Seminar 6. Walrasian Equilibrium in a Barter Economy

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Suppose we have a good-exchange economy,

- I is the set of all the individuals (consumers) in the economy,
- The prices of all n commodities is expressed by a vector $p = (p_1, p_2, \ldots, p_n)$,
- Every consumer has some endowments in the form of commodities expressed by a vector $e^i = (e_1^i, e_2^i, \ldots, e_n^i)$,
- $p \cdot e^i$ is the income of consumer i,

Assume (Assumption 5.1 on pp.203) that every consumer has a utility function u^i, which is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}_+^n$.

- By solving consumer i's' utility maximization problem, consumer i's Marshallian demand function is $x^i(p, p \cdot e^i) = (x_1^i, x_2^i, \ldots, x_n^i)$

General Equilibrium: When demand equal to supply in **every market** (market for every commodity), we would say that the system of markets is in General Equilibrium.

We use **Excess Demand** to describe "demand equal to supply".

DEFINITION 5.4 Aggregate Excess Demand (Jehle & Reny pp.204)
The aggregate excess demand function for good k is the real-valued function,

$$z_k(p) \equiv \Sigma_{i \in I} x_k^i(p, p \cdot e^i) - \Sigma_{i \in I} e_k^i$$

Where,

- $\Sigma_{i \in I} x_k^i(p, p \cdot e^i)$ is the summation of all consumers' Marshallian demand for commodity k,
- $\Sigma_{i \in I} e_k^i$ is the total amount of commodity k in this economy.

When $z_k(p) > 0$, the aggregate demand for good k exceeds the aggregate endowment of good k and so there is excess demand for good k. When $z_k(p) < 0$, there is excess supply of good k. That's why $z_k(p)$ is called "Excess Demand" for k.

The **aggregate excess demand function** is a vector-valued function,

$$z_(p) \equiv [z_1(p), z_2(p), \ldots, z_n(p)]$$

When $\exists\, p^* \in \mathbb{R}^n_{++}$ s.t. $z(p^*) = 0$, we say Walrasian Equilibrium (WE) exists. A WE in a barter economy includes a price vector p^* and an allocation (Marshallian demand given p^*) vector $x(p^*, p^* \cdot e)$.

THEOREM 5.2 Properties of Aggregate Excess Demand Functions (pp.204).
If for each consumer i, u^i satisfies Assumption 5.1, then for all $p \gg 0$,

1. Continuity: $z(.)$ is continuous at p.
2. Homogeneity: $z(\lambda p) = z(p)\ \ \forall \lambda > 0$.
3. Walras' law: $p \cdot z(p) = 0$.

THEOREM 5.5 Existence of Walrasian Equilibrium
If each consumer's utility function satisfies Assumption 5.1, and $\Sigma_{i=1}^I e^i \gg 0$, then there exists at least one price vector, $p^* \gg 0$, such that $z(p^*)$.

1 Jehle & Reny 5.4 - Excess demand function and GE

Derive the excess demand function $z(p)$ for the economy in Example 5.1. Verify that it satisfies Walras' law.

Example 5.1 on pp.211
In a simple two-person economy, consumers 1 and 2 have identical CES utility functions,

$$u^i(x_1, x_2) = x_1^\rho + x_2^\rho, \quad i = 1, 2$$

where $\rho \in (0, 1)$.
The initial endowments are $e^1 = (1, 0), e^2 = (0, 1)$.

Does WE exist?
Yes. The requirements of Theorem 5.5 are satisfied.

- $\Sigma_{i=1}^2 e^i = (1, 0) + (0, 1) = (1, 1) \gg 0$
- $u^i(x_1, x_2) = x_1^\rho + x_2^\rho$ is strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$ when $\rho \in (0, 1)$

How to find WE?
We let $z(p) = 0$ to find p.

How to find WEA?
By substituting p^* and $y^* = p^* e$ into $x(p, y)$.

GE: Everyone maximizes its own utility/profit
&
Markets clearing.

