Carleton University

Department of Electronics

ELEC4700

Assignment 2 - Finite Difference Method

Author: Nicholas Wicklund #101007831

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Contents

1 Introduction

The purpose of this assignment is to explore the finite difference method and use it to solve electrostatic potential problems using Laplace's equation.

2 Part 1: Electrostatic Potential Problems - Simple Cases

This case models the voltage across a simple unrestricted rectangle with two types of boundary conditions. The figure below shows the rectangular region being modelled.

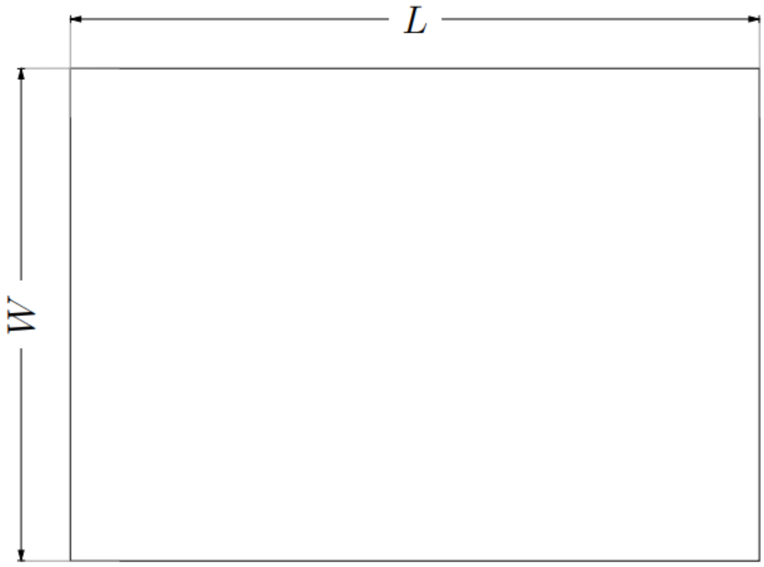

Figure 1: Simple Rectangular Region [1]

The ratio of length to width is 3/2 and in this case the chosen length and width were 75 and 50 respectively. To model this voltage the finite difference model is applied to Laplace's equation as follows [2]:

$$\Delta^2 V = 0 \tag{1}$$

$$\frac{df(x)}{dx} = \frac{f(x + -f(x)}{\Delta x} \tag{2}$$

$$\frac{V_{m+1,n} - 2V_{m,n} + V_{m-1,n}}{(\Delta x)^2} + \frac{V_{m,n-1} - 2V_{m,n} + V_{m,n+1}}{(\Delta y)^2} = 0 \tag{3}$$

Assuming that $\Delta x = \Delta y$:

$$V_{m,n} = \frac{V_{m+1,n} + V_{m-1,n} + V_{m,n+1} + V_{m,m-1}}{4} \tag{4}$$

2.1 Part A - V = V_o at x = 0 and V = 0 at x = L Top and Bottom Not Fixed

To solve this part of the assignment the left side of the region was set to 1V (Vo), the right side of the region was set to 0V and the top and bottom were not fixed. Since the top and bottom were not fixed the Finite Difference equation above was modified for these boundary's to only consider the three surrounding indexes instead of the usual four. This means that the 3 surrounding indices are divided by 3 instead of 4. The final result of this electrostatic potential problem can be seen in the figures below.

