Cardboard Comfortable When it Comes to Crashing

Jeffrey Giansiracusa
Ernie Esser
Simon Pai

翻译[0]：周吕文 zhou.lv.wen@gmail.com，孙达超

扫码听课

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[0]仅限本人 (周吕文)“10 天美赛高强度培训” 课程使用, 请勿传播! 若发现翻译问题，请邮件告知，谢谢。免费试读材料仅提供前 10 页翻译。

Abstract

A scene in an upcoming action movie requires a stuntman on a motorcycle to jump over an elephant. Cardboard boxes will be used to cushion his landing. To protect the nervous stuntman and the motorcycle, we investigate various configurations of the stunt. We formulate a model for the energy required to crush a box based on size, shape, and material. We also summarize the most readily available boxes on the market. We choose a maximum safe deceleration rate of 5 g based on comparison with airbag rigs used professionally for high fall stunts. To ensure that the stuntman lands on the box rig (rather than missing it completely) we analyze the uncertainty in his trajectory and extract the landing point uncertainty. We go on to construct a numerical simulation of the impact and motion through the boxes based on our earlier box crush energy calculations. After analyzing the sensitivity and stability of this simulation, we use it to examine the effectiveness of various configurations for the box stack (including different box sizes, types of boxes, and stacking patterns). Our findings indicate that 200 kg is the most desirable combined mass of the motorcycle and stuntman. A 300 kg mass is marginal and 400 kg is too heavy. We also conclude that a launch ramp angle of 20° is optimal when considering safety, camera angle, and clearance over the elephant. Our results show that a stack constructed of $(30\ \text{inch})^3$ boxes with vertical mattress walls spaced periodically is optimal when considering construction time, cost, and cushioning capacity. We recommend that this stack be constructed in dimensions 4 meters high, 4 meters wide, and 24 meters long. It will consist of approximately 1100 boxes and cost $4300 in materials. The stuntman's wages are uncertain but fortunately the elephant works for peanuts.

即将上映的动作电影里的某一场景, 需要一位特技演员骑着摩托车飞跃一只大象. 在他着陆时, 需要用纸箱来进行缓冲. 为了保护紧张的特技演员以及他的摩托车，我们研究了多种不同情况下的特技表演. 根据纸箱的尺寸, 形状和材料, 我们建立了一个模型来描述压碎一个纸箱所需要的能量. 同时，我们也列出了市场上最容易买到的纸箱. 通过与高楼坠落特技中所用的安全气垫对比，我们将最大安全减速度确定为 5g. 为了确保特技演员能够着陆在纸箱上 (而不是完全错过纸箱), 我们分析了其运动轨迹的不确定性, 并由此得出着陆点的不确定性. 接着, 基于先前对压碎纸箱所需能量的计算, 我们建立了一个数值仿真来模拟特技演员通过纸箱时的运动及所受冲击. 对模拟的灵敏度和稳定性进行分析之后, 我们用它来检验各种配置的纸箱堆效果 (包括不同的纸箱尺寸, 类型, 以及堆叠方式). 研究结果表明摩托车和特技演员的总质量最佳取值为 200kg. 300kg 的质量是临界值，而 400kg 就过重了. 考虑到安全, 摄影视角, 以及大象上方间距, 我们得出最佳发射角为 20°. 我们的研究结果表明: 在综合考虑纸箱的堆放时间, 成本和缓冲效果下, 用边长 30 英寸的立方体纸箱，周期性间隔地加入竖直的扁纸箱构造出的纸箱堆最佳. 我们推荐纸箱堆的尺寸为 4 米高, 4 米宽, 24 米长. 这大约需要 1100 个纸箱, 材料花费约 4300 美元。特技演员的薪水还不确定, 不过好在大象只需吃点花生就行了.

