Cobb–Douglas Production Functions

1 mathematical tricks

- the derivative of αx^{β} with respect to x is $\alpha\beta x^{\beta-1}$
- $x^{\alpha}x^{\beta} = x^{\alpha+\beta}$ (for any α and β)
- $\frac{1}{x^{\alpha}} = x^{-\alpha}$
- if $m = n^{B}$, then $n = m^{1/B}$

2 the production function

A production function $y = f(x_1, x_2)$ is a *Cobb–Douglas* production function if it can be written in the form

$$y = Ax_1^a x_2^b \tag{1}$$

where A, a and b are positive constants.

So the partial derivatives of a Cobb–Douglas production function are :

$$MP_1 = \frac{\partial y}{\partial x_1} = aAx_1^{a-1}x_2^b \tag{2}$$

$$MP_2 = \frac{\partial y}{\partial x_2} = bAx_1^{a}x_2^{b-1} \tag{3}$$

The absolute value of the slope of an isoquant is the *technical rate of substitution*, or TRS. This TRS equals MP_1/MP_2 so that (2) and (3) imply that

$$TRS = \frac{MP_1}{MP_2} = \frac{ax_1}{bx_2} \tag{4}$$

Equations (2) and (3) imply that the Cobb–Douglas technology is *monotonic*, since both partial derivatives are positive. Equation (4) demonstrates the technology is *convex*, since the (absolute value) of the TRS falls as x_1 increases and x_2 decreases.

3 returns to scale

Suppose that all inputs are scaled up by some factor t. The new level of output is

$$f(tx_1, tx_2) = A(tx_1)^a(tx_2)^b = t^{a+b}Ax_1^ax_2^b \tag{5}$$

Notice from equation (5) that $f(tx_1, tx_2) = t^{a+b}f(x_1, x_2)$. We have *increasing returns to scale* if $f(tx_1, tx_2) > tf(x_1, x_2)$ whenever $t > 1$. So here we have increasing returns to scale if $t^{a+b} > t$, which is the same thing as $a + b > 1$.

Similarly, decreasing returns to scale arise if $f(tx_1, tx_2) < tf(x_1, x_2)$ whenever $t > 1$. Here decreasing returns to scale occur if $t^{a+b} < t$, or $a + b < 1$.

4 profit maximization

A firm in perfect competition seeks to maximize its profit

$$\pi = pf(x_1, x_2) - w_1x_1 - w_2x_2 \tag{6}$$

This profit maximum is achieved when $p\frac{\partial f}{\partial x_1} = w_1$ and $\frac{\partial f}{\partial x_2} = w_2$. From equations (2) and (3), with a Cobb–Douglas production function, profit is maximized if

$$paAx_1^{a-1}x_2^b = w_1 \tag{7}$$

$$pbAx_1^ax_2^{b-1} = w_2 \tag{8}$$

Equations (7) and (8) describe the input choices which maximize profit, but they do constitute two equations in two unknowns. Some substitutions are needed to derive the actual profit–maximizing choices of x_1 and x_2 as explicit functions of w_1, w_2 and p.

If both sides of (7) are multiplied by x_1, and both sides of (8) are multiplied by x_2, then

$$pa[Ax_1^ax_2^b] = w_1x_1 \tag{9}$$

$$pb[Ax_1^ax_2^b] = w_2x_2 \tag{10}$$

The expression in square brackets in (9) and (10) is just y. So

$$apy = w_1 x_1 \tag{11}$$

$$bpy = w_2 x_2 \tag{12}$$

As well, equations (11) and (12) imply that

$$x_2 = \frac{b}{a}\frac{w_1}{w_2}x_1 \tag{13}$$

Plugging in for x_2 from (13) into (7) yields

$$paAx_1^{a-1}[\frac{b}{a}\frac{w_1}{w_2}x_1]^b = w_1 \tag{14}$$

which can be written

$$pAa^{1-b}b^b w_1^{b-1}w_2^{-b}x_1^{a+b-1} = 1 \tag{15}$$

or

$$x_1^{1-a-b} = pAa^{1-b}b^b w_1^{b-1}w_2^{-b} \tag{16}$$

Taking both sides to the power $1/(1-a-b)$ yields

$$x_1 = A^{1/(1-a-b)}a^{(1-b)/(1-a-b)}b^{b/(1-a-b)}w_1^{-(1-b)/(1-a-b)}w_2^{-b/(1-a-b)}p^{1/(1-a-b)} \tag{17}$$

which is the profit–maximizing firm's demand for input #1, as a function of the prices of the 2 inputs, and of the price of the output.

