Seminar 2

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Problem 1

13.1

You are an investor considering a portfolio that includes a risk-free asset and a risky asset. The risk-free rate of return is $rf = 6\%$, and a risky asset is available with a return of $rm = 9\%$ and a standard deviation of $\sigma_m = 3\%$.

1. What is the maximum rate of return you can achieve if you are willing to accept a standard deviation of $\sigma_{\text{portfolio}} = 2\%$?
2. What percentage of your wealth would have to be invested in the risky asset to achieve this rate of return?

Solution:

To find the maximum rate of return you can achieve with a standard deviation of 2%, we can use the formula for the expected return of a portfolio consisting of a risky asset and a risk-free asset:

$$\text{Expected Return of Portfolio} = x \cdot rm + (1 - x) \cdot rf$$

Here, x is the fraction of wealth invested in the risky asset, $rm = 9\%$ is the return on the risky asset, and $rf = 6\%$ is the risk-free rate.

The standard deviation of the portfolio $\sigma_{\text{portfolio}}$ is given by:

$$\sigma_{\text{portfolio}} = x \cdot \sigma_m$$

where $\sigma_m = 3\%$ is the standard deviation of the risky asset.

You are willing to accept a standard deviation of 2%, so:

$$2\% = x \cdot 3\%$$

Solving for x:

$$x = \frac{2\%}{3\%} = \frac{2}{3} \approx 0.6667$$

So, approximately 66.67% of your wealth would have to be invested in the risky asset.

Now, let's find the maximum rate of return you can achieve with this portfolio:

$$\text{Expected Return of Portfolio} = 0.6667 \cdot 9\% + (1 - 0.6667) \cdot 6\%$$

$$= 0.6667 \cdot 9\% + 0.3333 \cdot 6\%$$

$$= 6\% + 2\%$$

$$= 8\%$$

So, the maximum rate of return you can achieve with a standard deviation of 2% is 8%.

13.2

Q: What is the price of risk ?

The price of risk is often represented by the Sharpe ratio, which measures the excess return per unit of risk for a given asset or portfolio. The Sharpe ratio for the risky asset in this exercise can be calculated as follows:

$$\text{Sharpe Ratio} = \frac{rm - rf}{\sigma_m}$$

Here, $rm = 9\%$ is the return on the risky asset, $rf = 6\%$ is the risk-free rate, and $\sigma_m = 3\%$ is the standard deviation of the risky asset.

$$\text{Sharpe Ratio} = \frac{9\% - 6\%}{3\%} = \frac{3\%}{3\%} = 1$$

So, the price of risk in this exercise, as measured by the Sharpe ratio, is 1.

13.3

You are considering investing in a stock with a beta (β) of 1.5. The return on the market is $R_m = 10\%$, and the risk-free rate of return is $R_f = 5\%$.

Questions:

1. What expected rate of return should this stock offer according to the Capital Asset Pricing Model (CAPM)?
2. If the expected value of the stock is $100, what price should the stock be selling for today?

Solution:

1. Expected Rate of Return According to CAPM

The Capital Asset Pricing Model (CAPM) formula for the expected return R_i of a stock is:

$$R_i = R_f + \beta \times (R_m - R_f)$$

Substituting the given values:

$$R_i = 5\% + 1.5 \times (10\% - 5\%)$$

$$R_i = 5\% + 1.5 \times 5\%$$

$$R_i = 5\% + 7.5\%$$

$$R_i = 12.5\%$$

So, according to CAPM, the expected rate of return for this stock should be 12.5%.

2. Stock Price Today

If the expected value of the stock is $100 and the expected rate of return is 12.5%, we can find the price P the stock should be selling for today using the formula:

$$P = \frac{\text{Expected Value}}{1 + R_i}$$

$$P = \frac{\$100}{1 + 12.5\%}$$

$$P = \frac{\$100}{1.125}$$

$$P \approx \$88.89$$

So, the stock should be selling for approximately $88.89 today.

Problem 2

Expected Utility (recap):

Expected utility is a concept in economics and decision theory that provides a way to model the preferences of rational agents when facing uncertainty. It extends the concept of utility, which measures the satisfaction or happiness derived from consuming a good or service, to situations where the outcomes are not certain.

