Monte Carlo Path Tracing

Today

- Path tracing starting from the eye
- Path tracing starting from the lights
- Which direction is best?
- Bidirectional ray tracing
- Random walks and Markov chains

Next

- Irradiance caching
- Photon mapping

The Rendering Equation

$$L(x,\omega) = L_e(x,\omega) + \int_{H^2} f_r(x,\omega' \to \omega)\, L(x^*(x,\omega'),-\omega') \cos\theta' \, d\omega'$$

$$L = L_e + K \circ L$$

Solving the Rendering Equation

Rendering Equation

$$L = L_e + K \circ L$$

$$(I - K) \circ L = L_e$$

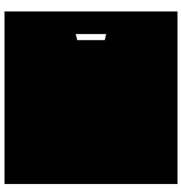
Solution

$$L = (I - K)^{-1} \circ L_e$$

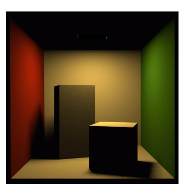
$$= (I + K + K^2 + K^3 + \cdots) \circ L_e$$

$$= (I + K(I + K(I + K \cdots))) \circ L_e$$

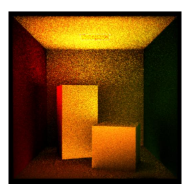
Successive Gathers



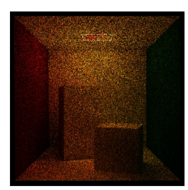
L_e



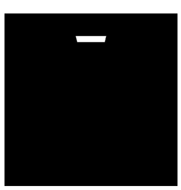
$K \circ L_e$



$K \circ K \circ L_e$

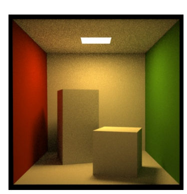


$K \circ K \circ K \circ L_e$

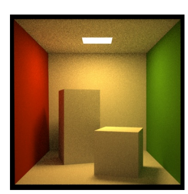


L_e

$L_e + K \circ L_e$

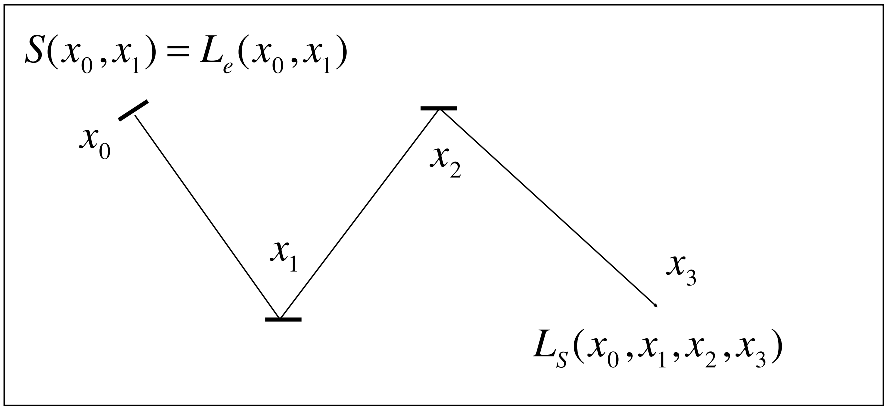


$L_e + \cdots K^2 \circ L_e$

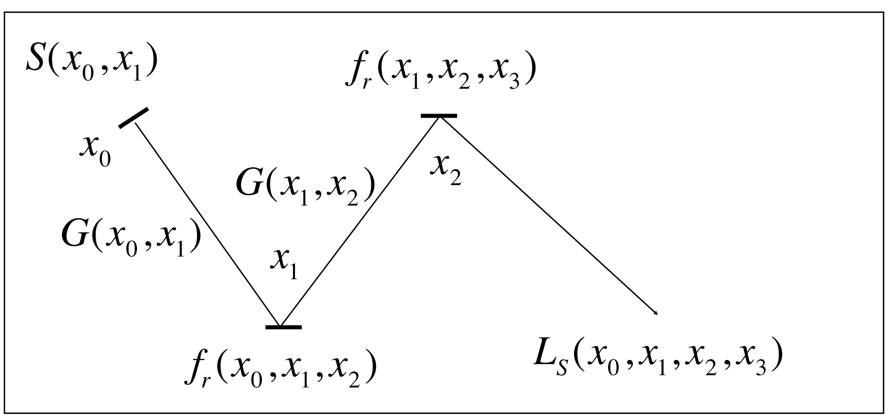


$L_e + \cdots K^3 \circ L_e$

Light Path



Light Path



$$L_S(x_0, x_1, x_2, x_3) = S(x_0, x_1) G(x_o, x_1) f_r(x_0, x_1, x_2) G(x_1, x_2) f_r(x_1, x_2, x_3)$$

