneous degree zero

a) ↳ compare MRTS at x and kx

$$MRTS_{12}(x) = \frac{\partial f(x_1,x_2)/\partial x_1}{\partial f(x_1,x_2)/\partial x_2} = \frac{f_1(x)}{f_2(x)} \qquad MRTS_{12}(kx) = \frac{\partial f(kx_1,kx_2)/\partial x_1}{\partial f(kx_1,kx_2)/\partial x_2} = \frac{f_1(kx_1)}{f_2(kx_2)} = \frac{f_1(x_1)}{f_2(x_2)}$$

$MRTS(x) = MRTS(kx)$ implies that the slope at x and kx are the same. So they are radially parallel ■

b) $$MRTS_{12}(x) = \frac{f_1(kx_1,kx_2)}{f_2(kx_1,kx_2)} = \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}$$

lets choose $k = 1/x_2$ then $$MRTS_{12}(x) = \frac{f_1(1/x_2 \cdot x_1, 1/x_2 \cdot x_2)}{f_2(1/x_2 \cdot x_1, 1/x_2 \cdot x_2)} = \frac{f_1(x_1/x_2, 1)}{f_2(x_1/x_2, 1)}$$

now our MRTS just depend on (x_1/x_2)

c) $MP_1 = f_1(x_1,x_2)$ and $MP_2 = f_2(x_1,x_2)$

given $R = x_2/x_1$ then $MP_1 = f_1(x_1, Rx_1)$ and $MP_2 = f_2(x_2/R, x_2)$

$\Leftrightarrow x_2 = Rx_1 \Leftrightarrow x_1 = x_2/R$

since $x_2 = Rx_1$ then $\dfrac{dx_2}{dx_1} = R$

so $\quad f(x_1,x_2) = f(x_1, Rx_1)$

and $\dfrac{dx_1}{dx_2} = \dfrac{1}{R}$

$$MP_1 \leq \frac{df}{dx_1} = \frac{df}{dx_1} + \frac{df}{dx_2} \cdot \frac{dx_2}{dx_1} = f_1 + f_2 \cdot R$$

$$MP_2 = \frac{df}{dx_2} = \frac{df}{dx_1} \cdot \frac{dx_1}{dx_2} + \frac{df}{dx_2} = f_2 \cdot \frac{1}{R} + f_1$$

then MP_1 is a linear function in R

if R increases given that f_1 and f_2 positive, MP_1 is nondecreasing

MP_2 is also a linear function in $(1/R)$

if R increases given that f_1 and f_2 positive, MP_2 is nonincreasing

d) for homothetic

$$f(x_1, x_2) = f(g(x_1, x_2))$$

↳ slope

$$-\frac{f_1'(g(x_1,x_2))}{f_2'(g(x_1,x_2))} = -\frac{f'(\cancel{g(x_1,x_2)}) \cdot g_1'(x_1,x_2)}{f'(\cancel{g(x_1,x_2)}) \cdot g_2'(x_1,x_2)} = -\frac{g_1'(x_1,x_2)}{g_2'(x_1,x_2)}$$

↳ MRTS is very similar with slope

$$\frac{f_1(kx_1, kx_2)}{f_2(kx_1, kx_2)} = \frac{g_1(kx_1, kx_2)}{g_2(kx_1, kx_2)} = \quad \text{and with } k = 1/x_1$$

$$= \frac{g_1(1, x_2/x_1)}{g_2(1, x_2/x_1)}$$ they depends on input proportions

↳

3.8 A Leontif production function has the form $y = \min\{\alpha x_1, \beta x_2\}$ for $\alpha > 0$ and $\beta > 0$. Carefully sketch the isoquant map for this technology and verify that the elasticity of substitution $\sigma = 0$, where defined

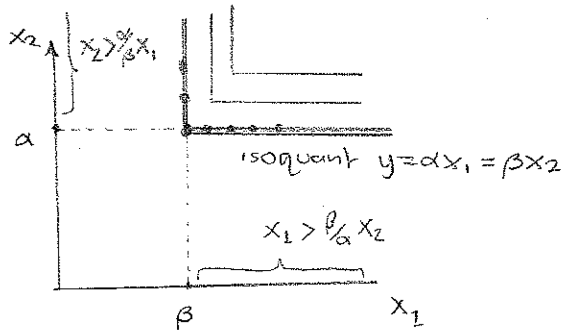
$\alpha x_1 = \beta x_2 \Leftrightarrow x_1/x_2 = \beta/\alpha$

↳ If $\alpha x_1 = \beta x_2$ then $y = \alpha x_1$ or βx_2

If $\alpha x_1 > \beta x_2 \Leftrightarrow x_1 > \beta/\alpha \, x_2$, the production will still unchange

this implies that they are isoquant

If $\alpha x_1 < \beta x_2 \Leftrightarrow x_2 > \alpha/\beta \, x_1$, the production is also the same. isoquant



↳ taking the total differential of the log of the factor ratio

$$d\ln\left(\frac{\beta x_2}{\alpha x_1}\right) = \frac{\beta}{x_2} dx_2 - \frac{\alpha}{x_1} dx_1$$

but the MRTS is not defined in the kinks as the function is discontinuous along all other segments of the isoquants the MRTS is zero. Therefore, the elasticity of substitution is only defined when the input ratio remains constant. In this case $\sigma = 0$

3.15 For the CES production function in the preceding exercise, prove the following claim made in the text

a) $\lim_{y \to 0} y = \prod_{i=1}^{n} x_i^{n}$ b) $\lim_{y \to \infty} y = \min\{x_1, \ldots x_n\}$

a) $y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho}$ where $\sum_{i=1}^{n} \alpha_i = 1$ and $0 < \rho < 1$

taking natural logaritm

$\ln y = \ln\left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho} = 1/\rho \ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}$

at $\rho = 0$, the value of the function is indeterminate

we can use L'Hospital rule

* L'Hospital $\lim_{\rho \to c} \frac{f(\rho)}{g(\rho)} = \lim_{x \to c} \frac{f'(\rho)}{g'(\rho)}$ with $f(\rho) = \ln\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]$

$g(\rho) = 1$

$f(\rho) = \ln\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]$

let $\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right] = u(\rho)$ then $f(\rho) = \ln(u(\rho))$

$$f'(\rho) = \frac{1}{u(\rho)} \cdot u'(\rho) = \frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln(x_i)}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}}$$

* $y = a^x$
$y' = a^x \ln a$

so if $u(\rho) = \alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} + \ldots \alpha_n x_n^{\rho}$

$u'(\rho) = \alpha_1 x_1^{\rho} \ln x_1 + \ldots$

$= \sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln(x_i)$

$$\text{Thus } \lim_{\rho \to 0} \ln y = \frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln(x_i)}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}} = \frac{\sum_{i=1}^{n} \alpha_i x_i^{0} \ln(x_i)}{\sum_{i=1}^{n} \alpha_i x_i^{0}} = \frac{\sum \alpha_i \ln(x_i)}{\sum \alpha_i} = \frac{\sum \ln x_i^{\alpha_i}}{1}$$

$$\lim_{\rho \to 0} \ln y = \sum \ln x_i^{\alpha_i} \Leftrightarrow \lim_{\rho \to 0} \ln y = (\ln x_1^{\alpha_1} + \ln x_2^{\alpha_2} + \ldots + \ln x_n^{\alpha_n})$$

$$\Leftrightarrow \lim_{\rho \to 0} \ln y = \ln(x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n}) = \ln\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)$$

$$\Leftrightarrow \lim_{\rho \to 0} y = \prod_{i=1}^{n} x_i^{\alpha_i}$$

b) we will use the theorem by

If we assume that $\alpha_i = \alpha_j$ then we will have

$$y = \left(\sum \alpha_i x_i^{\rho}\right)^{1/\rho} = (\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} + \ldots \alpha_n x_n^{\rho})^{1/\rho}$$
$$= \alpha^{1/\rho}(x_1^{\rho} + x_2^{\rho} + \ldots x_n^{\rho})^{1/\rho}$$

3.16. Suppose that f satisfies Assumption 3.1

a) Show that the minimization problem (3.1) has a solution x^+, for all $(w,y) \geqslant 0$ for all $(w,y) \geqslant 0$ such that $y \in f(\mathbb{R}^n_+)$

b) Show that the solution to (3.1) in unique, if, in addition, $w \gg 0$.

b) uniqueness. Lets x' is the solution for (3.1)

Suppose there is another x'' which can be a solution, and $x'' \neq x'$

$$\min wx \quad s.t \quad y = f(x)$$

FOC $\quad \mathcal{L} = w_i x_i - \lambda (f(x) - y)$

$\hookrightarrow \quad \frac{\partial \mathcal{L}}{\partial x} = w_i - \lambda \frac{\partial f(x^*)}{\partial x^*} = 0 \quad \Rightarrow \quad w_i^* = \lambda \frac{\partial f(x^*)}{\partial x^*}$

then $\quad \frac{\partial f(x'')}{\partial x_i} = \frac{w_i}{\lambda} \quad$ and $\quad \frac{\partial f(x')}{\partial x_i} = \frac{w_i}{\lambda}$

thus $\quad \frac{\partial f(x')}{\partial x_i} = \frac{\partial f(x'')}{\partial x_i} \quad$ which implies $x' = x''$

3.6

$x^\circ = (x_1, x_2) \quad , \quad x' = (\alpha x_1, \alpha x_2)$

a)

$$x^\circ = -\frac{f_1'(x_1, x_2)}{f_2'(x_1, x_2)}$$

$$x' = -\frac{f_1'(\alpha x_1, \alpha x_2)}{f_2'(\alpha x_1, \alpha x_2)}$$

$$-\frac{f_1'(x_1, x_2)}{f_2'(x_1, x_2)} = -\frac{f_1'(\alpha x_1, \alpha x_2)}{f_2'(\alpha x_1, \alpha x_2)}$$

b) $MRTS_{12}(x) = \dfrac{f_1'(x_1, x_2)}{f_2'(x_1, x_2)} = \dfrac{f_1'(\frac{x_1}{x_2}, 1)}{f_2'(\frac{x_1}{x_2}, 1)}$ multiply $(\frac{x_1}{x_2})$

f_{11}

c) $MP_1 = f_1(x_1, x_2) = f_1(x_1, Rx_1) = f_1' + f_2' \times R$

$MP_2 = f_2(x_1, x_2) = f_2(\frac{x_1}{R}, x_2) = \frac{f_1'}{R} + f_2'$

d) $f(x_1, x_2) = f(g(x_1, x_2))$

$$-\frac{f_1'(g(x_1, x_2))}{f_2'(g(x_1, x_2))} = -\frac{f'(g(x_1, x_2)) \cdot g_1'(x_1, x_2)}{f'(g(x_1, x_2)) \cdot g_2'(x_1, x_2)} = -\frac{g_1'(x_1, x_2)}{g_2'(x_1, x_2)}$$

$$x^\circ(x_1, x_2) \quad -\frac{g_1'(x_1, x_2)}{g_2'(x_1, x_2)}$$

$$x'(\alpha x_1, \alpha x_2) \quad -\frac{g_1'(\alpha x_1, \alpha x_2)}{g_2'(\alpha x_1, \alpha x_2)} = -\frac{g_1'(x_1, x_2)}{g_2'(x_1, x_2)}$$

$MRTS_{12}(x)$

$$= \frac{g_1'(x_1, x_2)}{g_2'(x_1, x_2)} = \frac{g_1'(1, \frac{x_2}{x_1})}{g_2'(1, \frac{x_2}{x_1})}$$

$MP_1 = \frac{x_2}{x_1} \qquad MP_2 \ \frac{x_2}{x_1}$

3.16

b) Uniqueness.

suppose $x^2 \neq x^1$

$\min w \cdot x \quad st \quad y = f(x)$

$$\mathcal{L} = w_i x_i - \lambda(f(x_i) - y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = w - \lambda \cdot \frac{\partial f(x^*)}{\partial x^*} = 0$$

$$\lambda \cdot \frac{\partial f(x^*)}{\partial x^*} = w_i$$

$$w_i^* = \lambda \frac{\partial f(x^*)}{\partial x_i}$$

Then, $\dfrac{\partial f(x^1)}{\partial x_i} = \dfrac{w_i}{\lambda}$, $\dfrac{\partial f(x^2)}{\partial x_i} = \dfrac{w_i}{\lambda}$

$$\Rightarrow \frac{\partial f(x^1)}{\partial x_i} = \frac{\partial f(x^2)}{\partial x_i}$$

Thus, $x^1 = x^2$,

3.16

$$c(w,y) = \min_{x \in \mathbb{R}^n_+} w \cdot x \quad \text{st} \quad f(x) \geq y.$$

a)

$y \in f(\mathbb{R}^n_+)$; $(w,y) \geq 0$

Notice ① $w \cdot x$ is continuous fn

② ~~compact~~ continuous fn has a minimum for a compact set.

pick $\bar{x} \in \mathbb{R}$ st $y = f(\bar{x})$

we have fixed y

- by def of minimizing

$$wx \leq w\bar{x}$$

thus.

$$\min_{x \in A} wx \quad \text{s.t} \quad x \in A$$

$$A \equiv \{ x \in \mathbb{R} \mid wx \leq w\bar{x} \text{ and } f(x) \geq y \}$$

claim A is closed and bounded $\Rightarrow$ compact.

$\forall x.$ $\{x \mid f(x) \geq y\}$ is closed

pick a sequence $x^n \to x$

s.t $f(x^n) \geq y \quad \forall n$

$f(x) < y$ by contradiction.

$\Rightarrow$ by cont of f, $\exists N$ s.t $f(x^N) < y$.

$A \equiv \{ wx \leq w\bar{x} \} \Rightarrow w_\ell x_\ell \leq w\bar{x} \Rightarrow x_\ell \leq \frac{w\bar{x}}{w_\ell} \Rightarrow$ Bounded $\forall \ell$

So, minimization is

$$\min_x wx \quad \text{s.t} \quad x \in A^*$$

$$A^* = \{ x \mid wx \leq \underbrace{w\bar{x}}_{\geq 0} \text{ and } f(x) \geq y \}$$ compact

$\Rightarrow \exists$ solution x^*

Adv Microeconomics H·W9

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#3.21

i) WTS : superadditivity of cost function :

$$c(w^1+w^2, y) \geq c(w^1, y) + c(w^2, y)$$

By definition of the cost function, there must be some cost minimizing bundles for each price vector :

$$c(w^1+w^2, y) \equiv \min_{x\geq 0} \{(w^1+w^2)\cdot x^*\} \quad s.t \quad f(x) = y$$

$$c(w^1, y) \equiv \min_{x\geq 0} \{w^1 \cdot x^1\} \quad s.t \quad f(x) = y$$

$$c(w^2, y) \equiv \min_{x\geq 0} \{w^2 \cdot x^2\} \quad s.t \quad f(x) = y$$

From cost minimization

$$c(w^1, y) \leq w^1 \cdot x^*$$

$$c(w^2, y) \leq w^2 \cdot x^*$$

Adding the last two equations gives :

$$w^1 \cdot x^* + w^2 \cdot x^* \geq c(w^1, y) + c(w^2, y)$$

$$\Rightarrow (w^1+w^2)\cdot x^* \geq c(w^1, y) + c(w^2, y)$$

$$\Rightarrow c(w^1+w^2, y) \geq c(w^1, y) + c(w^2, y)$$

ii) WTS : Cost function is non-decreasing in w :

Suppose $\Delta w \geq 0$, that is $\Delta w_i \geq 0$ for all i and there exists i s.t $\Delta w_i > 0$

For $\Delta w_i = 0$ we have $c(\Delta w_i, y) = 0$

Then by superadditivity of cost function :

$$c(w + \Delta w, y) \geq c(w, y) + \cancelto{0}{c(\Delta w, y)}$$

Thus, $\quad c(w+\Delta w, y) \geq c(w, y)$

#3.30.

i) If Average Cost is increasing with respect to y,

then $LRMC(y) > LRAC(y)$.

By definition, $LRAC(w,y) = \frac{C(w,y)}{y}$ so,

$$\left(\frac{C(w,y)}{y}\right)' = \frac{C'(w,y)y - C(w,y)}{y^2} = \frac{1}{y}\left(\underbrace{\frac{C'(w,y)}{y}}_{MC} - \underbrace{\frac{C(w,y)}{y}}_{AC}\right) > 0$$

$\Leftrightarrow MC > AC$

ii) If Average Cost is decreasing with respect to y,

then $LRMC(y) < LRAC(y)$

similarily,

$$\left(\frac{C(w,y)}{y}\right)' = \frac{C'(w,y)y - C(w,y)}{y^2} = \frac{1}{y}\left(\underbrace{\frac{C'(w,y)}{y}}_{MC} - \underbrace{\frac{C(w,y)}{y}}_{AC}\right) < 0$$

$\Leftrightarrow MC < AC$

iii) If Average Cost is constant with respect to y.

then $LRMC(y) = LRAC(y)$

$$\left(\frac{C(w,y)}{y}\right)' = \frac{C'(w,y)y - C(w,y)}{y^2} = \frac{1}{y}\left(\underbrace{\frac{C'(w,y)}{y}}_{MC} - \underbrace{\frac{C(w,y)}{y}}_{AC}\right) = 0$$

$\Leftrightarrow MC = AC$

$\blacksquare$

3.42

$$\underset{y\geq 0}{Max} \{ py - w_1 x_1 - w_2 x_2 \} \qquad s.t \quad y = x_1^{\alpha} x_2^{\beta}$$

Substitute in expression for y to turn this into an unconstrained maximization problem:

$$\underset{x_i \geq 0}{Max} \{ p(x_1^{\alpha} x_2^{\beta}) - w_1 \cdot x_1 - w_2 \cdot x_2 \}$$

FOC: wrt x_1 : $\alpha p x_1^{\alpha-1} x_2^{\beta} - w_1 = 0$

wrt x_2 : $\beta p x_1^{\alpha} x_2^{\beta-1} - w_2 = 0$

then

$$x_1 = \left[\left(\frac{w_1}{\beta}\right)^{1-\beta} \left(\frac{w_2}{\alpha}\right)^{\beta} \right]^{\frac{1}{\alpha+\beta-1}}$$

$$x_2 = \left[\left(\frac{w_1}{\beta}\right)^{\alpha} \cdot \left(\frac{w_2}{\alpha}\right)^{1-\alpha} \right]^{\frac{1}{\alpha+\beta-1}}$$

Substitute the expressions for x_1 and x_2 above into the profit function:

$$\pi(p, w_1, w_2) = p(x_1^{\alpha} x_2^{\beta}) - w_1 x_1 - w_2 x_2$$

$$= p\left[\left(\frac{w_1}{\beta}\right)^{\alpha} \left(\frac{w_2}{\alpha}\right)^{\beta} \right]^{\frac{1}{\alpha+\beta-1}} - w_1 \left[\left(\frac{w_1}{\beta}\right)^{1-\beta} \left(\frac{w_2}{\alpha}\right)^{\beta} \right]^{\frac{1}{\alpha+\beta-1}} - w_2 \left[\left(\frac{w_1}{\beta}\right)^{\alpha} \left(\frac{w_2}{\alpha}\right)^{1-\alpha} \right]^{\frac{1}{\alpha+\beta-1}}$$

For a well-defined production function we must have $y' > 0$ and $y'' < 0$

FOC wrt x_1 : $\alpha x_1^{\alpha-1} x_2^{\beta} > 0$

SOC wrt x_2 : $\alpha(\alpha-1) x_1^{\alpha-2} \cdot x_2^{\beta} < 0$

These two conditions require $\alpha > 0$ and $\alpha < 1$.

Similarly from FOC and SOC wrt x_2

we need $\beta > 0$ and $\beta < 1$

$\blacksquare$

3.50

$$\pi(p,w) = \max_{y,x} \; p\cdot y - w\cdot x \qquad \text{s.t } T(y,x)=0$$

① increasing in P

Suppose $p' > p$ for all outputs, then

$p'\cdot y > p\cdot y$ and add $(-wx)$ to both side,

$p'\cdot y - wx > p\cdot y - wx$

$\Rightarrow \pi(p', w) > \pi(p,w)$

② decreasing in W

Suppose $w' > w$ for all inputs, then

$w'\cdot x > w\cdot x \rightarrow -w'\cdot x < -w\cdot x$ add $(p\cdot y)$ to both side,

$p\cdot y - w'x < p\cdot y - wx$

$\Rightarrow \pi(p, w') < \pi(p,w)$

③ Homogeneous of degree one in (p,w)

WTS : $\pi(tp, tw) = t\cdot\pi(p,w)$ for all $t \geq 0$

Let y be profit maximizing output vector at p, w

so that $p\cdot y - wx \geq p\cdot y' - w\cdot x'$ for all y' in Y and x' in X.