1.1 Excess demand function $z(p)$

From Example 1.11 on pp.26, we know the **Marshallian demands** of consumer i for commodity 1 and commodity 2 are:

$$x_1^i(p, y^i) = \frac{p_1^{r-1} y^i}{p_1^r + p_2^r},$$

$$x_2^i(p, y^i) = \frac{p_2^{r-1} y^i}{p_1^r + p_2^r}.$$

where $r = \frac{\rho}{\rho-1}$, $i = 1, 2$.
Given any price vector $p = (p_1, p_2)$, and initial endowment $e^1 = (1, 0), e^2 = (0, 1)$, we know the income of the two consumers are

$$y^1 = p(1, 0)' = p_1$$

$$y^2 = p(0, 1)' = p_2$$

According to Deffinition 5.4, we have aggregated excess demand for commodity 1:

$$\begin{aligned}
z_1(p) &= \Sigma_{i=1}^2 x_1^i(p, p \cdot e^i) - \Sigma_{i=1}^2 e_1^i \\
&= [x_1^1(p, p_1) + x_1^2(p, p_2)] - (e_1^1 + e_1^2) \\
&= (\frac{p_1^{r-1} p_1}{p_1^r + p_2^r} + \frac{p_1^{r-1} p_2}{p_1^r + p_2^r}) - (1 + 0) \\
&= \frac{p_1^{r-1}(p_1 + p_2)}{p_1^r + p_2^r} - 1
\end{aligned}$$

Similarly, the aggregated excess demand for commodity 2 is:

$$
\begin{aligned}
z_2(p) &= \Sigma_{i=1}^2 x_2^i(p, p\cdot e^i) - \Sigma_{i=1}^2 e_2^i \\
&= [x_2^1(p, p_1) + x_2^2(p, p_2)] - (e_2^1 + e_2^2) \\
&= (\frac{p_2^{r-1} p_1}{p_1^r + p_2^r} + \frac{p_2^{r-1} p_2}{p_1^r + p_2^r}) - (1+0) \\
&= \frac{p_2^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1
\end{aligned}
$$

Thus, the **Aggregated Excess Demand Function** is vector:

$$
z(p) = (z_1(p), z_2(p)) = (\frac{p_1^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1, \frac{p_2^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1)
$$

Note Aggregated Excess Demand Function $z(p)$ is a vector, and each element corresponds to one commodity.

1.2 Walras' law

- Walras' law: $p \cdot z(p) = 0$.

$$
\begin{aligned}
p \cdot z(p) &= (p_1, p_2)(z_1(p), z_2(p))' \\
&= p_1[\frac{p_1^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1] + p_2[\frac{p_2^{r-1}(p_1+p_2)}{p_1^r + p_2^r} - 1] \\
&= [\frac{p_1^r(p_1+p_2)}{p_1^r + p_2^r} - p_1] + [\frac{p_2^r(p_1+p_2)}{p_1^r + p_2^r} - p_2] \\
&= \frac{(p_1^r + p_2^r)(p_1+p_2)}{p_1^r + p_2^r} - p_1 - p_2 \\
&= 0
\end{aligned}
$$

2 Jehle & Reny 5.5 - WEA and Edgeworth box

In Example 5.1, calculate the consumers' Walrasian equilibrium allocations and illustrate in an Edgeworth box. Sketch in the contract curve and identify the core.

2.1 WEA

We already have $z(p)$. Now let $z(p) = 0$ to find p^*.

When $(z_1(p), z_2(p)) = (0,0)$, we have: (Note I omitted star below for simplicity, but you should know only p^* leads to $z(p) = 0$)

$$\frac{p_1^{r-1}(p_1+p_2)}{p_1^r+p_2^r} = 1, \quad \frac{p_2^{r-1}(p_1+p_2)}{p_1^r+p_2^r} = 1$$