Solving the Rendering Equation

One path

$$L_S(x_0,x_1,x_2,x_3) = S(x_0,x_1)G(x_o,x_1)f_r(x_0,x_1,x_2)G(x_1,x_2)f_r(x_1,x_2,x_3)$$

Solution is the integral over all paths

$$L(x_{k-1},x_k) = \sum_{k=1}^{\infty} \int_{M^2} \cdots \int_{M^2} L_S(x_0,\cdots,x_{k-2},x_{k-1},x_k)\,dA(x_0)\cdots dA(x_{k-2})$$

Solve using Monte Carlo Integration

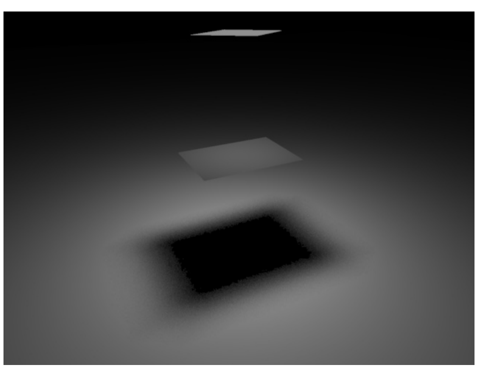
Question: How to generate a random path?

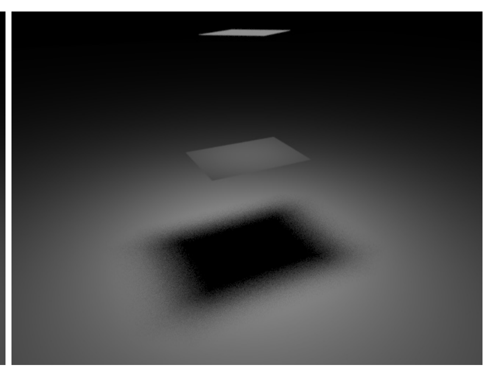
Path Tracing from the Eye

Path Tracing: From Camera

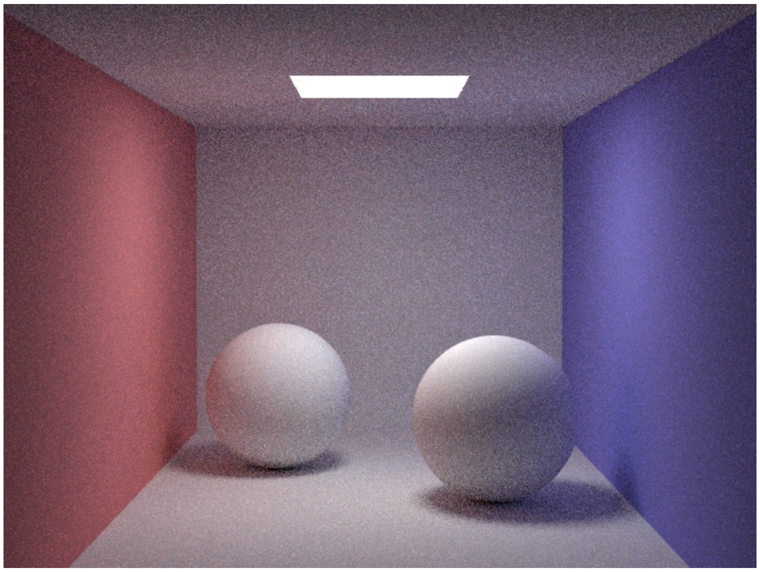
```
Step 1. Choose a camera ray r given the
   (x, y, u, v, t ) sample
      weight = 1;
Step 2. Find ray-surface intersection
Step 3.
   if hit light
      return weight * Le(r);
   else
      weight *= reflectance(r)
      Choose new ray r' ~ BRDF(O|I)
      Go to Step 2.
```