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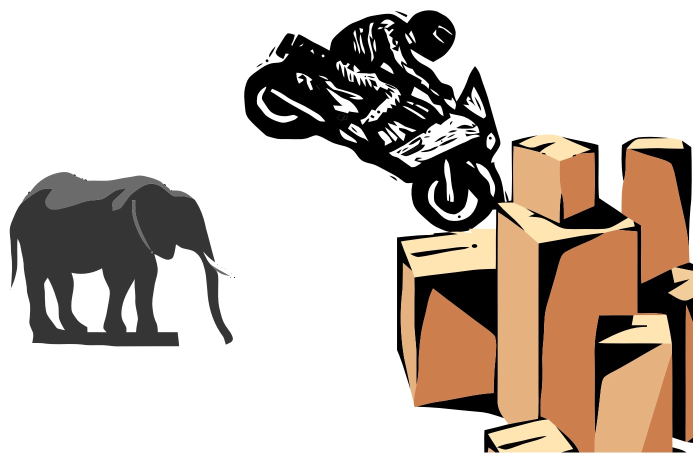

Figure 1: The death-defying leap over the elephant!

1 Introduction / 引言

The American movie-going public's appetite for action and danger on the screen steadily increases year by year. The trend is clear: big budget + big action = big payoff. Somewhere in the darker alleys of Hollywood a desperate director decides to make a name for himself by pulling off the biggest action sequence ever. He plans to film a motorcycle jumping over the biggest animal on land–the elephant!

Bureaucracy being what it is these days, Hollywood officials have hired us to ensure that neither the stuntman nor the elephant is injured (of course, they could care less about the director). The answer to keeping the elephant safe is simple: the elephant just stands in place and does nothing. The safety of the stuntman is a more difficult problem.

Airbag rigs are commonly rented for high fall stunts [?]. However, airbags are designed only to catch humans. Catching the motorcycle in the airbag

在美国，大众电影消费者对动作和心跳场景的口味一年比一年浓. 这个趋势是显而易见的: 高预算 + 大动作 = 巨额回报. 这不, 在好莱坞的某个僻暗的角落里, 一位绝望的导演决定通过推出史上最大规模的系列动作电影来重拾名誉. 他计划拍摄一个摩托车跃过陆地上最大动物-大象！

好莱坞官员这些日子似乎开始办正事了, 他们聘用我们帮忙分析，确保这次的特技演员和大象均不会受伤(相比之下, 他们当然没那么关心导演了). 要想保护大象安全, 答案很简单：让它乖乖地站在那什么也别做就行. 然而, 特技演员的安全则是一个比较复杂的问题.

在拍高楼坠落特技, 安全气垫是最常用. 然而, 气垫的设计只是用来接住特技演员的. 用来接摩托车会有损

would risk damaging it, and replacement costs might well exceed the budget of our unnamed director. The alternative is to use a cardboard box rig–a stack of boxes that will crush and absorb the impact of the motorcycle and stuntman.

We are now faced with the following:

1. The primary objective is to safely catch the stuntman and motorcycle.
2. As a secondary objective, we wish to minimize the cost and size of the box rig.

Our approach is:

1. First we investigate the relationship between the size/shape/material of a box and the work required to crush it. We call this quantity the *crush energy* of a box.
2. We review the most commonly available cardboard box types. We restrict our consideration to only these types because custom boxes are much more expensive than standard ones.
3. By comparison with an airbag rig, we estimate the maximum acceptable deceleration that the stuntman can experience during landing.
4. We analyze the trajectory of the motorcycle and the uncertainty in its landing location. This determines the proper placement of the box rig and how large an area it must cover to safely catch the motorcycle and stuntman.
5. Using the crush energy formula, we estimate the number of boxes that must be crushed in order to arrest the motion of the motorcycle and stuntman.
6. We formulate a numerical simulation of the motorcycle as it enters the

坏气垫的潜在危险，而且更换的费用可能会超出这位不知名导演的预算. 一种替代方案就是用一个纸箱堆-一堆纸箱在被压碎的同时将吸收掉摩托车和演员带来的冲击.