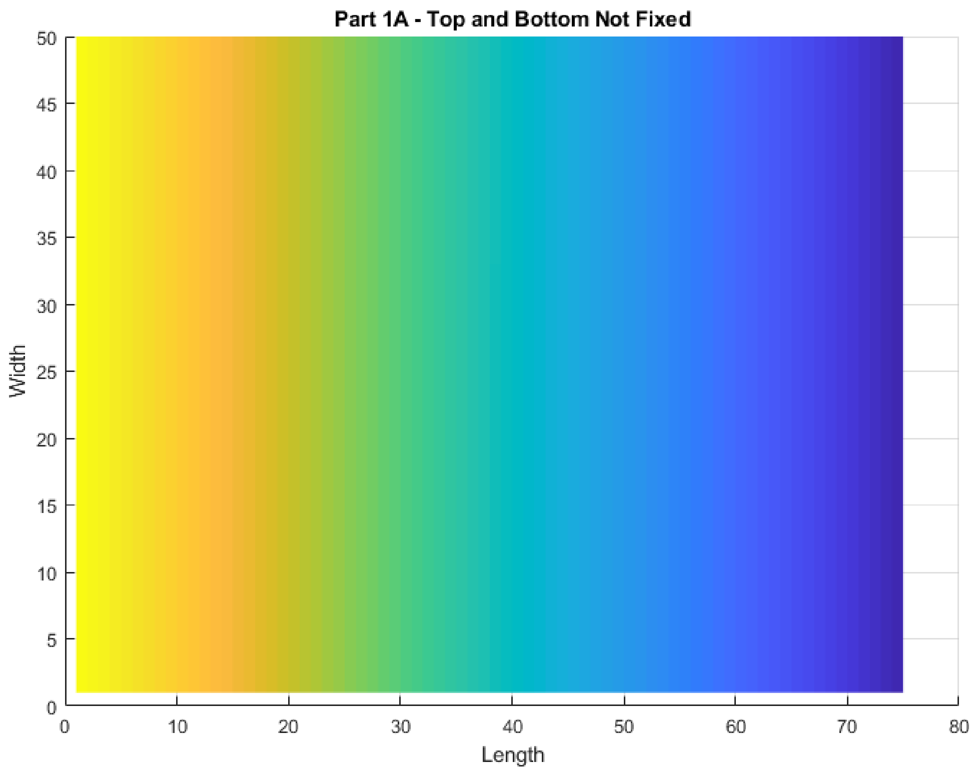

Figure 2: 2D Voltage Graph

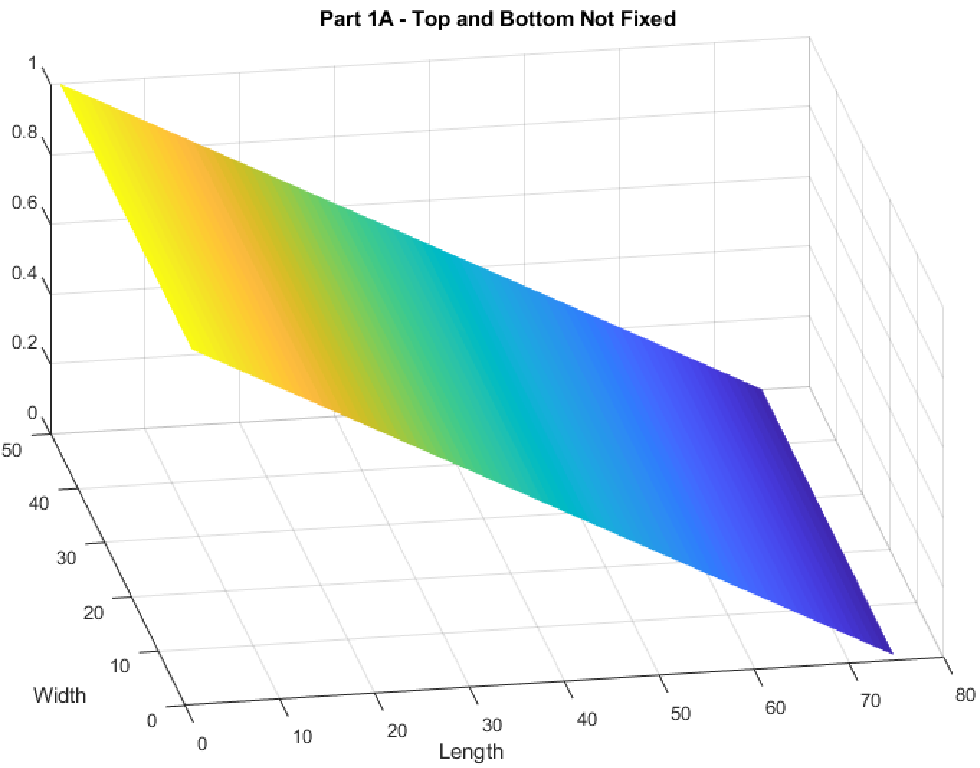

Figure 3: 3D Voltage Graph

2.2 Part B - V = V_o at x = 0, x = L and V = 0 at y = 0 and y = W

This problem is solved using both the finite difference method used above and an analytic method defined in the assignment manual. The analytic method is then compared against the finite difference method using a number of different mesh sizes.

2.2.1 Finite Difference Method

Using the same finite difference method outlined in Part 1A, the Voltage distribution can be seen in the following figures:

