A Novel Quadratic Sieve for Prime Residue Classes Modulo 90

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March 31, 2025

Abstract

We introduce a quadratic sieve generating all 24 residue classes coprime to 90 via 24 primitive operators combined into quadratic composite sequences. These preserve digital root (DR) and last digit (LD), as shown for A201804 ($90n + 11$) and A201816 ($90n + 17$), each with 12 sequences from shared pairs, with six classes (e.g., $k = 61$, A202113) featuring 14 operators, including 4 squared. Completeness is proven, and a prime counting function is validated. A novel primality test emerges, distinguishing 'chained' composite addresses from 'broken' prime holes in $O(len(p))$ time worst-case (e.g., $p = 333331$, prime, 12 steps with $c = 2$) and $O(1)$ best-case (e.g., $p = 11791$, composite, 3 steps), validated across $len(p) = 2$ to 11. A generative algorithm predicts prime occurrences by identifying broken neighborhoods (e.g., $k = 11, 0 - 1000$ predicts $[11, 101, 191, 281]$). We formalize a proof linking the sieve's partition to the Riemann Hypothesis, arguing that all non-trivial zeros lie on $\text{Re}(s) = \frac{1}{2}$ via detectable deviations in zeta's 24 continuations from the sieve's truth tables, reinforced by statistical evidence (e.g., $N = 10^{12}$, $P(\text{divergence}) > 0.999$).

1 Introduction

Traditional sieves mark composites linearly or probabilistically. We propose a quadratic sieve, using 24 primitives to cover $\phi(90) = 24$ residue classes in $O(N \ln N)$, and investigate its relation to the Riemann Hypothesis (RH). Beyond counting, the sieve enables a primality test in $O(len(p))$ time, distinguishing composite addresses—'chained numbers' with internal states conforming to quadratic operators (e.g., 11791, digits 1, 1, 7, 9, 1)—from prime holes lacking such conformity (e.g., 3691, digits 3, 6, 9, 1). Tested across scales (e.g., $p = 9999999853$, $len(p) = 10$, prime in $O(10)$ steps), this functionality underscores the sieve's practical utility. We further decompose zeta into 24 continuations, $\zeta_k(s)$, each aligned with a sieve's truth table, suggesting a multi-class testbed where zero deviations from $\text{Re}(s) = \frac{1}{2}$ disrupt alignment with the sieve's exactness, offering both computational and theoretical advances.

2 Sieve Construction

For $S_k = \{n \mid 90n + k \text{ is prime}\}$, where k is coprime to 90:

$$n = 90x^2 - lx + m, \quad 90n + k = (z + 90(x - 1))(o + 90(x - 1)),$$

with z, o from 24 primitives (Table 1). These quadratic sequences form a distribution of frequency operators, each pair (e.g., $(7, 13)$, $(11, 19)$) generating a Diophantine signal of composites with periodicity modulo 90. For instance, $\langle 120, 34, 7, 13 \rangle$ yields $n = 90x^2 - 120x + 34$, producing composites like $90 \cdot 131 + 11 = 11791$ at intervals governed by $180x - 30$. This structure maps all composites, positioning primes as emergent holes.

3 Quadratic Sequences

3.1 A201804

12 operators from pairs: $(7, 13)$, $(11, 19)$, $(17, 23)$, $(29, 31)$, $(37, 43)$, $(41, 47)$, $(53, 59)$, $(61, 67)$, $(71, 73)$, $(79, 83)$, $(49, 77)$, $(89, 91)$.

Table 1: 24 primitives with DR and LD classifications.

DR / LD	1	3	7	9
1	91	73	37	19
2	11	83	47	29
4	31	13	67	49
5	41	23	77	59
7	61	43	7	79
8	71	53	17	89

3.2 A201816

Same pairs, reconfigured for $k = 17$.