For the first commodity:

$$\begin{aligned}
\frac{p_1^{-r}}{p_1^{-r}} \frac{p_1^{r-1}(p_1+p_2)}{p_1^r+p_2^r} &= 1 \\
\frac{p_1^{-1}(p_1+p_2)}{(p_1/p_1)^r+(p_2/p_1)^r} &= 1 \\
\frac{1+p_2/p_1}{1+(p_2/p_1)^r} &= 1 \\
1+\frac{p_2}{p_1} &= 1+(\frac{p_2}{p_1})^r \\
(\frac{p_2}{p_1})^{r-1} &= 1
\end{aligned}$$

We know $r-1 = \frac{\rho}{\rho-1} - 1 = \frac{1}{\rho-1} \neq 0$, $p \gg 0$. Then $\frac{p_2}{p_1} = 1$
Similarly, for the second commodity, we have $\frac{p_1}{p_2} = 1$
To conclude, the WE price p^* is $(p_1^*.p_2^*)$ $s.t.$ $p_1^* = p_2^*$. Let's just denote the $p_1^* = p_2^* = a$, the demands under the price p^* are:

$$x_1^i(p,a) = \frac{a^{r-1}a}{a^r+a^r} = 0.5,$$

$$x_2^i(p,a) = \frac{a^{r-1}a}{a^r+a^r} = 0.5.$$

$i = 1,2$. The WEA is thus $x^* = ((0.5,0.5),(0.5,0.5))$

- Only relative price $\frac{p_1}{p_2}$ matters, since you can always "rescale" the prices;
- To describe WE, you need to denote both p^* and WEA.

2.2 Edgeworth box

Contract curve The curve that links the two consumers' indifference curves' tangent point.
Core Given some endowment e, the core of the economy is the set of all feasible allocations that are not against ("blocked") by any consumers (a formal definition is on pp.200-201).

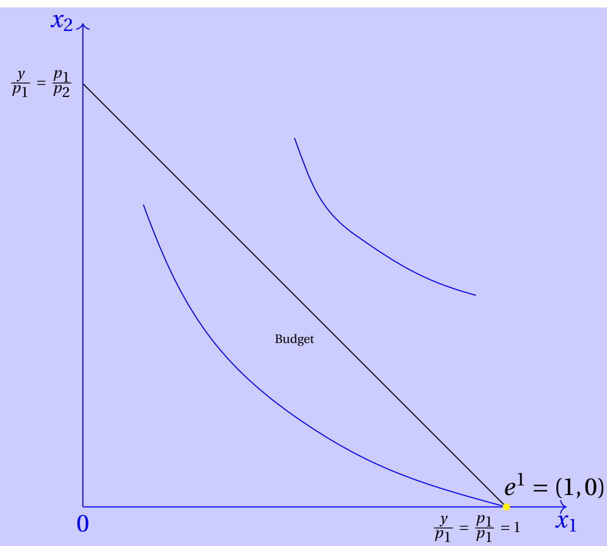

Figure 1: Consumer 1

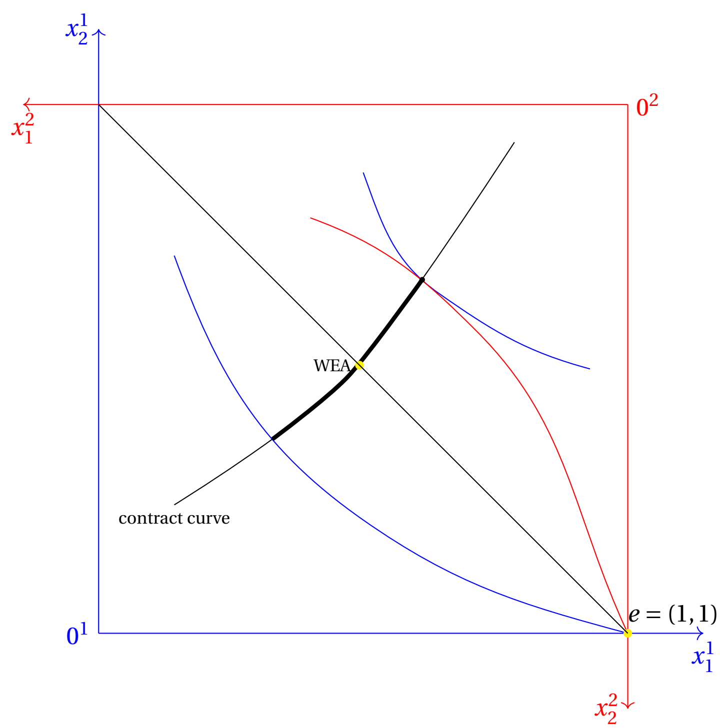

Figure 2: The core

3 Jehle & Reny 5.11 - Pareto-efficient allocations and WEA

Consider a two-consumer, two-good exchange economy. Utility functions and endowments are