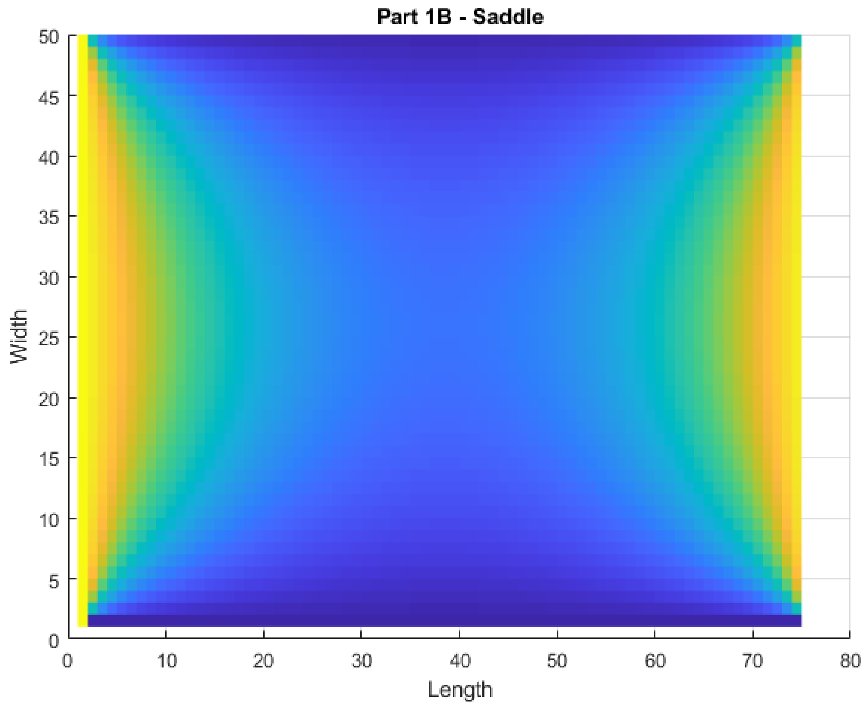

Figure 4: 2D Voltage Graph

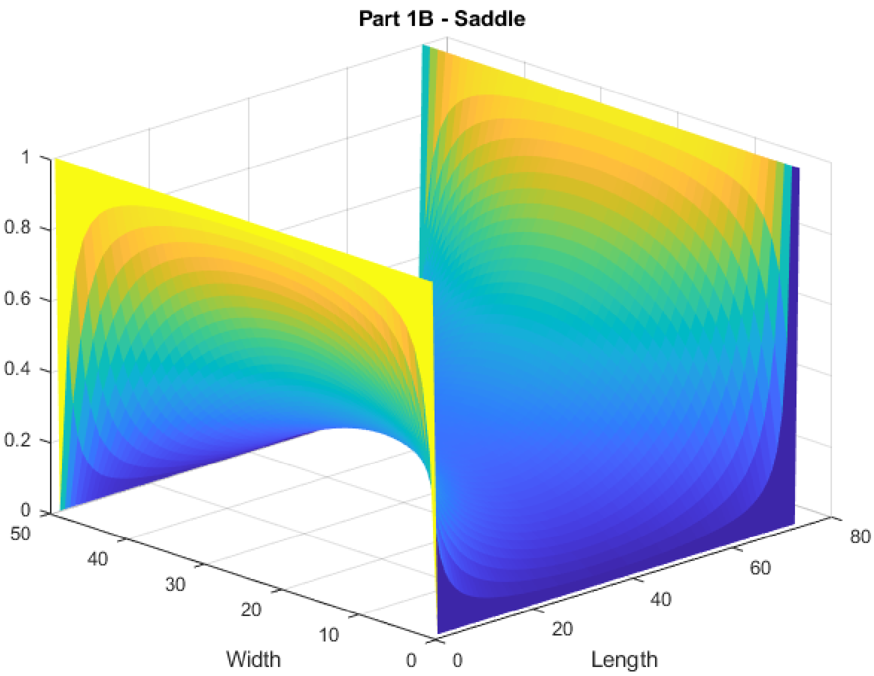

Figure 5: 3D Voltage Graph

2.2.2 Analytical Method

The analytical solution uses the following series equation to solve for V(x,y).

$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \frac{cosh(\frac{n\pi x}{a})}{cosh(\frac{n\pi b}{a})} sin(\frac{n\pi y}{a}) \tag{5}$$

Using this equation the following 3D plot was produced after 100 summations.

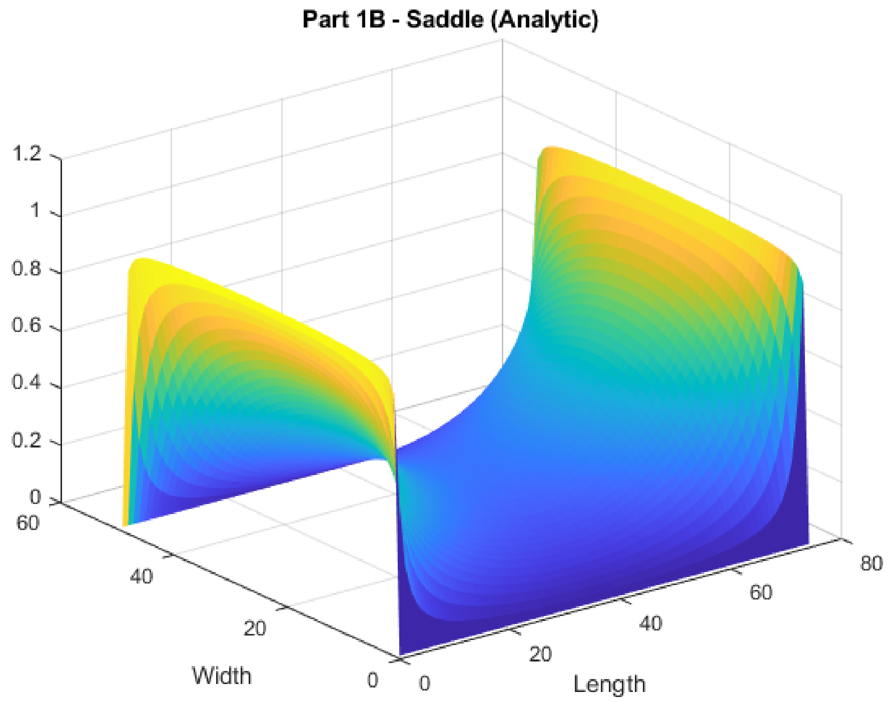

Figure 6: 3D Voltage Graph

2.2.3 Comparison

The following plot is a difference comparison between the finite difference solution and the analytical solution when L = 75 and W = 50 (mesh size).

