Technische Universität Kaiserslautern
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December 15th, 2011

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Wintersemester 2011/12

Multicriteria Optimization

Solution Sheet 9

IN-CLASS EXERCISES
(To be done in the tutorial on December 20th, 2011)

Problem 9.1 – Sum-Max-SPP

Let $G = (V, A)$ be a directed graph, $s, t \in V$ and denote by $\mathcal{P}^{s,t}$ the set of all s-t paths in G, where a path $P \in A^{|P|}$ is defined by its edges. Consider the following shortest path problem with one sum and $p - 1$ bottleneck objectives:

$$\min \left(\sum_{a \in P} c^1(a),\ \max_{a \in P} c^2(a),\ \ldots,\ \max_{a \in P} c^p(a) \right) \qquad \text{(Sum-Max-Spp)}$$

$$\text{s.t.} \quad P \in \mathcal{P}^{s,t}.$$

a) Find a good upper bound on the number of nondominated points for Sum-Max-Spp.

b) Find an algorithm to find a complete minimal set of efficient solutions. You do not have to give a rigorous proof of the correctness. *Hint: Try to first find a solution for* $p = 2$*, then generalize.*

c) How can we solve the problem if there are bottleneck objectives only?

Solution 9.1:

For a path $P \in \mathcal{P}^{s,t}$, we define $c^1(P) = \sum_{a \in P} c^1(a)$ and $c^j(P) = \max_{a \in P} c^j(P)$ for $j = 2, \ldots, p$. For $\hat{c} \in \mathbb{R}^{p-1}$, we define

$$\mathcal{P}^{s,t}_{\hat{c}} = \left\{ P \in \mathcal{P}^{s,t} :\ c^j(P) \leqslant \hat{c}_{j-1} \text{ for } j = 2, \ldots, p \right\}.$$

a) Let $\mathcal{X}'$ be a minimal complete set of efficient solutions. For each $P \in \mathcal{X}'$, it must hold that

$$c^1(P) = \min \left\{ c^1(P') :\ P' \in \mathcal{P}^{s,t} \text{ with } c^2(P') \leqslant c^2(P), \ldots, c^p(P') \leqslant c^p(P) \right\},$$

i. e., P must be a minimizer of c^1 within $\mathcal{P}^{s,t}_{(c^2(P),\ldots,c^p(P))}$. Otherwise, if there were a P' with $c^1(P') \leqslant c^1(P)$ satisfying all the bottlenecks defined by P, that path would dominate P. Thus each configuration of bottleneck values contributes at most one element to $\mathcal{X}'$.

Let

$$C^j = \{v \in \mathbb{R} : \exists\, a \in A \text{ with } c^j(a) = v\}$$

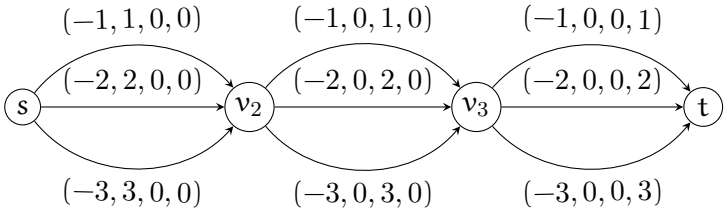
be the set of distinct cost values of the j-th objective function. Then the set of possible configurations of the bottleneck objectives is

$$C^1 \times C^2 \times \cdots \times C^p$$

and thus $\prod_{j=2}^{p} \left|C^j\right|$ is an upper bound for $|\mathcal{X}'| = |\mathcal{Y}_N|$. Note that $\left|C^j\right| \leqslant |A|$, so a weaker bound not involving the C^j would be $|\mathcal{Y}_N| \leqslant |A|^{p-1}$.

We now give an infinite class of graphs for which the first bound is tight. Assume $|A| = k \cdot (p-1)$ for some $k \in \mathbb{N}$ and define $G = (V, A)$ as a line graph with parallel arcs as follows:

- $V = (v_1 = s, v_2, \ldots, v_p = t)$
- For each $i \in \{2, \ldots, p\}$ there are k parallel arcs (v_{i-1}, v_i) which we denote by $a^{i,j}$ for $j = 1, \ldots, k$.
- For an arc $a^{i,j} \in A$, the cost vector is defined by $c^1(a) = -j$, $c^i(a) = j$, $c^l(a) = 0$ for $l \notin \{1, i\}$. An example for $p = 4$ and $k = 3$ is given below.