Penumbra: Trees vs. Paths


4 eye rays per pixel
16 shadow rays per eye ray


64 eye rays per pixel
1 shadow ray per eye ray

Path Tracing



10 paths / pixel

M. Fajardo Arnold Path Tracer

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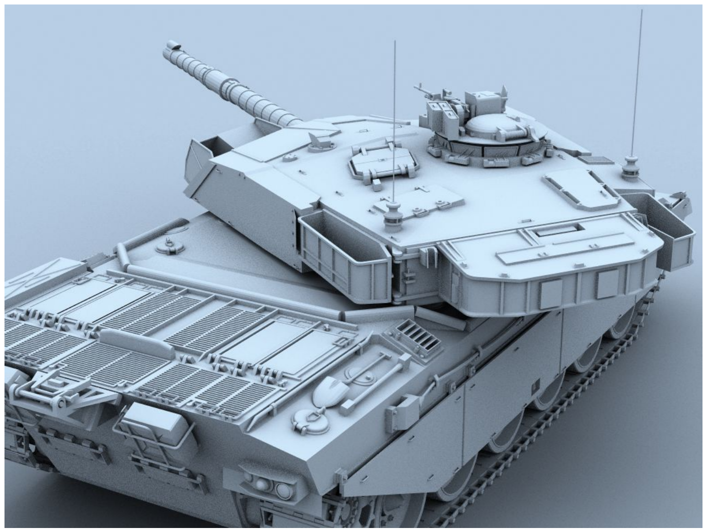
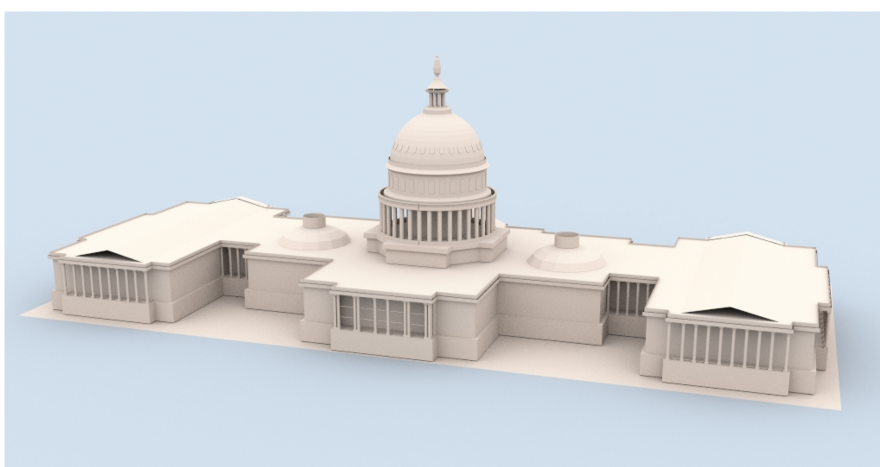
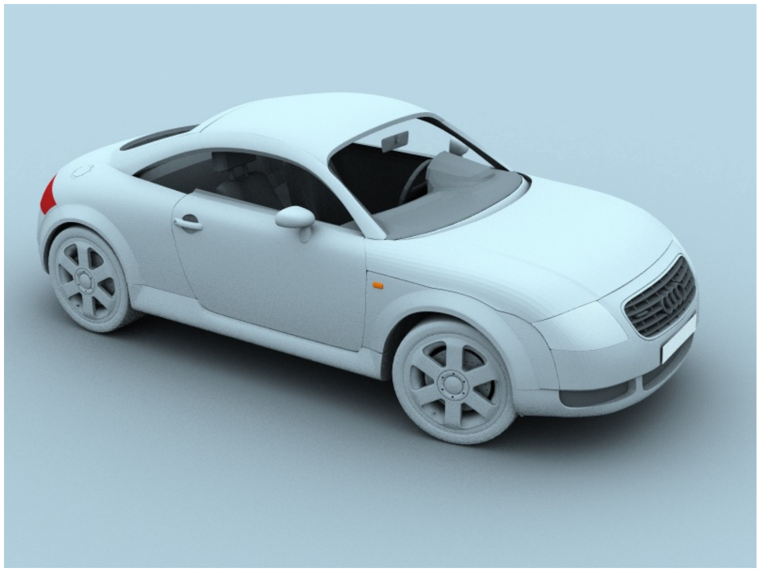