It follows that for $t \geq 0$,

$$tp\cdot y - tw\cdot x \geq tp\cdot y' - tw\cdot x'$$

Hence, y also maximize profit at tp, tw

$$\Rightarrow \pi(tp, tw) = tp\cdot y - tw\cdot x = t\cdot\pi(p,w)$$

④ Convex in (p, w)

Let y^0, x^0 max profit at p^0, w^0

y^1, x^1 max profit at p^1, w^1

Define

y^*, x^* max profit at p^λ, w^λ

with $p^\lambda = \lambda p^0 + (1-\lambda) p^1$

$w^\lambda = \lambda w^0 + (1-\lambda) w^1$

By definition,

$$\pi(p^0, w^0) = p^0 y^0 - w^0 x^0 \geq p^0 y^* - w^0 x^*$$

$$\pi(p^1, w^1) = p^1 y^1 - w^1 x^1 \geq p^1 y^* - w^1 x^*$$

Then,

$$\begin{aligned}
\lambda \cdot \pi(p^0, w^0) + (1-\lambda)\pi(p^1, w^1) &\geq \lambda[p^0 \cdot y^* - w^0 x^*] + (1-\lambda)[p^1 y^* - w^1 x^*] \\
&\geq [\lambda(p^0 \cdot y^*) + (1-\lambda)(p^1 y^*)] + [\lambda(w^0 x^*) + (1-\lambda)(w^1 x^*)] \\
&\geq [\lambda p^0 + (1-\lambda)p^1] y^* + [\lambda w^0 + (1-\lambda) w^1] x^* \\
&\geq p^\lambda \cdot y^* + w^\lambda x^* \\
&\geq \pi(p^\lambda, y^\lambda) \\
&= \pi(\lambda p^0 + (1-\lambda)p^1, \lambda w^0 + (1-\lambda) w^1)
\end{aligned}$$

This implies convex. ▨

⑤ Differentiable in $(p, w) \gg 0$

An expression is called "well defined" if its definition assigns it a unique value.

3.6 Let $f(x_1,x_2)$ be nondecreasing and homogeneous of degree one. i) Show that the isoquants of f are radially parallel, with equal slope at all points along any given ray from the origin. b) Use this to demonstrate that the marginal rate of technical substitution depends only on input proportions. c) Further, show that MP_1 is nondecreasing and MP_2 is nonincreasing in input proportions, $R \equiv x_2/x_1$. d) Show that the same is true when the production function is homothetic.

↳ since $f(x_1,x_2)$ is non decreasing

given $x_1' > x_1$ or $x_2' > x_2$ $\quad f(x_1',x_2') \geq f(x_1,x_2)$

$f(x_1,x_2)$ is homogeneous degree one

then $f(kx_1,kx_2) = kf(x_1,x_2)$ and $f_1(kx_1,kx_2) = f_1(x_1,x_2)$

$f_2(kx_1,kx_2) = f_2(x_1,x_2)$

$\partial f/\partial x_1$, $\partial f/\partial x_2$ homogeneous degree zero

a) ↳ compare MRTS at x and kx

$$MRTS_{12}(x) = \frac{\partial f(x_1,x_2)/\partial x_1}{\partial f(x_1,x_2)/\partial x_2} = \frac{f_1(x)}{f_2(x)} \qquad MRTS_{12}(kx) = \frac{\partial f(kx_1,kx_2)/\partial x_1}{\partial f(kx_1,kx_2)/\partial x_2} = \frac{f_1(kx_1)}{f_2(kx_2)} = \frac{f_1(x_1)}{f_2(x_2)}$$

$MRTS(x) = MRTS(kx)$ implies that the slope at x and kx are the same. So they are radially parallel

b) $$MRTS_{12}(x) = \frac{f_1(kx_1,kx_2)}{f_2(kx_1,kx_2)} = \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}$$

lets choose $k = 1/x_2$ then $$MRTS_{12}(x) = \frac{f_1(\frac{1}{x_2}\cdot x_1, \frac{1}{x_2}\cdot x_2)}{f_2(\frac{1}{x_2}x_1, \frac{1}{x_2}x_2)} = \frac{f_1(x_1/x_2, 1)}{f_2(x_1/x_2, 1)}$$

now our MRTS just depend on (x_1/x_2)

c) $MP_1 = f_1(x_1,x_2)$ and $MP_2 = f_2(x_1,x_2)$

given $R = x_2/x_1$ then $MP_1 = f_1(x_1, Rx_1)$ and $MP_2 = f_2(x_2/R, x_2)$

$\Leftrightarrow x_2 = Rx_1 \Leftrightarrow x_1 = x_2/R$

so $f(x_1,x_2) = f(x_1,Rx_1)$ $\quad$ since $x_2 = Rx_1$ then $\frac{dx_2}{dx_1} = R$

and $\frac{dx_1}{dx_2} = \frac{1}{R}$

$$MP_1 \leq \frac{df}{dx_1} = \frac{df}{dx_1} + \frac{df}{dx_2}\cdot\frac{dx_2}{dx_1} = f_1 + f_2\cdot R$$

$$MP_2 = \frac{df}{dx_2} = \frac{df}{dx_1}\cdot\frac{dx_1}{dx_2} + \frac{df}{dx_2} = f_2\cdot\frac{1}{R} + f_1$$

then MP_1 is a linear function in R

If R increases given that f_1 and f_2 positive, MP_1 is nondecreasing

MP_2 is also a linear function in $(1/R)$

If R increases given that f_1 and f_2 positive, MP_2 is nonincreasing

d) for homothetic

$$f(x_1, x_2) = f(g(x_1, x_2))$$

↳ slope

$$-\frac{f_1'(g(x_1,x_2))}{f_2'(g(x_1,x_2))} = -\frac{\cancel{f'(g(x_1,x_2))} \cdot g_1'(x_1,x_2)}{\cancel{f'(g(x_1,x_2))} \cdot g_2'(x_1,x_2)} = -\frac{g_1(x_1,x_2)}{g_2(x_1,x_2)}$$

↳ MRTS is very similar with slope

$$\frac{f_1(kx_1, kx_2)}{f_2(kx_1, kx_2)} = \frac{g_1(kx_1, kx_2)}{g_2(kx_1, kx_2)} = \quad \text{and with } k = 1/x_1$$

$$= \frac{g_1(1, x_2/x_1)}{g_2(1, x_2/x_1)} \quad \text{they depends on input proportions}$$

↳

3.8 A Leontif production function has the form $y = \min\{\alpha x_1, \beta x_2\}$ for $\alpha > 0$ and $\beta > 0$. Carefully sketch the isoquant map for this technology and verify that the elasticity of substitution $\sigma = 0$, where defined

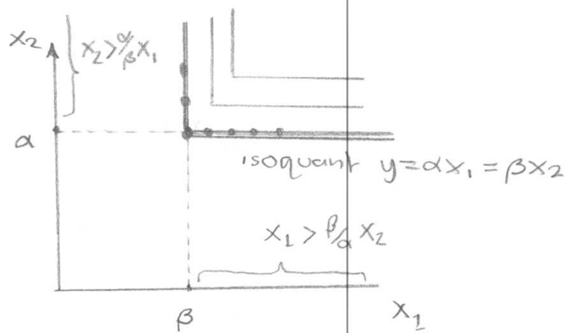
$\alpha x_1 = \beta x_2 \Leftrightarrow x_1/x_2 = \beta/\alpha$

↳ If $\alpha x_1 = \beta x_2$ then $y = \alpha x_1$ or βx_2

If $\alpha x_1 > \beta x_2 \Leftrightarrow x_1 > \beta/\alpha \, x_2$, the production will still unchange

this implies that they are isoquant

If $\alpha x_1 < \beta x_2 \Leftrightarrow x_2 > \alpha/\beta \, x_1$, the production is also the same. isoquant



↳ taking the total differential of the log of the factor ratio

$$d \ln\left(\frac{\beta x_2}{\alpha x_1}\right) = \frac{\beta}{x_2} dx_2 - \frac{\alpha}{x_1} dx_1$$

but the MRTS is not defined in the kinks as the function is continuous but not differentiable along all other segments of the isoquants the MRTS is zero. Therefore, the elasticity of substitution is only defined when the input ratio remains constant. In this case $\sigma = 0$

3.15 For the CES production function in the preceding exercise, prove the following claims made in the text

a) $\lim_{y \to 0} y = \prod_{i=1}^{n} x_i^{n}$ b) $\lim_{y \to \infty} y = \min\{x_1, \dots x_n\}$

a) $y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho}$ where $\sum_{i=1}^{n} \alpha_i = 1$ and $0 < \rho < 1$

taking natural logaritm

$\ln y = \ln\left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho} = 1/\rho \ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}$

at $\rho = 0$, the value of the function is indeterminate

we can use L'Hospital rule

* L'Hospital $\lim_{\rho \to c} \frac{f(\rho)}{g(\rho)} = \lim_{x \to c} \frac{f'(\rho)}{g'(\rho)}$ with $f(\rho) = \ln\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]$

$g(\rho) = 1$

$f(\rho) = \ln\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]$

let $\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right] = u(\rho)$ then $f(\rho) = \ln(u(\rho))$

$f'(\rho) = \frac{1}{u(\rho)} \cdot u'(\rho)$

$= \frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln(x_i)}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}}$

* $y = a^x$

$y' = a^x \ln a$

so if $u(\rho) = \alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} + \dots \alpha_n x_n^{\rho}$

$u'(\rho) = \alpha_1 x_1^{\rho} \ln x_1 + \dots$

$= \sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln(x_i)$

Thus $\lim_{\rho \to 0} \ln y = \frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln(x_i)}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}} = \frac{\sum_{i=1}^{n} \alpha_i x_i^{0} \ln(x_i)}{\sum_{i=1}^{n} \alpha_i x_i^{0}} = \frac{\sum \alpha_i \ln(x_i)}{\sum \alpha_i} = \frac{\sum \ln x_i^{\alpha_i}}{1}$

$\lim_{\rho \to 0} \ln y = \sum \ln x_i^{\alpha_i} \Leftrightarrow \lim_{\rho \to 0} \ln y = (\ln x_1^{\alpha_1} + \ln x_2^{\alpha_2} + \dots + \ln x_n^{\alpha_n})$

$\Leftrightarrow \lim_{\rho \to 0} \ln y = \ln(x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}) = \ln\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)$

$\Leftrightarrow \lim_{\rho \to 0} y = \prod_{i=1}^{n} x_i^{\alpha_i}$

b) we will use the theorem by

If we assume that $\alpha_i = \alpha_j$ then we will have

$y = \left(\sum \alpha_i x_i^{\rho}\right)^{1/\rho} = (\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} + \dots \alpha_n x_n^{\rho})^{1/\rho}$

$= \alpha^{1/\rho}(x_1^{\rho} + x_2^{\rho} + \dots x_n^{\rho})^{1/\rho}$

by Hardy's theorem 4

$$\lim_{r \to -\infty} M_r(a) = \operatorname{Min} a$$

which means that if we think that $M_r(x_1, \dots x_n) = \left(\sum \alpha_i x_i^{\,p}\right)^{1/p}$

than $\lim_{p \to -\infty} M_r(x_1, x_2 \dots x_n) = \min(x_1, x_2, \dots x_n)$

3.16. Suppose that f satisfies Assumption 3.1 (f is continuous, strictly increasing, strictly quasiconcave)

a) Show that the minimization problem (3.1) has a solution x^*, for all $(w,y) \geq 0$ for all $(w,y) \geq 0$ such that $y \in f(\mathbb{R}^n_+)$

b) Show that the solution to (3.1) in unique, if, in addition, $w \gg 0$.

↳ Strategy. $p \to q$ we use contrapositive

Suppose that X^* is a solution but not for all $(w,y) \geq 0$ such that $y \in f(\mathbb{R}^n_+)$

$(\exists x^\circ)$ x° is also a solution for $\exists (w,y) \geq$ st $y \in f(\mathbb{R}^n_+)$

and $x^\circ \neq x^*$. Let $x^\circ = k x^*$ for $k > 0$

↳ since they are both the solution for (w,y) that means that

$c(w, y(x^\circ)) = c(w, y(x^*))$

given fixed w it means

$$y(x^\circ) = y(x^*) \Leftrightarrow y(kx^*) = y(x^*)$$

which contradict that y is strictly increasing

↳ let $x^t = tx^* + (1-t)x^\circ$ and since both x^* and x° are in the same budget their combination x^t will also in the same budget line

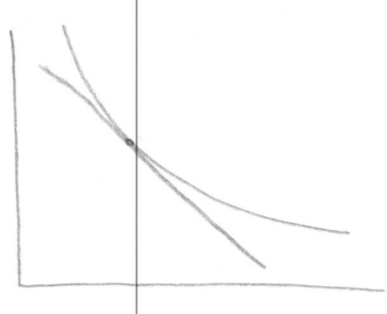
then $y(x^t) \ngtr \min[y(x^*), y(x^\circ)]$

otherwise x^t will be chosen

since they are in the same budget line

But this will also contradict that y is strictly quasiconcave

↳ to prove continuity we will just use theorem 2.41, Theorem of the Maximum Value. That theorem ensure us that if f is continuous we will have a solution x^* for all $(w,y) \geq 0$



b) uniqueness. Lets x' is the solution for (3.1)

Suppose there is another x'' which can be a solution, and $x'' \neq x'$

$$\min wx \quad s.t \quad y = f(x)$$

FOC $\mathcal{L} = w_i x_i - \lambda (f(x_i) - y)$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial x} = w_i - \lambda \frac{\partial f(x^*)}{\partial x^*} = 0 \Rightarrow w_i^* = \lambda \frac{\partial f(x^*)}{\partial x^*}$$

then $\dfrac{\partial f(x'')}{\partial x_i} = \dfrac{w_i}{\lambda}$ and $\dfrac{\partial f(x')}{\partial x_i} = \dfrac{w_i}{\lambda}$

thus $\dfrac{\partial f(x')}{\partial x_i} = \dfrac{\partial f(x'')}{\partial x_i}$ which implies $x' = x''$

Econ 206
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4.9 Stackelberg Duopoly
Market demand ; $p = 100 - (q_1 + q_2)$
Firm costs ; $c_1 = 10 q_1$; $c_2 = q_2^2$

(a) Calculate market price and each firm's profit assuming that firm 1 is the leader and firm 2 the follower ...

Revenue for firm 2 (R_2):

$$R_2 = p \cdot q_2$$
$$= [100 - (q_1 + q_2)] \cdot q_2$$
$$= 100 q_2 - q_2 q_1 - q_2^2$$

Marginal Revenue for firm 2 (MR_2)

$$MR_2 = \frac{\partial R_2}{\partial q_2} = 100 - q_1 - 2q_2$$

In this market we have the condition : Marginal revenue = Marginal cost

so, $MR_2 = MC_2 = \frac{\partial c_2}{\partial q_2}$

$$= 100 - q_1 - 2q_2 = 2q_2$$
$$\boxed{q_2 = 25 - \frac{q_1}{4}} \rightarrow \text{Firm 2 reaction function} \quad \ldots (1)$$

Firm 1 : $R_1 = p \cdot q_1$
$$= [100 - (q_1 + q_2)] \cdot q_1$$
$$= 100 q_1 - q_1^2 - q_1 q_2 \rightarrow \text{substituting with (1)}$$
$$= 75 q_1 - \cancel{3/4} q_1^2$$

$$MR_1 = MC_1$$
$$= 75 - 3/2 q_1 = 10$$
$q_1 = 130/3$, then we will have $q_2 = 85/6$

Price : $p = 100 - (q_1 + q_2)$
$$= 100 - (130/3 + 85/6)$$
$$= 255/6$$

Profit : $\pi_i = p \cdot q_i - c_i$
$$\pi_1 = 255/6 \cdot 260/6 - 2600/6 = 50.700/36$$
$$\pi_2 = 255/6 \cdot 85/6 - 7225/6 = 14.450/36$$

b) Firm 2 is the leader and firm 1 is the follower...

$R_1 = p \cdot q_1$

$= 100q_1 - q_1^2 - q_1 q_2$

$MR_1 = 100 - 2q_1 - q_2$

$MR_1 = MC_1 = 10$

$100 - 2q_1 - q_2 = 10$

$\boxed{q_1 = 45 - \frac{1}{2}q_2}$

Firm 1's reaction function

$R_2 = p \cdot q_2$ — substitute with firm 1's reaction function

$= 100q_2 - q_1 q_2 - q_2^2$

$= 100q_2 - (45 - \frac{1}{2}q_2)q_2 - q_2^2$

$= 55q_2 - \frac{1}{2}q_2^2$

$MR_2 = MC_2 = 2q_2$

$55 - q_2 = 2q_2$

$\boxed{q_2 = \frac{110}{6}}$

$\downarrow$

$\boxed{q_1 = \frac{215}{6}}$

Price : $p = 100 - q_1 - q_2$

$= \dfrac{600 - 215 - 110}{6}$

$= 275/6$

Profit : $\pi_i = p \cdot q_i - c_i$

$\pi_1 = 275/6 \cdot 215/6 - 2150/6 = 46.225/36$

$\pi_2 = 275/6 \cdot 110/6 - 1100/6 = 18.150/36$

c)

	π_1	π_2
Firm 1 leader	50.700/36	14.450/36
Firm 2 leader	46.225/36	18.150/36

From this table, each firm wants to be the leader for the higher profit.

d) If both firms assume they are leaders, the outputs are:

$q_1 = 260/6 \quad q_2 = 110/6$

$p = 100 - q_1 - q_2$

$= \dfrac{600 - 260 - 110}{6}$

$= \dfrac{230}{6}$

$\pi_1 = p \cdot q_1 - c_1 = 44.200/36$

$\pi_2 = p \cdot q_2 - c_2 = 13.200/36$

4.9 d) Cournot - Nash Equilibrium

$R_1 = 100 \cdot q_1 - q_1^2 - q_1 q_2$ $\qquad$ $R_2 = 100 q_2 - q_1 q_2 - q_2^2$

$MR_1 = 100 - 2q_1 - q_2 = MC_1 = 10$ $\qquad$ $MR_2 = 100 - q_1 - 2q_2 = 2q_2$

$q_1 = 45 - q_2/2 \; \ldots (1)$ $\qquad$ $4q_2 + q_1 = 100 \; \ldots (2)$

combining (1) and (2) we'll have.. (substituting for q_1)

$$4q_2 + (45 - q_2/2) = 100$$

$$7q_2 = 110$$

$$\boxed{q_2 = 110/7} \longrightarrow \boxed{q_1 = 260/7}$$

$p = 100 - q_1 - q_2 = 330/7$

$\pi_i = p q_i - c_i$

$\pi_1 = 67.600/49$

$\pi_2 = 24.200/49$

to summarize a), b), c) and d) :

	Stackelberg			Cournot Nash
	Firm 1 - Leader	Firm 2 - leader	Both leaders	
q_1	43	36	43	37
q_2	14	18	18	16
π_1	1408	1284	1228	1380
π_2	401	504	367	494
p	43	46	38	47
$\pi_1 + \pi_2$	1809	1788	1595	1874

4.11) Cournot market, each identical firm has cost function $c(q) = k + cq$, where $k > 0$ is fixed cost.

a) What will p, q, π be with J firms in the market?

market demand $= p = a - b\sum_{k=1}^{J} q^k$

then,

$$\pi^j (q^1, \dots, q^J) = p \cdot q^j - c(q^j)$$

$$= \left(a - b\sum_{k=1}^{J} q^k\right) \cdot q^j - (k + c \cdot q^j)$$

$$= aq^j - bq^{j^2} - b\sum_{k \neq j} q^k q^j - k - cq^j \quad \dots (*)$$

We want to find a vector of outputs $(\bar{q}_1, \dots, \bar{q}_J)$ s.t each firm's output choice is profit maximizing given the output choices of other firms (Cournot - Nash Equilibrium).

So, if $(\bar{q}_1, \dots, \bar{q}_J)$ is a Cournot-Nash equilibrium, $\bar{q}^j$ must maximize $(*)$ when $q_k = \bar{q}_k$ for all $k \neq j$.

$$\frac{\partial \pi^j}{\partial \bar{q}^j} = a - 2b\bar{q}^j - b\sum_{k \neq j} \bar{q}^k - c = 0$$

$$b\bar{q}^j = a - c - b \cdot \sum_{k=1}^{J} \bar{q}$$

all firms produced the same # of output in equilibrium.

$$b\bar{q} = a - c - bJ\bar{q} \implies \bar{q} = \frac{a-c}{b(J+1)} \implies \sum \bar{q} = \frac{J(a-c)}{b(J+1)}$$

$$p = a - \frac{J(a-c)}{J+1}$$

$$\pi^j = \left(a - \frac{J(a-c)}{(J+1)}\right)\left(\frac{(a-c)}{b(J+1)}\right) - k - c\left(\frac{a-c}{b(J+1)}\right)$$

$$\vdots$$

$$= \frac{(a-c)^2}{b(J+1)^2} - k$$

b) Free entry and exit implies $\pi = 0$

from a) we have $\pi^J = \dfrac{(a-c)^2}{b(J+1)^2} - k = 0$

$$(J+1)^2 = \frac{(a-c)^2}{kb}$$

$$J = \frac{(a-c)}{\sqrt{kb}} - 1$$

4.12 Bertrand Duopoly (Section 4.2.2)

- Market demand : $q = \alpha - \beta p$
- No fixed cost
- Identical marginal cost

The Bertrand equilibrium :

$$q_i(p_i, p_j) \begin{cases} q(p_i) & \text{if } p_i < p_j \\ \frac{1}{2} q(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

a) & b) By assuming $p_i \geq MC_i$, fixed cost will not have any effect on marginal cost, this means that since both firms have identical marginal cost, then the Bertrand equilibrium will be $p_1 = p_2$ and $q_1 = q_2 = \frac{1}{2} q$

c) if $MC_1 < MC_2$, then firm 1 could determine it's price less than firm 2's MC ($p_1 < MC_2$). Thus, firm 1 will supply all of the goods ($q_1 = q$ and $q_2 = 0$).