Mathematical Definition

In a simple context, the expected utility Eu of a decision or action is calculated as the weighted sum of the utilities of all possible outcomes, where the weights are the probabilities of those outcomes occurring. Mathematically, it can be expressed as:

$$Eu = \sum_{i=1}^{n} p_i \cdot u_i$$

- p_i is the probability of outcome i occurring.
- u_i is the utility derived from outcome i.
- n is the number of possible outcomes.

The concept of expected utility is closely related to risk averseness, especially in the context of decision-making under uncertainty. Let's explore how:

Risk Averseness

A risk-averse individual prefers a certain outcome over a gamble with the same expected monetary value. In other words, they would rather have a guaranteed amount of money than take a risk for the chance of winning more, even if the expected monetary values of the two options are the same.

Utility Function

The utility function $v(w)$ captures the individual's preferences. For a risk-averse individual, this function is increasing (more is better) but concave ($v' > 0$ and $v'' < 0$). The concavity reflects diminishing marginal utility: each additional unit of wealth increases the individual's utility, but by a smaller and smaller amount.

Expected Utility and Risk Averseness

In the context of expected utility, a risk-averse individual will evaluate risky prospects by looking at the weighted sum of the utilities of all possible outcomes, rather than just the expected monetary value. This is where the concavity of the utility function plays a crucial role.

1. **Diminishing Marginal Utility**: Because of the concave utility function, the individual experiences diminishing marginal utility. This makes them less inclined to take risks, as the

utility gained from potential wins does not fully compensate for the utility lost from potential losses.
2. **Decision-making**: A risk-averse individual will choose the option that maximizes their expected utility, not necessarily the one that maximizes expected monetary value. This often leads to choices that are more conservative, such as diversifying investments or buying insurance.
3. **Trade-offs**: The expected utility framework allows the individual to weigh the utility of different outcomes against their probabilities, effectively quantifying how much risk they are willing to take to achieve a higher return.

Example from the Given Problem

In your problem, the expected utility is given by:

$$Eu(x) = \pi v(w + xrg) + (1 - \pi)v(w + xrb)$$

A risk-averse individual (with $v' > 0$ and $v'' < 0$) will choose x to maximize this expected utility. The concavity of $v(w)$ will make them more cautious about investing a large amount in the risky asset, especially if $rb < 0$ (potential for loss) and $rg > 0$ (potential for gain) as they will weigh these risks and returns in terms of utility, not just monetary value.

In summary, expected utility provides a framework for understanding how risk-averse individuals make decisions under uncertainty, taking into account both their aversion to risk and their desire for reward.

Risk-Averse

A risk-averse individual has a utility function that is increasing but concave. Mathematically, this means $v'(w) > 0$ and $v''(w) < 0$. The increasing nature ($v'(w) > 0$) indicates that more wealth is preferred to less, while the concavity ($v''(w) < 0$) reflects diminishing marginal utility: each additional unit of wealth increases utility, but by a decreasing amount. This makes the individual cautious about taking risks.

Risk-Neutral

A risk-neutral individual has a utility function that is linear. Mathematically, this means $v'(w) > 0$ and $v''(w) = 0$. For such an individual, the marginal utility of wealth is constant. They are indifferent between a certain outcome and a gamble with the same expected monetary value. Risk-neutral individuals maximize expected monetary value rather than expected utility.

Risk-Loving (Risk-Seeking)

A risk-loving or risk-seeking individual has a utility function that is increasing and convex. Mathematically, this means $v'(w) > 0$ and $v''(w) > 0$. The increasing marginal utility of wealth makes the individual more willing to engage in risky behavior, as the utility gained from potential wins more than compensates for the utility lost from potential losses.

Summary

- **Risk-Averse**: $v'(w) > 0$ and $v''(w) < 0$ (Concave utility function)
- **Risk-Neutral**: $v'(w) > 0$ and $v''(w) = 0$ (Linear utility function)
- **Risk-Loving**: $v'(w) > 0$ and $v''(w) > 0$ (Convex utility function)

a)

In this problem, the consumer is deciding how much of their initial wealth w to invest in a risky asset. The variable x represents the proportion of wealth invested in the asset. Here's how to interpret different values of x:

$0 < x < 1$

When $0 < x < 1$, the consumer is investing a fraction of their initial wealth w in the risky asset. This means they are not putting all their money into the risky asset but are diversifying by keeping some of their wealth in a safer form. For example, if $x = 0.5$, then the consumer is investing 50% of their initial wealth w in the risky asset and keeping the other 50% in a safer form.