M. Fajardo Arnold Path Tracer

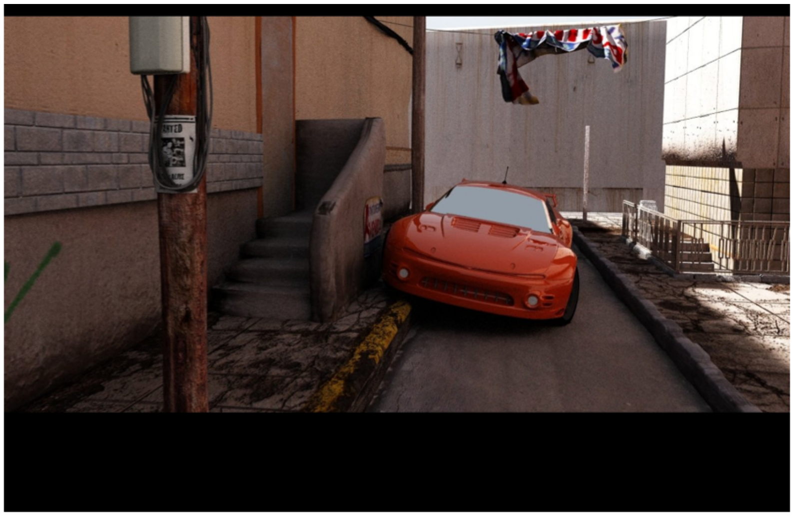


M. Fajardo Arnold Path Tracer



M. Fajardo Arnold Path Tracer

M. Fajardo Arnold Path Tracer



Street scene 1

1536x654, 16 paths/pixel, 2 bounces, 250,000 faces, 18 min., dual PIII-800

How Many Bounces?

Avoid paths that carry little energy

- Terminate when the weight is low

Photons with similar power is a good thing

- Think of importance sampling
- Integrand is f(x)/p(x) which is constant

Russian Roulette

Terminate photon with probability *p*

Adjust weight of the result by 1/(1-p)

$$E(X) = p \cdot 0 + (1-p)\frac{E(X)}{1-p} = E(X)$$

Intuition:

Reflecting from a surface with R=.5

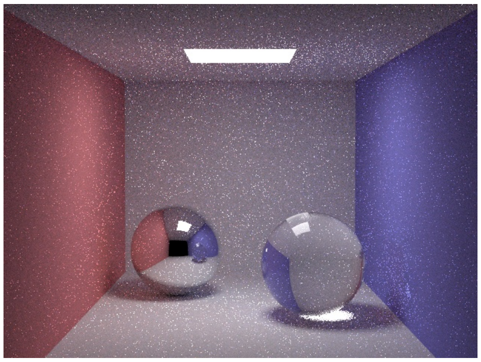
100 incoming photons with power 2 W

1. **Reflect 100 photons with power 1 W**
2. **Reflect 50 photons with power 2 W**

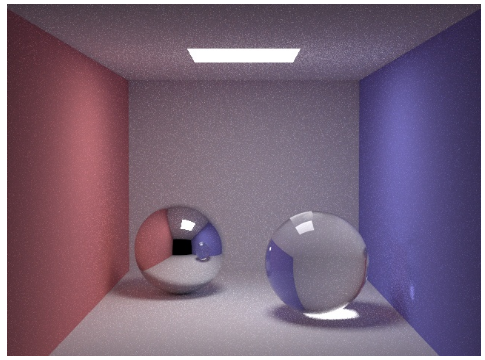
Path Tracing: Include Direct Lighting

```
Step 1. Choose a camera ray r given the
   (x,y,u,v,t) sample
     weight = 1;
     L = 0
Step 2. Find ray-surface intersection
Step 3.
     L += weight * Sum (fr * Le(light))
     weight *= reflectance(x)
     Choose new ray r' ~ BRDF pdf(r)
     Go to Step 2.
```