4.15 $q^j = (p^j)^{-2} \left(\sum_{\substack{i=1 \\ i \neq j}}^{j} p_i^{-1/2} \right)^{-2}$ and $c(q) = cq + k$

a) $\frac{\partial q^j}{\partial p^j} = \underbrace{-2}_{\ominus} \underbrace{(p^j)^{-3}}_{\oplus} \underbrace{\left(\Sigma p_i^{-1/2} \right)^{-2}}_{\oplus} < 0$

the demand is negatively sloped

Constant - own - price elasticity:

$\frac{\partial q^j}{\partial p^j} \cdot \frac{p^j}{q^j} = -2 (p^j)^{-3} \cancel{\left(\Sigma p_i^{-1/2} \right)^{-2}} \cdot \frac{p^j}{(p^j)^{-2} \cancel{\left(\Sigma p_i^{-\frac{1}{2}} \right)^{-2}}}$

$= -2 \left(\frac{p^j}{p^j} \right)^{-3} = -2$

All goods are substitutes for each other:

$\frac{\partial q^j}{\partial p^i} = (-2) \cdot \left(-\frac{1}{2}\right) \cdot (p^j)^{-2} \left(\sum_{\substack{i=1 \\ i \neq j}}^{j} p_i^{-\frac{1}{2}} \right)^{-3} \cdot p_i^{-\frac{3}{2}}$

$= (p^j)^{-2} \left(\Sigma p_i^{-\frac{1}{2}} \right)^{-3} \cdot p_i^{-\frac{3}{2}} > 0 \quad i \neq j$

since $\frac{\partial q^j}{\partial p^i} > 0$, all goods are substitutes for each other

b) $(p_1, p_2, \ldots, p_J)$ suppose all firms raise the price by a factor k, where $k > 1$ $(kp_1, kp_2, \ldots, kp_J)$

$q^j = (p^j)^{-2} \left(\Sigma p_i^{-\frac{1}{2}} \right)^{-2}$

$q^{j'} = (kp^j)^{-2} \left(\Sigma (kp_i)^{-\frac{1}{2}} \right)^{-2}$

$q^{j'} = k^{-1} (p^j)^{-2} \left(\Sigma p_i^{-\frac{1}{2}} \right)^{-2}$

$k > 1, \quad q^{j'} < q^j$

PROBLEM SET ECON206 Advanced Microeconomics

Sangho Lee, Wishnu Mahraddika, Angsoka Paundralingga

4.16 Suppose that a consumer's utility function over all goods, $u(q,\mathbf{x})$, is continuous, strictly increasing, and strictly quasiconcave, and that the price p of the vector of goods, $\mathbf{x}$, is fixed. Let m denote the composite commodity $\mathbf{p}\ \mathbf{x}$, so that m is the amount of income spent on x. define the utility function u over two goods q and m as follows

$$\bar{u}\,(q,m) \equiv \text{Max } u(q,\,\mathbf{x}) \text{ s.t. } \mathbf{p}\ \mathbf{x} \leq\ m$$

a. Show that u(q,m) is strictly increasing and strictly quasiconcave. If you can, appeal to a theorem that allows you to conclude that it is also continuous.

b. Sow that if $q(p,\mathbf{p},y)$ and $x(p,\mathbf{p},y)$ denote the consumer's Marshallian demands for q and x then, $q(p,\mathbf{p},y)$ and $m(p,\mathbf{p},y)=\mathbf{p}\ \mathbf{x}(p,\mathbf{p},y)$ solve

$$\text{Max } \bar{u}\,(q,m) \text{ s.t. } pq + m \leq\ y$$

And that the maximized value of $\bar{u}$ is $v(p,\mathbf{p},y)$.

c. Conclude that when the prices of all but one good are fixed, one can analyze the consumer's problem as if there were only two goods, the good whose price is not fixed, and the composite commodity, "money spent on all other goods."

a) $\bar{u}(q,x)$ is continuous

↳ Let fix $m=\bar{m}$ and $q_2 > q_1$

$\bar{U}(q_1,\bar{m}) \equiv \max\limits_{x\in A} U(q_1,X)$ with $A=\{x \mid p\cdot x \leq \bar{m}\}$

let x^* is the solution for this problem

$\bar{U}(q_1,\bar{m}) = U(q_1, X^*)$ with $x^* \in A$

since $q_2 > q_1$ and $U(q,x)$ is strictly increasing

thus $U(q_2, x^*) > U(q_1, x^*)$

$\max\limits_{x\in A} \bar{U}(q_2, x^*) > \max\limits_{x\in A} U(q_1, x^*)$

$\bar{U}(q_2,\bar{m}) > \bar{U}(q_1,\bar{m})$

which implies $\bar{U}(q,m)$ strictly increasing in q

↳ Let fix $q=\bar{q}$ and assume $m_2 > m_1$

$\bar{U}(\bar{q},m_1) \equiv \max\limits_{x\in A} U(\bar{q},x)$ $A=\{x \mid p\cdot x \leq m_1\}$

suppose $x' \in A$ is the solution of maximisation problem

$= U(\bar{q}, x')$ $x' \in A$ which means $p\cdot x' \leq m_1$

$\bar{U}(\bar{q},m_2) \equiv \max\limits_{x\in B} U(\bar{q},X)$ $B=\{x \mid p\cdot x \leq m_2\}$

suppose $x'' \in B$ is the solution of maximisation problem

$= U(\bar{q}, x'')$ $x'' \in B$ which means $p\cdot x'' \leq m_2$

Given that $m_1 < m_2 \Leftrightarrow px' < px'' \Leftrightarrow x' < x''$

since $u(\bar{q}, x)$ is strictly increasing then $x' < x''$ implies $u(\bar{q}, x') < u(\bar{q}, x'')$

$$u(\bar{q}, x') < u(\bar{q}, x'')$$

$$\max_{x' \in A} u(\bar{q}, x') < \max_{x'' \in B} u(\bar{q}, x'')$$

$$\bar{u}(\bar{q}, m_1) < \bar{u}(\bar{q}, m_2)$$ $\bar{u}(q, m)$ is strictly increasing in m

↳ Proof that $\bar{u}(q, m)$ is quasi concave

Let $\bar{u}(q_1, m_1) = \max_{x \in A} u(q_1, x)$ $\quad A = \{x \mid p \cdot x \leq m_1\}$

$= u(q_1, x_1^*)$ and $x_1^* \in A$

Let $\bar{u}(q_2, m_2) = \max_{x \in B} u(q_2, x)$ $\quad B = \{x \mid p \cdot x \leq m_2\}$

$= u(q_2, x_2^*)$ and $x_2^* \in B$

~~c~~reate a combinatorial of x_1 and x_2

$x^t = t x_1^* + (1-t) x_2^*$ so $q^t = t q_1 + (1-t) q_2$, $m^t = t m_1 + (1-t) m_2$, $t \in (0,1)$

$p \cdot x^t = p \cdot t x_1^* + p \cdot (1-t) x_2^*$

since $x_1^* \in A$, $x_2^* \in B$

thus $p \cdot x_1^* \leq m_1$ and $p x_2^* \leq m_2$

$p \cdot x^t \leq t m_1^* + (1-t) m_2^* = m^t$

let $x^t \in C = \{x \mid p \cdot x \leq m_t\}$

$\bar{u}(q^t, m^t) = \max_{x \in C} u(q^t, x) \geq u(q^t, x^t)$ $\quad (x^t \in C)$

$\bar{u}(q^t, m^t) > \min(\bar{u}(q_1, x_1^*), \bar{u}(q_2, x_2^*))$

b) Given that $\bar{U}(q,m) \equiv \max_{x} u(q,x)$ subject to $p \cdot x \leq m$ we have to prove

$$V(p_q, p, y) \equiv \max_{q,m} \bar{U}(q,m) \text{ subject to } p \cdot q + m \leq y$$

with $q(p_q, p, y)$ and $x(p_q, p, y)$ is the Marshallian demands for q and x

↳ write down the Lagrangian for

$$\max_{x} u(q,x) \quad \text{s.t } p \cdot x \leq m$$

$$\mathcal{L} = u(q,x) + \lambda(m - px)$$

since $\bar{U}$ is the solution

↳ $\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial U}{\partial q}$ since $\bar{U}(q,m)$ then $\frac{\partial \bar{U}}{\partial q}$ $\frac{\partial U}{\partial q} = \frac{\partial \bar{U}}{\partial q}$ ... (1)

we don't need $\frac{\partial U}{\partial q} = 0$ because $\bar{U}(q,m)$ doesn't necessarry implies q is the max

↳ $\frac{\partial \mathcal{L}}{\partial m} = \frac{\partial \bar{U}}{\partial m} \Leftrightarrow \lambda = \frac{d\bar{U}}{dm}$ .... (2)

FOC ↳ $\frac{\partial \mathcal{L}}{\partial x_i} = 0 \Leftrightarrow \frac{\partial u}{\partial x_i} - \lambda p \Leftrightarrow \frac{\partial u}{\partial x_i} = \lambda p$ .... (3)

↳ substitute (3) to (2)

$$\frac{\partial u}{\partial x_i} = \lambda p = \left(\frac{\partial \bar{U}}{\partial m}\right) p \Leftrightarrow \frac{\partial \bar{U}}{\partial m} = \frac{1}{p}\frac{\partial U}{\partial x_i} \quad \text{.... (4)}$$

↳ by using (4) and (1), create an identity

$$\boxed{\frac{\frac{\partial U}{\partial q}}{\frac{\partial U}{\partial x_i} \cdot \frac{1}{p_i}} = \frac{\frac{\partial \bar{U}}{\partial q}}{\frac{\partial \bar{U}}{\partial m}}} \Leftrightarrow p_i \cdot \frac{\partial U}{\partial q} \Big/ \frac{\partial U}{\partial x_i} = \frac{\partial \bar{U}}{\partial q} \Big/ \frac{\partial \bar{U}}{\partial m} \quad \text{.... (5)}$$

↳ write down the Lagrangian for

$$\max_{q,x} u(q,x) \quad \text{s.t. } p_q \cdot q + p \cdot x \leq y$$

$$\mathcal{L} = u(q,x) + \lambda(y - p_q \cdot q - px)$$

FOC ↳ $\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial U}{\partial q} - \lambda p_q = 0 \Leftrightarrow \frac{\partial U}{\partial q} = \lambda p_q$ .... (6)

↳ $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial U}{\partial x} - \lambda p = 0 \Leftrightarrow \frac{\partial U}{\partial x} = \lambda p$ .... (7)

↳ the price ratio between p_q and p_i

$$p_q / p_i = \frac{\partial U}{\partial q} \Big/ \frac{\partial U}{\partial x_i} \quad \text{.... (8)}$$

From (8) and (5) $p_i \frac{\partial U}{\partial q} \Big/ \frac{\partial U}{\partial x_i} = \cancel{p_i} \cdot p_q / \cancel{p_i} = p_q = \frac{\partial U}{\partial q} \Big/ \frac{\partial U}{\partial x_i}$

Let q^* & x^* are the maximum value for $\max_{q,x} \bar{U}(q,m)$ s.t $pq + m \leq y$

$$\mathcal{L} = \bar{U}(q,m) + \lambda(y - P_q q - m)$$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial q} \equiv \frac{\partial \bar{U}}{\partial q} - \lambda P_q = 0 \Leftrightarrow \frac{\partial \bar{U}}{\partial q} = \lambda P_q \quad \cdots \quad (9)$$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial m} = \frac{\partial \bar{U}}{\partial m} - \lambda = 0 \Leftrightarrow \frac{\partial \bar{U}}{\partial m} = \lambda \quad \cdots \quad (10)$$

substitute (9) and (10)

$$\frac{\partial \bar{U}}{\partial q} \Big/ \frac{\partial \bar{U}}{\partial m} = P_q \quad \cdots \quad (11)$$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial \lambda} = y - P_q \cdot q - m = 0 \quad \cdots \quad (12)$$

using (11) and (12) we know that this lead to the same conclusion with number (8).

c) Since $v(p_y, p, y) \equiv \max \bar{U}(q,m)$ s.t $pq + m \leq y$

equalsto $v(p_y, P, y) \equiv \max U(q, x)$ st $p.q + p \cdot x = p.q + m \leq y$

than we can conclude that when the prices of all but one good are fixed, we can analyze the consumer's poblems as if there were only two gods

4.19 A consumer has preferences over the single good x and other goods y represented by the utility function $u(x,y) = \ln(x) + y$. Let the price of x be p, the price of y be unity and let income be $m > 1$.

(a) Derive the Marshallian demand for x and y.

(b) Derive the indirect utility function.

(c) Use the Slutsky equation to decompose the effect of an-own price change on the demand for x into an income and substitution effect.

(d) Suppose that the price of x rises from p^0 to $p^1 > p^0$. Find CV, EV and change in the consumer surplus connected with price increase.

(e) Illustrate your findings with two diagrams, one giving indifference curves and budget constraints, other giving Hicksian and Marshallian demands.

a. Setup the Lagrangean and change the notation $x_1 \equiv x$, $x_2 \equiv y$

$p_1 \equiv p_x$, $p_2 \equiv p_y = 1$

$$U(x_1, x_2) = \ln x_1 + x_2 \quad \text{st} \quad p_1x_1 + p_2x_2 \leq m$$

$$\mathcal{L} = \ln x_1 + x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

↳ FOC

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 \leq 0 \quad (\text{or} = 0 \text{ if } x_1 > 0) \quad \dots (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda p_2 \leq 0 \quad (\text{or} = 0 \text{ if } x_2 > 0) \quad \dots (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1x_1 - p_2x_2 \leq 0 \quad (\text{or} = 0 \text{ if } x > 0) \quad \dots (3)$$

↳ Given that $x_1 > 0$ (since $1/x_1$ will be undefined if $x_1 = 0$)

$\lambda > 0$ (since $1 - \lambda p_2 \leq 0 \iff 1 \leq \lambda p_2$ while $p_2 > 0$)

we have $\frac{1}{x_1} - \lambda p_1 = 0$ and $m - p_1x_1 - p_2x_2 = 0$

$$\frac{1}{x_1} = \lambda p_1 \iff p_1x_1 = \frac{1}{\lambda}$$

using (2) we have $1 - \lambda p_2 \leq 0 \iff 1 \leq \lambda p_2 \iff \frac{1}{\lambda} \leq p_2$ thus $p_1x_1 \leq p_2$

or $\boxed{p_1x_1 = p_2}$ if $x_2 > 0$

↳ Marshallian demand for x_2

$$p_2x_2 + p_1x_1 = m \iff x_2 = \frac{m - p_1x_1}{p_2}$$

if $x_2 > 0$ then $p_1x_1 = p_2$ $\quad x_2(p_1, p_2, m) = \frac{m - p_2}{p_2}$

$$\text{thus} \quad x_2(p_1, p_2, m) \begin{cases} \frac{m - p_2}{p_2} & , \text{if } m \geq p_2 \\ 0 & , \text{if } m < p_2 \end{cases}$$

Marshallian demand for x_1.

$$x_1 = \frac{p_1x_1}{p_1} = \frac{p_2}{p_1} \quad , \text{if } x_2 > 0 \text{ or } m \geq p_2$$

$$x_1 = \frac{m - p_2x_2}{p_1} = \frac{m - 0}{p_1} = \frac{m}{p_1} \quad \text{if } x_2 = 0 \text{ . or } m < p_2$$

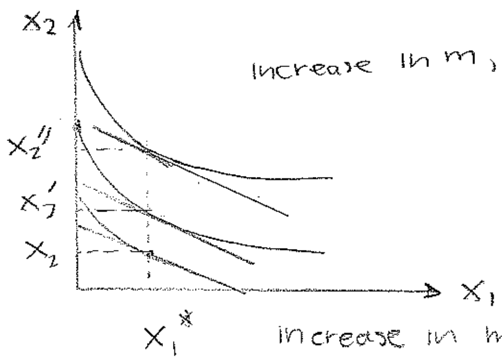
↳ since $p_2 = 1$ and $m > 1 = p_2$ then

$$x_1(p_1, p_2, m) = p_2/p_1 = 1/p_1$$

$$x_2(p_1, p_2, m) = \frac{m - p_2}{p_2} = \frac{m-1}{1} = m - 1$$

d)

e) since the income effect for X_1 is zero and the income effect X_2 is negative

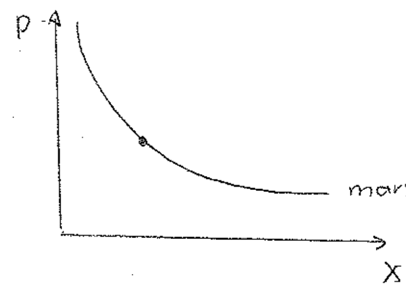


increase in m, increases demand of X_2

increase in m doesn't change the demand of x_1

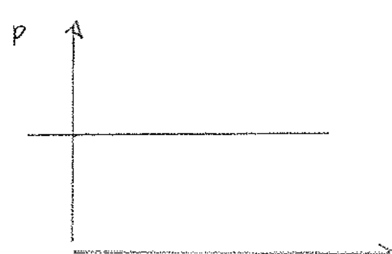
for good 2, the substitution effect is zero which means

Demand for good 1



marshallion $x_1 = 1/p_1$

Demand for good 2



b. Indirect utility function

↳ plug in the marshallian demand to the direct utility function

$$u(x_1, x_2) = \ln x_1 + x_2$$

with $x_1 = 1/p_1$ and $x_2 = m-1$

$$v(p_1, p_2, m) = \ln(1/p_1) + (m-1)$$

c. Slutsky equation

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - x_i \frac{\partial x_i}{\partial m}$$

for $x_1 = 1/p_1$ $\quad \frac{dx_1}{dm} = 0 \quad \frac{dx_1}{dp_1} = -\frac{1}{p_1^2}$ thus $\frac{dx_1^h}{dp_1} = \frac{dx_1}{dp_1} + x_1 \frac{dx_1}{dm}$

$$\frac{dx_1^h}{dp_1} = -\frac{1}{p_1^2} + x_1 \cdot 0 = -\frac{1}{p_1^2}$$

for $x_2 = m-1$ $\quad \frac{dx_2}{dm} = 1 \quad \frac{dx_2}{dp_2} = 0$ thus $\frac{dx_2^h}{dp_2} = \frac{dx_2}{dp_2} + x_2 \frac{dx_2}{dm}$

$$\frac{dx_2^h}{dp_2} = 0 + (m-1)\cdot 1 = m-1$$

↳ for x_1 income effect $\frac{dx_1}{dm} x_1 = 0$

"increase in income doesn't change the demand for x_1"

substitution effect $\frac{dx_1^h}{dp_1} = -\frac{1}{p_1^2}$ ....,

"increase in price of good 1 will increase the hicksian demand with a same utility"

↳ for x_2 income effect $\frac{dx_2}{dm} x_2 = m-1$

"increase in income will increase the demand for x_2"

substitution effect $\frac{dx_2^h}{dp_2} = 0$

"increase in price of good 2 doesn't change the hicksian demand while keeping the level of utility function

4.26 A competitive industry is in long-run equilibrium. Market demand is linear. $p = a - bQ$. Where $a > 0$, $b > 0$, and Q is market output. Each firm in the iundusty has the same technology with cost function. $c(q) = k^2 + q^2$

a. What is the long-run equilibrium price? (Assume what's necessary of the parameters to ensure that this is positive and less than a.)
b. Suppose that the government imposes a per-unit tax. $t > 0$. On every producing firm in the industry describe what would happen in the long run to the number of firms in the industry. What is the posttax market equilibrium price? (Again, assume whatever is necessary to ensure that this is positive and less than a.)
c. Calculate the long-run effect of this tax on consumer surplus. Show that the deadweight loss from this tax exceeds the amount of tax revenue collected by the government in the posttax market equilibrium.