$$u^1(x_1, x_2) = (x_1 x_2)^2 \quad and \quad e^1 = (18, 4)$$
$$u^2(x_1, x_2) = ln(x_1) + 2ln(x_2) \quad and \quad e^2 = (3, 6)$$

1. Characterise the set of Pareto-efficient allocations as completely as possible.
2. Characterise the core of this economy.
3. Find a Walrasian equilibrium and compute the WEA.
4. Verify that the WEA you found in part (c) is in the core.

3.1 Pareto-efficient allocations

DEFINITION 5.1 Pareto-Efficient Allocations (Jehle & Reny pp.199)
A feasible allocation, $x \in F(e)$, is Pareto efficient if there is no other feasible allocation, $y \in F(e)$, such that $y^i \succsim^i x^i$ for all consumers, i, with at least one preference strict.

- Feasible: $F(e) = \{x | \Sigma_{i \in I} x^i = \Sigma_{i \in I} e^i\}$, possible and no waste.
- No improvement without harming anyone.

Note that the question is a "social planner" point of view. That is, imagin you can allocate the endowments freely between the two consumers, and there is no price, no exchange, simply transfer.

We can use the fact that only the points (allocations) on the contract curve in a Edgeworth box are both feasible and pareto-efficient.
Denote the allocation for the two consumers is (x^1, x^2), where $x^1 = (x_1^1, x_2^1), x^2 = (x_1^2, x_2^2)$. **Feasible implies:**

$$\begin{cases} x_1^1 + x_1^2 = e_1^1 + e_1^2 = 18 + 3 = 21 \\ x_2^1 + x_2^2 = e_2^1 + e_2^2 = 4 + 6 = 10 \end{cases} \tag{1}$$

Besides, all points on the contract curve are tangent points of indifference curves representing u^1 and u^2 $\Rightarrow$ given any allocation (x^1, x^2) on the contract curve, the **slopes (MRS)** of the two indifference curves passing through the allocation are the same, i.e. $MRS_{12}^1|_{x^1} = MRS_{12}^2|_{x^2}$

$$MRS_{12}^1 = \frac{\partial u^1(x)/\partial x_1}{\partial u^1(x)/\partial x_2} = \frac{2x_1 x_2^2}{2x_1^2 x_2} = \frac{x_2}{x_1}$$

$$MRS_{12}^2 = \frac{\partial u^2(x)/\partial x_1}{\partial u^2(x)/\partial x_2} = \frac{1/x_1}{2/x_2} = \frac{x_2}{2x_1}$$

Thus

$$MRS_{12}^1|_{x^1} = MRS_{12}^2|_{x^2} \Rightarrow \frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^2} \tag{2}$$

Equition 1 and equition 2 describe the pareto-efficient allocation (Note we don't assume any utility level here like what the solution sketch did).

You can also use's Paolo's method to solve:

$$\max_{(x^1,x^2)\in F(e)} u^1(x_1^1, x_2^1), \;\; s.t. \;\; u^2(x_1^2, x_2^2) \geq \bar{u}$$

The intuition is the same: pareto-efficient means maximizing consumer 1's utility without harming consumer 2's utility.

3.2 Core

In a Edgeworth box, the core of the economy is the part of **contract curve intersecting with the "lens-shaped" area**, which is constructed by the two indifference curve passing through the initial allocation.

The core is a refinement of the Pareto-efficient allocation in the previous subqustion in the view of consumers. That is, the consumers are only willing to exchange their endowments and make a deal inside the core.