Variance Decreases with N



10 rays per pixel



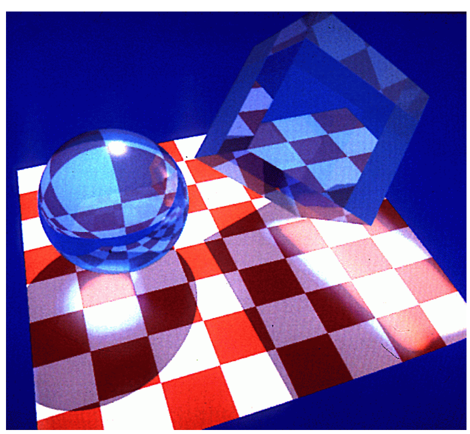
100 rays per pixel

From Jensen, Realistic Image Synthesis Using Photon Maps

Fixed Sampling (Not Random Enough)

Light Ray Tracing

Early Example [Arvo, 1986]



"Backward" ray tracing

Path Tracing: From Lights

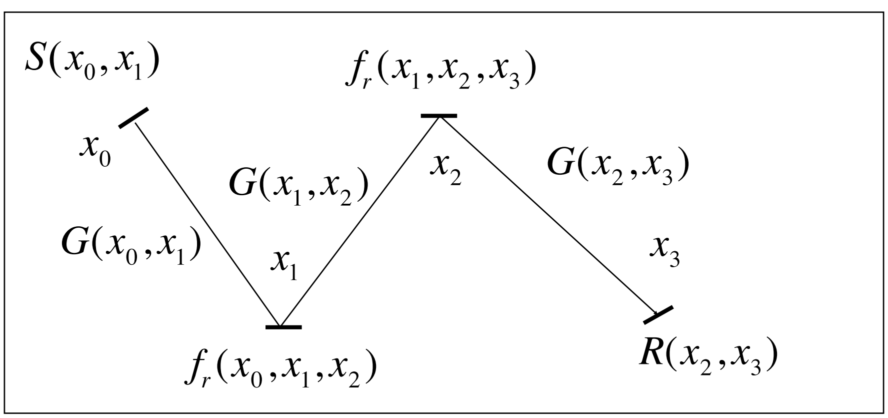
```
Step 1. Choose a light ray.
   Choose a ray from the light source
   distribution function
   x ~ p(x)
   d ~ p(d|x)
   r = (x, d)
   weight = Φ;
```


Path Tracing: From Lights

```
Step 1. Choose a light ray
Step 2. Find ray-surface intersection
Step 3. Reflect or transmit
   u = Uniform()
   if u < reflectance(x)
      Choose new direction d ~ BRDF(O|I)
      goto Step 2
   else u < reflectance(x)+transmittance(x)
      Choose new direction d ~ BTDF(O|I)
      goto Step 2
   else // absorption=1-reflectance-transmittance
      terminate on surface; deposit energy
```