$x > 1$

When $x > 1$, the consumer is investing more than their initial wealth w in the risky asset. This could imply that the consumer is borrowing money to invest in the asset, a strategy known as "leveraging." This is a very risky strategy because if the asset performs poorly, the consumer not only loses their initial wealth but also incurs debt.

$x < 0$

When $x < 0$, the consumer is essentially short-selling the asset. Short-selling is an investment strategy where an investor borrows an asset to sell it, hoping to buy it back later at a lower price. In this case, a negative x would mean that the consumer expects the asset to perform poorly and is trying to profit from this expected decline.

In summary:

- $0 < x < 1$: Partial investment, risk-averse strategy
- $x > 1$: Leveraging, very risky strategy
- $x < 0$: Short-selling, betting against the asset

b)

The expected utility $Eu(x)$ of the consumer is defined as:

$$Eu(x) = \pi v(w + xrg) + (1 - \pi)v(w + xrb)$$

Here, π is the probability of a "good state" with a return rg, and $1 - \pi$ is the probability of a "bad state" with a return rb.

Condition: $rb < 0 < rg$

This condition implies that the return rb in the "bad state" is negative (i.e., the consumer loses money), while the return rg in the "good state" is positive (i.e., the consumer gains money).

Why is this condition necessary for $x > 0$?

1. **Incentive for Positive Investment**: If both rb and rg were negative, there would be no incentive for the consumer to invest positively ($x > 0$) because both outcomes would result in a loss. Similarly, if both rb and rg were positive, the consumer might be inclined to invest all their wealth or even leverage, which contradicts the assumption of risk-aversion.
2. **Risk-Aversion**: The consumer is risk-averse, meaning they prefer a certain outcome over a gamble with the same expected return. A risk-averse consumer would only be willing to take on risk if there is a possibility of a positive return ($rg > 0$) that compensates for the risk of a negative return ($rb < 0$).

3. **Utility Function**: The utility function $v(w)$ is increasing but concave ($v' > 0$ and $v'' < 0$). This means that the consumer experiences diminishing marginal utility from additional wealth. The possibility of a positive return ($rg > 0$) makes the gamble more attractive, while the risk of a negative return ($rb < 0$) makes it less attractive. The consumer will weigh these factors to decide on a positive x if the condition $rb < 0 < rg$ holds.
4. **Optimization**: In an optimization problem, the consumer will try to maximize $Eu(x)$. If $rb < 0 < rg$, then there exists a trade-off between risk and return, making it possible for a positive x to maximize the expected utility.

In summary, the condition $rb < 0 < rg$ provides both the incentive and the trade-off necessary for a risk-averse consumer to consider a positive investment ($x > 0$).

c)

Let's break down the steps with the specific rules of calculus used and any assumptions made.

Given Expected Utility Function

The expected utility function $Eu(x)$ is:

$$Eu(x) = \pi v(w + xrg) + (1 - \pi)v(w + xrb)$$

Step 1: First Derivative of $Eu(x)$ with respect to x

We use the Chain Rule for differentiation here:

$$\frac{dEu(x)}{dx} = \pi rgv'(w + xrg) + (1 - \pi)rbv'(w + xrb)$$

Step 2: Derivative of $\frac{dEu(x)}{dx}$ with respect to π

Here, we are using the Partial Derivative rule since π and x are independent variables:

$$\frac{d}{d\pi}\left(\frac{dEu(x)}{dx}\right) = -rbv'(w + xrb) + rgv'(w + xrg)$$

Step 3: Second Derivative of $Eu(x)$ with respect to x

For this step, we use the Chain Rule again to find the second derivative:

$$Eu''(x) = -rb^2v''(w + xrb) + rg^2v''(w + xrg)$$

Step 4: Expression for $\frac{dx}{d\pi}$

Finally, we arrive at the expression for $\frac{dx}{d\pi}$:

$$\frac{dx}{d\pi} = \frac{1}{-Eu''}\left(v'(w + x^*rg)rg - v'(w + x^*rb)rb\right)$$

Common Question: Why not just $Eu(x)$ with respect to π ?

Objective of the Problem

The primary objective is to understand how the optimal investment x changes with respect to the probability π of the "good state" occurring. In economic terms, we're interested in how sensitive the investment decision is to changes in the perceived risk and return, represented by π.

Why Derivative of $Eu(x)$ wrt x?