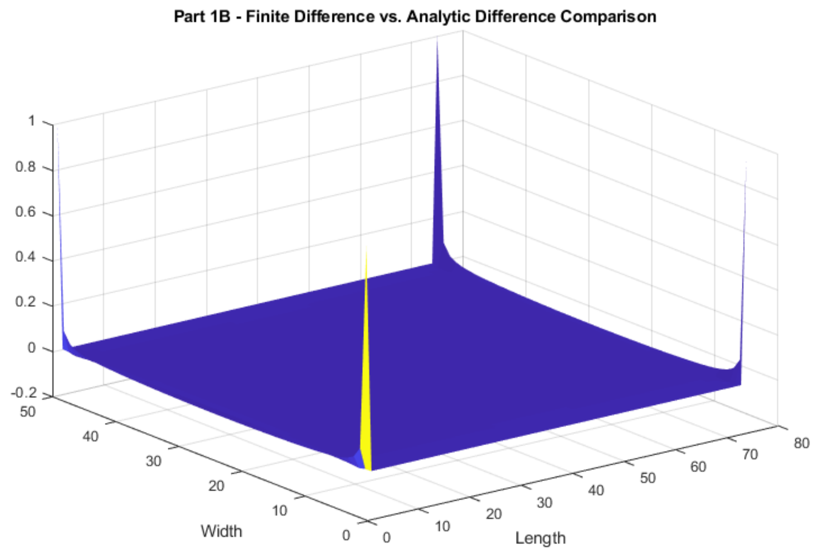

Figure 7: Finite Difference - Analytic Voltage Graph

The peaks at the corners are the spots where the finite difference solution differs from the analytical solution. I noticed that these peaks shrink significantly as the mesh density (dependent on L and W values) is increased suggesting that a greater mesh density provides a more accurate solution.

The advantage of using finite difference over the analytical solution are that there is much less complex math involved to find the solution as the finite difference method is easily applied. The disadvantage is that the finite difference solution is less accurate than the analytical solution.

3 Part 2 - Bottleneck Case

This part of the assignment uses the finite difference method from part 1 to solve for the current flow in the rectangular region shown below:

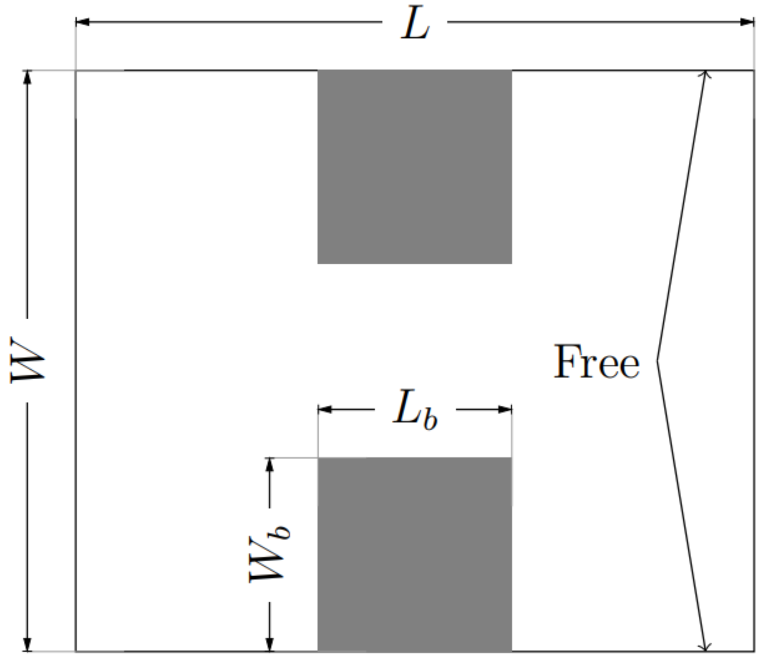

Figure 8: Part 2 - Rectangular Region [1]

The boxes in this figure are modelled using Laplace's equation and a map of conductivity's as follows:

$$\Delta(\sigma_{x,y}\Delta V) = 0 \tag{6}$$

This modifies the finite difference equation in Part A to be the following:

$$V_{m,n} = \frac{V_{m+1,n} + V_{m-1,n} + V_{m,n+1} + V_{m,m-1}}{rxm + rxp + rym + ryp} \tag{7}$$

Where rxm, ryp, rym and ryp represent the conductivity's in the nodes surrounding the node being calculated.

3.1 Part A - Current flow and Associated Plots

Provided that the length and width values are 75 and 50 respectively, the conductivity inside the boxes is 10^{-2}, the conductivity outside the boxes is 1, the length of the box is 37.5, and the width between the boxes is 10, the current flow between the two contracts (Left and Right barriers) is 0.2675 amps.

The following plots depict the conductivity $\sigma(x, y)$, voltage V(x,y), electric field E(x,y), and current density J(x,y) given the parameters listed above.