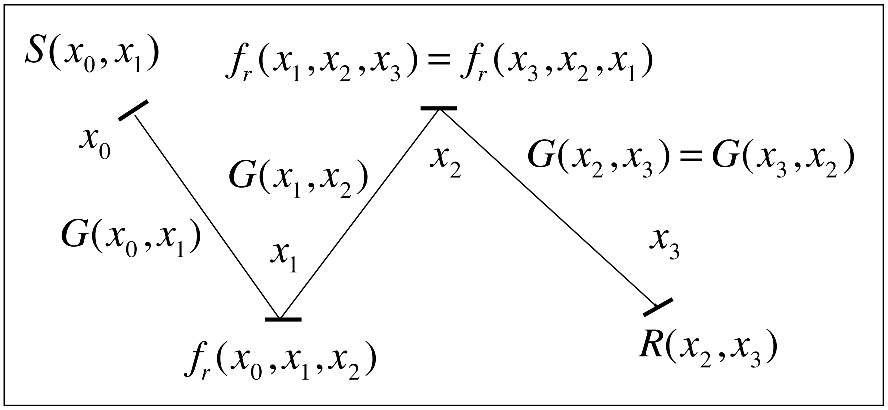
Bidirectional Path Tracing

Symmetric Light Path



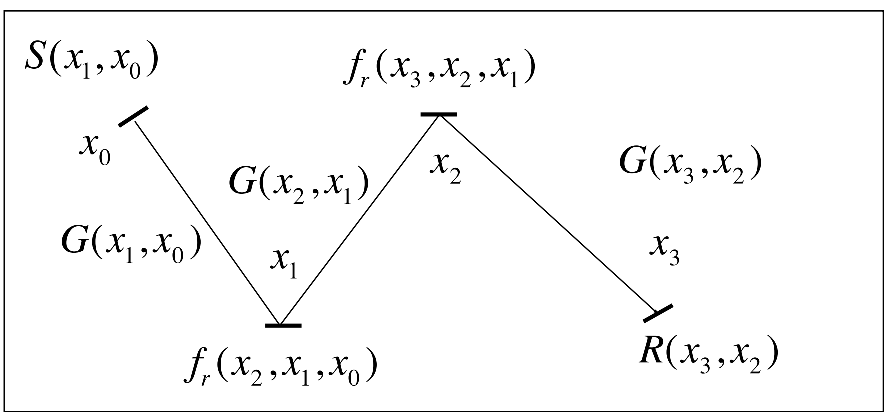
$$M = S(x_0, x_1) G(x_o, x_1) f_r(x_0, x_1, x_2) G(x_1, x_2) f_r(x_1, x_2, x_3) G(x_2, x_3) R(x_2, x_3)$$

Symmetric Light Path



$$M = S(x_0,x_1)G(x_0,x_1)f_r(x_0,x_1,x_2)G(x_1,x_2)f_r(x_1,x_2,x_3)G(x_2,x_3)R(x_2,x_3)$$

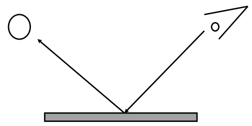
Symmetric Light Path



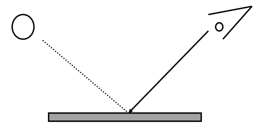
$$M = R(x_3,x_2)G(x_3,x_2)f_r(x_3,x_2,x_1)G(x_2,x_1)f_r(x_2,x_1,x_0)G(x_1,x_0)S(x_1,x_0)$$

Bidirectional Ray Tracing

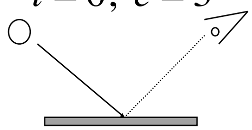
$$k = l + e$$



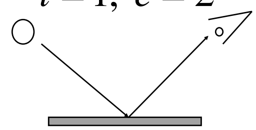
$l = 0,\ e = 3$



$l = 1,\ e = 2$



$l = 2,\ e = 1$



$l = 3,\ e = 0$

$$k = 3$$

Path Pyramid

$k = 3$

$k = 4$

$k = 5$

$k = 6$

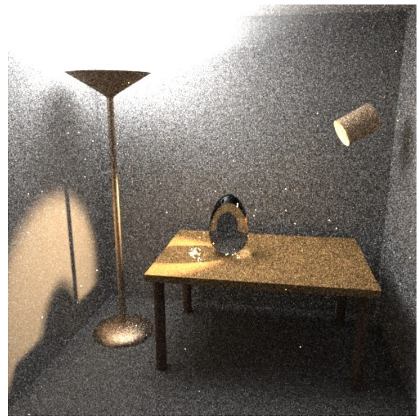
$(l = 2, e = 1)$

$(l = 5, e = 1)$

l **From Veach and Guibas** e

Comparison

Same amount of time



Bidirectional path tracing
25 rays per pixel

Path tracing
56 rays per pixel

From Veach and Guibas

Which Direction?