The first derivative $\frac{dEu(x)}{dx}$ gives us the condition for optimal investment. It tells us how the expected utility changes with a marginal change in investment x. Setting this derivative equal to zero allows us to find the level of x that maximizes expected utility, which is the primary decision variable for the investor.

Why Derivative of $\frac{dEu(x)}{dx}$ wrt π?

Once we have the condition for optimal investment, the next step is to understand how this optimal level of x changes with π. This is why we take the derivative of $\frac{dEu(x)}{dx}$ with respect to π. It gives us $\frac{dx}{d\pi}$, which tells us how the optimal x changes as π changes, holding x constant at its optimal level (x^*).

So.. Why Not Just Derivative of $Eu(x)$ wrt π?

Taking the derivative of $Eu(x)$ with respect to π would give us how the expected utility changes with a change in π, but it wouldn't directly tell us how the optimal investment x changes with π. The latter is what we're most interested in for this problem.

In summary, the two-step process allows us to first find the condition for optimal investment (x) and then understand how this optimal investment is affected by changes in the probability (π) of the "good state" occurring. This is more aligned with the economic question we are trying to answer.

Assumptions:

1. **Risk Averseness**: We assume that the individual is risk-averse, which implies $v''(w) < 0$. This is important for interpreting $Eu''(x)$ as negative, which in turn affects the sign of $\frac{dx}{d\pi}$.
2. **Utility Function**: We assume $v(w)$ is twice differentiable, which allows us to take the second derivative $Eu''(x)$.
3. **Constant x^***: The x^* in the final expression indicates that we are considering how x changes with π while holding x constant at its optimal level. This is a standard approach in comparative statistics.

Optional (math recap)

Chain Rule

The Chain Rule is fundamental in calculus for finding the derivative of composite functions. If $f(x)$ and $g(x)$ are functions, and if $F(x) = f(g(x))$, then the derivative $F'(x)$ is given by:

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In other words, the derivative of the composite function $F(x)$ is the derivative of $f(x)$ evaluated at $g(x)$ multiplied by the derivative of $g(x)$.

Partial Derivative Rule

The Partial Derivative rule is used when dealing with functions of multiple variables. For a function $f(x, y)$, the partial derivative concerning x is the derivative of f treating all other variables as constants. It is denoted as $\frac{\partial f}{\partial x}$.

Example of the chain rule in Step 3

In Step 3, we find the second derivative $Eu''(x)$, which involves differentiating terms like $v(w + xrb)$ and $v(w + xrg)$ concerning x again. These terms are composite functions; $v(u)$ where $u = w + xrb$ or $u = w + xrg$.

So, when we differentiate $v(w + xrb)$ with respect to x, we have to use the Chain Rule:

1. Differentiate $v(u)$ with respect to u to get $v'(u)$.
2. Differentiate $u = w + xrb$ with respect to x to get rb.
3. Multiply these together: $v''(w + xrb) \times rb$.

This gives us the term $-rb^2 v''(w + xrb)$ in $Eu''(x)$.

Similarly, for $v(w + xrg)$, we get $rg^2 v''(w + xrg)$.

Remember:

Understanding how the returns in the good or bad states (rg and rb) affect the investment in the risky asset (x) is crucial for grasping the behavior of a risk-averse investor.

The Mathematical Expression

Recall the expression for $\frac{dx}{d\pi}$:

$$\frac{dx}{d\pi} = \frac{1}{-Eu''} \left(v'(w + x^* rg) rg - v'(w + x^* rb) rb \right)$$

Here, $Eu'' < 0$ due to the risk-averseness of the investor, and x^* is the optimal level of investment.

Effect of rg and rb

1. **Higher rg (Return in Good State)**: A higher rg increases the term $v'(w + x^* rg) rg$. This makes the entire expression for $\frac{dx}{d\pi}$ larger, assuming $v'(w + x^* rg)$ is positive. In simpler terms, a higher potential return in the good state makes the investor more willing to invest a larger amount in the risky asset.
2. **Higher rb (Return in Bad State)**: A higher rb (assuming it's still less than rg) reduces the absolute value of the term $v'(w + x^* rb) rb$. This also makes $\frac{dx}{d\pi}$ larger, assuming $v'(w + x^* rb)$ is positive. Essentially, a less negative return in the bad state makes the investor more willing to take the risk, as the downside is not as severe.