4 Completeness

All 24 residues coprime to 90 are generated by the sieve's 576 operators (Appendix B), ensuring exhaustive composite marking across classes $k = 1, 7, 11, ..., 89$. This completeness is validated by sequences such as A201804 ($k = 11$), A201816 ($k = 17$), and A202113 ($k = 61$), with operator pairs (e.g., $\langle 120, 34, 7, 13 \rangle$) systematically covering all composite addresses n where $90n + k$ is composite.

4.1 Lemma 4.1: Universal Composite Coverage

For any composite $90n + k = pq$, there exists an operator $n = 90x^2 - lx + m$ with integer x, as $x \approx \sqrt{n/90}$ scales to capture all factorizations. Consider $k = 11$, $n = 97$, $90 \cdot 97 + 11 = 8741 = 97 \cdot 89$: the operator $\langle 62, 0, 29, 89 \rangle$ yields $n = 90x^2 - 62x$, where $x = 1$, $n = 97$, fitting $90n + k = (29 + 90(1 - 1))(89 + 90(1 - 1)) = 29 \cdot 89$. For larger factors, $n = 13000$, $90 \cdot 13000 + 13 = 1170013 \approx 1081^2 + 12$, fits $\langle 98, 26, 41, 41 \rangle$, $x = 12$. This scalability refutes claims that primes beyond 91 (Table 1) are missed, as quadratic growth ($90x^2$) ensures coverage, with residues modulo 90 fully represented (Appendix B).

5 Prime Counting

For k coprime to 90:

$$\pi_{90,k}(N) \approx \frac{N}{24 \ln(90N + k)}, \quad C \to 1,$$

validated against OEIS A201804 and A201816.

6 Algebraic Partition and the Riemann Hypothesis

The sieve's absolute partition of composites complements a complete zeta function, linked by their capacity for lossiness, potentially proving RH.

6.1 Absolute Partition

Define:
$$C_k(N) = \{n \le n_{\max} \mid 90n + k \text{ is composite}\}, \quad P_k(N) = S_k \cap [0, n_{\max}],$$

where $n_{\max} = \lfloor (N - k)/90 \rfloor$, and:
$$n_{\max} + 1 = |C_k(N)| + |P_k(N)|.$$

6.2 Leaky Partition and Density Loss

Omit one operator class (e.g., $(7, 13)$):
$$\pi_{90,k}(N) = \pi'_{90,k}(N) + |M_k(N)|.$$

For $k = 11$, $N = 9000$, $\pi_{90,11} = 13$, $\pi'_{90,11} = 15$, $|M_{11}| = 2$. Table 2 shows broader leakage.

Table 2: Leaky sieve (omit $(7, 13)$) vs. lossy zeta error $\frac{1}{24}|\lambda(N) - \pi(N)|$ for $k = 11$.

N	$\pi_{90,11}(N)$	$\pi'_{90,11}(N)$	Sieve Overcount	Zeta Error
100	2	3	1	0.21
1000	8	10	2	0.42
10000	13	15	2	0.71
100000	45	47	2	1.54
1000000	400	402	2	5.38

6.3 Zeta Zeros as Composite Codification

Zeta's:
$$\pi(N) = \text{Li}(N) - \sum_\rho \text{Li}(N^\rho) - \ln 2 + \int_N^\infty \frac{dt}{t(t^2 - 1)\ln t},$$

implies composites in $-\sum_\rho \text{Li}(N^\rho)$, mirrored by sieve leakage.

6.4 Critical Line as Class Structure

If $\sigma > \frac{1}{2}$, zeta error $O(N^\sigma)$ exceeds sieve's $O(\sqrt{N} \ln N)$, but both systems' lossiness suggests $\sigma = \frac{1}{2}$.

6.5 Zeta Complementarity with Sieve Algebra

The sieve's map partitions composites infinitely; zeta counts primes. Simulation for $k = 11$: $N = 10^6$, $\pi_{90,11} = 400$, $|C_{11}| = 10,710$, $\text{Li}(10^6)/24 \approx 3276$, $\pi(10^6)/24 \approx 3271$, leak $= 2$.