Solve a linear system $Mx = b$

Solve for a single x_i**?**

Solve the reverse equation

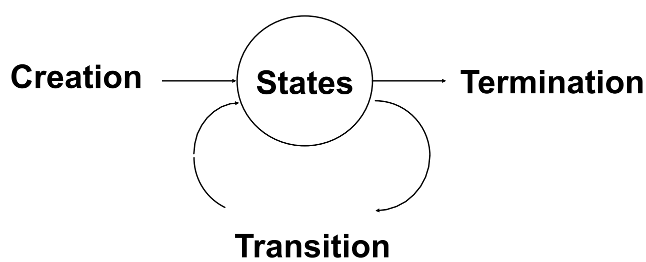
- **Source** x_i
- **Estimator** $<(x_i + Mx_i + M^2 x_i + \cdots), b>$

More efficient than solving for all the unknowns

[von Neumann and Ulam]

Discrete Random Walk

Discrete Random Process



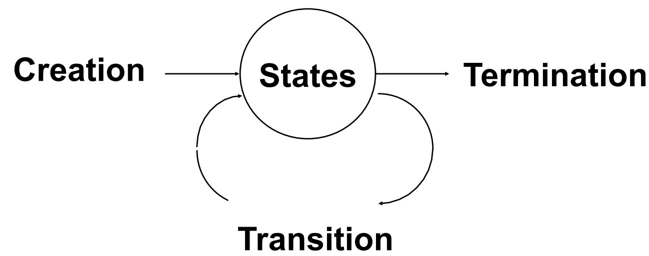
Assign probabilities to each process

p_i^0 : probability of creation in state i

$p_{i,j}$: probability of transition from state $i \rightarrow j$

p_i^* : probability of termination in state i $\quad p_i^* = 1 - \sum_j p_{i,j}$

Discrete Random Process



Equilibrium number of particles in each state

$$P_i = \sum_j p_{i,j} P_j + p_i^0 \qquad M_{i,j} = p_{i,j}$$

$$P = MP + p^0$$

Equilibrium Distribution of States

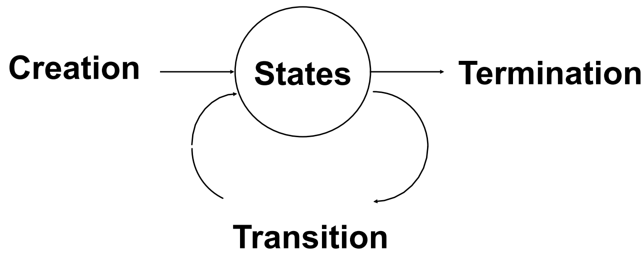
Total probability of being in states P

Solve this equation

$$(I - M)P = p^0$$

$$P = (I - M)^{-1} p^0$$
$$= \left(I + M + M^2 + \cdots\right) p^0$$

Discrete Random Walk



1. Generate random particles from sources.
2. Undertake a discrete random walk.
3. Count how many terminate in state i

[von Neumann and Ulam; Forsythe and Leibler; 1950s]

Monte Carlo Algorithm

Define a random variable on the space of paths

Path: $\alpha_k = (i_1, i_2, \ldots, i_k)$

Probability: $P(\alpha_k)$

Estimator: $W(\alpha_k)$

Expectation:

$$E[W] = \sum_{k=1}^{\infty} \sum_{\alpha_k} P(\alpha_k) W(\alpha_k)$$

Monte Carlo Algorithm

Define a random variable on the space of paths

Probability: $P(\alpha_k) = p^0_{i_1} \times p_{i_1,i_2} \cdots p_{i_{k-1},i_k} \times p^*_{i_k}$

Estimator: $W_j(\alpha_k) = \dfrac{\delta_{i_k,j}}{p^*_{i_k}}$

Estimator

Count the number of particles terminating in state *j*

$$E[W_j] = \sum_{k=1}^{\infty} \sum_{i_k} \cdots \sum_{i_1} (p^0_{i_1} p_{i_1,i_2} \cdots p_{i_{k-1},i_k} p^*_{i_k}) \frac{\delta_{i_k,j}}{p^*_j}$$
$$= \left[p^0 \right]_j + \left[M p^0 \right]_j + \left[M^2 p^0 \right]_j \cdots$$

Equilibrium Distribution of States

Total probability of being in states P

$$P = \left(I + M + M^2 + \cdots\right)p^0$$

Note that this is the solution of the equation

$$(I - M)P = p^0$$

Thus, the discrete random walk is an unbiased estimate of the equilibrium number of particles in each state