Contract curve implies the allocations in the core must satisfy equition 1 and equition 2 we just solved. In addition, the **"lens-shaped" area** means:

$$\begin{cases} u^1(x_1^1, x_2^1) = (x_1^1 x_2^1)^2 \geq u^1(18,4) = (18\times 4)^2 = 72^2 \\ u^2(x_1^2, x_2^2) = ln(x_1^2) + 2ln(x_2^2) \geq u^2(3,6) = ln3 + 2ln6 = ln108 \end{cases}$$

Simplify:

$$\begin{cases} x_1^1 x_2^1 \geq 72 \\ x_1^2 (x_2^2)^2 \geq 108 \end{cases} \tag{3}$$

To sum up, equition 1, 2 and 3 all together describe the core of the economy.

3.3 WEA

We can follow what we did in exercise 5.4 and 5.5 to find WEA (we need to assume price vector $p = (p_1, p_2)$):

1. Solve Marshallian demands $x^i(p, y)$ by Lagrangian
2. Construct Aggregated Excess Demand Function $z(p)$
3. Solve WE p^* by letting $z(p) = 0$
4. Substitute p^* and the income given p^* back to $x^i(p, y)$, you get WEA.

Step 1. Marshallian demands $x^i(p, y)$

Compare this exercise with the 2019 exam question 1 a) (e) WEA for Deb and Frank. How to calculate Deb's Marshallian demand?

(1) Consumer 1's utility maximization problem:

$$L^1 = (x_1 x_2)^2 + \lambda(18p_1 + 4p_2 - p_1 x_1 - p_2 x_2)$$

FOC:

$$\begin{cases} \frac{\partial L^1}{\partial x_1} = 0 \Rightarrow 2x_2^2 x_1 = \lambda p_1 \\ \frac{\partial L^1}{\partial x_2} = 0 \Rightarrow 2x_1^2 x_2 = \lambda p_2 \\ 18p_1 + 4p_2 = p_1 x_1 + p_2 x_2 \end{cases}$$

Take ratio of the first two conditions,

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2} x_1$$

Substituting into budget constraint,

$$18p_1 + 4p_2 = p_1 x_1 + p_2 \frac{p_1}{p_2} \Rightarrow x_1^1 = \frac{9p_1 + 2p_2}{p_1}$$

Thus

$$x_2^1 = \frac{9p_1 + 2p_2}{p_2}$$

(2) Consumer 2's utility maximization problem:

$$L^2 = ln x_1 + 2 ln x_2 + \lambda(3p_1 + 6p_2 - p_1 x_1 - p_2 x_2)$$

FOC:

$$\begin{cases} \frac{\partial L^2}{\partial x_1} = 0 \Rightarrow \frac{1}{x_1} = \lambda p_1 \\ \frac{\partial L^2}{\partial x_2} = 0 \Rightarrow \frac{2}{x_2} = \lambda p_2 \\ 3p_1 + 6p_2 = p_1 x_1 + p_2 x_2 \end{cases}$$

Take ratio of the first two conditions,

$$\frac{x_2}{2x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{2p_1}{p_2}x_1$$

Substituting into budget constraint,

$$3p_1 + 6p_2 = p_1x_1 + p_2\frac{2p_1}{p_2}x_1 \Rightarrow x_1 = \frac{p_1 + 2p_2}{p_1}$$

Thus

$$x_2 = \frac{2p_1 + 4p_2}{p_2}$$

To sum up, the **Marshallian demands** for the two consumers are:

$$\begin{cases} x_1^1 = \frac{9p_1+2p_2}{p_1} \\ x_2^1 = \frac{9p_1+2p_2}{p_2} \end{cases}, \begin{cases} x_1^2 = \frac{p_1+2p_2}{p_1} \\ x_2^2 = \frac{2p_1+4p_2}{p_2} \end{cases} \tag{4}$$