我们现在面临以下问题:

1. 首要目标是要能安全地接住演员和摩托车.
2. 其次, 我们希望花费的最少, 以及纸箱堆尺寸最小.

我们的解决途径是:

1. 首先，我们研究纸箱的尺寸/形状/材料及其被压碎所需功之间的关系. 我们称这个量为纸箱的损坏能量.
2. 我们查了几种最常用的纸箱类型. 并把对问题的考虑仅仅限制在这些类型中, 原因是专门定制的纸箱比标准的贵得多.
3. 类比气垫装置 (高楼坠落特技), 我们对特技演员在着陆时所能承受的最大加速度进行了估计.
4. 我们分析了摩托车的飞行轨迹, 以及其着陆点的不确定性. 这些可以帮助我们确定纸箱堆合适的放置位置和所需覆盖面积, 以保证安全的接住摩托车和特技演员.
5. 利用纸箱的损坏能量公式, 我们估算了为了制动摩托车和演员的运动所需损坏纸箱的数量.
6. 我们建立了一个针对摩托车进入纸箱堆时的数值

box rig. Using this model we analyze the effectiveness of various types of boxes and stacking arrangements. We also compare low, medium, and high trajectory jumps.

模拟模型. 利用这个模型, 我们分析了多种类型的纸箱, 堆叠方式的效果. 同时, 我们还对比了低, 中, 高三种轨迹的飞跃.

7. As an alternative to catching the stuntman while he is still sitting on the motorcycle, we analyze the possibility of having the stuntman bail out in mid-air and land separately from the motorcycle.

7. 除了特技演员和摩托车一起着陆的情况外, 我们还分析了另一种可能：特技演员与摩托车在半空中分离，各自着陆.

8. Based on the results from our simulation, cost and construction considerations, and safety requirements, we make a set of strong recommendations regarding placement, size, construction, and stacking type of the box rig.

8. 基于数值仿真的结果, 纸箱堆的费用和搭建因素、以及安全性要求, 我们给出一套关于纸箱堆的堆放位置, 尺寸, 构造和堆叠方式的建议.

2 The energy absorbed by crushing cardboard / 撞毁纸箱被吸收的能量

To calculate the ability of a cardboard box to absorb the impact of the stuntman and motorcycle we estimate the energy required to crush the box. This estimate is based on a combination of physical considerations and experimental box crushing.

- **Assumption:** The primary source of energy absorption is in the breakdown of the box walls due to edge compressive forces.

Commercial cardboard is rated by Edge Crush Test (ECT), which reports the pounds per inch of edge compressive force parallel to the flute (The flute of corrugated cardboard is the wavy layer between the two wall layers. Here, *'parallel to the flute'* means the direction parallel to the flute wavefronts.) which the cardboard can withstand before breaking. This can be interpreted as the force against the edge per unit length of crease created [?, ?]. Note that once a crease has formed, very little work is required to further bend the cardboard.

To understand how the formation of wall creases relates to the process of crushing a box, we conducted several experiments by dropping a crush-test dummy on a box. See Fig. 2 for illustration of experimental setup.

We observed that:

为了计算纸箱吸收来自摩托车和特技演员冲击的能力, 我们首先估计撞毁纸箱所需要的能量. 这个估算是基于物理学理论和纸箱撞毁实验来实现的.

- **假设:** 纸箱被损毁时能量吸收的主要部分是源自侧面的压缩力.

商用纸箱是按照边压强度 (ETC) 来排等级的, 边压强度是指每英寸宽度的棱面纸板毁坏前在平行于凹槽方向 (瓦楞纸板的凹槽是指层两层纸板间的波浪形纸芯夹层. 这里, 平行于凹槽 指的是平行于波浪形纸芯的波阵面.) 所能承受的压力 (单位: 磅). 这可以被理解为纸箱侧面为反抗形成单位长度折皱的支撑力. 值得注意的是, 一旦纸箱上形成了一个折皱, 要压弯整个纸箱只需要做很小的功.