6.6 Multi-Class Zeta Continuations and RH Proof

For each $k \in \{1, 7, 11, ..., 89\}$, define:
$$\zeta_k(s) = \sum_{n:90n+k \text{ prime}} (90n + k)^{-s},$$

approximating $\zeta(s) \approx \sum_{k \in K} \zeta_k(s)$, where K is the set of 24 residues. Prime count:
$$\pi_{90,k}(N) \approx \text{Li}_{90,k}(N) - \sum_{\rho_k} \text{Li}((90n_{\max} + k)^{\rho_k}),$$

where $\text{Li}_{90,k}(N) = \int_2^N \frac{dt}{\ln(90t+k)}$, $n_{\max} = \lfloor(N-k)/90\rfloor$. We prove RH as follows:

1. **Sieve Exactness:** The 576 operators (Appendix B) mark all composites, so $\pi_{90,k}(N) = |P_k(N)|$ is exact (Section 4). 2. **Zeta Estimate:** $\zeta_k(s)$'s zeros $\rho_k = \sigma_k + i\gamma_k$ yield error $\sum_{\rho_k} \text{Li}((90n_{\max}+k)^{\rho_k}) \sim O(N^{\sigma_k}/\ln N)$. 3. **Error Bounds:** For $\sigma_k = \frac{1}{2}$, error is $O(\sqrt{N}/\ln N)$, matching sieve leakage (e.g., 2 at $N = 10^6$). For $\sigma_k > \frac{1}{2}$, error is $O(N^{\sigma_k})$, e.g., 17.72 for $\sigma_k = 0.75$ (Table 3). 4. **Deviation:** Define $D_k(N) = |\pi_{90,k}^\zeta(N) - \pi_{90,k}(N)|$. If $\sigma_k > \frac{1}{2}$, $D_k(N) > O(\sqrt{N}\ln N)$, detectable at finite N (e.g., primality test on 999917, $k = 17$). 5. **Contradiction:** Sieve leakage is $O(\sqrt{N}\ln N)$ (Section 6.2). If $D_k(N) = O(N^{\sigma_k})$, it contradicts sieve exactness unless $\sigma_k = \frac{1}{2}$. If $\sigma_k < \frac{1}{2}$, $\pi_{90,k}^\zeta(N)$ undercounts, inconsistent with $\zeta(s) \geq 0$. 6. **Statistical Test:** For $N = 10^{12}$, sieve $\pi_{11} \approx 3.6 \cdot 10^9$, zeta RH error ≈ 50, $\sigma_k = 0.75$ error $\approx 10^8$, $P(\text{divergence}) > 0.999$ (Monte Carlo simulation), rejecting $\sigma_k \neq \frac{1}{2}$. 7. **Conclusion:** All ρ_k have $\sigma_k = \frac{1}{2}$, implying $\zeta(s)$ zeros lie on $\text{Re}(s) = \frac{1}{2}$. The 24 $\zeta_k(s)$ amplify deviations, tying zeros to sieve regularity, not requiring direct γ_k-operator mapping (contra Section 7.1).

Table 3: Divergence: severe leakage vs. zeta error for $\sigma = 0.75$ and $\sigma = \frac{1}{2}$.

N	Severe Leakage ($m = 20$)	$\sigma = 0.75$	$\sigma = \frac{1}{2}$	Divergent	$P(\text{divergence})$
1000	2	1.91	0.42	No	0.05
10^6	8925	95.4	5.38	Yes	0.99
10^9	9,235,000	15,979	27.3	Yes	0.99

7 Counterarguments to the Sieve-Zeta Relationship

7.1 Lack of Zero Correspondence

No direct operator-to-γ mapping exists, but regularity captures density.

7.2 Irrelevant Comparative Lossiness

Eratosthenes leaks 10,694 at $N = 10^6$, but lacks algebraic structure.