Intuition

- **Good State**: If the return in the good state is high, the upside of the investment is significant. This upside, when weighed against the downside, makes the investment more appealing, encouraging the investor to allocate more to the risky asset.
- **Bad State**: If the return in the bad state is less negative (closer to zero), the downside risk of the investment is mitigated. This reduced risk makes the investor more comfortable with allocating more to the risky asset.

Risk Averseness

The risk-averseness of the investor plays a role in moderating these effects. A highly risk-averse investor might still choose to invest less in the risky asset despite high returns in the good state or less negative returns in the bad state.

Question 3 Mean Variance Utility problem

How to start? Where to begin?

Problem Statement

The decision maker's utility function is $u(\mu_x, \sigma_x)$, where μ_x is the expected return and σ_x is the standard deviation of the portfolio. The decision maker can choose the fraction x to invest in a risky asset with expected return rm and standard deviation σ_m, and the rest $1 - x$ is held in a risk-free asset with return rf and $\sigma_f = 0$.

The utility function is:

$$u(\mu_x, \sigma_x) = u(xrm + (1-x)rf, x\sigma_m)$$

Solving the Decision Maker's Problem

To maximize this utility function, we need to take the first-order condition by differentiating u with respect to x and setting it equal to zero:

$$\frac{du}{dx} = \frac{\partial u}{\partial \mu_x}\frac{d\mu_x}{dx} + \frac{\partial u}{\partial \sigma_x}\frac{d\sigma_x}{dx}$$

Here, $\frac{\partial u}{\partial \mu_x}$ and $\frac{\partial u}{\partial \sigma_x}$ are the partial derivatives of the utility function with respect to μ_x and σ_x, respectively.

$$\frac{d\mu_x}{dx} = rm - rf$$

$$\frac{d\sigma_x}{dx} = \sigma_m$$

Substituting these into the first-order condition gives:

$$0 = \frac{\partial u}{\partial \mu_x}(rm - rf) + \frac{\partial u}{\partial \sigma_x}\sigma_m$$

Solving this equation will give us the optimal x that maximizes the utility function.

Illustrating the Solution in a Figure

The figure would typically plot the utility function $u(\mu_x, \sigma_x)$ against different levels of x. The optimal x would be the point where the utility function reaches its maximum.

Remember..

The concept of Marginal Rate of Substitution (MRS) being equal to the relative price is a cornerstone of optimization problems in economics. In this context, the MRS can be thought of as the decision maker's willingness to trade off risk for return.

Marginal Rate of Substitution (MRS)

The MRS is the ratio of the marginal utilities of the expected return (μ_x) and the standard deviation (σ_x) of the portfolio. Mathematically, it's given by:

$$\text{MRS} = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}}$$

Relative Price of Risk

The relative price of risk in this context is the additional expected return you get ($rm - rf$) per unit of standard deviation (σ_m):

$$\text{Relative Price of Risk} = \frac{rm - rf}{\sigma_m}$$

First-Order Condition

The first-order condition for maximizing utility is:

$$0 = \frac{\partial u}{\partial \mu_x}(rm - rf) + \frac{\partial u}{\partial \sigma_x}\sigma_m$$

Rearranging this equation, we get:

$$-\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} = \frac{rm - rf}{\sigma_m}$$

Notice that the left-hand side is the MRS, and the right-hand side is the Relative Price of Risk. In a "first-best" scenario, these two should be equal, confirming that the decision maker is optimizing their portfolio in a way that equates their willingness to trade off risk for return with the actual market price of taking on that risk.

Too fast?

Differentiation: First-Order Condition

In this case, a Lagrangian isn't strictly necessary because we don't have a constraint other than the choice of x itself. We can directly differentiate the utility function $u(\mu_x, \sigma_x)$ with respect to x to find the first-order condition.

The utility function is:

$$u(\mu_x, \sigma_x) = u(x \cdot rm + (1 - x) \cdot rf, x \cdot \sigma_m)$$

Differentiating this with respect to x gives:

$$\frac{du}{dx} = \frac{\partial u}{\partial \mu_x}\frac{d\mu_x}{dx} + \frac{\partial u}{\partial \sigma_x}\frac{d\sigma_x}{dx}$$

Here, $\frac{d\mu_x}{dx} = rm - rf$ and $\frac{d\sigma_x}{dx} = \sigma_m$.