a) market demand $p = a - bQ$ with $c(q) = k^2 + q^2$

long run equilibrium price $p = MC = \frac{\partial c(q)}{\partial q} = 2q$

to ensure that $p > 0$ we need that $q > 0$ and so with J identical firms

so that $Q = J \cdot q > 0$ and $bQ < a \Leftrightarrow Q < a/b$

so $0 < Q < a/b$

b) unit tax $t > 0$

$c(q') = c(q) + q \cdot t = k^2 + q^2 + qt$ so $MC' = 2q + t$

$p = MC' = 2q' + t = a - bQ' \Leftrightarrow 2q' + t = a - bq'J' \Leftrightarrow (2 + bJ')q' = a - t$

$\Leftrightarrow q^t = \frac{(a-t)}{2+bJ'} \Leftrightarrow J^t = \left[\frac{(a-t)}{q^0} - 2\right] \cdot \frac{1}{b}$

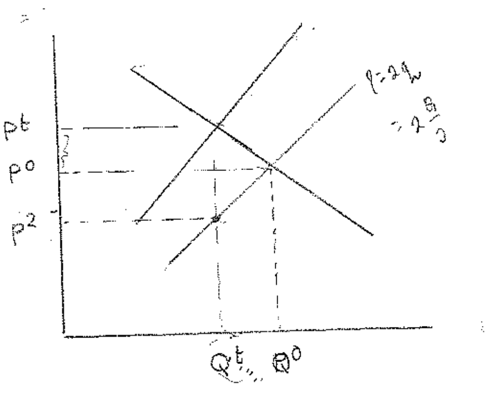
before tax $MC = 2q = a - bQ = a - bJq \Leftrightarrow 2q + bJq = a \Leftrightarrow q(2+bJ) = a$

$\Leftrightarrow q^0 = \frac{a}{2+bJ} \Leftrightarrow J^0 = \left[\frac{a}{q} - 2\right] \cdot \frac{1}{b}$

$$J^t = \left[\frac{a}{q'} - \frac{t}{q'} - 2\right] \frac{1}{b} = \left[\frac{a}{q'} - 2\right]\frac{1}{b} - \left[\frac{t}{q} \cdot \frac{1}{b}\right] = J^0 - \left[\frac{t}{q} \cdot \frac{1}{b}\right]$$

the number of firms will be reduced by t/qb

c)



after tax

↳ find p^t and Q^t

$Q^t = q^t \cdot J = \left[\frac{a-t}{2+bJ}\right] \cdot J$

$p^t = a - bQ^t = a - b\left[\frac{a-t}{2+bJ}\right] \cdot J = \frac{2a + \cancel{abJ} - \cancel{abJ} + tbJ}{2+bJ}$

$= \frac{2a + tbJ}{2 + bJ}$

↳ find p^0 and Q^0

$Q^0 = q^0 \cdot J = \left[\frac{a}{2+bJ}\right] \cdot J$

$p^0 = a - bQ^0 = a - b\left[\frac{a}{2+bJ}\right] \cdot J = \frac{2a + \cancel{abJ} - \cancel{abJ}}{2+bJ}$

$= \frac{2a}{2+bJ}$

↳ find p^2

p^2 is the price when Q^t for every firms

supply curve before tax $p = 2q = 2Q/J$

after tax $p = 2q + t$

$$p^2 = 2\frac{Q}{J} = 2\frac{Q^t}{J} = \frac{2}{\cancel{J}}\left[\frac{a-t}{2+bJ}\right]\cdot \cancel{J} = \frac{2a-2t}{2+bJ}$$

$$\text{Total tax} = Q^t \cdot t = \left[\frac{a-t}{2+bJ}\right]\cdot J \cdot t = (a-t)\left[\frac{Jt}{2+bJ}\right]$$

$$\text{Dead Weight Loss} = \tfrac{1}{2}(Q^0 - Q^t)\cdot(p^t - p^2)$$

$$= \frac{1}{2}\left[\frac{aJ}{2+bJ} - \frac{aJ-tJ}{2+bJ}\right]\cdot\left[\frac{2a+tbJ}{2+bJ} - \frac{2a-2t}{2+bJ}\right]$$

$$= \frac{1}{2}\left[\frac{tJ}{2+bJ}\right]\cdot\left[\frac{tbJ+2t}{2+bJ}\right] = \frac{1}{2}\left[\frac{Jt}{2+bJ}\right]\cdot t$$

So total tax : Deadweight Loss $= (a-t)\left[\frac{Jt}{2+bJ}\right] : \frac{1}{2}t\left[\frac{Jt}{2+bJ}\right]$

$= a-t : \frac{1}{2}t$

d) $c(q^{t_3}) = c(q) + t = k^2 + q^2 + t$ so $MC' = 2q$ Let t_3, Q_3, J_3 is lumpsump tax per firms number of firms & quantity

the market supply curve doesn't shift so producer will still produce $p = 2q$

since $Q^t \cdot t = \left[\frac{a-t}{2+bJ_t}\right]\cdot J_t t$

total revenue from tax is equal to total firms times tax per firm

$$\Leftrightarrow \quad J_t t \cdot \left[\frac{a-t}{2+bJ_t}\right] = J_3 \cdot t_3$$

If $t_3 = t$ the higher t, the lower J_3

which implies in the long run the number of firms will decrease

If $J_3 = J^t$ $\quad t\cdot\left[\frac{a-t}{2+bJ^t}\right] = t_3$

the higher J^t, the lower t_3 compare to t

e) lump-sump tax on consumer

4.27 A per-unit tax, $t > 0$ is levied on the output of a monopoly. The monopolist faces demand, $q = p^{-\epsilon}$where $\epsilon > 1$, and has constant average costs. Show that the monopolist will increase price b more than the amount of the per-unit tax.

↳ Average cost constant

$$\frac{c(q)}{q} = \alpha \Leftrightarrow c(q) = \alpha q \quad \text{which implies} \quad \frac{d(c(q))}{dq} = MC = \alpha$$

$$AC = MC = \alpha$$

↳ demand function

$$q(p) = p^{-\varepsilon} \Leftrightarrow p = q^{-1/\varepsilon}$$

↳ profit function

$$\Pi = TR - TC = p \cdot q - c(q) = q^{-1/\varepsilon} \cdot q - c(q) = q^{\frac{\varepsilon-1}{\varepsilon}} - c(q)$$

FOC for profit maximization

$$\frac{d\Pi}{dq} = 0 \Leftrightarrow \frac{\varepsilon-1}{\varepsilon} q^{\left(\frac{\varepsilon-1}{\varepsilon}-1\right)} - \alpha = 0 \Leftrightarrow \frac{\varepsilon-1}{\varepsilon} q^{-1/\varepsilon} - \alpha = 0 \Leftrightarrow \frac{\varepsilon-1}{\varepsilon} q^{-1/\varepsilon} = \alpha$$

$$\Leftrightarrow q = \left[\left(\frac{\varepsilon}{\varepsilon-1}\right)\alpha\right]^{-\varepsilon}$$

substitute to p

$$\Leftrightarrow p = q^{-1/\varepsilon} = \left[\left(\frac{\varepsilon}{\varepsilon-1}\right)\alpha\right]^{-1/\cancel{\varepsilon} \cdot -\cancel{\varepsilon}} = \left[\frac{\varepsilon}{\varepsilon-1}\right]\alpha$$

↳ since $\varepsilon > 1$ thus $\frac{\varepsilon}{\varepsilon-1} > 1$

↳ per unit tax will increase α to $\alpha + t$ so

$$p' = \frac{\varepsilon}{\varepsilon-1}(\alpha+t) > \frac{\varepsilon}{\varepsilon-1}\alpha$$

$$p' > p$$

with p' is a price after tax

Econ - 206 Homework # 13

- Angsoka (Oka) Paundralingga
- Song Ho Lee
- Wishnu Mahraddika

5.6 a) Prove : $x(p^*)$ WEA $\longrightarrow x(p^*) \in F(e)$

by contradiction :

Assume $x(p^*) \notin F(e)$

by definition of $F(e)$

$$F(e) : \{ x \mid \sum_{i \in I} x^i = \sum_{i \in I} e^i \}$$

so $x(p^*) \notin F(e)$ implies $\sum x^i < \sum e^i$

$x(p^*) = (x^1(p^*, p^* e^1), \ldots, x^I(p^*, p^* \cdot e^I))$

since

$\sum x^i < \sum e^i$, then $x^1(p, p \cdot e^1) + x^2(p, p \cdot e^2) + \ldots < e^1 + e^2 + \ldots$

$= p^*[x^1(p, p \cdot e^1) + \ldots + x^I(p, p \cdot e^I)] < p^*[e^1 + e^2 + \ldots + e^I]$

$= \sum p^* x < \sum p^* e$

$= \underbrace{\sum p^* (\sum x^i - \sum e^i) < 0}_{\text{which violates Walras' law}}$

b) If $u^i(x^i) > u^i(\hat{x}^i) \longrightarrow p \cdot x^i \geq p \cdot \hat{x}^i$

given that u^i is strictly increasing

then $x^a > x^b \longrightarrow u(x^a) > u(x^b)$

let

$x^i > \hat{x}^i$ then we will have $u(x^i) > u(\hat{x}^i)$

If we multiply x^i and $\hat{x}^i$ with p, we will have :

$p x^i > p \hat{x}^i$ ∎

5.10 Pareto efficient $\rightarrow$ there is no other bundle other than x^* such that

$$u(x^i) > u(x^{i*}) \quad \forall i$$

It is equivalent with :

$\bar{x}^i$ solves the problem $\max_{x^i} u^i(x^i)$ s.t. $u^j(x^j) \geq u^j(\bar{x}^j)$

$$\Sigma x_1^i = \Sigma e_1^i \quad i = 1, 2$$

$$\Sigma x_2^i = \Sigma e_2^i \quad i = 1, 2$$

Thus, no other x'^i will fulfill $u^i(x'^i) > u^i(\bar{x}^i) \quad \forall i$

or, more formally : (next page)

Formally 5.10 part b

$\bar{x} \in F(e)$ is Pareto efficient $\Leftrightarrow$ $\bar{x}^i$ solves the maximization problem

For n goods and I consumer

($\Rightarrow$) Use contradiction

assume $\bar{x}^i$ is not the solution of

$$\max_{x^i} u^i(x^i) \quad \text{s.t} \quad u^j(x^j) \geq u^j(\bar{x}^j) \quad i \neq j$$
$$\sum_{b=1}^{n} x_i^b = \sum_{b=1}^{n} e_i^b \quad \forall i$$

so we must have another $\tilde{x}^i$ which is the solution

and $u^i(\hat{x}^i) \geq u^i(\bar{x}^i) \quad \forall i$

and $\hat{x}$ is still feasible, $\sum x_i = \sum e_i$

So since $U(\hat{x}) \geq U(\bar{x})$ $\hat{x} > \bar{x}$ which means

$$\sum x_i^b < \sum e_i^b$$

which contradict that $\bar{x} \in F(e)$

($\Leftarrow$) use contradiction

assume $\bar{x} \notin F(e)$ which means $\{x \mid \sum x^i < \sum e^i\}$

then it will contradict the assumption that $\bar{x}^i$ solves the maximization problem. Given that utility is increasing in x we will have another bundle that can maximize the problem while binding the constraint

$$\{x \mid \sum x^i = \sum e^i\}$$

5.11 $u^1(x_1, x_2) = (x_1 x_2)^2 \quad e^1 = (18, 4)$

$u^2(x_1, x_2) = \ln x_1 + 2 \ln x_2 \quad e^2 = (3, 6)$

(a) Characterize the set of Pareto - Efficient allocations:

① $\Sigma x_t = \Sigma e_t$

$x_1^1 + x_1^2 = 18 + 3 = 21$

$x_2^1 + x_2^2 = 4 + 6 = 10$

$MRS_{12}^1 = MRS_{12}^2 \rightarrow \frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^1}$

(b) Characterize the core

$x_1^1 x_2^1 \geq 72$

$\ln(x_1^2) + 2\ln(x_2^2) \geq \ln(108)$

(c) Find a Walrasian equilibrium and compute the WEA

Consumer 1's demand $\Rightarrow$

$L = (x_1 x_2)^2 + \lambda(p.e - p_1 x_1 - p_2 x_2)$

$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 x_2^2 - p_1 \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_1^2 x_2 - p_2 \lambda = 0 \end{aligned}\right\} p_1 x_1 = p_2 x_2$

$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 = p \cdot e \quad \therefore x_1 = \frac{p \cdot e}{2p_1} \quad x_2 = \frac{p \cdot e}{2p_2}$

Consumer 2's demand $\Rightarrow$

$L = \ln x_1 + 2\ln x_2 + \lambda(p.e - p_1 x_1 - p_2 x_2)$

$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{1}{x_1} - p_1 \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= \frac{2}{x_2} - p_2 \lambda = 0 \end{aligned}\right\} 2p_1 x_1 = p_2 x_2$

$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 = p \cdot e \quad \therefore x_1 = \frac{p \cdot e}{3p_1} \quad x_2 = \frac{2p \cdot e}{3p_2}$

$(p\cdot e)^1 = 18p_1 + 4p_2 \qquad (p\cdot e)^2 = 3p_1 + 6p_2$

we can rewrite the demand as:

Consumer 1 = $x_1^1 = \dfrac{18p_1 + 4p_2}{2p_1} = \dfrac{9p_1 + 2p_2}{p_1} \qquad x_2^1 = \dfrac{18p_1 + 4p_2}{2p_2} = \dfrac{9p_1 + 2p_2}{p_2}$

Consumer 2 = $x_1^2 = \dfrac{3p_1 + 6p_2}{3p_1} = \dfrac{p_1 + 2p_2}{p_1} \qquad x_2^2 = \dfrac{6p_1 + 12p_2}{3p_2} = \dfrac{2p_1 + 4p_2}{p_2}$

$$x_1^1 + x_2^1 = \frac{10p_1 + 4p_2}{p_1} = 21 \Rightarrow 11p_1 = 4p_2$$

$$x_1^2 + x_2^2 = \frac{11p_1 + 6p_2}{p_1} = 10 \Rightarrow 11p_1 = 4p_2$$

$$\left.\right\} \frac{p_1^*}{p_2^*} = \frac{4}{11}$$

Consumer 1 $\rightarrow x_1^1 = \frac{29}{2} \quad x_2^1 = \frac{58}{11}$

Consumer 2 $\rightarrow x_1^2 = \frac{26}{4} \quad x_2^2 = \frac{52}{11}$

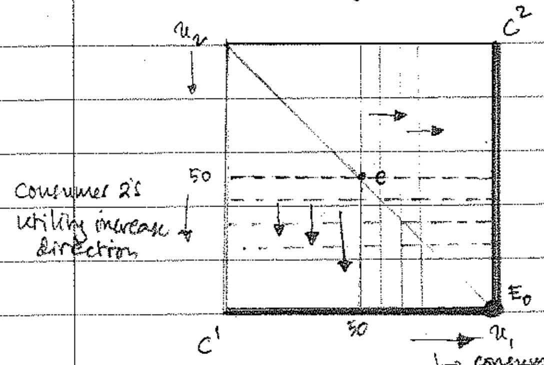
(d) verify the equilibrium in (c) is in the core

$$x_1^1 \cdot x_2^1 = \frac{29}{2}\cdot\frac{58}{11} = \frac{841}{11} = 76{,}4 > 72$$

$$\ln(x_2^1) + 2\ln(x_2^2) = \ln\left(\frac{26}{4}\right) + 2\ln\left(\frac{52}{4}\right) = \ln(145.26) > \ln(108)$$

5.15. There are 100 units of x_1 and 100 units of x_2. Consumers 1 and 2 are each endowed with 50 units of each good. Consumer 1 says, "I love x_1, but I can take or leave x_2." Consumer 2 says, "I love x_2, but I can take or leave x_1."

(a) Draw an Edgeworth box for these traders and sketch their preferences.



- Consumer 1 has strong preference for x_1 and indifferent about x_2 → Consumer 1's preference is vertical, and he/she gets more utility by consuming more of x_1.

- Consumer 2 has strong preference for x_2 and indifferent about x_1 → his/her preference is horizontal, and he/she gets more utility by consuming more of x_2.

- Consumer 1 will be able to consume more of x_1 by exchanging his/her endowed x_2 with consumer 2's endowed x_1 (visa versa). This will enable each consumer to maximize their utility where consumer 1 only consumes x_1 and consumer 2 only consumes x_2 (at point E_0)

(b) Identify the core of this economy: Point E_0

(c) Find all Walrasian equilibria for this economy: Point E_0, where consumer 1/2 consumes x_1/x_2 only.

$p_1 = p_2$ and consumer 1 = (100, 0)

consumer 2 = (0, 100)

Econ 206

Hw. #14

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5.19 ; 5.26 ; 5.27 ; 5.33

5.19 An exchange economy has three consumers and three goods. Consumers' utility function and endowments are as follows:

$u^1(x_1, x_2, x_3) = \min(x_1, x_2)$ $\quad e^1 = (1, 0, 0)$

$u^2(x_1, x_2, x_3) = \min(x_2, x_3)$ $\quad e^2 = (0, 1, 0)$

$u^3(x_1, x_2, x_3) = \min(x_1, x_3)$ $\quad e^3 = (0, 1, 0)$

Find a Walrasian equilibrium and the associated WEA for this economy!

The demand for each person for each good:

$x_1^1 = x_2^1 = p_1 / p_1 + p_2$, and $x_3^1 = 0$

$x_2^2 = x_3^2 = p_2 / p_2 + p_3$, and $x_1^2 = 0$

$x_1^3 = x_3^3 = p_3 / p_1 + p_3$, and $x_2^3 = 0$

In Walrasian equilibrium:

good 1: $x_1^1 + x_1^2 + x_1^3 = e_1^1 + e_1^2 + e_1^3$

$p_1/p_1+p_2 + 0 + p_3/p_1+p_3 = 1 + 0 + 0$ $\quad \cancel{p_1/p_1+p_2} + p_3/p_1+p_3 = \cancel{p_1/p_1+p_2} + p_2/p_2+p_3$

good 2: $x_2^1 + x_2^2 + x_2^3 = e_2^1 + e_2^2 + e_2^3$ $\quad p_3/p_1+p_3 = p_2/p_2+p_3$

$p_1/p_1+p_2 + p_2/p_2+p_3 + 0 = 0 + 1 + 0$ ..(*)

$2p_2/p_2+p_3 = 1$

good 3: $x_3^1 + x_3^2 + x_3^3 = e_3^1 + e_3^2 + e_3^3$ $\quad 2p_2 = p_2 + p_3$

$0 + p_2/p_2+p_3 + p_3/p_1+p_3 = 0 + 0 + 1$ $\quad \boxed{p_2 = p_3}$ (**)

Using (*) and (**):

$p_1/p_1+p_2 + p_2/p_2+p_3 = 1$

$p_1/p_1+p_2 + \cancel{p_2}/2\cancel{p_2} = 1$

$p_1/p_1+p_2 = 1/2 \longrightarrow 2p_1 = p_1 + p_2$

$\boxed{p_1 = p_2}$ $\quad \boxed{p_1 = p_2 = p_3}$ (***)

Thus, the equilibrium:

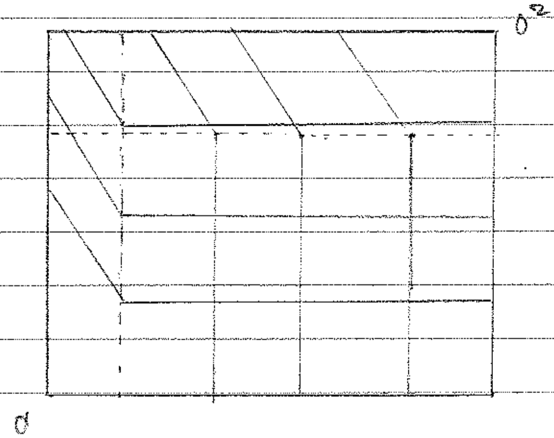
$x_1^1 = x_2^1 = p_1/p_1+p_2 = 1/2$, and $x_3^1 = 0$

$x_2^2 = x_3^2 = p_2/p_2+p_3 = 1/2$, and $x_1^2 = 0$

$x_1^3 = x_3^3 = p_3/p_1+p_3 = 1/2$, and $x_2^3 = 0$

with the price $p_1 = p_2 = p_3$.