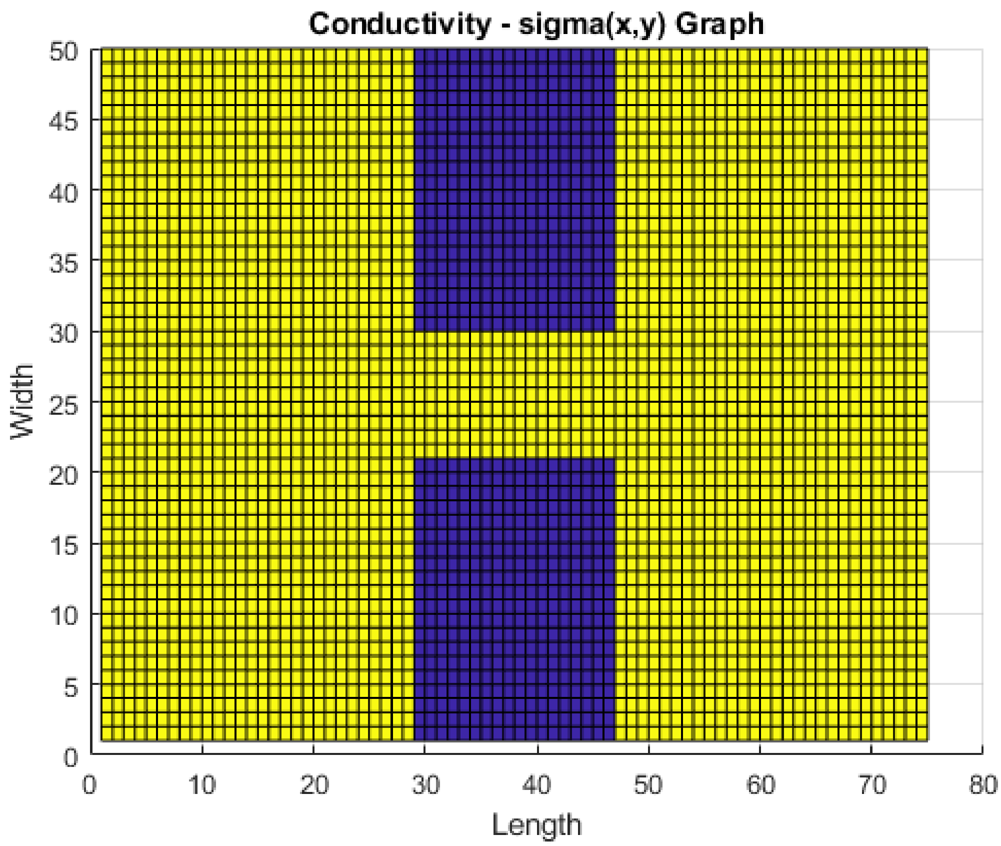

Figure 9: Part 2 - Conductivity Map

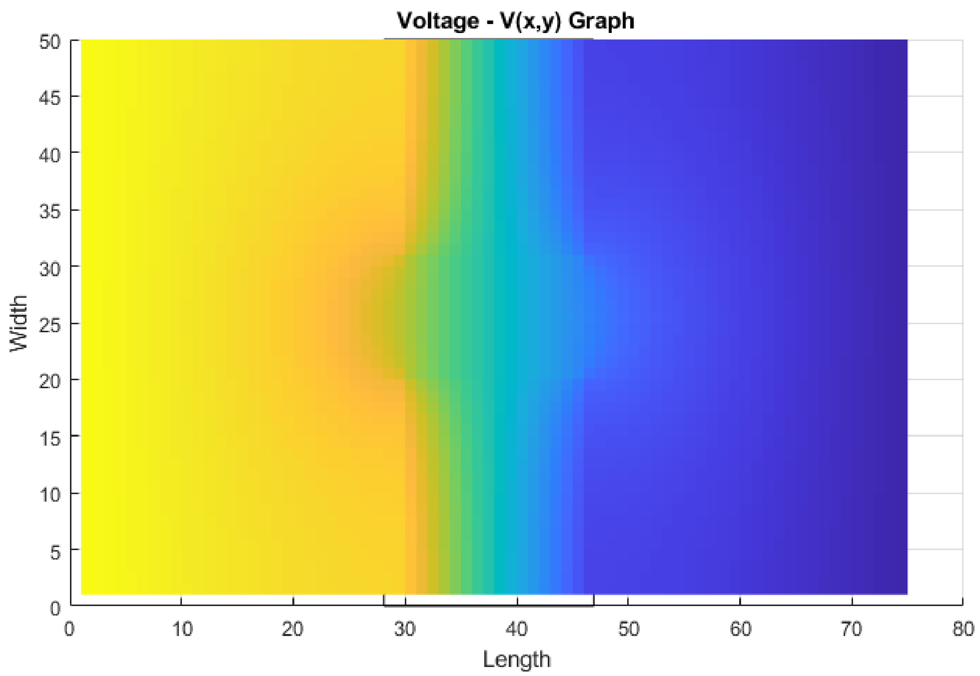

Figure 10: Part 2 - 2D Voltage Graph

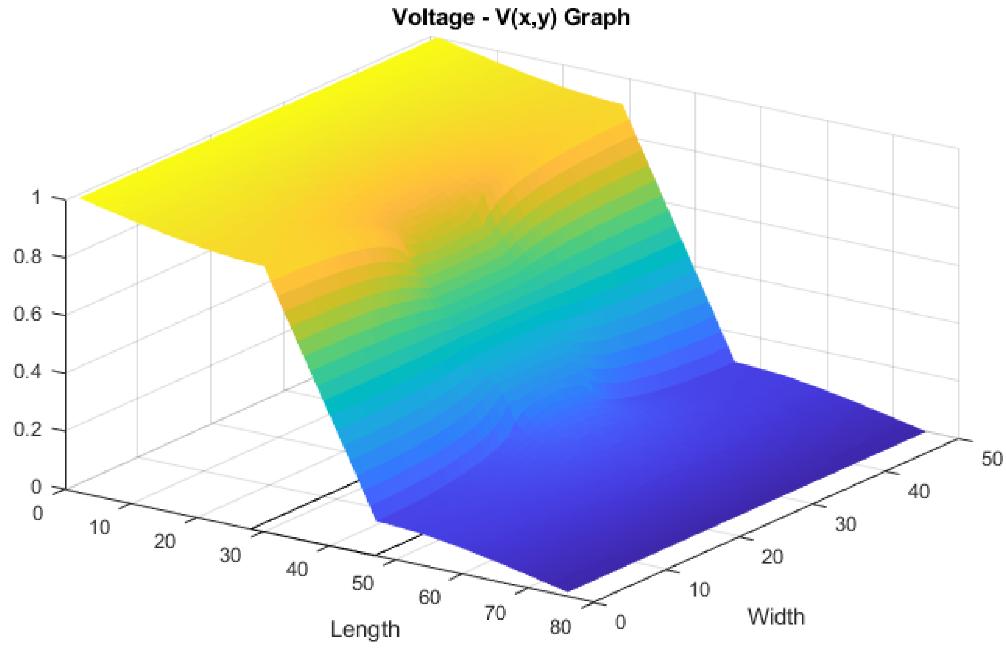

Figure 11: Part 2 - 3D Voltage Graph

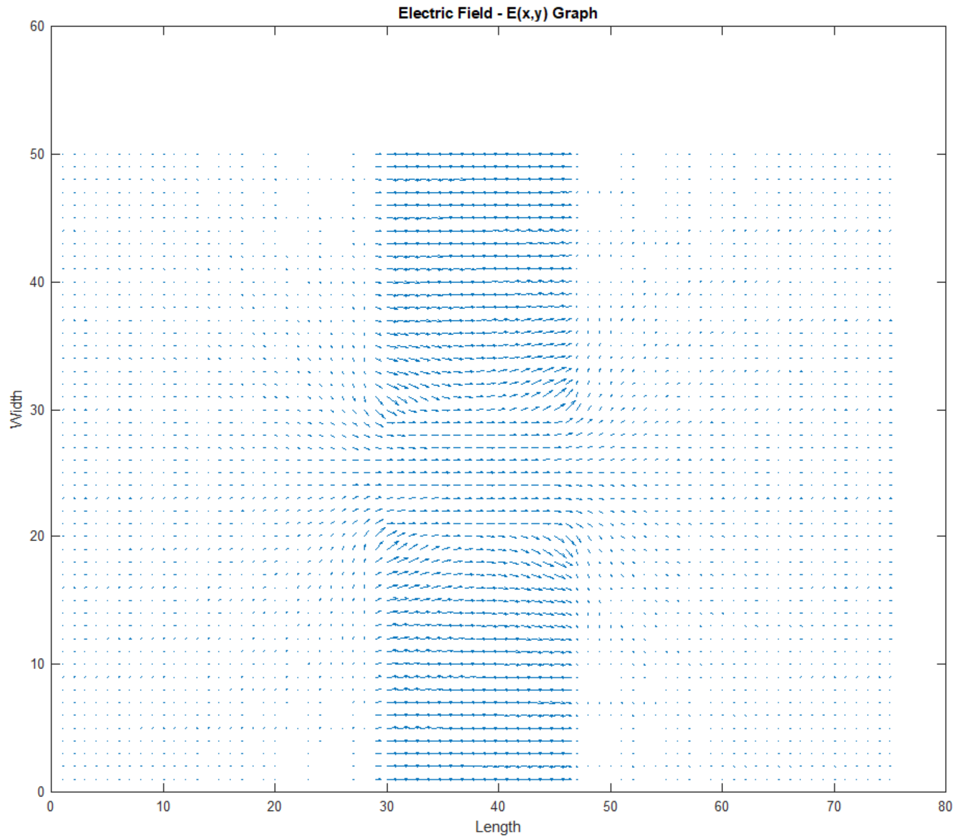

Figure 12: Part 2 - Electric Field Quiver Graph