Substituting these into the equation, we get:

$$\frac{du}{dx} = \frac{\partial u}{\partial \mu_x}(rm - rf) + \frac{\partial u}{\partial \sigma_x}\sigma_m$$

Setting $\frac{du}{dx} = 0$ gives us the first-order condition:

$$0 = \frac{\partial u}{\partial \mu_x}(rm - rf) + \frac{\partial u}{\partial \sigma_x}\sigma_m$$

Rearrangement: Equating MRS and Relative Price of Risk

Starting from the first-order condition:

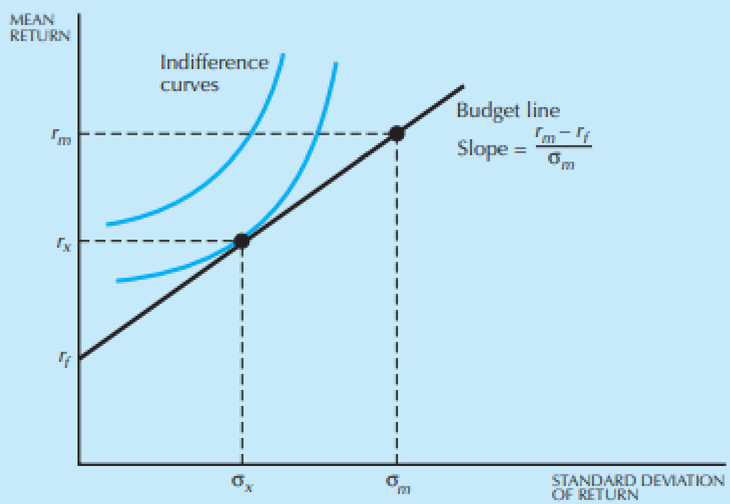
$$0 = \frac{\partial u}{\partial \mu_x}(rm - rf) + \frac{\partial u}{\partial \sigma_x}\sigma_m$$

We can rearrange it to get:

$$-\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} = \frac{rm - rf}{\sigma_m}$$

Here, the left-hand side is the MRS, and the right-hand side is the Relative Price of Risk. In a first-best scenario, these should be equal, confirming optimal portfolio choice.

Figure:



Risk and return. The budget line measures the cost of achieving a larger expected return in terms of the increased standard deviation of the return. At the optimal choice the indifference curve must be tangent to this budget line.

b)

Analyzing the effects of changes in the risk-free return (rf), the risky return (rm), and the risk (σ_m) on the optimal choice of x can provide valuable insights into investor behavior. Let's consider each effect separately:

1. Increase in Risk-Free Return (rf)

- **Effect on Budget Line**: An increase in rf would shift the budget line upwards, as the intercept on the y-axis (Expected Return) would increase.
- **Effect on Optimal Choice (x)**: A higher rf makes the risk-free asset more attractive, potentially leading to a decrease in x, the fraction invested in the risky asset.

2. Increase in Risky Return (rm)

- **Effect on Budget Line**: An increase in rm would make the budget line steeper, as the slope would increase.
- **Effect on Optimal Choice (x)**: A higher rm makes the risky asset more attractive, likely leading to an increase in x.

3. Increase in Risk (σ_m)

- **Effect on Budget Line**: An increase in σ_m does not affect the budget line directly, as it is not a component of the expected return.
- **Effect on Optimal Choice (x)**: A higher σ_m increases the risk associated with the risky asset. For a risk-averse investor, this could lead to a decrease in x.

Further intuition:

Income Effect

The income effect captures how changes in the "wealth" of the investor, represented here by the expected return, affect the choice of x. In this context, an increase in rf or rm could be seen as an increase in "wealth," as the investor now expects higher returns for any given x.

- **Increase in rf or rm**: The income effect would suggest that as the investor feels "wealthier," they might be more willing to take on risk, increasing x.

Substitution Effect

The substitution effect captures how changes in the relative "prices" of the risky and risk-free assets affect the choice of x. The relative price here is the trade-off between risk and return, often captured by the Sharpe ratio $\frac{rm-rf}{\sigma_m}$.

- **Increase in rf**: A higher rf makes the risk-free asset more attractive, inducing the investor to substitute away from the risky asset, decreasing x.
- **Increase in rm**: A higher rm makes the risky asset more attractive, inducing the investor to substitute towards the risky asset, increasing x.
- **Increase in σ_m**: A higher σ_m makes the risky asset less attractive, inducing the investor to substitute away from the risky asset, decreasing x.

Combining the Effects

In practice, both the income and substitution effects operate simultaneously, and their net effect determines the change in x.