5.26.



$$u^1 = \min\{x_1, c\} + x_2$$
$$u^2 = \min\{x_2, c\} + x_1$$

5.27) a) Suppose both consumers can not trade with each other :

$$u^i(x_1, x_2) = \ln(x_1) + \ln(x_2) \quad , \quad i = 1,2$$

$$u^1 = \ln x_1^1 + \ln x_2^1$$

$$= \ln x_1^1 + \ln(10 - x_1^1)$$

The FOC : $\dfrac{\partial u^1}{\partial x_1^1} = 0 \Rightarrow \dfrac{1}{x_1^1} - \dfrac{1}{10 - x_1^1} = \dfrac{10 - x_1^1 - x_1^1}{x_1^1(10 - x_1^1)} = 0$

$\therefore x_1^1 = 5 \; ; \; x_2^1 = 5$ (since goods are storable)

Thus, $u^1 = 2 \ln 5$

Similarly for consumer 2 :

$$x_1^2 = x_2^2 = 12.5$$

Thus, $u^2 = 2 \ln 12.5$

b) $p_1 \cdot 10 = p_1 x_1^1 + p_2 x_2^1$ ... (1)

$p_1 \cdot 20 + p_2 \cdot 5 = p_1 x_1^2 + p_2 x_2^2$ ... (2)

Assume $p_1 = 1$ (spot price per unit in period 1),

then (1) becomes and, (2) becomes :

$x_1^1 = 10 - p_2 x_2^1$ (1a) $\qquad x_1^2 = 20 + p_2(5 - x_2^2)$ ... (2a)

Thus, by (1a), $u^1 = \ln x_1^1 + \ln x_2^1 = \ln(10 - p_2 x_2^1) + \ln x_2^1$

FOC : $\dfrac{\partial u^1}{\partial x_2^1} = \dfrac{-p_2}{10 - p_2 x_2^1} + \dfrac{1}{x_2^1} = \dfrac{10 - p_2 x_2^1 - p_2 x_2^1}{x_2^1(10 - p_2 x_2^1)} = 0$

$p_2 x_2^1 = 10/2$

$x_2^1 = 5/p_2$, substitute to (1a) $x_1^1 = 5$

For consumer 2, by (2a), we have $u^2 = \ln x_1^2 + \ln x_2^2 = \ln(20 + p_2(5 - x_2^2)) + \ln x_2^2$

Using similar steps, we have $x_2^2 = 10/p_2 + 2.5$, $x_1^1 = 10 + 2.5 p_2$

Since $\sum x_j^i = \sum e_j^i \;\; \forall j$, thus $x_1^1 + x_1^2 = e_1^1 + e_1^2$

$$5 + 10 + 2.5 p_2 = 10 + 20 \longrightarrow 2.5 p_2 = 15 \; ; \; p_2 = 6$$

$$\therefore p_2/p_1 = 6/1 = 6$$

5.33 a. Edgeworth box with envy free and not fair (not Pareto efficient) allocation :

Let consumer 1 have the following preference :

$$x_1^1 \succsim^1 x_1^2 \quad \text{and} \quad x_2^1 \succsim^1 x_2^2$$

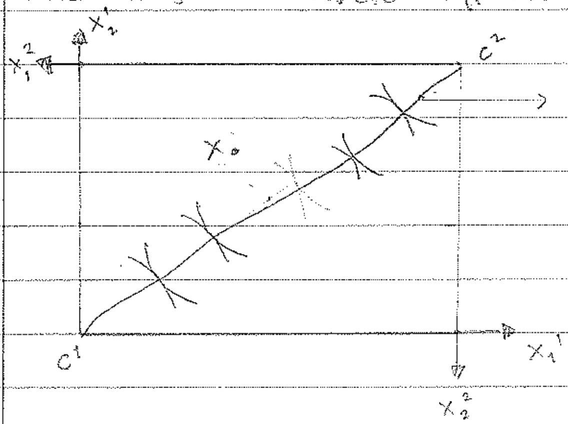
and consumer 2's preference :

$$x_1^2 \succsim^2 x_1^1 \quad \text{and} \quad x_2^2 \succsim^2 x_2^1$$

But this allocation is not included in the Pareto Efficient set

$$x = ((x_1^1, x_1^2), (x_2^1, x_2^2)) \notin \text{Pareto Efficient set}$$

so allocation x is envy free for consumer 1 and 2, but it is not fair since it is not Pareto Efficient.



b. By assumption 5.1

Utility u is continuous, strongly increasing and strictly quasiconcave.

let $p = (p_1, p_2, \ldots, p_n) \gg 0$

thus this maximization problem has a unique solution

max $u(x_i)$ s.t $x_i \in \mathbb{R}^n$

$$p(x_i) \leq p(e_i) \text{ where } \sum e_i > 0$$

uniqueness follows from the strict quasiconcavity of u

Theorem of maximum ensure that there is a unique solution, and the solution is Pareto Efficient.

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5.10

Proof. Consider the welfare maximization problem

$$\max_{x} W = \sum_{i \in \mathcal{I}} \alpha^i u^i(x^i) \quad \text{s.t} \quad \sum_{i \in \mathcal{I}}^{n} x_j^i \leqslant \sum_{i \in \mathcal{I}} e_j^i \quad \text{for } j=1,\dots,n$$

$$\mathcal{L}: (\alpha^1 u^1(x^1) + \alpha^2 u^2(x^2) + \dots \alpha^n u^n(x^n)) + \theta^j[(e_1^1 + e_1^2 + \dots e_1^n) - (x_1^1 + x_1^2 + \dots x_1^n)]$$

FOC $\dfrac{\partial \mathcal{L}}{\partial x^i} = \alpha^i \dfrac{\partial u^i(x^i)}{\partial x^i} - \theta_j = 0$

so for each agent i we have $\alpha^i \nabla u^i(x^i)$, a vector of MRS

with $\theta = (\theta_1, \dots \theta_n)$, n vector of numbers θ

$$\alpha^i \nabla u^i(x^i) = \theta$$

* since the utility function is strictly concave, we will have some set of weights α^i for $i \in \mathcal{I}$ and n-vector of numbers $\theta = (\theta_1, \dots \theta_n)$ such that $\alpha^i \nabla u^i((x^i)^*) = \theta$
  - x^* (a vector of n goods) and x^* is a WEA satisfies the constraints, and x^* maximizes W subject to the constraint

The

b Use part a to proof the First Welfare Theorem

First Welfare Theorem

If x^* is a WEA $\Rightarrow$ x^* is a Pareto efficient

From part a, we have

$$\alpha^i \nabla u^i (x^i) = \theta$$

and that there is exist one weight α^i $\nearrow$ inverse of marginal utility

if we take $\alpha^i = [\nabla u^i(x^i)]^{-1} \cdot \theta$

↳ utility maximization problem for every person i

$$\max \; u^i(x_j^i) \;\; s.t. \;\; \sum_{j=1} p_j e_j^i - \sum_{j=1} p_j x_j^i$$

and the $\mathcal{L}^i = u^i(x_j^i) + \lambda [\sum p_j e_j^i - \sum p_j x_j^i]$

$$\text{FOC} \quad \frac{\partial u^i(x_j^i)}{\partial x_j^i} = \lambda_i p_j \quad \Leftrightarrow \quad \lambda_i = \frac{1}{p_j} \cdot \frac{\partial u^i(x_j^i)}{\partial x_j^i}$$

so we can think the vector θ as $1/p_j$ or a vector of prices

then x^* is also a Pareto efficient

for part b

Since we know that x^* solve the maximization problem

~~the~~ the same x will also Pareto efficient

5.11

11

a) Borda rule satisfies U, WP, and D

Let's assume that P_1 is a dictador

$$\begin{array}{c} \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \begin{array}{c} P_1 \\ \begin{pmatrix} x \\ y \\ z \\ a \end{pmatrix} \end{array} \begin{array}{c} P_2 \\ \begin{pmatrix} z \\ x \\ y \\ a \end{pmatrix} \end{array} \begin{array}{c} P_3 \\ \begin{pmatrix} x \\ a \\ y \\ z \end{pmatrix} \end{array} \begin{array}{c} P_4 \\ \begin{pmatrix} x \\ y \\ a \\ z \end{pmatrix} \end{array} \Rightarrow \begin{array}{c} \text{Social Welfare} \\ \begin{pmatrix} x \\ y \\ z \\ a \end{pmatrix} \begin{array}{l} 4+3+4+4 = 15 \\ 3+2+2+3 = 10 \\ 2+4+1+1 = 8 \\ 1+1+3+2 = 7 \end{array} \end{array}$$

Borda count:

$x \succ y = 4$	$y \succ x = 0$	$z \succ x = 1$	$a \succ x = 0$
$x \succ z = 3$	$y \succ z = 3$	$z \succ y = 1$	$a \succ y = 1$
$x \succ a = 4$	$y \succ a = 3$	$z \succ a = 2$	$a \succ z = 2$
$B(x) = 11$	$B(y) = 6$	$B(z) = 4$	$B(a) = 3$

(D) so the social welfare : $x \succ y \succ z \succ a$, since P_1 is the dictactor preference for P_1 $x \succsim_1 y \succsim_1 z \succsim_1 a$

(WP) for P_1 $x \succsim_1 y$

P_2 $x \succsim_2 y$

P_3 $x \succsim_3 a \succsim_3 y$ which implies $x \succsim_3 y$

P_4 $x \succsim_4 y$

and the social welfare is $x \succ y$. So it satisfies WP

(U) we have $P_1 : z \succsim_1 a$ and $P_4 : a \succsim_4 z$

which impliess that Borda rule satisfies all of possible pair b Show that Borda count does not satisfy IIA

example. number of voters

5	4	4	3
x	y	z	y
z	$\boxed{x}$	y	z
y	$\boxed{z}$	x	x

$B(x) = 9 + 5 = 14$ //
$x \succ z = 5 + 4 = 9$
$x \succ y = 5$

$B(y) = 8 + 7 = 15$ //
$y \succ x = 4 + 4 = 8$
$y \succ z = 4 + 3 = 7$

$B(z) = 7 + 9 = 16$ //
$z \succ x = 3 + 4 = 7$
$z \succ y = 5 + 4 = 9$

winner of Borda rule
$z \succ y \succ x$

if we modify this

5	4	4	3
x	$\boxed{x}$	z	y
z	y	y	z
y	$\boxed{z}$	x	x

$B(x) = 9 + 9 = 18$ //
$x \succ z = 5 + 4 = 9$
$x \succ y = 5 + 4 = 9$

$B(y) = 4 + 7 = 11$ //
$y \succ x = 4 \quad 4$
$y \succ z = 4 + 3 = 7$

$B(z) = 7 + 9 = 16$ //
$z \succ x = 3 + 4 = 7$
$z \succ y = 5 + 4 = 9$

winner of Borda rule
$x \succ z \succ y$

which means that the Borda count does not satisfy IIA since non-winner x turned into a winner while no one reversed their relative preference of x vs original winner z.

* By IIA, if x is preferred to z out of the choice set $\{x, z\}$, then introducing a third alternative y, thus expanding the choice set to $\{x, y, z\}$, must not make z preferable to x

5.12

12

Proof.

by minimal Liberalism, (L^*)

There are at least two individuals k and j AND two pairs of (distinct) alternatives $\{w, z\}$ and $\{x, y\}$ with

$w \neq z$ and $x \neq y$

such that if $w \succ_k z$ THEN $w \succ z$ AND

IF $z \succ_k w$ THEN $z \succ w$ AND

IF $x \succ_j y$ THEN $x \succ y$ AND

IF $y \succ_j x$ THEN $y \succ x$

Suppose we have this kind of preferences which satisfies (U)

individu:	k	j	everyone else
	x	z	x
	w	y	z
	z	x	w
	y	w	y

$x \succ w$

$z \succ y$

o) so we have $w \succ_k z$ and $y \succ_j x$

o) from everyone else $x \succ_\ell w$ and $z \succ_\ell y$ for all ℓ (including k and j)

By L^* we have $w \succ z$ and $y \succ x$

By WP we also have $x \succ w$ and $z \succ y$.

$w \succ z$, $z \succ y$, $y \succ x$ and $x \succ w$ which contradict the transitivity. $w \succ w$

∴ this implies we can not have L^*, WP, and U together.

7.3) a.

1 \ 2	L	M	R
U	2,1	1,1	0,0
C	1,2	3,1	2,1
D	2,-2	1,-1	-1,-1

for (1)

$u_1(D,L) = 2 \leq u_1(U,L) = 2$

$u_1(D,M) = 1 \leq u_1(U,M) = 1$

$u_1(D,R) = -1 < u_1(U,R) = 0$

thus U weakly dominates strategy over D

for 2nd player $U_2(U,R) = 0 < U_2(U,M) = 1$

$U_2(C,R) = 1 \leq U_2(C,M) = 1$

$U_2(D,R) = -1 \leq U_2(D,M) = -1$

thus M weakly dominates strategy over R

↳ first attempt, ① eliminate D followed by ② R [R is still weakly dominated]

1 \ 2	L	M
U	2,1	1,1
C	1,2	3,1

$U_2(U,L) = 1 \geq U_2(U,M) = 1$

$U_2(C,L) = 2 > U_2(C,M) = 1$

L weakly dominates M

③ eliminate M

1 \ 2	L
U	2,1
C	1,2

$U_1(U,L) = 2 > U_1(C,L) \Rightarrow$ outcome (U,L)

↳ second attempt, ① eliminate R

1 \ 2	L	M
U	2,1	1,1
C	1,2	3,1
D	2,-2	1,-1

we don't have any strategy which is weakly / strictly dominates other

So in our first attempt

1 \ 2	L	H ⑤	R ③	
U	2,1	1,1	0,0	
~~C~~	~~1,2~~	~~3,1~~	~~2,1~~	④
~~D~~	~~2,-2~~	~~1,1~~	~~-1,-1~~	①

(U,L) is the equilibrium if we eliminate D firstly

If we chose to eliminate R

1 \ 2	L	H	R ①
U	2,1	1,1	0,0
C	1,2	3,1	2,1
D	2,-2	1,1	-1,-1

there is no equilibrium

(b) Definition:

Strictly Dominated in S for player i

a strategy $\bar{s}_i$, for player i is strictly dominated by strategy $\hat{s}_i$

$$\text{IF } u_i(\hat{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

which implies

Strictly dominated in S for player $-i$

$$\text{IF } u_{-i}(s_i, \hat{\hat{s}}_{-i}) > u_{-i}(s_i, \bar{\bar{s}}_{-i}) \; \forall s_i \in S_i$$

or $\bar{\bar{s}}_{-i}$ is strictly dominated by strategy $\hat{\hat{s}}_{-i}$

Suppose in round n, we have $\bar{s}_i$ in S (strictly dominated) for player i
and $\bar{\bar{s}}_i$ in S (strictly dominated) for player $-i$

↳ to proof that the order of elimination (choose to eliminate first $\bar{s}_i$ or to eliminate $\bar{\bar{s}}_{-i}$ first) doesn't change the end result
we just need to show that:

IF we eliminate $\bar{s}_i$ in round n, we will also eliminate $\bar{\bar{s}}_{-i}$ in round $n+1$
and vice versa

↳ choose to eliminate $\bar{s}_i$ in round n

then $\bar{\bar{s}}_{-i}$ is still strictly dominated for player $-i$

$$u_{-i}(s_i, \hat{\hat{s}}_{-i}) > u_{-i}(s_i, \bar{\bar{s}}_{-i}) \quad \forall \; s_i \in S_i - \bar{s}_i$$

Thus $\bar{\bar{s}}_{-i}$ will then be eliminated in round $n+1$

IN GENERAL

Suppose that a strategy S_i is strictly dominated in round n by some undominated strategy $\hat{S}_i$

↳ Let's assume S_i is not eliminated in round n (because there are other $\bar{S}_{-i}$ which is also strictly dominated for other player $-i$)

↳ It will still be dominated by $\hat{S}_i$ in round $n+1$

↳ and then S_i will be eliminated eventually

7.4. a) check whether pure strategy L strictly dominates pure strategy M

	L	M
U	3,$\boxed{0}$	0,$\boxed{-3}$
D	2,$\boxed{4}$	4,$\boxed{5}$

If player 1 choose U $\quad U_2(U,L) > U_2(U,M)$

BUT $\quad 0 > -3$

If player 1 choose D $\quad U_2(D,L) < U_2(D,M)$

$4 < 5$

IT DOESN'T

check whether pure strategy R strictly dominates pure strategy M

	M	R
U	0,$\boxed{-3}$	0,$\boxed{-4}$
D	4,$\boxed{5}$	-1,$\boxed{8}$

If player 1 choose U $\quad U_2(U,M) > U_2(U,R)$

$-3 > -4$

BUT

If player 1 choose D $\quad U_2(D,M) < U_2(D,R)$

$5 < 8$

IT DOESN'T

b) check whether pure strategy M strictly dominated by mix strategy L & R with probability $1/2$

for player 2

	L	R
U	0	-4
D	4	8

mix

$(\frac{1}{2} \times 0) + (\frac{1}{2} \times -4) = -2$

$(\frac{1}{2} \times 4) + (\frac{1}{2} \times 8) = 6$

	M	Mix
U	-3	-2
D	5	6

if player 1 choose U $\quad U_2(U,M) < U_2(U,\text{Mix})$

$-3 < -2$

if player 1 choose D $\quad U_2(D,M) < U_2(D,\text{Mix})$

$5 < 6$

YES, IT DOES!

7.5. a) No pure strategy strictly dominates any other

Let's assume number of player just 2

1 \ 2	1	2	3 . . .	99	100
1	0,0	1,-1	1,-1	1,-1	1,-1
2	-1,1	0,0	1,-1	1,-1	1,-1
3	-1,1		0,0	⋮	⋮
⋮	⋮			1,-1	1,-1
99	-1,1	-1,1	.....	0,0	1,-1
100	-1,1	-1,1	---		0,0

for player 2

IF player 1 choose 1

$U_2(1,1) > U_2(1,s_2) \quad \forall s_2 \in [2,100]$

$0 > -1$

AND

IF player 1 choose 2

$U_2(2,1) > U_2(2,s_2) \quad \forall s_2 \in [2,100]$

$1 > \{0,1\}$

BUT

IF player 1 choose 3

$U_2(3,1) = U_2(3,2)$

$1 = 1$

In other word

for player i $\quad U_i(\hat{s}_i, s_{-i}) \geq U_i(s_i, s_{-i}) \quad \forall (s_i, s_{-i}) \in S$ with $s_i \neq \hat{s}_i$

b) Mixed strategy that strictly dominates 100, $\quad U_2(s_i,100) = \begin{cases} -1, & i \leq 99 \\ 0, & i = 100 \end{cases}$

utility for player 2

Player 1	1	2	3 . . . 99	100
1	0	-1	-1 ... -1	-1
2	1	0	-1 .... -1	-1
3	1	1	0 -1 ... -1	-1
⋮			-1	-1
99	1	1	1 ---- 0	-1
100	1	1	1 1	0

1 (0) and 99 (-1)

1 (1), 1 (0) and 98 (-1)

2 (1), 1 (0) and 97 (-1)

98 (1), 1 (0) and 1 (-1

99 (1), 1 (0)

If we use mixed strategy in which player 2 chooses number between 1 and 99 each with probability $1/99$

Player 1	MIX	100
1	$[(1/99)\times 0\times(-1)]+[(1/99)\times 1\times(0)]+[(1/99)\times 98\times(-1)] = -98/99$	-1
2	$[(1/99)\times 1\times(-1)]+[(1/99)\times 1\times(0)]+[(1/99)\times 97\times(-1)] = -96/99$	-1
3	$[(1/99)\times 2\times(1)]+[(1/99)\times 1\times(0)]+[(1/99)\times 96\times(-1)] = -94/99$	-1
⋮		⋮
i	$[(1/99)\times(i-1)\times(1)] + 0 + [(1/99)\times(100-i-1)\times(-1)]$ $= (1/99)\times(2i-100)$	-1

$$U_2(s_i, MIX) = (1/99)\times(2i-100)$$

for $i=1$, $U_2(s_1, MIX) = U_2(1, MIX) = -98/99$

$i=99$, $U_2(s_{99}, MIX) = U_2(99, MIX) = 98/99$

so $U_2(s_i, MIX)$ is in $[-98/99, 98/99]$ for $i \leq 99$

which is bigger than -1

for $i=100$, $U_2(s_{100}, MIX) = U_2(100, MIX) = 101/99 = 1\,2/99 > 0$

thus $U_2(s_i, MIX) > U_2(s_i, 100) \quad \forall s_i \in S$

(c) 99 is not strictly dominated

By definition

99 is strictly dominated if FOR player i

$$u_i(99, s_{-i}) < u_i(s_i, s_{-i}) \quad \forall (s_i, s_{-i}) \in S \text{ with } s_i \neq 99$$

Proof by counterexample

1 \ 2	1	2	3	 99	100
3	1	1	0	-1 .... -1	-1

(**) (*)

If player 1 chooses 3

$U_2(3, 99) < U_2(3, s_i)$

$-1 < 1$ or $-1 < 0$

but this is only true for $s_i \leq 3$ (**)

For $s_i > 3$ (*)

$U_2(3, 99) = U_2(3, s_i)$

$-1 = -1$

d Show that iterative elimination of strictly dominated strategies yields the unique choice of 1 for each of the N players, and that this requires 99 rounds of elimination

↳ If we have 2 player and use a similar strategy in part b

MIXED strategy in which player i chooses number between 1 and a each with probability $1/a$

↳ THEN that mixed strategy will STRICTLY DOMINATES $a+1$

↳ for player 2 $\quad U_2(S_i, MIX) > U_2(S_i, 100) \quad \forall S_i \in S$

$U_1(MIX, S_i) > U_1(MIX, S_i) \quad \forall S_i \in S$

with $U_2(S_i, MIX) = (1/a)(2i-(a+1))$

$U_1(MIX, S_i) = (1/a)(2i-(a+1))$

↳ Elimination iteratively

In round 1, for example, we will eliminate 100 by having $a=99$, $W_i^1 \subseteq \{1, \dots 99\}$

In round 2, we will eliminate 99 by having $a=98$, $W_i^2 \subseteq \{1, \dots 98\}$

In round 3, we will eliminate 98 by having $a=97$, $W_i^3 \subseteq \{1, \dots 97\}$

⋮

In round k, we will eliminate $(101-k)$ by having $a=100-k$, $W_i^k \subseteq \{1, \dots 100-k\}$

After 99 rounds

In round 99, we will eliminate $(101-99)=2$ by having $a=1$. $W_i^{99} \subseteq \{1\}$

↳ Thus the procedure yields the unique choice of 1

FORMALLY, to proof that there is "<u>one unique choice of 1</u>"

Let the only actions that survive the IESD (Iterative elimination of strictly domination) be a_1^* and a_2^*. In this case there is only 2 player

Use contradiction strategy

Assume (a_1^*, a_2^*) not a Nash equilibrium $(a_1^*, a_2^*) \notin N(G)$

Then we will have another strategy which have better utility. For player 1

$$u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*) \text{ for some } a_1' \in A_1$$

Since there is only (a_1^*, a_2^*) is the only strategy that survive, a_1' must have been strictly dominated by some other action $a_1'' \in A_1$ at some round of elimination process, so

$$u_1(a_1'; a_2) < u_1(a_1'', a_2) \text{ for each } a_2 \in A_2 \text{ not yet eliminated}$$

Precisely we will also have

$$u_1(a_1', a_2^*) < u_1(a_1''; a_2^*)$$

Since by hypothesis a_2^* is never eliminated

IF $a_1'' = a_1^*$ THEN $u_1(a_1', a_2^*) < u_1(a_1^*, a_1^*)$ contradict our assumption

IF not, then $u_1(a_1', a_2^*) < u_1(a_1'', a_2^*) < u_1(a_1''', a_2^*)$

But once again if $a_1''' = a_1^*$ it will again contradict our assumption.