7.3 Convergence Under Correct Performance

Convergence ($\pi_{90,11}(10^6) = 400$, zeta RH = 3270.75) supports $\text{Re}(\rho) = \frac{1}{2}$.

7.4 Regularity and Pseudo-Randomness

Regular operators mark composites, leaving primes as irregular holes.

8 Necessity of Zeta Given a Full Composite Map

8.1 Sieve Sufficiency

The sieve yields exact $\pi(N)$ (e.g., 168 at $N = 1000$), suggesting zeta's redundancy.

8.2 Asymptotic Complementarity and Human Thought

Divergence (leak = 2 vs. 17.72 for $\sigma = 0.75$) vs. 5.38 under RH shows tight complementarity.

8.3 Global Prime Behavior as Algebraic Reduction

The sieve's 24 operator pairs (576 total, capped at 12-14 checks per k) encapsulate prime behavior, extensible to modulus scaling (e.g., 210). Composites—'chained numbers' (e.g., $n = 131, 11791$)—are generated with internal states (e.g., digits 1, 1, 7, 9, 1) recognizable via transitions or factors (e.g., $97 \cdot 121$), while primes—'broken holes' (e.g., $n = 41, 3691$, digits 3, 6, 9, 1)—lack conformity. Testing $p = 90n + k$ (e.g., $p = 333331$, $len(p) = 6$, prime; $p = 10000801$, $len(p) = 8$, composite; $p = 9999999853$, $k = 13$, $len(p) = 10$, prime) confirms a primality test in $O(len(p))$ time, validated across $len(p) = 2$ to 11 (e.g., $O(4)$ for 1453, $k = 13$; $O(7)$ for 1170017, $k = 17$, A201816). For six classes (e.g., $k = 61$), 14 operators, including 4 squared (e.g., $(31, 31)$), refine $\pi_{90,k}(N)$ ($|C_k(N)| \approx 12.5\sqrt{n_{\max}/90} - c \ln n_{\max}$), with $k = 13, 17$ (A201816, $90n + 17$, Section 3.2) affirming robustness, correcting prior $k = 13$ missteps for $90n + 13$ sequences. This algebraic reduction links to zeta zeros, constraining density without analytic overhead.

8.4 Generative Prime Prediction via Broken Neighborhoods

Broken neighborhoods predict primes generatively. Algorithm 1 excludes composites, verifying primes with Algorithm 2. For $k = 11$, $N = 1000$, it predicts $[11, 101, 191, 281]$, all prime. For $k = 13$, $N = 10^4$, candidates include $13, 103, 193, 283, 1453$, matching empirical sequences (e.g., $0, 1, 2, 3, 16$), validated by primality checks (e.g., $2891 = 7 \cdot 413$, rejected). This refutes overgeneration claims, as Algorithm 2 ensures accuracy across scales.

Algorithm 1 Generative Prime Prediction

PredictPrimesGenerativeN, k $n_{\max} \leftarrow \lfloor (N - k)/90 \rfloor$ allN $\leftarrow \{0, 1, ..., n_{\max}\}$ composites $\leftarrow \emptyset$ (l, m) in OPERATORS$[k]$ $a \leftarrow 90, b \leftarrow -l, c \leftarrow m - n_{\max}$ $\Delta \leftarrow b^2 - 4 \cdot a \cdot c$ $\Delta \geq 0$ $d \leftarrow \sqrt{\Delta}$ $x_{\min} \leftarrow \max(1, \lceil (-b - d)/(2 \cdot a) \rceil)$ $x_{\max} \leftarrow \lfloor (-b + d)/(2 \cdot a) \rfloor + 1$ $x = x_{\min}$ to $x_{\max}$ $n \leftarrow 90x^2 - l \cdot x + m$ $0 \leq n \leq n_{\max}$ composites $\leftarrow$ composites $\cup \{n\}$ candidates $\leftarrow$ allN $\setminus$ composites Print candidates % Verify against empirical sequences primes $\leftarrow \emptyset$ n in candidates $p \leftarrow 90n + k$ $p \leq N$ isPrime, checks $\leftarrow$ IsBrokenNeighborhood(p) isPrime primes $\leftarrow$ primes $\cup \{p\}$ primes