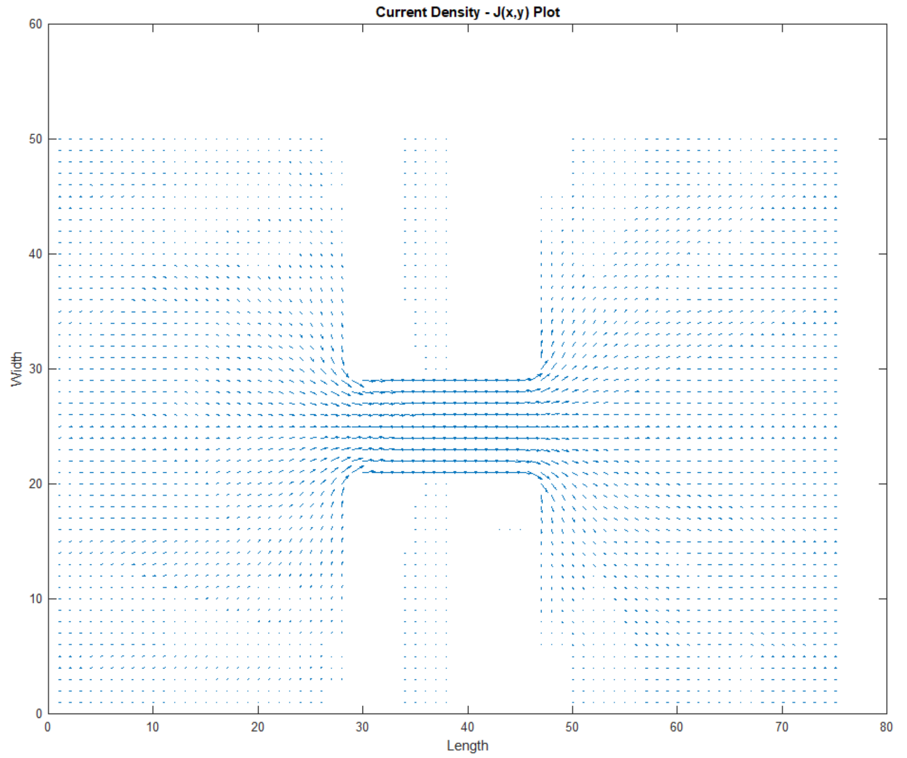

Figure 13: Part 2 - Current Density Quiver Graph

3.2 Part B - Mesh Density

As the mesh density is increased by increasing the length and width of the rectangular region, the current also increases as shown in the figure below:

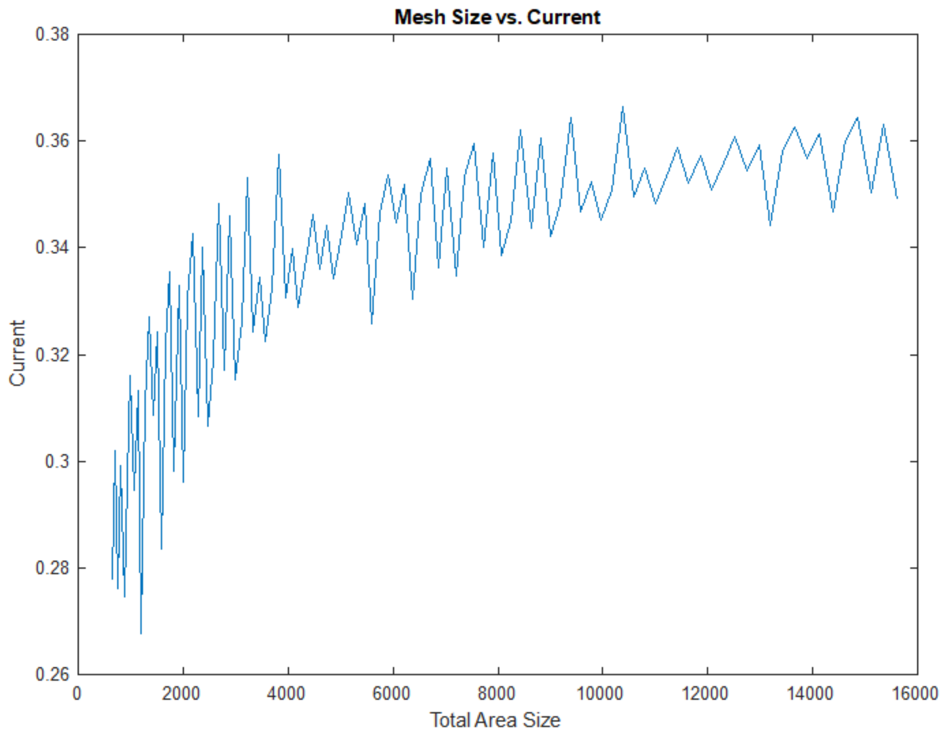

Figure 14: Part 2 - Current vs. Mesh Density

From the figure above it can be seen that the current initially increases as the mesh density increases and eventually levels off around 0.35 amps. This indicates that as the mesh density increases the accuracy also increases because the current converges on a final value.