为了了解在压碎纸箱的过程中纸箱壁的折皱是如何形成的, 我们进行了多次实验, 实验是往纸箱上投下假人来测试. 实验的装置如图2所示.

我们观察到:

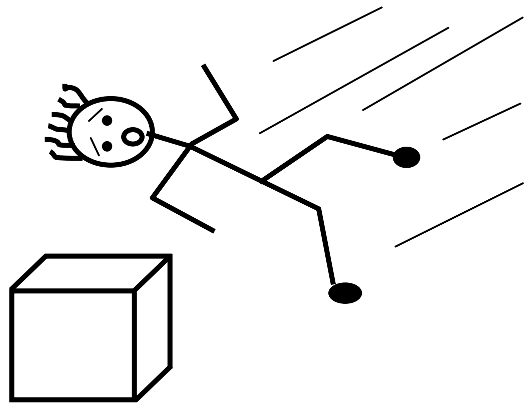

Figure 2: Experimental apparatus for crushing boxes. We dropped a crush-test dummy (i.e. team member) onto several boxes and observed how the structure broke down each time.

- The first wall-creases typically form in the first %15 of the stroke distance.
- These creases extend across two faces of the box (A schematic of one such crease is illustrated in Fig. 3).
- Once these have formed the box deforms further with comparatively little resistance because additional creases are created by torque forces rather than edge compressive forces.
- The primary creases each have length approximately equal to the diagonal length of the face.

The work done in crushing the box is given by the average force applied times the distance through which it was applied. This and the above experimental qualitative results lead us to write the following equation for energy absorbed by a box of dimension $l_x \times l_y \times l_z$ being crushed in the z-direction.

- 纸箱的最初形成的折皱通常在首次下压约 15% 的距离时出现.
- 纸箱的折皱会延伸至两个对着的面 (图3中显示了这样一道折皱的示意图).
- 一旦折皱形成, 继续使纸箱变形, 压垮时收到的阻力就相当小了: 因为新增的折皱主要由扭力产生，而不再是对侧面的压力.
- 纸箱最初的两道折皱长度都近似等于其上表面对角线长.

压碎纸箱过程中所做的功由平均施力乘上施力距离给出. 由此并结合上面实验的定性结果, 我们引出如下方程, 来计算三维尺寸为 $l_x \times l_y \times l_z$ 的纸箱在 z 轴正方向上被压时吸收的能量.

$$\underbrace{E}_{\text{功或能量}} = \underbrace{ECT \times \left(2\sqrt{l_x^2 + l_y^2}\right)}_{\text{作用力}} \times \underbrace{l_z \times 0.15}_{\text{作用距离}} \tag{2.1}$$

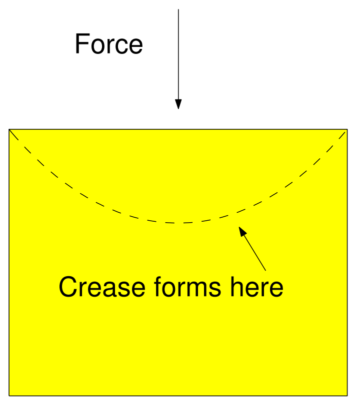

Figure 3: The first crease forms in a curve across the side faces as the box is compressed from above.

As a reality check, we compute the crush energy for a standard 8.5 in × 17 in × 11 in box with ECT 20 lbs/in and C-flute (the type commonly used to store paper). With these numerical values, Eq. 2.1 gives an energy of 187 Joules. This corresponds roughly to a 140 lbs person sitting on the box and nearly flattening it.

- **Comparison with experiment:** Crush-test dummy results confirm that this is indeed a good estimate.

In addition to the crush energy, energy can also be absorbed in the process of flattening the flute within the cardboard walls. However, the pressure required to do this is ∼150 kPa [?] and the surface area involved is more than a square meter, so a quick calculation show that the stuntman would decelerate too quickly if his kinetic energy were to be transferred into flattening boxes. We therefore ignore this additional flattening effect.