Otherwise we continue this way and eventually reach the desired contradiction since A_1 is a finite set by hypothesis.

e) There are $N=3$ players, and one applies the procedure of iterative weak dominance (PIWD) then $W_i^1 = \{1, 2, \dots 14\}$, $W_i^2 = \{1, 2\}$, $W_i^3 = \{1\}$ $\forall i$

Procedure: we have to find the smallest number for S_1 given S_2 and S_3

let x is for player 1, y is for player 2, and z is for player 3

IF player 2 chooses 100 for the beginning

Then player 1 win if x is closer to $1/3$ of the average than z

$$\frac{1}{3}\text{ average} - x < z - \frac{1}{3}\text{average} \Leftrightarrow \frac{2}{3}\text{ average} - x < z$$

$$\frac{2}{3}\left(\frac{x+y+z}{3}\right) - x < z$$

$$\Leftrightarrow \frac{2}{9}x + \frac{2}{9}\cdot y + \frac{2}{9}z - x < z$$

$$\Leftrightarrow \frac{2}{9}\cdot y - \frac{7}{9}x < \frac{7}{9}z \Leftrightarrow \boxed{2y - 7x < 7z}$$

since $y = 100$ then $200 - 7x < 7z$

To avoid tie, we also have $z > x$

$$z > x \Leftrightarrow 7z > 7x \Leftrightarrow -7z < -7x \Leftrightarrow 200 - 7z < 200 - 7x$$

Thus $200 - 7z < 200 - 7x < 7z$

$$\Leftrightarrow 200 - 7z < 7z \quad \text{or } 2y \dots$$

$$\Leftrightarrow 200 < 14z \Leftrightarrow z > 14.28$$

~~then~~ or $\boxed{2y < 14z}$

for player 1, he/she will remove his/her weakly dominated strategy which is for $x > 14$ then $W_i^1 = \{1, 2, \dots 14\}$

Second iteration

So in this case lets think that player 2 chooses 14, once again, we have to find x and z

$$2y < 14z \Leftrightarrow 2 \cdot 14 < 14z \Leftrightarrow z > 2$$

So once again he/she will remove his/her weakly dominated strategy for $x > 2$ then $W_i^2 = \{1, 2\}$

Third iteration

Let player 2 chooses 2, once again, we have to find x and z

$$2y < 14z \Leftrightarrow 2 \cdot 2 < 14z \Leftrightarrow z > 1$$

then his last choices $W_i^3 = \{1\}$

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Advanced Microeconomics H·W

#7.6

By definition,

(p) Strategy surving iterative weak dominance

$U_i(\hat{S_i}, S_{-i}) \geq U_i(S_i, S_{-i})$ for $\forall S_i \in S$, $S_i \neq \hat{S_i}$

(q)

$\Rightarrow$ Strategy surving iterative strict dominance

$U_i(\hat{S_i}, S_{-i}) > U_i(S_i, S_{-i})$ for $\forall S_i \in S$, $S_i \neq \hat{S_i}$

proof using contradiction!

$p \rightarrow q$ assume $p \rightarrow \sim q$.

Assume $\hat{S_i}$ does not survive from strict dominance

then, $\exists \bar{S_i}$ s.t

$$U_i(\bar{S_i}, S_{-i}) > U_i(\hat{S_i}, S_{-i})$$

This is contradiction for the first definition.

Thus, any strategy surving iterative weak dominance survives iterative strict dominance.

#7.8

Cheng, shows that any symmetric finite game has a symmetric Nash equilibrium in "Note on Equilibrium in Symmetric Games (2004)"?

proof)

For each pure strategy $s \in S$, define a continuous function of a mixed strategy σ by

$$g_s(\sigma) \equiv \max(0, u(s, \sigma) - u(\sigma, \sigma)).$$

In words, $g_s(\sigma)$ is the gain, if any, of unilaterally deviating from the symmetric mixed profile of all playing σ to playing pure strategy s.

Next define,

$$y_s(\sigma) \equiv \frac{\sigma_s + g_s(\sigma)}{1 + \sum_{t \in S} g_t(\sigma)}$$

The set of functions $y_s(\cdot)$ $\forall s \in S$ defines a mapping from the set of mixed strategies to itself.

We first show that the fixed points of $y(\cdot)$ are equilibria.

Of all the pure strategies in the support of σ, one, say w, must be worst, implying $u(w, \sigma) \leq u(\sigma, \sigma)$ which implies that $g_w(\sigma) = 0$.

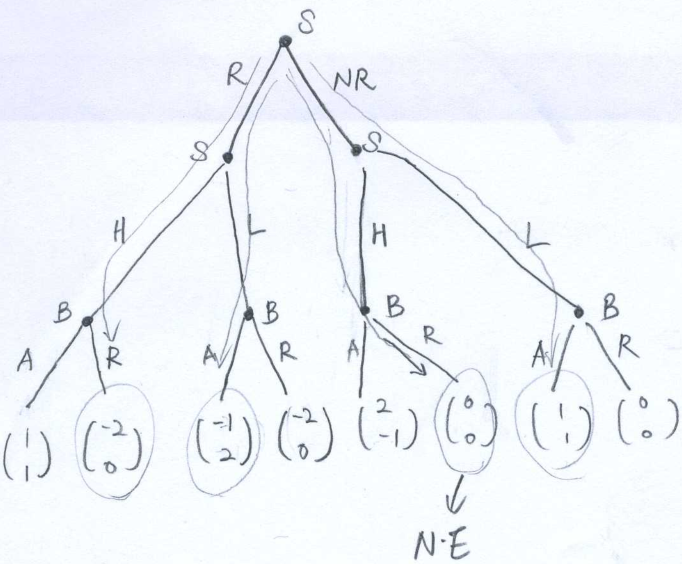
Assume $y(\sigma) = \sigma$. Then, y must not decrease σ_w. The numerator is σ_w so the denominator must be 1, which implies that for all $s \in S$, $g_s(\sigma) = 0$ and so all playing σ is an equilibrium.

Conversely, if all playing σ is an equilibrium then all the g's vanish, making σ a fixed point under $y(\cdot)$.

Finally, since $y(\cdot)$ is a continuous mapping of a compact, convex set, it has a fixed point by Brouwer's Theorem. $\boxtimes$

7.16

< Buyer - Seller Game >

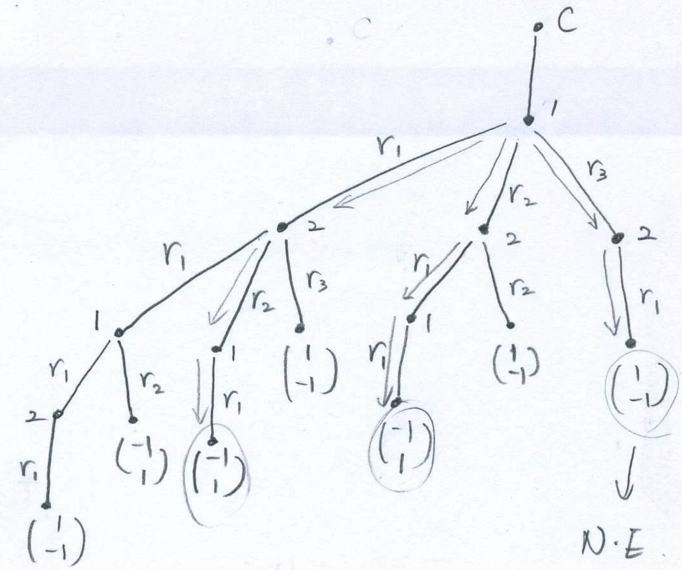


Buyer : If P↑ → Reject!

P↓ → Accept!

Seller : NR and High price.

< Take-away Game >

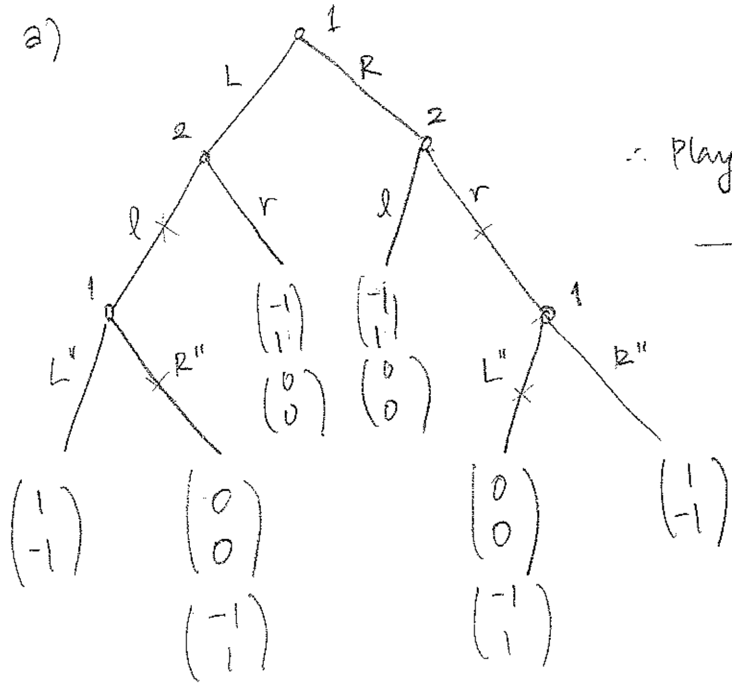


P_2 : If P_1 choose r_2 then P_2 choose r_1 $\binom{-1}{1}$

If P_1 choose r_1 then P_2 choose r_2 $\binom{-1}{1}$

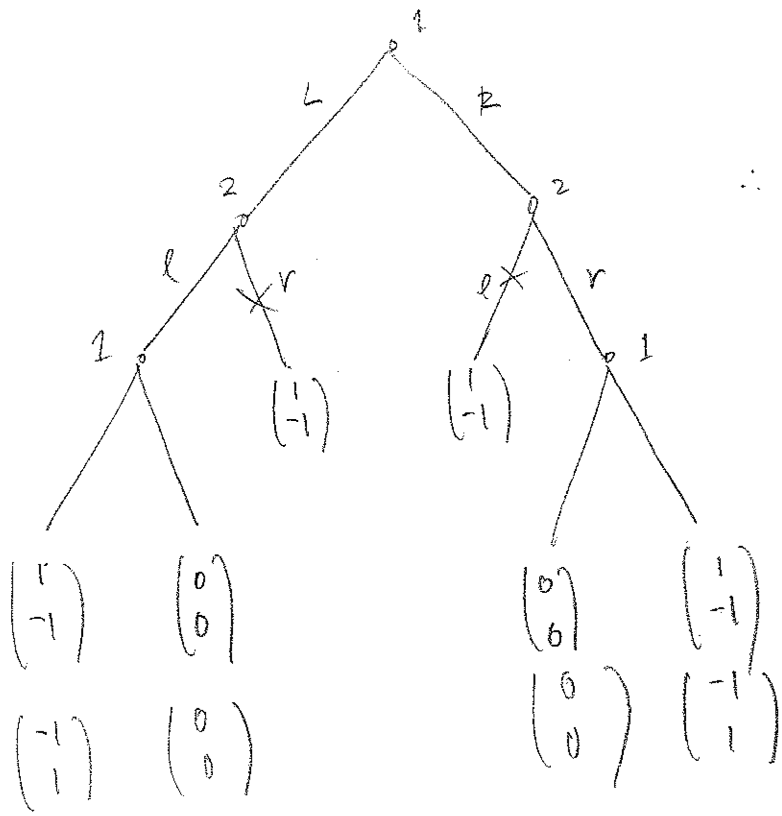
P_1 : P_1 always choose r_3 $\binom{1}{-1}$ Nash equilibrium.

7.19 a)



∴ Player 2 can ensure a win

—— " —— a draw



∴ Player 1 can ensure a win

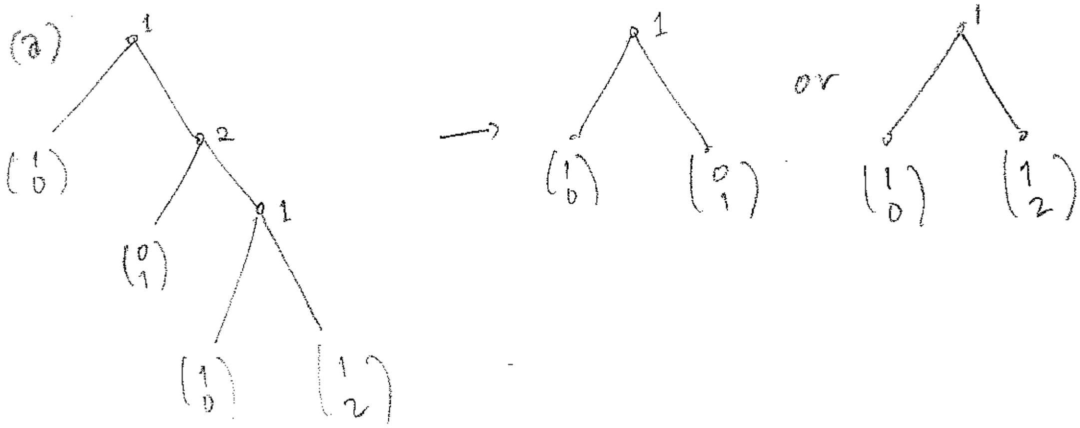
Player 1 can ensure a draw

7.19 b) generalize a) to some well-known parlor games
(tic-tac-toe)

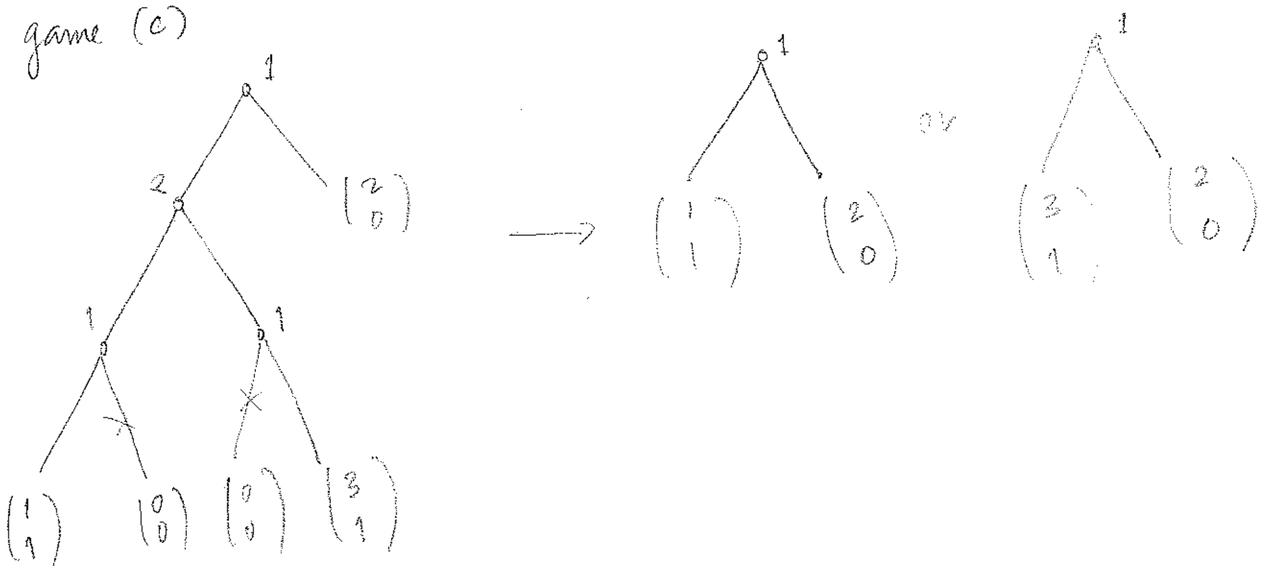
- there is no way to ensure a win

7.18. (a) Which games admit multiple backward induction strategies?

game (a)



game (c)



7.18 (b)

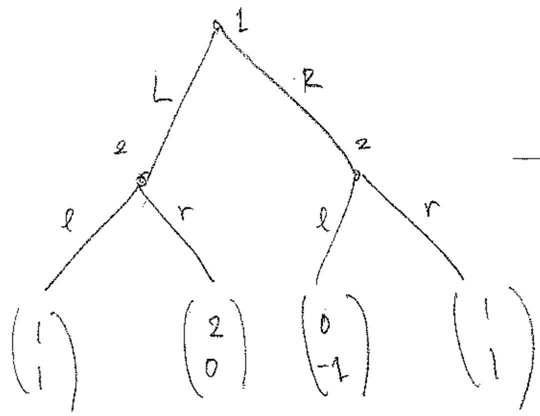
from game (a) we see that the backward induction strategies are not unique

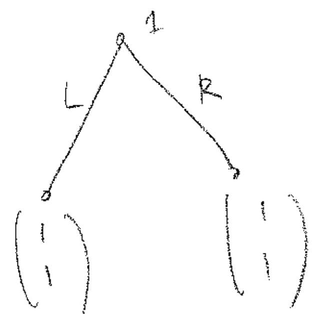
Formal proof : If no player is indifferent between any pair of end nodes, then the backward induction strategy is unique

By contradiction, suppose there are more than 1 backward induction strategy. Let $\bar{s}_i$ and $\bar{\bar{s}}_i$ two of these strategies for player i.

Then, there exist a node x in the information set of i th player (after $k = 0, 1, \ldots, k$ inductions) such that $u_x(\bar{s}_i) = u_x(\bar{\bar{s}}_i)$

(c) example : b.i strategies not unique with unique payment vector



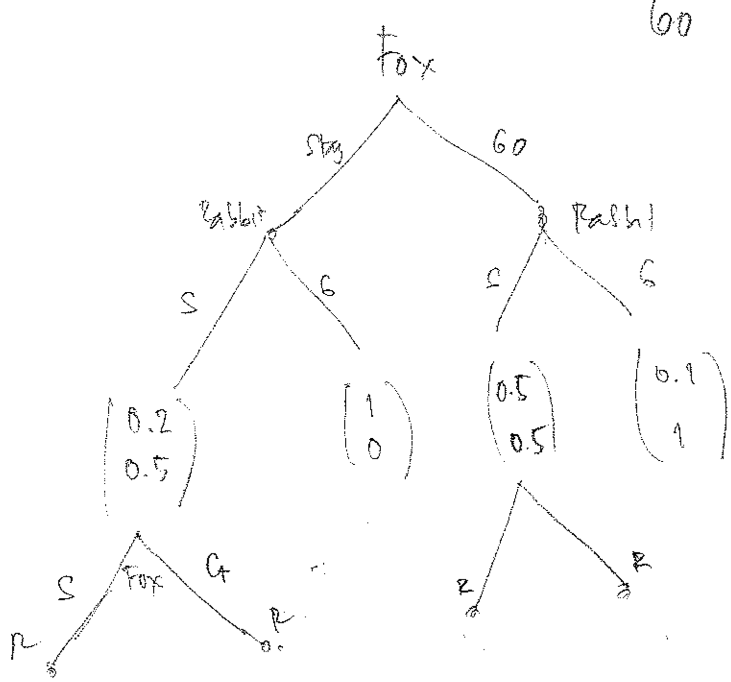


7.21 an example of a finite game of imperfect information and perfect recall in which there is no "last" information set. That is, for every information set, there is a node x, within it such that $(x, a) \in X$ is not an end node for some action a.