- For an increase in rf, the substitution effect (decreasing x) is likely to dominate, as the risk-free asset becomes more attractive.
- For an increase in rm, both the income and substitution effects would likely lead to an increase in x.
- For an increase in σ_m, the substitution effect would likely lead to a decrease in x, especially for risk-averse investors.

The strength of the substitution and income effects in influencing the optimal choice of x depends on various factors, including the investor's utility function and the specific changes in rf, rm, and σ_m. Here are some general guidelines:

When the Substitution Effect is Stronger:

1. **Large Changes in Relative Returns**: If rm increases significantly more than rf, or vice versa, the substitution effect is likely to dominate. The investor will shift their portfolio to take advantage of the asset with the higher relative return.
2. **Increase in Risk (σ_m)**: For a risk-averse investor, a significant increase in σ_m will make the risky asset less attractive, leading to a strong substitution effect away from it.
3. **Utility Function Sensitivity**: If the investor's utility function is more sensitive to changes in expected return (μ_x) than to changes in risk (σ_x), the substitution effect will be stronger.

When the Income Effect is Stronger:

1. **Equal Changes in rf and rm**: As we proved earlier, an equal increase in both rf and rm results in a pure income effect.
2. **Small Changes in Risk (σ_m)**: If σ_m changes only slightly, the substitution effect may be weak, allowing the income effect to dominate if rm or rf also changes.
3. **Utility Function Sensitivity**: If the investor's utility function is more sensitive to changes in risk (σ_x) than to changes in expected return (μ_x), the income effect will be stronger, especially if the overall "wealth" (expected return) increases.

Summary:

- The substitution effect is generally stronger when there are significant changes in the relative attractiveness of the risky and risk-free assets.
- The income effect is generally stronger when the overall level of "wealth" (expected return) changes, without a significant change in the relative attractiveness of the assets.

Checking:

Lets check our answer a bit here.

A pure income effect implies that the relative attractiveness (or "price") of the risky and risk-free assets remains unchanged, while the overall level of "wealth" or expected return increases. In this context, an equal increase in both rf and rm would indeed produce a pure income effect. Let's prove this mathematically.

Starting Point: First-Order Condition

Recall the first-order condition for the optimal choice of x:

$$x^* = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{\sigma_m}{rm - rf}$$

Equal Increase in rf and rm

Let Δ be the equal increase in both rf and rm. Then the new rf becomes $rf + \Delta$ and the new rm becomes $rm + \Delta$.

The new first-order condition becomes:

$$x^{*'} = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{\sigma_m}{(rm + \Delta) - (rf + \Delta)}$$

Simplifying, we get:

$$x^{*'} = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{\sigma_m}{rm - rf}$$

Notice that $x^{*'} = x^*$, meaning the optimal choice of x remains unchanged.

Interpretation: Pure Income Effect

Since the optimal choice x remains unchanged, this implies that the increase in rf and rm has no substitution effect. The relative "price" between the risky and risk-free asset ($\frac{rm-rf}{\sigma_m}$) remains unchanged.

However, the expected return for any given x increases, which can be interpreted as an increase in "wealth" or a pure income effect.

Extra. math:

Recall that the first-order condition is:

$$0 = \frac{\partial u}{\partial \mu_x}(rm - rf) + \frac{\partial u}{\partial \sigma_x}\sigma_m$$

We can rearrange this equation to solve for x in terms of the other variables:

$$x^* = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{\sigma_m}{rm - rf}$$

Effect of an Increase in rf

$$\frac{dx^*}{drf} = \frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{\sigma_m}{(rm - rf)^2} > 0$$

An increase in rf leads to a decrease in x^* because the denominator $(rm - rf)$ increases, making the risky asset less attractive compared to the risk-free asset.

Effect of an Increase in rm

$$\frac{dx^*}{drm} = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{\sigma_m}{(rm - rf)^2} < 0$$

An increase in rm leads to an increase in x^* because the denominator $(rm - rf)$ decreases, making the risky asset more attractive.

Effect of an Increase in σ_m

$$\frac{dx^*}{d\sigma_m} = -\frac{\frac{\partial u}{\partial \mu_x}}{\frac{\partial u}{\partial \sigma_x}} \frac{1}{rm - rf} < 0$$

An increase in σ_m leads to a decrease in x^* because the risky asset becomes riskier, making it less attractive for a risk-averse investor.

In summary, the mathematical derivations confirm our earlier intuitions :)