- Any successful box rig configuration must dissipate all of the kinetic energy of the stuntman and motorcycle through box crushing alone.

为了做一个实例验证, 我们计算了尺寸为 8.5 英寸 × 17 英寸 × 11 英寸, ECT 为 20 磅/英寸的 C-楞型标准纸箱 (常用作储存纸张) 的损毁能量. 将这些参数代入式 (2.1) 中, 得出能量为 187 焦耳. 这个大致相当于一个 140 磅的人坐在该纸箱上, 并几乎将它压平.

- 实验对比: 假人撞压测试的结果证实这是个很好的估算.

除了纸箱损毁能量外, 把纸板的波浪形纸芯夹层彻底压平也会吸收能量. 然而彻底压平纸板需要的压力值高达 150kPa, 且受力表面积会大于 1 平方米, 因此通过一个简单的计算就能得出: 如果特技演员的动能被用来彻底压平纸箱, 这种情况下特技演员将具有过大的减速度. 因此，我们忽略这个彻底压平纸箱而额外吸收的能量.

- 任何成功的纸箱堆的结构都必须使特技演员和摩托车的所有动能仅被消耗在撞毁纸箱的过程中.

3 Commonly available box types / 常用的纸箱类型

Minimization of costs is an important concern in this stunt. The cardboard box rig will consist of perhaps hundreds of boxes, and wholesale box prices can range up to $10 or $20 per unit (for larger boxes), so we therefore restrict our attention to only those box types which are most commonly available. We investigate the box types listed in table 1.

在这个特技表演中，如何把成本花费降到最低是一个重要问题. 纸箱堆也许会包含数百个纸箱, 纸箱的批发单价能高达 10 到 20 美元 (较大的纸箱), 因此我们将注意力只集中在那些最常用的纸箱类型上. 我们将查到的纸箱类型结果列于表1中.

Table 1: Commonly available box types. Information from [?, ?].

Type	Size (in)	ECT rating (lbs/in)	Price per box
A	$10 \times 10 \times 10$	32	$0.40
B	$20 \times 20 \times 20$	32	$1.50
C	$20 \times 20 \times 20$	48	$3.50
D	$30 \times 30 \times 30$	32	$5.00
E	$44 \times 12 \times 12$	32	$1.75
F	$80 \times 60 \times 7$	32	$10.00

We will use this table later to evaluate the cost of various box rig configurations.

后面我们会用该表来评估不同结构的纸箱堆的花费.

4 Some quick estimates / 一些快速的估计

We now make a few rough calculations and estimates. We will use these results to set safety tolerances and as a guide in working with the more complex numerical simulation that we later formulate.

现在, 我们要做一些粗略计算和估计. 这些结果将用来设置安全性容差, 并作为后面建立更复杂的数值仿真的向导.

4.1 Maximum safe acceleration / 最大安全加速度

To determine acceptable forces and accelerations for the stuntman as he enters the cardboard box rig, we compare the box rig with other cushioning devices. In the stunt rigging business it is common practice to use an air bag for high-falls of up to 30 meters. Airbags rated for falls of up to 30 meters

为了确定杂技演员进入纸箱堆时可以承受的力和加速度大小, 我们把纸箱堆与其他缓冲装置做一下对比. 在特技表演的防具中, 常用气垫保护从高达 30 米处跳下的演员. 承受 30 米高空坠落的气垫大约需要 4 米厚.

are approximately 4 meters deep.

