ECON 3220/4220 Term Paper

15th September 2022

1 Production in the long and short run

Consider the production function $f(x_1, x_2) = \sqrt{x_1 x_2}$.

a. Demonstrate that returns to scale are constant.

Solution: $f(tx_1, tx_2) = \sqrt{tx_1 tx_2} = t\sqrt{x_1 x_2} = tf(x_1, x_2)$.

b. Derive the marginal products

$$MP_1 = \frac{1}{2}\sqrt{\frac{x_2}{x_1}}$$
$$MP_2 = \frac{1}{2}\sqrt{\frac{x_1}{x_2}}$$

and discuss how they depend on levels of inputs. Demonstrate that the equality $f(x_1, x_2) = \sum_{i=1}^{2} MP_i(x_1, x_2) x_i$ holds.

Solution: Straightforward by differentiation and substitution. The characteristics may be seen directly from the functions.

b. Derive the marginal rate of technical substitution

$$MRTS_{12} = \frac{x_2}{x_1},$$

and explain that the isoquants are decreasing and convex.

Solution: First part follows from the definition. The second part may be argued informally by realising that $\frac{x_2}{x_1}$ is decreasing along an isoquant; more formally, it follows from inverting $\sqrt{x_1 x_2} = y$ to get $x_2 = \frac{y^2}{x_1}$ and differentiate.

c. Suppose that the amount of input 2 is fixed, i.e. $x_2 = \bar{x}_2$. Explain that economies of scale of the short-run production function $f(x_1, \bar{x}_2)$ are decreasing.

Solution: Follows from the discussion in part b. In particular, $f(tx_1, \bar{x}_2) = \sqrt{tx_1 \bar{x}_2} = \sqrt{t}\sqrt{x_1 x_2} < tf(x_1, x_2)$ or $\frac{\partial^2 f}{\partial x_1^2} = -\frac{1}{4}\sqrt{\frac{\bar{x}_2}{x_1^3}} < 0$.

2 Costs in the long and short run

a. Derive the conditional input demand functions of a firm with the production function $f(x_1, x_2) = \sqrt{x_1 x_2}$ and which faces input prices $w_1, w_2 > 0$

$$x_1(w_1, w_2, y) = y\sqrt{\tfrac{w_2}{w_1}}$$
$$x_2(w_1, w_2, y) = y\sqrt{\tfrac{w_1}{w_2}}$$

and discuss how they depend on prices and output.

Solution: Use the the first-order condition $MRTS_{12} = \frac{w_1}{w_2}$ and the production function to solve for cost-mimising input. The characteristics follow by inspection of the functions.

b. Derive the cost function

$$c(w_1, w_2, y) = 2y\sqrt{w_1 w_2}$$

and discuss how costs depend on input prices and output.

Solution: Straigthforward by subsitution. The characteristics follow by inspection of the functions. It should be noted that the cost function is linear in output.

c. Suppose again that the amount of input 2 is fixed, i.e. $x_2 = \bar{x}_2$. Derive the short-run conditional input demand and cost functions

$$x_1^s(w_1, \bar{x}_2, y) = \tfrac{y^2}{\bar{x}_2}$$
$$c^s(w_1, \bar{x}_2, y) = w_1\tfrac{y^2}{\bar{x}_2} + w_2\bar{x}_2$$

and discuss how these depend on input prices and output. Explain that short-run costs equals long-run costs when output corresponds to the efficient long-run conditional input demand, i.e. $y = \sqrt{\frac{w_2}{w_1}}\bar{x}_2$, and discuss the economic implication of this result. Illustrate the result in a figure.

Solution: Straigthforward by subsituting x_1 from the short-run production function. The characteristics follow by inspection of the function. It should be noted that the cost function is convex in output. The last result follows by substitution for $y = \sqrt{\frac{w_2}{w_1}}\bar{x}_2$ in the two cost functions. In the figure, long-run costs increase linearly while short-run costs lies above long-run costs, are convex and tangential to long-run costs at $y = \sqrt{\frac{w_2}{w_1}}\bar{x}_2$.

d. Derive long-run and short-run average costs

$$\tfrac{c}{y} = 2\sqrt{w_1 w_2}$$
$$\tfrac{c^s}{y} = w_1\tfrac{y}{\bar{x}_2} + w_2\tfrac{\bar{x}_2}{y}$$

and demonstrate that average short-run costs reach a minimum and equal long-run average costs when $y = \bar{x}_2 \sqrt{\frac{w_2}{w_1}}$. Illustrate the average costs functions in a figure.

Solution: Straigthforward; the second result follows from minimisation of $\frac{c^s}{y}$ with respect to y. In the figure, average long-run costs are constant, while short-run average costs are U-shaped, lies above average long-run costs and with equality of the two at $y = \bar{x}_2 \sqrt{\frac{w_2}{w_1}}$.

3 Profits in the long and short run

Assume the firm above faces a fixed output price p and aims to maximise profits, $py - w_1 x_1 - w_1 x_2$, where y denotes the amount of output.

a. Solve the short-run profit-maximisation problem

$$\max_y py - c^s (w_1, \bar{x}_2, y),$$

and derive the short-run output supply and input demand functions

$$\begin{aligned} y^s (w_1, \bar{x}_2, y) &= \tfrac{p}{2w_1} \bar{x}_2 \\ x_1^s (w_1, \bar{x}_2, y) &= \left(\tfrac{p}{2w_1} \right)^2 \bar{x}_2 \end{aligned}$$

Verify that the second-order condition is satisfied.

Solution: Straightforward maximisation.

d. Explain why the long-run profit-maximisation problem has a solution only if $p \leq \frac{c}{y} = 2\sqrt{w_1 w_2}$ with $y = 0$ if $p < \frac{c}{y} = 2\sqrt{w_1 w_2}$.

Solution: From the observation that $\pi (p, w_1, w_2) = py - c (w_1, w_2, y) \lesseqqgtr 0$ when $p \lesseqqgtr \frac{c}{y} = 2\sqrt{w_1 w_2}$.