Step 2. Aggregated Excess Demand Function $z(p)$

$z(p) = (z_1(p), z_2(p))$. For commodity 1, we have

$$z_1(p) = x_1^1 + x_1^2 - (e_1^1 + e_1^2) = \frac{9p_1 + 2p_2}{p_1} + \frac{p_1 + 2p_2}{p_1} - (18+3) = \frac{10p_1 + 4p_2}{p_1} - 21$$

$$z_2(p) = x_2^1 + x_2^2 - (e_2^1 + e_2^2) = \frac{9p_1 + 2p_2}{p_2} + \frac{2p_1 + 4p_2}{p_2} - (4+6) = \frac{11p_1 + 6p_2}{p_2} - 10$$

Step 3. $z(p) = 0$

$$z_1(p) = \frac{10p_1 + 4p_2}{p_1} - 21 = 0 \Rightarrow \frac{p_1^*}{p_2^*} = \frac{4}{11}$$

$$z_1(p) = \frac{11p_1 + 6p_2}{p_2} - 10 = 0 \Rightarrow \frac{p_1^*}{p_2^*} = \frac{4}{11}$$

We have $\frac{p_1^*}{p_2^*} = \frac{4}{11}$

Step 4. Back to $x^i(p, y)$ **with** $\frac{p_1^*}{p_2^*}$

Substituting $\frac{p_1^*}{p_2^*} = \frac{4}{11}$ back to equition 4,

$$\begin{cases} x_1^{1*} = \frac{9p_1^*+2p_2^*}{p_1^*} = 9 + 2\frac{p_2^*}{p_1^*} = \frac{29}{2} \\ x_2^{1*} = \frac{9p_1^*+2p_2^*}{p_2^*} = 9\frac{p_1^*}{p_2^*} + 2 = \frac{58}{11} \end{cases}, \begin{cases} x_1^{2*} = \frac{p_1+2p_2}{p_1} = 1 + 2\frac{p_2^*}{p_1^*} = \frac{13}{2} \\ x_2^{2*} = \frac{2p_1+4p_2}{p_2} = 2\frac{p_1^*}{p_2^*} + 4 = \frac{52}{11} \end{cases} \tag{5}$$

Therefore the WEA is

$$[(\frac{29}{2}, \frac{58}{11}), (\frac{13}{2}, \frac{52}{11})]$$

An equivalent alternative in Paolo's sketch (to avoid writing down Lagrangian) is to use $\frac{p_1^*}{p_2^*} = MRS_{12}^1|_{x^{1*}} = MRS_{12}^2|_{x^{2*}}$ and budget constraint directly.

3.4 WEA is in the core

As we have showed, any allocation in the core must satisfy equition 1, 2 and 3. Let's copy the three equitions below and verify them one by one:

1. Feasible

$$\begin{cases} x_1^1 + x_1^2 = e_1^1 + e_1^2 = 18 + 3 = 21 \\ x_2^1 + x_2^2 = e_2^1 + e_2^2 = 4 + 6 = 10 \end{cases}$$

Obviously, WEA $[(\frac{29}{2}, \frac{58}{11}), (\frac{13}{2}, \frac{52}{11})]$ satisfies the condition $(\frac{29}{2} + \frac{13}{2} = 21, \frac{58}{11} + \frac{52}{11} = 10)$.

2. Contract curve

$$MRS_{12}^1|_{x^1} = MRS_{12}^2|_{x^2} \Rightarrow \frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^2}$$

$$\frac{x_2^1}{x_1^1} = \frac{\frac{58}{11}}{\frac{29}{2}} = \frac{4}{11}$$

$$\frac{x_2^2}{2x_1^2} = \frac{\frac{52}{11}}{2 \times \frac{13}{2}} = \frac{4}{11}$$

3. Lens-shaped area

$$\begin{cases} x_1^1 x_2^1 \geq 72 \\ x_1^2 (x_2^2)^2 \geq 108 \end{cases}$$

$$x_1^1 x_2^1 = \frac{29}{2}\frac{58}{11} = \frac{29^2}{11} \approx 76.45 \geq 72$$

$$x_1^2 x_2^2 = \frac{13}{2}(\frac{52}{11})^2 \approx 145.26 \geq 108$$

To conclude, the WEA we calculated is in the core.