Pay off matrix

Rabbit \ Fox	Stay	Go
Stay	0.2^n / 0.5^n	0.5^n / 0.5^n
Go	1 / 0	0.1 / 1

$\begin{pmatrix} \text{Fox} \\ \text{Rabbit} \end{pmatrix}$



a) Proof :

* Definition :

↳ subgame ; $G^x = [(T^x, \prec^x), (N, t^x), (u_i^x)_{i \in N}]$

is a subgame beginning at x with

1. IF $s_i \in S_i$ is a normal form strategy for i in G. let $s_i^x = s_i \mid X_i \cap T^x$
2. IF $s \in S$, let $s^x = s^x = (s_i^x)_{i \in N}$

S_i^x be the set of all such s_i^x, and S^x that of all s^x

↳ T . a finite set of nodes

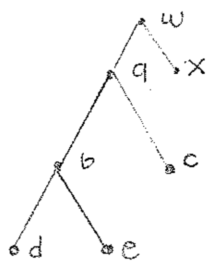
↳ $\prec$: precedence, a strict partial ordering of T

$P(t) = \{x \in T \mid x \prec t\}$ set of predecessors of t

↳ $(T, \prec)$ is an ARBORESCENCE , $\forall t \in T$, $P(t)$ is completely ordered by $\prec$

ie $(x_1, x_2 \in P(t)) \Rightarrow x_1 \prec x_2,\ x_1 = x_2$ or $x_2 \prec x_1$

Example



$T = \{a, b, c, d, e, x, w\}$

$P(d) = \{a, b, w\}$

↳ A tree is an arborescene with exactly one initial node w

$\forall t \in T \setminus \{w\}$, $p_1(t) = \max P(t)$. immediate predecessor of t

so $(p_1(t) \prec t)$ AND $(x \prec t) \Rightarrow (x = p_1(t)$ or $x \prec p_1(t))$

$F(t) = p_1^{-1}(t) = \{t' \mid p_1(t') = t\}$: immediate successor of t

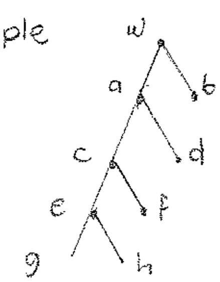
$Z = \{z \in T \mid F(z) = \phi\}$: set of terminal nodes

$X = T \setminus Z$: set of strategic nodes

$Y = T \setminus W$: set of non-initial nodes

$Z(x) = \{z \in Z \mid x \prec z\}$: set of terminal successor of x

Example



$Z = \{b, d, f, g, h\}$, terminal nodes

$X = \{w, a, c, e\}$, strategic nodes

$W = \{w\}$, initial nodes

$Y = \{a, b, c, d, e, f, g, h\}$, non-initial nodes

$Z(e) = \{g, h\}$ $\qquad$ $P_1(g) = \{e\}$

$Z(a) = \{d, f, g, h\}$ $\qquad$ $F(e) = \{g, h\}$

↳ An extensive form game of perfect information is a tuple

$$G = [(T, <), (N, \iota), (u_i)_{i \in N}]$$

with $(T, <)$ is a tree

N : finite set of agents

ι : $X \to N$ player function

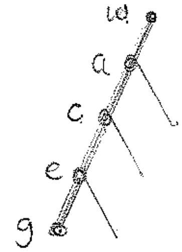
u_i : $Z \to \mathbb{R}$ is player i's utility function.

↳ a pure strategy for player 1

function $s_i : X_i \to T$ with $s_i(x) \in F(x) = \{ t' \mid P_1(t') = x \}$

↳ $X_i = \iota^{-1}(i) = \{ x \in X \mid \iota(x) = i \} \subset X$: all the nodes where player i decides

EXAMPLE



$t_0(s) = w$

$t_1(s) = s(w) = a$

$t_2(s) = s(s(w)) = c$

$\vdots$

$t_{\ell(s)}(s) = z(s) = g$

strategy profile s

↳ terminal node with $\ell(s)$ = integer such that $t_{\ell(s)}(s) \in Z$

in this example $\ell(s) = 3$

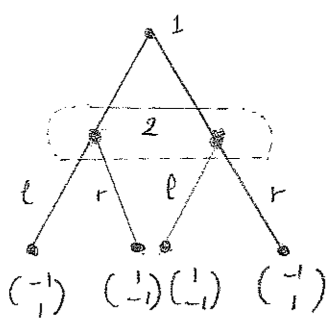
ANSWER

- Let $x \in X$ such that $G^{x'}$ has a proper subgame subgame perfect equilibrium (p.s.s.p.e) for all $x' \in X$ with $x < x'$
- Let $F(x) = \{ t_1, \dots \text{l dots}, \dots t_J \}$
- For those t_j in X, let s_j^t be a p.s.s.p.e of G_j^t
- Define $z(t_j)$ to be t_j if $t_j \in Z$, and otherwise let $z(t_j) = z(s_j^t)$.
- Choose $$s_{\iota(x)}(x) \in \arg\max_{t \in F(x)} u_\iota(x)(z(t_j))$$

We have defined all components of a strategy vector $s : T^x \cap X \to T$

Obviously, s is a Nash equilibrium of G^x, so s is a p.s.s.p.e of G starting from x.

b. Example



finite, extensive form game having no pure strategy subgame perfect equilibrium

7.30

Argue that for finite extensive form games, if a behavioral strategy, b, is completely mixed,

7.a) every information set is reached with positive probability

given that $P(x|b)$ is a probability that node x is reached given the behavioral strategy b, then $p(x|b)$ is bounded $(0,1]$

$$p(x) = \frac{p(x|b)}{\sum_{y \in I} p(y|b)}$$

and since $\sum_{y \in I} p(y|b) = 1$, we can make sure that

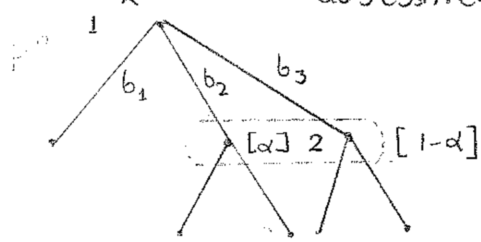
$$p(x) = \frac{p(x|b)}{1} \geqslant 0$$

b) the assessment (p, b) is consistent IFF p is derived from b using Bayes rule.

($\Rightarrow$) IF assessment (p, b) is consistent THEN p is derived from b using Bayes rule

Let assessment (p, b) is consistent

an n sequence of assessment $(p^n, b^n) \rightarrow (p, b)$



$b_1 + b_2 + b_3 = 1$ $\qquad \sum_i b_i = 1$

with b_i is the probability of behavioral strategy b_i. $b_i \in (0, 1)$

out of the n plays, on average, the left most choices of player 1 would occur $b_1 \times n$ times and the other two choices would occur $b_2 n + b_3 n = (b_2 + b_3) n$

Therefore, 2's information set would be reached $(b_2 + b_3) n$ times,

Out of these $(b_2 + b_3) n$, the left most node is reached $b_2 \cdot n$ times and the right most node is reached $b_3 \cdot n$ times

Thus, from a frequency point of view, given that 2's information set has been reached, the probability of the leftmost node is

$$\alpha^n = \frac{b_2 \cdot \cancel{n}}{(b_2+b_3)\cancel{n}} = \frac{b_2}{(b_2+b_3)} = \alpha$$

the rightmost node

$$(1-\alpha)^n = \frac{b_3 \cdot \cancel{n}}{(b_2+b_3)\cancel{n}} = \frac{b_3}{(b_2+b_3)} = (1-\alpha)$$

which actually follows Bayes

$$\alpha = \frac{P(\alpha \mid b_2)}{P(\alpha \mid b_2) + P(1-\alpha \mid b_3)} = \frac{b_2}{b_2+b_3} \qquad \square$$

($\Leftarrow$) IF p is derived from b using Bayes rule THEN assessment (p,b) is consistent

Given that p is derived from b using Bayes

$$p(x) = \frac{P(x \mid b)}{\sum_{y \in I} P(y \mid b)}$$

that means that $p(x)$ is a function of b.

Given that $b^n \to b$ since b is mixed behavior strategy

↳ use the Cauchy sequence property. (p.46 Reed) IF b^n converges to a finite limit, then b^n is a cauchy sequence

↳ so the only thing we need to do is to prove that $p^n \to p$ since p is $f(b)$ and b^n is a cauchy sequence

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall m, n)\left[(b^n - b) \leq \delta \Rightarrow (f(b^n) - f(b)) \leq \varepsilon\right]$$

This statement is true since $f^n = f(b^n)$ is continuous function

$\square$

7.32. Prove that if an assesment is consistent, then it satisfies Bayes' rule in every subgame.

Strategy : using mathematical induction

(i) prove that an assesment is consistent, then it satisfies Bayes' rule in one subgame

Using an answer from 7.30, we have proven that part.

(ii) given that it is true for n subgame, we have to proof that it is also true for n+1 subgame

Backward induction to find SPE (subgame perfect equilibrium) is actually a mathematical induction

by definition : A strategy profile b in Γ is a SPE, if its restriction $b|_x$ is a NE in subgame $\Gamma(x)$ for every subgame $\Gamma(x)$ of Γ

•) in perfect information games, from each decision node begins a subgame

IN Round n 1. take any node x all of whose followers are terminal nodes Z; let the player in x choose a best action $b_i(x)$

2. replace these nodes x by the utility vector that follows when $b_i(x)$ is chosen at x

3. If $x = x^*$ then stop; otherwise go back to stage 1

IN Round n+1, it will still be true for $b_i(x)$

8.6 Akerlof

a) Argue that in a competitive equilibrium under asymmetric information we must have $E(\theta | p) = p$

↳ In equilibrium, the buyers get weakly positive expected utility

$$E[\theta | p] \geq p$$

↳ If, however, $p < E[\theta | p]$. then it will make all buyers want to buy.
↳ But the number of demand will outnumber the supply.
↳ So, $E[\theta | p] = p$ in equilibrium

b) Show that if $u_s(p, \theta) = p - \theta/2$, THEN every $p \in (0, \frac{1}{2}]$ is an equilibrium price

Setup: $\theta \in [0, 1]$ index the quality of the car, uniformly distributed

Seller sells his car of quality θ with price p, $u_s(p, \theta)$

Buyer buys a car of quality θ with price p, $\theta - p$

They both receives 0 if they cannot buy or sell the car

↳ given p, the seller who will sell his car has $u_s(p, \theta) = p - \theta/2 \geq 0$

$p - \theta/2 \geq 0 \iff \theta \leq 2p$ and for buyer $\theta - p \Leftrightarrow \theta = p$

↳ Let $b = \min\{2p, 1\}$ ← since $\theta \in [0, 1]$

$$E[\theta | p] = \frac{\int_0^b \theta \, d\theta}{\int_0^b 1 \, d\theta} = \frac{\frac{1}{2}\theta^2 \Big]_0^b}{\theta \Big]_0^b} = \frac{\frac{1}{2}b^2 - 0}{b - 0} = \frac{1}{2}b$$

For $b = 2p$ $\quad E[\theta | p] = \frac{1}{2}(2p) = p$

For $b = 1$ $\quad E[\theta | p] = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$$E[\theta | p] = \begin{cases} p & \text{for } p \leq 1/2 \\ 1/2 & \text{for } p > 1/2 \end{cases}$$

Thus $E[\theta | p] = \min\{\frac{1}{2}, p\}$

since in the equilibrium $E[\theta | p] = p$, $p \in [0, \frac{1}{2}]$

c) Find the equilibrium price when $U_s(p,\theta) = p - \sqrt{\theta}$. Describe the equilibrium in words. In particular, which cars are traded in equilibrium?

Once again given p, only sellers with $U_s(p,\theta) \geq 0$ will sell

$$U_s(p,\theta) = p - \sqrt{\theta} \geq 0 \iff \theta \leq p^2$$

Let $b = \min\{p^2, 1\}$

$$E[\theta|p] = \frac{\int_0^b \theta \, d\theta}{\int_0^b 1 \, d\theta} = \frac{\frac{1}{2}\theta^2\Big]_0^b}{\theta\Big]_0^b} = \frac{\frac{1}{2}(b^2) - 0}{b - 0} = \frac{\frac{1}{2}b^2}{b} = \frac{1}{2}b$$

For $b = p^2$ $\quad E[\theta|p] = \frac{1}{2}p^2$

For $b = 1$ $\quad E[\theta|p] = \frac{1}{2}\cdot 1 = \frac{1}{2}$

Thus $E[\theta|p] = \min\{\frac{1}{2}p^2, \frac{1}{2}\}$

$\frac{1}{2}p^2 \leq \frac{1}{2}$ For $p^2 \leq 1 \iff p \leq 1$

so $$E[\theta|p] = \begin{cases} \frac{1}{2}p^2 &, \text{for } p \leq 1 \\ \frac{1}{2} &, p > 1 \end{cases}$$

In the equilibrium $E[\theta|p] = p$

for $p \leq 1$ $\quad E[\theta|p] = p$

$$\tfrac{1}{2}p^2 = p \iff p = 2$$

$p = 2$ and $p \leq 1$ is contradictive, and only $p = 0$ the only possible price ~~in~~ equilibrium

In this equilibrium, the market completely collapse. Only those cars that nobody is willing to pay a strictly price for are "traded"

For $p > 1$ $\quad E[\theta|p] = p$

$$\tfrac{1}{2} = p$$

the equilibrium condition does not hold

d) Find an equilibrium price when $U_s(p,\theta) = p - \theta^3$. How many equilibria are there in this case

with the same setup, only sellers with $U_s(p,\theta) \geq 0$ will sell

$$U_s(p,\theta) = p - \theta^3 \geq 0 \iff \theta \leq p^{1/3}$$

Let $b = \min\{p^{1/3}, 1\}$ we will have $E[\theta|p] = \frac{1}{2}b$

For $b = p^{1/3}$ $\quad E[\theta|p] = \frac{1}{2}p^{1/3}$

For $b = 1$ $\quad E[\theta|p] = \frac{1}{2}\cdot 1 = \frac{1}{2}$

Thus $E[\theta|p] = \min\{\frac{1}{2}p^{1/3}, \frac{1}{2}\}$

$\frac{1}{2}p^{1/3} \leq \frac{1}{2}$ For $p^{1/3} \leq 1 \iff p \leq 1$

So $E[\theta|p] = \begin{cases} \frac{1}{2}p^{1/3} & \text{for } p \leq 1 \\ \frac{1}{2} & \text{for } p > 1 \end{cases}$

In the equilibrium $E[\theta|p] = p$

for $p \leq 1$ $\quad E[\theta|p] = p$

$$\tfrac{1}{2}p^{1/3} = p \iff p = \pm \frac{1}{2\sqrt{2}}, 0$$

$p = -\frac{1}{2\sqrt{2}}$ is irrelevant

so we have two equilibrium price $p = 0$ and $p = \frac{1}{2\sqrt{2}}$

for $p > 1$ $\quad E[\theta|p] = p$

$$\tfrac{1}{2} = p$$

$p = \frac{1}{2}$ and $p > 1$ is contradictive, so the equilibrium condition does not hold

e) Discuss the Pareto efficient for the preceeding outcome and also the pareto improvements whenever possible

Pareto efficient IF the total surpluses (buyer's plus seller's) from trade is non-negative

part b). Equilibrium price is $p \in [0, \frac{1}{2}]$

$U_s = p - \frac{\theta}{2}$ and $U_b = \theta - p$

T Surpluses $= U_s + U_b = (\cancel{p} - \frac{\theta}{2}) + (\theta - \cancel{p}) = \frac{\theta}{2} > 0$ for $\forall \theta \in [0,1]$

The Pareto optimum is to have all cars traded

* if $p = 0$, the only seller who are willing to sell the car is the one who have $\theta = 0$ (remember $\theta \leq 2p$)

\+ if $p = \frac{1}{2}$, all cars tracted $\quad$ ($\theta \leq 2p \Leftrightarrow 1 \leq 2 \cdot \frac{1}{2} = 1$)

part c)

T Surpluses $= U_s + U_b = (\cancel{p} - \sqrt{\theta}) + (\theta - \cancel{p}) = \theta - \sqrt{\theta}$

but for $\forall \theta \in [0,1]$ $\quad \theta - \sqrt{\theta} \leq 0$

Pareto optimum is to have no car traded.
since the equilibrium price is $p = 0$ (no car traded),
even if the market collapses completely, efficiency is still achieved

part d)

Total surpluses $= U_s + U_b = (p - \theta^3) + (\theta - p)$

$$\theta - \theta^3 \geq 0 \text{ for all } \theta \in [0,1]$$

The Pareto optimum is to have all cars traded.
but with $p=0$ and $p = \frac{1}{2\sqrt{2}}$, some cars are not traded.
So they are Pareto dominated by an allocated to the buyers.

Adv Microeconomics H.W

Sang Ho Lee

8.10

($\Rightarrow$)

Consider the following screen game.

- Two insurance firms $j = A, B$
- Two types of consumer $i = l, h$

The insurance companies move first. Since these do not know wether they will face a low or a high risk consumer then they make me following offers

- $\varphi_A = (\varphi_l^B, 0) \Rightarrow$ An insurance policy tailered for the low-risk consumer φ_l^B and null policy $(0,0)$
- $\tilde{\varphi}_A = (\varphi_h^B, 0) \Rightarrow$ An insurance policy tailered for the high-risk consumer φ_h^A and null policy $(0,0)$
- $\varphi_B = (\varphi_l^B, 0) \Rightarrow$ The policy tailored for the low-risk φ_l^B and the null.
- $\tilde{\varphi}_B = (\varphi_h^B, 0) \Rightarrow$ " high-risk φ_h^B and the null.

Each insurance company makes the offers $\varphi_j, \tilde{\varphi}_j$ under the belief that the low risk consumer will choose the low-risk policy, and the high-risk consumer will choose the high-risk policy.

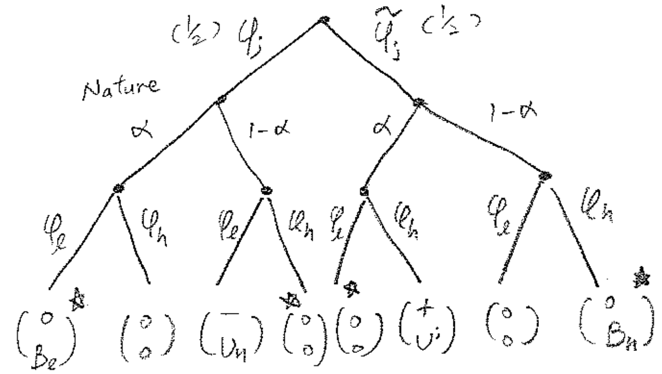
$$\Rightarrow \quad C_l(\varphi^j, \tilde{\varphi}) = (j, \varphi_l)$$

$$C_h(\varphi^j, \tilde{\varphi}) = (j, \varphi_h)$$

This game will represent a subgame perfect equilibrium if it induces a Nash equilibrium

$\Rightarrow$ no player has an incentive to deviate because if it decides to unilaterally deviate from its prefered (equilibrium) strategy, then it payoff would decrease.

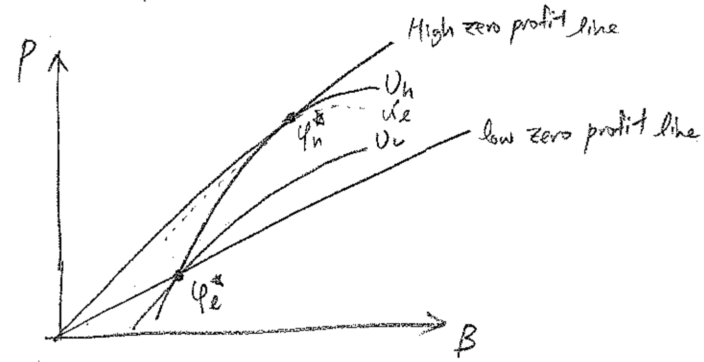
Graphically



(☆) The joint subgame perfect strategy is:

$$\{\varphi_j, \tilde{\varphi}_j, C_l(j, \varphi_l), C_h(j, \varphi_h)\}$$

By looking at the graph, the strategy (☆) is a Nash equilibrium and a subgame perfect equilibrium



$\Rightarrow$ an agency's j information set always yields the highest utility. and since the sequence of behavior strategies just converge to themselves

$\rightarrow$ the assessments are obviously consistent.