3.3 Part C - Narrowing the Bottleneck

By narrowing the bottleneck and therefore decreasing resistance between the two contacts the current increases due to Ohm's law shown in the equation below:

$$V = IR => I = \frac{V}{R} \tag{8}$$

This can be seen in the graph below that plots the current vs the Bottleneck Gap.

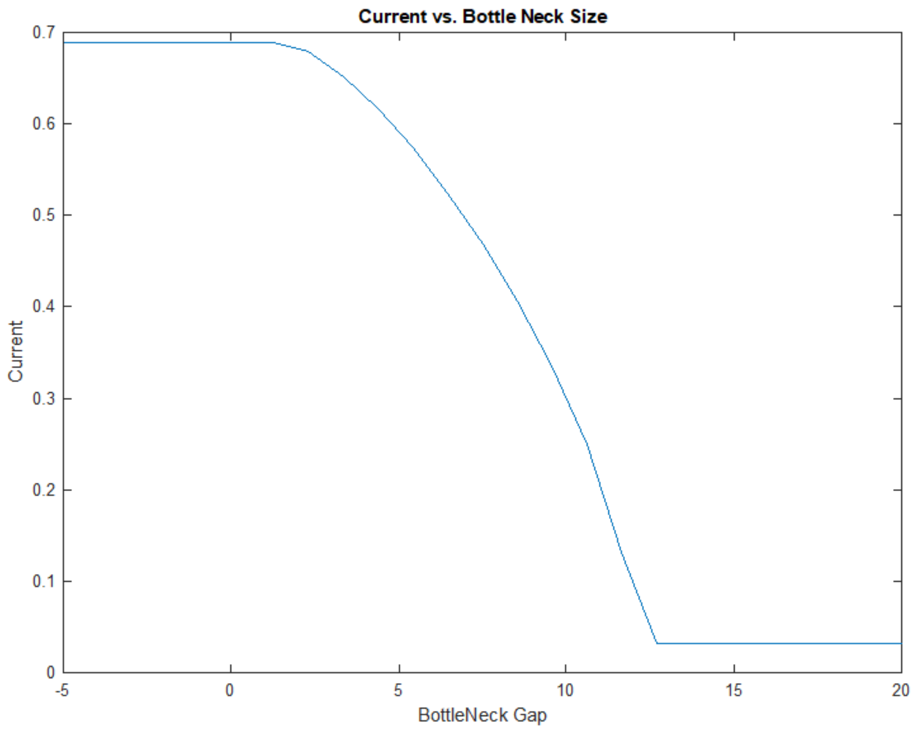

Figure 15: Part 2 - Current vs. Bottleneck

3.4 Part D - Varying the Conductivity of the Box

By varying the conductivity of the boxes this increases the current since increasing conductivity decreases resistance and therefore increases current as shown below:

$$R = \frac{L}{\sigma A} \tag{9}$$

$$I = \frac{V\sigma A}{L} \tag{10}$$

This can be seen in the graph below that plots the current vs. various connectivity's.

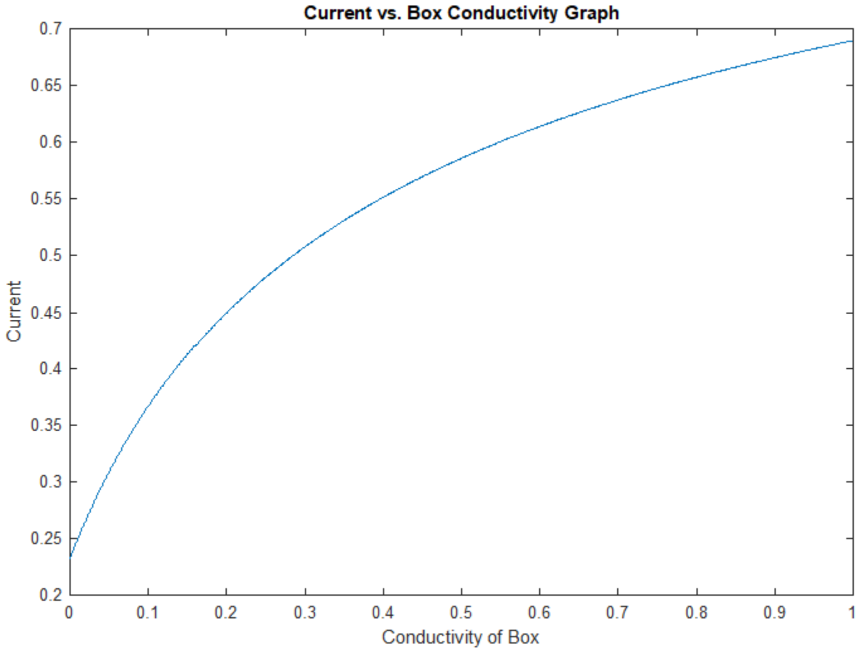

Figure 16: Part 2 - Current vs. Conductivity

References

[1] T. Smy, "ELEC 4700 Assignment - 2 Finite Difference Method." Feb 26, 2021

[2] T. Smy, "Conduction, Classical Solid, electrons, drift, resistance, hall effect, non-metals" Feb 26, 2021