Assume a stuntman falls from 30 meters above the airbag. Gravity accelerates him from rest to a velocity v. At this point he strikes the airbag and is decelerated completely, so we have

设想一位特技演员从 30 米高度坠落到气垫上时. 重力的作用令他从静止加速到速度 v. 此时, 他撞上气垫, 然后完全减速至 0, 因此我们有:

$$\sqrt{2gd_{\text{fall}}} = \sqrt{2a_{\text{bag}}h_{\text{bag}}}$$

where d_{fall} is the fall distance, a_{bag} is the deceleration rate the stuntman experiences in the airbag, h_{bag} is the height of the airbag, and g is the acceleration due to gravity. Thus

其中 d_{fall} 为下落距离, a_{bag} 为特技人在气垫上承受的减速度, h_{bag} 为气垫的高度, g 为重力加速度. 因此

$$a_{\text{bag}} = \frac{d_{\text{fall}}}{h_{\text{bag}}}g = \frac{30\text{m}}{4\text{m}}g = 7.5g$$

We therefore conclude:

- **When using an airbag, the stuntman experiences an average acceleration of at most** $7.5g$. This provides and upper bound on the maximum acceleration that a person can safely withstand.
- **However, with the airbag stunt the stuntman is able to land in a position that distributes forces evenly across his body.** In our stunt the stuntman will be landing in the box rig while still on the motorcycle. This will result in greater chance for injury under high deceleration.
- **We choose** $5g$ **as our maximum safe deceleration.** A box rig configuration which results in a higher acceleration will be rejected as unsafe.

因此我们得到以下结论:

- **当使用气垫时, 特技演员承受的最大平均 (负) 加速度为** $7.5g$. 这提供了一个人所能承受的安全加速度的上限.
- **但是, 借助气垫能够让演员在着陆时全身受力均匀.** 在我们的特技表演中, 特技演员将骑着摩托车着陆到纸箱堆. 在比较大的加速度下, 这将会增加特技演员受伤的风险.
- **我们选取** $5g$ **作为最大安全加速度.** 任何会导致更大加速度的纸箱堆结构都是不安全的, 将不会被使用.

4.2 Displacement and energy estimates, a reality check 制动距离及能量的估计, 一个实例验证

If the deceleration of the stuntman and motorcycle is constant through the boxes then we can estimate the distance required to bring him to rest. Since any deviation from constant acceleration will increase either the stopping distance or the peak deceleration, this will give us a lower bound on the stopping distance and hence a lower bound on the required dimensions of the box rig.

如果特技演员和摩托车在纸箱中减速过程中, 加速度是一个常数的话, 我们可以估计让演员停下所需的制动距离. 然尔任何偏离恒定加速度的扰动都会增加制动距离或者加速度的最大值, 但这可以为我们提供制动距离的下限, 并由此得到所需纸箱堆在尺寸上的下限.

Suppose the stuntman enters the rig at time $t = 0$ with velocity v_0

假设这位特技演员在 $t = 0$ 时刻飞入纸箱堆, 速

and experiences a constant deceleration a until he is brought to rest at time $t = t_f$. The stuntman's velocity $v(t)$ is then given by $v(t) = v_0 - at$. Since the stuntman is at rest at time t_f, we have

$$t_f = v_0/a$$

Let $x(t)$ be the displacement from the point of entry as a function of time. Since $x(0) = 0$,

$$x(t) = v_0 t - (1/2)at^2$$

and so the total distance traveled through the boxes is

$$\Delta x = x(t_f) = \frac{v_0^2}{a} - \left(\frac{1}{2}\right)(a)\left(\frac{v_0}{a}\right)^2 = \frac{v_0^2}{2a}$$

Therefore we arrive at:

- Given an impact velocity $v_0 \approx 20m/s$ and deceleration bounded by 5 g, the stuntman will require *at least* 4 meters to come to rest.

Conversely, if we instead have an idea of what the stopping distance should be, we can easily compute that the constant deceleration required to stop in a distance Δx is

$$\frac{v_0^2}{2g\Delta x}.$$

Using the calculation for the energy dissipated by crushing a box we can estimate how many boxes must be crushed to dissipate the energy of the incoming projectiles, i.e.: stuntman and motorcycle. The energy that must be dissipated in the boxes is roughly equal to the kinetic energy that the motorcycle and stuntman enter with. (Since the box rig should only be three or four meters high, the potential energy is a much smaller fraction of the total energy.) Thus for $v_0 = 20m/s$ and a mass of 200 kilograms, the change in energy is 40,000 Joules. From Eq. 2.1 we calculate that the crush energy of a standard (30 inch)^3 box is 633 Joules.