Thus, If the joint strategy is a subgame perfect equilibrium then the joint strategy is a sequential equilibrium.

8.12

(a) $u(w - p(e)) = d(e) + \bar{u}$ (from P. 365, 8.12)

$d(0) < d(1)$

$d(0) + \bar{u} < d(1) + \bar{u}$

$u(w - p(0)) < u(w - p(1))$

That is

$\cancel{w} - p(0) < \cancel{w} - p(1)$

$p(0) > p(1)$

(b) Company wants low effort.

Choose p, B s.t.

$u_0(w - p - l + B) \geq \bar{u}_0$

$u_0(w - p - l + B) \geq u_1(w - p - l + B) + d(1)$

Company has bargaining power

$\rightarrow$ constraints binding

$u_0(w - p - l + B) = \bar{u}_0$

$u_1(w - p - l + B) + d(1) = \bar{u}$

$u_0(w - p - l + B) = u_1(w - p + l + B) + d(1)$

$\Rightarrow u_0(w - p - l + B) > u_1(w - p - l + B)$

8.13

(a) Monotone likelihood ratio property:

Ratio of probability of a loss with zero effort divided by probability of a loss with full effort

$\pi_\ell(0) / \pi_\ell(1)$ is strictly increasing in ℓ.

$$\frac{1/3}{\frac{2}{3}} = \frac{1}{2} \quad \text{and} \quad \frac{2/3}{1/3} = 2$$

(b) Expected utility in case of low effort:

$$u(e=0) = \frac{1}{3}\sqrt{100} + \frac{2}{3}\sqrt{49} = 8$$

Expected utility in case of high effort:

$$u(e=1) = \frac{2}{3}\sqrt{100} + \frac{1}{3}\sqrt{49} - \frac{1}{3} = \frac{26}{3}$$

Since EU is higher with high effort, Consumer will choose high effort. Thus, her reservation utility $(\bar{u})$ is $26/3$

high effort level is optimal.

(C) Follows directly from (b).

High effort level will be chosen.

(d) The insurance company maximises

$$p - \sum_\ell \pi_\ell(e) B_\ell = p - \frac{1}{3}B$$

because there is only on level of loss

Maximization is subject to the constraint that the consumer is willing to buy insurance:

Paticipation constraint (PC):

$$\frac{2}{3}\sqrt{100-p} + \frac{1}{3}\sqrt{49-p+B} - \frac{1}{3} \geq \frac{26}{3}$$

for the case of high effort.

Because the consumer is risk averse she will buy full insurance. Because the company has all the bargaining power, the participation constraint is binding.

Thus, as $B=51$,

$$\frac{2}{3}\sqrt{100-p} + \frac{1}{3}\sqrt{49-p+51} - \frac{1}{3} \geq \frac{26}{3}$$

$$= \sqrt{100-p} - \frac{1}{3} \geq \frac{26}{3} \Rightarrow p = 19$$

Then, Expected profit is 2.

Similarly we can obtain for low effort that

$$\frac{1}{3}\sqrt{100-p} + \frac{2}{3}\sqrt{49-p+51} \geq 8$$

$$\Rightarrow p = 36$$

expected profit is $36 - 34 \approx 2$

(e)

If the firm offers full insurance, at whatever price, when effort is not observable the consumer maximizes EU by minimizing the cost of effort.

Thus, she will supply zero effort.

Calculate the EU for a low-effort customer which states to exert high-effort.

$$u(e=0) = \sqrt{100-p} > \sqrt{100-p} - \frac{1}{3}.$$

(f) profits of the company subject to a participation and an incentive constraint.

$$\max\left(p - \frac{2}{3}\cdot 0 - \frac{1}{3}B\right) \quad \text{s.t}$$

$$PC: \frac{2}{3}\sqrt{100-p} + \frac{1}{3}\sqrt{49-p+B} - \frac{1}{3} \geq \frac{26}{3}$$

$$IC: \frac{2}{3}\sqrt{100-p} + \frac{1}{3}\sqrt{49-p+B} - \frac{1}{3} \geq \frac{1}{3}\sqrt{100-p} + \frac{2}{3}\sqrt{49-p+B}$$

Because the company has the bargaining power, both constraints must be binding for a risk averse consumer.

solving 2 equation for $B, p \Rightarrow p = \frac{116}{9}$, $B = \frac{100}{3}$

$\boxed{\text{Thus, expected cost of damage refund: } \frac{100}{9}, \text{ expected profit: } \frac{16}{9}}$

(g) The company would make a loss for low effort. For high effort and symmetric information price can be conditional on effort. There is full insurance and the profit is 2. With asymmetric information there is only partial insurance and profits are lower $\frac{16}{9}$.

Consumer utility is always at the reservation level.

9.3 a. From (9.2)

$$\frac{du(r,v)}{dr} = (N-1)F^{(N-2)}(r)\,f(r)\,(v-\hat{b}(r)) - \underbrace{F^{(N-1)}(r)\,\hat{b}'(r)}_{\circledast}$$

by (9.3)

$$(N-1)F^{N-2}(v)f(v)\hat{b}(v) + F^{N-1}(v)\hat{b}'(v) = (N-1)\,v\,f(v)\,F^{N-2}(v)$$

$$\Leftrightarrow F^{N-1}(v)\,\hat{b}'(v) = (N-1)F^{N-2}(v)\,f(v)\cdot v - (N-1)F^{N-2}(v)\,f(v)\,\hat{b}(v)$$

$$\Leftrightarrow \underbrace{F^{N-1}(v)\,\hat{b}'(v)}_{*} = [(N-1)F^{N-2}(v)\,f(v)]\,(v-\hat{b}(v))$$

Thus, by change v with r

$$\Leftrightarrow F^{N-1}(r)\,\hat{b}'(r) = [(N-1)F^{N-2}(r)f(r)]\,(r-\hat{b}(r))$$

and change (9.2)

$$\frac{du(r,v)}{dr} = \left[(N-1)F^{(N-2)}(r)\,f(r)\right](v-\hat{b}(r)) - \left[(N-1)F^{(N-2)}(r)\,f(r)\right](r-\hat{b}(r))$$

$$= \left[(N-1)F^{(N-2)}(r)\,f(r)\right](v-r)$$

b) The sign of $\frac{du(r,v)}{dr}$ will depend on $(v-r)$ since $F(r)$ and $f(r) \geq 0$

IF $v > r$ THEN $\frac{du(r,v)}{dr} = \oplus$

IF $v < r$ THEN $\frac{du(r,v)}{dr} = \ominus$

AND $v = r$, THEN $\frac{du(r,v)}{dr} = 0$

thus $u(v,r)$ is maximized when $v = r$

9.4. F: distribution function
f: probability density function

v: bidder value

VNM $u = (v-b)^{1/\alpha}$, utility of winning an object with $b < v$

the bigger b, the less the utility

absolute risk aversion for the utility (ARA)

$$u'(v) = \frac{d\left[(v-b)^{1/\alpha}\right]}{dv} = \tfrac{1}{\alpha}(v-b)^{(1/\alpha - 1)}$$

$$u''(v) = \frac{d\left[\tfrac{1}{\alpha}(v-b)^{1/\alpha - 1}\right]}{dv} = \tfrac{1}{\alpha}\left(\tfrac{1}{\alpha}-1\right)(v-b)^{1/\alpha - 2}$$

$$ARA = -\frac{u''(v)}{u'(v)} = \frac{\cancel{\tfrac{1}{\alpha}}\left(\tfrac{1}{\alpha}-1\right)\cancel{(v-b)}^{1/\alpha-2}}{\cancel{\tfrac{1}{\alpha}}\;\cancel{(v-b)}^{1/\alpha-1}}$$

$$= -\left(\frac{1}{\alpha}-1\right)\frac{1}{(v-b)} = \left(1-\tfrac{1}{\alpha}\right)(v-b)^{-1}$$

the higher ARA, the higher risk aversion

thus if $\alpha = 1$ $\quad ARA = 0 \rightarrow$ risk neutral

$\alpha > 1$ $\quad ARA > 0 \rightarrow$ risk averse

↳ Order Statistics

- Let $X_1, X_2, \ldots X_n$ be n independently draws from a distribution F with associated density f. Let $Y_1^{(n)}, Y_2^{(n)}, \ldots Y_n^{(n)}$ be a rearrangement of these so that $Y_1^{(n)} \geq Y_2^{(n)} \geq \ldots \geq Y_n^{(n)}$, $k = 1, 2, \ldots n$ are referred to as order statistics.
- Let $F_k^{(n)}$ denote the distribution of $Y_k^{(n)}$, with corresponding probability density function $f_k^{(n)}$
- for n fixed, $Y_k^{(n)}$ can simply be written Y_k

↳ Highest Order Statistics

- The event that $Y_1 \leq y$ is the same as $\forall k, X_k \leq y$ since each X_k is an independent draw from the same distribution F, we have that

$$F_1(y) = \underbrace{F(y) \cdot F(y) \cdot \ldots F(y)}_{n \text{ person}} = F(y)^n$$

- given that $f(x)$ = probability density function and $F(x)$ = cumulative density function

$$F(x) = \int_0^x f(y)\,dy \iff f(x) = \frac{dF(x)}{dx}$$

$$f_1(y) = \frac{dF_1(y)}{dy} = \frac{d\left[F(y)^n\right]}{dy} = \frac{d\left[F(y)^n\right]}{dF(y)} \cdot \frac{dF(y)}{dy} = n F(y)^{n-1} f(y)$$

a) Let $u(r,v)$ the expected utility from bidding $\hat{b}_\alpha(r)$

VNM utility $= (v-b)^{1/\alpha}$

Expected utility (of winning) $= F^1(r) \cdot \underbrace{u(\cdot)}_{0} + F^2(r) \cdot \underbrace{u(\cdot)}_{0} + \ldots F^{N-1}(r)(v-\hat{b}_\alpha(r))^{1/\alpha}$

$$u(r,v) = F^{N-1}(r)(v-\hat{b}_\alpha(r))^{1/\alpha}$$

Find the maximum (use number 9.3)

$$\frac{\partial u(r,v)}{dr} = (N-1)\underbrace{F^{N-2}(r)}_{\oplus}\underbrace{f(r)}_{\oplus}(v-r)$$

$$\frac{\partial u(r,v)}{dr} = 0 \quad \text{for} \quad v = r$$

b) given that

$$u(r,v) = F^{N-1}(r)(v-\hat{b}_\alpha(r))^{1/\alpha}$$

$$[u(r,v)]^\alpha = \left[F^{N-1}(r)(v-\hat{b}_\alpha(r))^{1/\alpha}\right]^\alpha = [F^\alpha(r)]^{N-1}(v-\hat{b}_\alpha(r))$$

$$\frac{\partial[u(r,v)^\alpha]}{\partial r} = \frac{\partial u(r,v)^\alpha}{d u(r,v)} \cdot \frac{\partial u(r,v)}{\partial r} = \alpha u(r,v)^{\alpha-1} \cdot (N-1) F^{N-2}(r) f(r)(v-r)$$

$$= \underbrace{\alpha}_{\oplus}\underbrace{(N-1)}_{\oplus}\underbrace{u(r,v)^{\alpha-1}}_{\oplus}\underbrace{F^{N-2}(r)}_{\oplus}\underbrace{f(r)}_{\oplus}(v-r)$$

so once again the sufficient condition to maximize $[u(r,v)^\alpha]$ is $r=v$

c) given that

$$[u(r,v)]^\alpha = F^\alpha(r)^{N-1}(v-\hat{b}_\alpha(r))$$

$$= F^\alpha(r)^{N-1}v - F^\alpha(r)^{N-1}\hat{b}_\alpha(r)$$

$$= F^\alpha(r)^{N-1}v - F^\alpha(r)^{N-1}E[X_i \mid X_i < r]$$

$$= F^\alpha(r)^{N-1}v - \int_0^r x \underbrace{f^{\alpha(N-1)}(x)\,dx}_{= dF^{\alpha(N-1)}x}$$

by integration by parts

$$= F^\alpha(r)^{N-1}v - xF^{\alpha(N-1)}(x)\Big]_0^r + \int_0^r F^{\alpha(N-1)}(x)\,dx$$

$$= F^{\alpha(N-1)}(r)[v-x] + \int_0^r F^{\alpha(N-1)}(x)\,dx$$

$$= F^{\alpha(N-1)}(r)\, v - F^{\alpha(N-1)}(r)\cdot r + \int_0^r F^{\alpha(N-1)}(x)\, dx$$

$$[u(r,v)]^{\alpha} = F^{\alpha(N-1)}(r)\,[v-r] + \int_0^r F^{\alpha(N-1)}(x)\, dx$$

we thus obtain

$$[u(b_\alpha(r), r)]^{\alpha} - [u(b_\alpha(v), r)]^{\alpha} = F^{\alpha(N-1)}(r)\,[v-r] - \int_r^v F^{\alpha(N-1)}(x)\, dx$$

regardless of whether $v \geq r$ or $v \leq r$, which implies b_α is a symmetric equilibrium strategy. Using integration by parts, the equilibrium bid can be rewritten as

$$b_\alpha(v) = v - \int_0^v \frac{F^{\alpha(N-1)}(x)\, dx}{F^{\alpha(N-1)}(v)} = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{\alpha(N-1)} dx$$

Second strategy

$$\circledast \quad \frac{d\,[F^{\alpha(N-1)}(x)]}{dx} = \frac{d\,[F^{\alpha(N-1)}(x)]}{d\,F(x)} \times \frac{dF(x)}{dx} = \alpha(N-1)\,F^{\alpha(N-1)-1}(x)\cdot f(x) \quad \text{--- (1)}$$

$$\Leftrightarrow \quad \alpha(N-1)\,F^{\alpha(N-1)-1}(x)\, f(x)\, dx = d(F^{\alpha(N-1)}(x)) \quad \text{--- (2)}$$

$$\circledast \quad \frac{d\,[F^{\alpha(N-1)}(x)\,\hat{b}(x)]}{dx} = \alpha(N-1)\,F^{\alpha(N-1)-1}(x)\,\hat{b}(x) + F^{\alpha(N-1)}(x)\,\hat{b}'(x) \quad \text{--- (3)}$$

Start from

$$[u(r,v)]^{\alpha} = F^{\alpha(N-1)}(r)\,(v - \hat{b}_\alpha(r))$$

differentiate wrt r

$$\frac{d\,[u(r,v)]^{\alpha}}{dr} = \alpha(N-1)\,F^{\alpha(N-1)-1}(r)\cdot f(r)\,(v - \hat{b}_\alpha(r)) - F^{\alpha(N-1)}(r)\,\hat{b}_\alpha'(r)$$

since for $r=v$, the value equals zero

$$\Leftrightarrow \quad 0 = \alpha(N-1)\,F^{\alpha(N-1)-1}(v)\, f(v)\,(v - \hat{b}_\alpha(v)) - F^{\alpha(N-1)}(v)\,\hat{b}_\alpha'(v)$$

$$\Leftrightarrow \quad 0 = \alpha(N-1)\,F^{\alpha(N-1)-1}(v)\, f(v)\cdot v - \alpha(N-1)\,F^{\alpha(N-1)-1}(v)\, f(v)\,\hat{b}_\alpha(v) - F^{\alpha(N-1)}(v)\,\hat{b}_\alpha'(v)$$

$$\Leftrightarrow \quad \alpha(N-1)\,F^{\alpha(N-1)-1}(v)\, f(v)\,\hat{b}_\alpha(v) + F^{\alpha(N-1)}(v)\,\hat{b}_\alpha'(v) = \alpha(N-1)\,F^{\alpha(N-1)-1}(v)\, f(v)\cdot v$$

the left hand side is equation (3)

$$\frac{d\,[F^{\alpha(N-1)}(v)\,\hat{b}_\alpha(v)]}{dv} = \alpha(N-1)\,F^{\alpha(N-1)-1}(v)\, f(v)\cdot v$$

integrate both sides

$$F^{\alpha(N-1)}(v)\,\hat{b}_\alpha(v) = \int_0^v \alpha(N-1)\,F^{\alpha(N-1)-1}(x)\, f(x)\cdot x\, dx + \text{constant}$$

bidder with value zero must bid zero, we conclude that the constant above must be zero. We also use eq (2) to simplify right hand side

$$F^{\alpha(N-1)}(v)\,\hat{b}_\alpha(v) = \int_0^v x \, dF^{\alpha(N-1)}(x)$$

$$\hat{b}_\alpha(v) = \frac{1}{F^{\alpha(N-1)}(v)} \times \int_0^v x \, dF^{\alpha(N-1)}(x)$$

Integration by part

$$\int_0^v x \, dF^{\alpha(N-1)}(x) = x \cdot F^{\alpha(N-1)}(x)\Big]_0^v - \int_0^v F^{\alpha(N-1)}(x)\, dx$$

$$= v F^{\alpha(N-1)}(v) - \int_0^v F^{\alpha(N-1)}(x)\, dx$$

Thus $\hat{b}_\alpha(v) = \frac{1}{F^{\alpha(N-1)}(v)} \times \left[v F^{\alpha(N-1)}(v) - \int_0^v F^{\alpha(N-1)}(x)\, dx \right]$

$$= v - \int_0^v \frac{F^{\alpha(N-1)}(x)}{F^{\alpha(N-1)}(v)}\, dx = v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{\alpha(N-1)} dx$$

(e) $\hat{b}_\alpha(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{\alpha(N-1)}$

From the text we have

$$R_{FPA} = N \int_0^1 \hat{b}(v) f(v) F^{N-1}(v) dv$$

$$R_{SPA} = N(N-1) \int_0^1 v F^{N-2}(v) f(v)(1-F(v)) dv$$

so we don't have $\hat{b}_\alpha(v)$ for the second price auction, thus the risk aversion makes no difference in a second price auction.

For $\alpha = 1$ $\quad R_{FPA} = R_{SPA}$

For $\alpha > 1$ $\quad \hat{b}_\alpha^*(v) > \hat{b}_\alpha(v)$

$$R_{FPA}^* = N \int_0^1 \hat{b}_\alpha^*(v) f(v) F^{N-1}(v) dv > R_{FPA} = N \int_0^1 \hat{b}_\alpha(v) f(v) F^{N-1}(v) dv$$

(f) $\lim \alpha \to \infty$

then since $\frac{F(x)}{F(v)}$ is less than 1 for $x < v$

$\lim_{\alpha \to \infty} \left(\frac{F(x)}{F(v)}\right)^{\alpha(N-1)} = 0$ then $\hat{b}_\alpha(v)$ approaches v

seller revenue

$$\lim_{\alpha \to \infty} R_{FPA} = N \int_0^1 \hat{b}_\alpha(v) f(v) F^{N-1}(v) dv = \underbrace{N \int_0^1 v f(v) F^{N-1}(n) dv}_{\text{the maximum of } v}$$

so the seller revenue will be equal to the maximum v value of all bidders

9.5 In a private values model, argue that it is weakly dominant strategy for a bidder to bid her value in a second price auction even if the joint distribution of the bidder's values exhibits correlation

Let $r_i = \max_{j \neq i} b_j$ the maximum bid submitted by the other players,

Suppose first that player i is considering bidding $b_i > v_i$. We have the following possibilities:

- If $r_i \geq b_i > v_i$, bidder i does not win the object (or he does so in a tie with other players - ie. which is a probability - measure zero event) in any case, his expected payoff is zero and it would be exactly the same had he bid v_i
- If $b_i > r_i > v_i$, player i wins the object but gets a payoff of $v_i - r_i < 0$; he would be strictly better off by bidding v_i and not getting the object (i.e. stay with a payoff of zero).
- If $b_i > v_i > r_i$, player i wins the object and gets a payoff of $v_i - r_i \geq 0$. However, his payoff would be exactly the same if he were to bid v_i.

The reasoning is similar for the case $b_i < v_i$:

- When $r_i \leq b_i$ or $r_i > v_i$, the bidder's expected payoff would be unchanged if he were to bid v_i instead of b_i. However, if $b_i < r_i < v_i$, the bidder forgoes a positive payoff by underbidding. He currently loses the object (and gets zero payoff) whereas had he bid v_i, he would have won it and obtained a payoff of $v_i - r_i > 0$.

This argument establishes that bidding one's valuation is a weakly dominant strategy. By the very definition of weak dominance, this means that bidding one's valuation is weakly optimal no matter what the bidding strategies of the other players are. It is, therefore, irrelevant how the other players' strategies are related to their own valuations, or to our strategy. In other words, bidding one's valuation is weakly optimal irrespectively of any correlation in the joint distribution of the player's values