We claim that $\mathcal{P}^{s,t}$ is a minimal efficient set, i. e. all paths are efficient and have different objective values. As for the latter, observe that the choice of the i-th arc of a path, $a^{i,j}$, uniquely determines $c^{i+1}(P)$, because $a^{i,j}$ is the only arc $a \in P$ with $c^{i+1}(a) \neq 0$. Because $c^{i+1}(a^{i,j})$ is different for all $j = 1, \ldots, k$, all paths have different costs in the bottleneck objectives.

To show that all paths are efficient, assume for the contrary that a path P is dominated by some other path $\hat{P} \neq P$. Let i be minimal such that $P \ni a^{i,j} \neq a^{i,\hat{j}} \in \hat{P}$, i. e. the paths differ in the $i-1$-st arc. Since $\hat{P}$ dominates P, we must have $\hat{j} < j$, otherwise $c^i(P) = j < c^i(\hat{P}) = \hat{j}$ and P would not be dominated.

But then $c^1(a^{i,\hat{j}}) > c^1(a^{i,j})$, so there must be at least one index $i' > i$ where $\hat{P}$ takes an arc with larger index: $P \ni a^{i',j'} \neq a^{i',\hat{j}'} \in \hat{P}$ with $\hat{j}' > j'$. This however implies $c^{i'}(P) < c^{(i')}(\hat{P})$, which is a contradiction to $\hat{P}$ dominating P.

b) We claim that Algorithm 1 computes a minimal efficient set for Sum-Max-Spp. Observe that the **for** loop in Line 8 traverses all possible configurations of the bottleneck costs. Thus, by the argumentation of part a), the total set of paths generated in Line 9 contains a minimal complete efficient set. It is left to show that the detection of dominated solutions in the following lines is correct; i. e. that in Line 24, we add exactly one path for each $y \in \mathcal{Y}_N$.

Lemma 1: *For* $\hat{c} \in C^2 \times \cdots \times C^p$, *let*

$$c^*(\hat{c}) = \min_{P \in \mathcal{P}^{s,t}_{\hat{c}}} c^1(P) \text{ and } P^*(\hat{c}) = \arg\min_{P \in \mathcal{P}^{s,t}_{\hat{c}}} c^1(P),$$

if such a path exists. Then the set

$$\mathcal{P}_{\texttt{Alg}} \left\{ P^*(\hat{c}) : \mathcal{P}^{s,t}_{\hat{c}} \neq \emptyset,\ c^*(\hat{c}') > c^*(\hat{c}) \text{ for all } \hat{c}' \leqslant \hat{c},\ \hat{c}' \neq \hat{c} \right\}$$

is a minimal complete set of efficient solutions for Sum-Max-Spp.

Proof. Let $P \in \mathcal{X}_E$ be an efficient path with objective value $c(P) = c^1(P), \ldots, c^p(P)$. Let $\hat{c} = (c^2(P), \ldots, c^p(P))$, then by part a), $c(P) = c^*(\hat{c})$. Assume that $\mathcal{P}_{\texttt{Alg}}$ does not contain any path P' with $c(P') = c(P)$. Then by definition of $\mathcal{P}_{\texttt{Alg}}$ there must be some $\hat{c}' \leqslant \hat{c}$ such that $c^*(\hat{c}') \leqslant c^*(\hat{c})$. But this means that $P^*(\hat{c}')$ dominates P, contradicting $P \in \mathcal{X}_E$. □

With the additional observation that for any $\hat{c}' \geqslant \hat{c}$ we necessarily have $c^*(\hat{c}') \leqslant c^*(\hat{c})$ (since constraints are relaxed) it is easy to see that Algorithm 1 computes $\mathcal{P}_{\texttt{Alg}}$ and is thus correct.

Its running time is, if we omit output cost in Line 24, $\mathcal{O}\left(\prod_{j=2}^{p} |C^p| \left(|V| \log |V| + |A|\right)\right)$: The shortest path computation in Line 9 can be accomplished in $\Theta\left(|V| \log |V| + |A|\right)$ using Dijkstra's algorithm with Fibonacci heaps. Disabling arcs that don't meet the bottleneck constraints $\hat{c}$ takes only linear time for each such $\hat{c}$.

c) When there are bottleneck objectives only, we could use a modification of Algorithm 1 to compute a minimal complete efficient set. Instead of the minimum c^1-values for each bottleneck configuration, however, the `Value` array only needs to store a boolean flag, whether there exists a path in $\mathcal{P}^{s,t}_{\hat{c}}$ or not. Thus in Line 9 we can use a linear-time connectedness test instead of shortest path computation. The following observations could be used to further reduce the running time:

- a path with objective value $c(P) = c^1(P), \ldots, c^p(P)$ is efficient if and only if there does not exist a path P' with $c(P') \leqslant c(P),\ c(P') \neq c(P)$;
- if $\mathcal{P}^{s,t}_{\hat{c}}$ is empty for some $\hat{c}$, then it is also empty for every $\hat{c}' \leqslant \hat{c}$;
- if $\mathcal{P}^{s,t}_{\hat{c}}$ is nonempty, that also holds for all $\hat{c}' \geqslant \hat{c}$.