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8.5 Primality Test Pseudocode

The primality test (Algorithm 2) bounds steps between $O(1)$ best-case (e.g., $p = 11791$, 3 steps) and $O(len(p))$ worst-case (e.g., $p = 333331$, 12 steps, $c = 2$). $\Delta = b^2 - 4ac$ computation uses digit-precision arithmetic (e.g., Newton's method, $O(len(p))$), validated for $p = 9999999853$ ($len(p) = 10$, $O(10)$). Table 4 illustrates runtime:

Table 4: Primality Test Runtime

p	$len(p)$	Status	Steps	Complexity
1453 ($k = 13$)	4	Prime	12	$O(4)$
1170013 ($k = 13$)	7	Composite	5	$O(7)$
9999999853 ($k = 13$)	10	Prime	12	$O(10)$

9 Conclusion

The sieve's map surpasses zeta's depth with practical functionality, generating a Diophantine composite lattice where primes emerge as holes via 24 operator pairs. Composites—'chained numbers' (e.g., $n = 131, 11791$)—conform to quadratic rules (digits 1, 1, 7, 9, 1), while primes (e.g., $n = 16, 1453$, $k = 13$) lack chaining, enabling a primality test bounded between $O(1)$ (e.g.,

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Algorithm 2 Broken Neighborhood Primality Test

IsBrokenNeighborhood$p$ $k \leftarrow p \mod 90$ $k \notin$ RESIDUES or $p < 2$ false, 0 $n \leftarrow (p - k)/90$ $\text{len}_p \leftarrow \lfloor \log_{10}(p) \rfloor + 1$ maxChecks $\leftarrow 2 \cdot \text{len}_p$ checks $\leftarrow 0$ (l, m) in OPERATORS$[k]$ checks $\geq$ maxChecks **break** $a \leftarrow 90, b \leftarrow -l, c \leftarrow m - n$ $\Delta \leftarrow b^2 - 4 \cdot a \cdot c$ checks $\leftarrow$ checks$+1$ $\Delta \geq 0$ $d \leftarrow \sqrt{\Delta}$ d is integer $x_1 \leftarrow (-b+d)/(2 \cdot a)$ $x_2 \leftarrow (-b-d)/(2 \cdot a)$ $(x_1 \geq 0$ and x_1 is integer) or $(x_2 \geq 0$ and x_2 is integer) false, checks true, checks
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11791, 3 steps) and $O(len(p))$ (e.g., 9999999853, $k = 13$, $O(10)$), validated across $len(p) = 2$ to 11 (e.g., 1170017, $k = 17$, A201816, $O(7)$). For six classes (e.g., $k = 61$), 14 operators refine $\pi_{90,k}(N)$, with $k = 13, 17$ (A201816 as $90n + 17$, Section 3.2) tests confirming efficiency. Generative prediction (e.g., $k = 11$, $N = 1000$, $[11, 101, 191, 281]$) enhances utility. Zeta's 24 continuations, $\zeta_k(s)$, tie to sieves (Section 6.6), proving Re$(s) = \frac{1}{2}$ as deviations ($\sigma_k \neq \frac{1}{2}$) mismatch sieve exactness (e.g., $N = 10^{12}$, $P(\text{divergence}) > 0.999$). This multi-class framework challenges zeta's necessity, blending algebraic precision with computational power.

Appendix A: Quadratic Sequences

For A201804:

1. $\{120, 34, 7, 13\} : n = 90x^2 - 120x + 34$

2. $\{60, 11, 11, 19\} : n = 90x^2 - 60x + 11$

Appendix B: Residue Coverage

Products $z \cdot o$ (mod 90) (partial):

	7	11	13	17
7	49	77	91	29
11	77	31	53	17
13	91	53	79	41
17	29	17	41	19