Substituting for $x-1$ from (17) into (13) (and some simplifying) gives the demand for input #2,

$$x_2 = A^{1/(1-a-b)}a^{a/(1-a-b)}b^{(1-a)/(1-a-b)}w_1^{-a/(1-a-b)}w_2^{-(1-a)/(1-a-b)}p^{1/(1-a-b)} \tag{18}$$

Equation (11) implies that $y = \frac{w_1 x_1}{ap}$, which (from (17) implies that

$$y = A^{1/(1-a-b)}a^{a/(1-a-b)}b^{b/(1-a-b)}w_1^{-a/(1-a-b)}w_2^{-b/(1-a-b)}p^{(a+b)/(1-a-b)} \tag{19}$$

which is the equation for the perfectly competitive firm's *supply curve* : its profit–maximizing quantity of output, as a function of the price p of the output (and of the prices of the inputs).

5 profit maximization and returns to scale

Since a and b are both positive, equation (19) indicates that the quantity supplied of output y, by the profit–maximizing perfectly competitive firm, will be increasing in the price p of its output, and decreasing in the prices w_1 and w_2 of its inputs, if — and only if —

$$1 - a - b > 0$$

Now the condition for the production technology to exhibit decreasing returns to scale was $a + b < 1$, which is the same thing as $1 - a - b > 0$.

So as long as the technology exhibits decreasing returns to scale, then the firm's supply curve slopes up, and its quantity of output is a decreasing function of all input prices. Equations (17) and (18) show that the (unconditional) demand for each input is decreasing in the price of that input, provided that the technology exhibits decreasing returns to scale.

[If $a + b = 1$, so that there are constant returns to scale, then equations (17), (18) and (19) are meaningless, since $1/(1 - a - b) = 1/0 = \infty$. If $a + b > 1$, so that there are increasing returns to scale, then it turns out that equations (7) and (8) don't actually define a maximum. So section 4 makes sense only if we have decreasing returns to scale.]

6 cost minimization

The cost minimization problem (chapter 20) takes the target level of output y as given, along with the unit prices w_1 and w_2 of the inputs.

The first–order conditions for cost minimization imply that $MP_1/MP_2 = w_1/w_2$, which (from equation (4)) here implies

$$\frac{a}{b}\frac{x_2}{x_1} = \frac{w_1}{w_2} \tag{20}$$

which is actually the same thing as condition (13) above.

The firm must actually meet its target level of output, so that it must choose input levels x_1 and x_2 such that $Ax_1^a x_2^b = y$. Using (13) to substitute for x_2, this requirement becomes

$$y = Ax_1^a[\frac{b}{a}\frac{w_1}{w_2}x_1]^b \tag{21}$$

or

$$y = Aa^{-b}b^b w_1^b w_2^{-b} x_1^{a+b} \tag{22}$$

or

$$x_1^{a+b} = A^{-1}a^b b^{-b} w_1^{-b} w_2^b y \tag{23}$$

Taking both sides of (23) to the power $1/(a+b)$,

$$x_1 = A^{-1/(a+b)} a^{b/(a+b)} b^{-b/(a+b)} w_1^{-b/(a+b)} w_2^{b/(a+b)} y^{1/(a+b)} \tag{24}$$

which is the *conditional input demand function* for input #1. Substituting from (24) into (13), the conditional input demand for input #2 is

$$x_2 = A^{-1/(a+b)} a^{-a/(a+b)} b^{a/(a+b)} w_1^{a/(a+b)} w_2^{-a/(a+b)} y^{1/(a+b)} \tag{25}$$

Equations (24) and (25) imply that

$$w_1 x_1 = A^{-1/(a+b)} a^{b/(a+b)} b^{-b/(a+b)} w_1^{a/(a+b)} w_2^{b/(a+b)} y^{1/(a+b)} \tag{26}$$

and

$$w_2 x_2 = A^{-1/(a+b)} a^{-a/(a+b)} b^{a/(a+b)} w_1^{a/(a+b)} w_2^{b/(a+b)} y^{1/(a+b)} \tag{27}$$

so that the total cost of producing y units in the cheapest possible way is

$$C(w_1, w_2, y) = w_1 x_1 + w_2 x_2 = B w_1^{a/(a+b)} w_2^{b/(a+b)} y^{1/(a+b)} \tag{28}$$

where the constant B is defined as

$$B \equiv A^{-1/(a+b)}([\frac{a}{b}]^{b/(a+b)} + [\frac{b}{a}]^{a/(a+b)}) \tag{29}$$

7 returns to scale again

The average cost is

$$AC = \frac{C(w_1, w_2, y}{y} = \frac{B w_1^{a/(a+b)} w_2^{b/(a+b)} y^{1/(a+b)}}{y} \tag{30}$$

But $y^{1/(a+b)}/y = y^{1/(a+b)-1} = y^{(1-a-b)/(a+b)}$ so that

$$AC = B w_1^{a/(a+b)} w_2^{b/(a+b)} y^{(a+b-1)/(a+b)} \tag{31}$$

The exponent $(1-a-b)/(a+b)$ on y in equation (31) is positive if $a+b < 1$, negative if $a+b > 1$ and zero if $a+b = 1$. This exponent being posiitve means that the average cost is increasing with the target output level y.

So the average cost decreases with output if and only if $a+b > 1$. This result confirms the results of section 3 : another definition of increasing returns to scale is that the average cost decline with the quantity of output.