Algorithm 1 Algorithm solving SUM-MAX-SPP

Input: Graph $G = (V, A)$ with p objective functions $c^1, \dots, c^p$.
Output: A minimal complete set $\mathcal{S}$ of efficient solutions for SUM-MAX-SPP.
for all $j = 2, \dots, p$ **do**
    Compute and C^j and sort it such that $C^j_1 \leqslant \dots \leqslant C^j_{|C_j|}$
end for
Allocate a $p-1$-dimensional array $\texttt{Value}\left[\left|C^2\right|, \dots, |C^p|\right]$
$S \leftarrow \emptyset$
for all $I \in \left[1, \left|C^2\right|\right] \times \dots \times \left[1, \left|C^j\right|\right]$, traversed in lexicographic order **do**
    $\hat{c} \leftarrow (C^2_{I_1}, \dots, C^p_{I_{p-1}})$
    Find a path $P = \arg\min_{P \in \mathcal{P}^{s,t}_{\hat{c}}} c^1(P)$
    if there is no such path ($\mathcal{P}^{s,t}_{\hat{c}}$ is empty) **then**
        $\texttt{Value}[I] \leftarrow \infty$
    else
        $\texttt{Value}[I] \leftarrow c^1(P)$
        $\texttt{dominated} \leftarrow \texttt{False}$
        for all $i = 1, \dots, p-1$ with $I_i > 1$ **do**
            $\hat{I} \leftarrow I$
            $\hat{I}_i = \hat{I}_i - 1$
            if $\texttt{Value}[\hat{I}] = c^1(P)$ **then**
                $\texttt{dominated} \leftarrow \texttt{True}$
                break
            end if
            if not dominated **then**
                $S \leftarrow S \cup \{P\}$
            end if
        end for
    end if
end for
return S

TURN-IN EXERCISES
(Please hand in by January 5th, 2012)

Problem 9.2

a) Using the multicriteria label-setting algorithm from the lecture, compute a minimal complete set of efficient paths from v_1 to v_7 in the graph of Figure 1.

b) Consider a directed graph $G = (V, A)$ with cost function $c : A \to \mathbb{R}^p$ not containing negative cycles, i. e., for all cycles Z in G we have $c(Z) = \sum_{a \in Z} c(a) \geqslant 0$.

    1. Prove the following statement: Let $P^{s,t}$ be an efficient s-t path on G. Then any subpath $P^{u,v}$ from u to v, where u and v are vertices on $P^{s,t}$, is an efficient path from u to v.

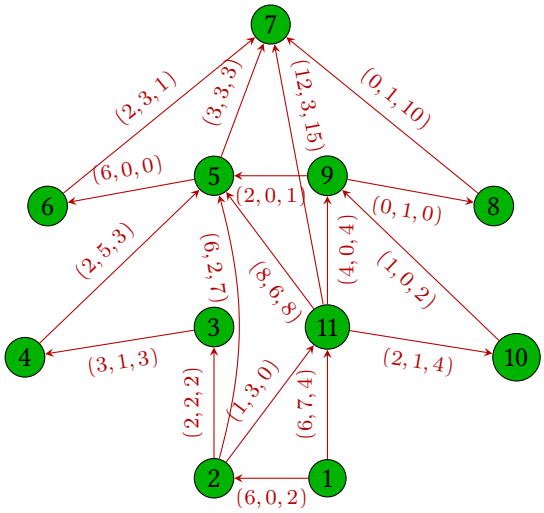


Figure 1: The graph for Problem 2

2. Show that the converse is not true in general, i. e. compositions of efficient paths are not necessarily efficient.

Problem 9.3 – Multicriteria FloydWarshall

Recall how the algorithm of Floyd and Warshall computes all-to-all shortest paths in a directed graph under the assumption that it does not contain negative dicycles. The objective of this exercise is to extend that algorithm so as to find a minimal complete efficient set for a multicriteria shortest path problem (which also must not contain any negative cycle in the sense of Problem 2).

To solve this exercise, assign a list of labels to each node, similar as for the multicriteria label-setting algorithm in the lecture. Then use the generalized Bellman principle of the previous exercise to restrict the set of possible efficient paths in the inner `FOR` loop of FloydWarshall.

Download of exercises at
`http://optimierung.mathematik.uni-kl.de/teaching/ws1112/multicriteria-opt.html`