度为 v_0, 加速度恒定为 a(加速度为负), 直到 $t = t_f$ 时减速到静止. 特技演员任一时刻的速度 $v(t)$ 由公式 $v(t) = v_0 - at$ 给出. 由于特技演员在 $t = t_f$ 时刻达到静止, 我们有

令 $x(t)$ 为从飞入纸箱堆后前进的距离, 是时间的函数. 由于 $x(0) = 0$,

所以在纸箱堆中运动的总距离为

因此我们得出:

- 在给定冲击速度 $v_0 \approx 20m/s$, 减速度限定为 $5g$ 的条件下, 特技演员需要至少 4 米的距离制动.

反过来, 如果我们知道是制动距离应该为多少, 我们就能很容易计算出在距离 Δx 中所需要的制动加速度常量为

利用对撞毁一个纸箱耗能的计算, 我们可以估计一共需要多少纸箱来消耗掉即将冲出的“导弹”, 也就是特技演员和摩托车. 在纸箱堆中必须消耗的能量可以粗略地认为等于摩托车和演员刚进入纸箱堆时的初始动能. (由于纸箱堆只能高至 3 到 4 米, 因此势能只占总能量很少一部分.) 因此对于初速度 $v_0 = 20m/s$, 总质量 200 千克的情况, 能量的变化为 40,000 焦耳. 从式 (2.1) 我们计算出一个尺寸为 30 立方英寸, 标准规格的纸箱的损毁能量为 633 焦耳.

$$\frac{40,000 \text{ Joules}}{633 \text{ Joules/box}} \approx 60 \text{ boxes}$$

- **Conclusion:** The incoming stuntman and motorcycle must crush about 60 boxes to come to a stop.

- **结论:** 正在飞来的特持演员和摩托车必需撞坏大约 60 个纸箱后, 才能停下来.

5 Trajectory Analysis and Cushion Location 轨迹分析及着陆范围

Regardless of how much energy it takes to crush the cardboard boxes, they won't dissipate any of the stuntman's energy unless he actually lands on them. It is therefore important to consider the trajectory of the stuntman and motorcycle through the air so we know where the box rig should be placed and what the uncertainty in landing location is.

We calculate trajectories by solving the following differential equation, where v is the speed, k is the drag coefficient, and $\vec{x}$ is the position:

无论有多少能量能撞毁多少纸箱, 除非这位演员能正好着陆在纸箱堆上, 否则特技演员的任何能量也不会被消耗. 所以说，关键在于分析特技演员和摩托车在空中的运动轨迹, 我们才能知道纸箱堆应该放置在什么地方, 以及演员着陆位置的不确定性范围.

我们通过求解下面的微分方程来计算轨迹, 其中 v 为速度，k 为阻力系数, $\vec{x}$ 空间位置:

$$(\vec{x})'' = -g\hat{z} - \frac{k}{m}|v|^2\hat{v}$$

MATLAB's ODE45 function was used to solve an equivalent system of first order equations. We use an air drag coefficient of $k = 1.0$ (this value is from Ref. [**?**]). We see immediately from figure **??** that it would be unwise to ignore air resistance.

- **Air drag effects alter the stuntman's landing position by up to several meters.** We therefore incorporate air resistance into all simulations.

MATLAB 的 ODE45 函数可以用来求解等效系统的一阶微分方程 (组). 我们取空气阻力系数 $k = 1.0$(此值来自参考文献 [**?**]). 从图**??**中我们可以明显看出, 忽略空气阻力是非常不明智的做法.

- **空气阻力的影响改变了演员的着陆位置, 造成了数米以上的变动.** 因此我们将空气阻力纳入所有的仿真计算中.

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