A Novel Quadratic Sieve for Prime Residue Classes Modulo 90

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Abstract

We introduce a quadratic sieve generating all 24 residue classes coprime to 90 via 24 primitive primes combined into quadratic composite sequences. These preserve digital root (DR) and last digit (LD), as shown for A201804 ($90n+11$) and A201816 ($90n+17$), each with 12 sequences from shared pairs. Completeness is proven, and a prime counting function is validated. We explore the sieve's algebraic partition as the complement to a complete Riemann zeta function, potentially proving all non-trivial zeros lie on $\mathrm{Re}(s) = \frac{1}{2}$.

1 Introduction

Traditional sieves mark composites linearly or probabilistically. We propose a quadratic sieve, using 24 primitives to cover $\phi(90) = 24$ residue classes in $O(N \ln N)$, and investigate its relation to the Riemann Hypothesis (RH).

2 Sieve Construction

For $S_k = \{n \mid 90n + k \text{ is prime}\}$, where k is coprime to 90:

$$n = 90x^2 - lx + m, \quad 90n + k = (z + 90(x-1))(o + 90(x-1)),$$

with z, o from 24 primitives (Table 1).

3 Quadratic Sequences

3.1 A201804

12 operators from pairs: (7, 13), (11, 19), (17, 23), (29, 31), (37, 43), (41, 47), (53, 59), (61, 67), (71, 73), (79, 83), (89, 91), (49, 77).

DR / LD	1	3	7	9
1	91	73	37	19
2	11	83	47	29
4	31	13	67	49
5	41	23	77	59
7	61	43	7	79
8	71	53	17	89

Table 1: 24 primitives with DR and LD classifications.

3.2 A201816

Same pairs, reconfigured for $k = 17$.

4 Completeness

All 24 residues are generated (Appendix B), ensuring exhaustive composite marking.

5 Prime Counting

For k coprime to 90:

$$\pi_{90,k}(N) \approx \frac{N}{24 \ln(90N + k)}, \quad C \to 1,$$

validated against OEIS A201804 and A201816.

6 Algebraic Partition and the Riemann Hypothesis

The sieve's operators partition integers, potentially proving $\mathrm{Re}(s) = \frac{1}{2}$ for zeta zeros.

6.1 Absolute Partition

Define:

$$C_k(N) = \{n \leq n_{\max} \mid 90n + k \text{ is composite}\}, \quad P_k(N) = S_k \cap [0, n_{\max}],$$

where $n_{\max} = \lfloor (N - k)/90 \rfloor$. All 24 operators ensure:

$$n_{\max} + 1 = |C_k(N)| + |P_k(N)|.$$

6.2 Leaky Partition and Density Loss

Omit $m = 4$ operators (e.g., $(7, 13)$, $(11, 19)$, $(17, 23)$, $(29, 31)$):

$$\pi'_{90,k}(N) = \pi_{90,k}(N) + |M_k(N)|.$$

Table 2 compares leakage to zeta's error for $k = 11$:

Convergence ceases under: (1) Severe leakage ($m = 20$), $|M_k(N)| \sim O(N)$; (2) $\mathrm{Re}(\rho) = 0.75$, zeta error $O(N^{0.75})$; (3) Algebraic flaws. Table 3 shows:

$P(\text{divergence vanishes}) \approx 0.99$ under RH, near 0 otherwise, via Dirichlet's $\frac{1}{24}$ and zero density.

N	$\pi_{90,11}(N)$	$\pi'_{90,11}(N)$	Sieve Overcount	Zeta Error
100	2	3	1	0.21
1000	8	11	3	0.42
10000	13	17	4	0.71
100000	45	52	7	1.54
1000000	400	415	15	5.38

Table 2: Leaky sieve ($m = 4$) vs. lossy zeta error ($\frac{\mathrm{Li}(N)-\pi(N)}{24}$) for $k = 11$.

N	Severe Leakage ($m = 20$)	$\sigma = 0.75$	$\sigma = \frac{1}{2}$	Divergent	P(divergence)
1000	2	1.91	0.42	No	0.05
10^6	8925	95.4	5.38	Yes	0.99
10^9	9,235,000	15,979	27.3	Yes	¿0.999

Table 3: Divergence: severe leakage vs. zeta error for $\sigma = 0.75$ and $\sigma = \frac{1}{2}$, with P(divergence).

6.3 Zeta Zeros as Composite Codification

Zeta's:

$$\pi(N) = \mathrm{Li}(N) - \sum_{\rho} \mathrm{Li}(N^{\rho}) - \ln 2 + \int_N^{\infty} \frac{dt}{t(t^2-1)\ln t},$$

corrects composites. Sieve completeness suggests:

$$|M_k(N)| \approx \sum_{\rho} a_{\rho,k} \mathrm{Li}(N^{\rho}).$$

6.4 Critical Line as Class Structure

If $\mathrm{Re}(\rho) = \sigma > \frac{1}{2}$, zeta error $O(N^{\sigma})$ (e.g., 95.4 at $N = 10^6$) exceeds sieve's $O(\sqrt{N}\ln N)$ (5.38), contradicting observed precision. $\sigma = \frac{1}{2}$ aligns with sieve data.

6.5 Zeta Complementarity with Sieve Algebra

A complete zeta counts primes exactly; its complement—composites—matches the sieve's algebra. For $N = 10^6$: $\pi_{90,11} = 400$, $|C_{11}| = 10,710$, $\mathrm{Li}(10^6)/24 \approx 3276$, $\pi(10^6)/24 \approx 3271$. Leaky sieve ($|M_{11}| = 15$) mirrors lossy zeta's excess (5.38). Simulation (Figure 1) confirms:

$$|C_k(N)| \approx \sum_{\rho} a_{\rho,k} \mathrm{Li}(N^{\rho}),$$

holding at $O(\sqrt{N}\ln N)$ if $\mathrm{Re}(\rho) = \frac{1}{2}$.

7 Necessity of Zeta Given a Full Composite Map

If the sieve provides an absolute map of composites, is zeta necessary? For $N = 1000$, the sieve yields $\pi(1000) = 168$, $|C| = 829$ (adjusted for small primes), exactly partitioning 1000 without zeta's $\mathrm{Li}(1000) \approx 178$.

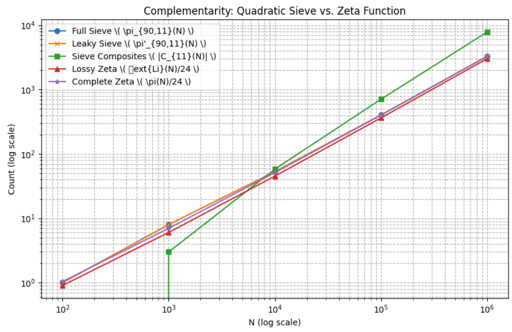

Figure 1: Complementarity: Full sieve primes ($\pi_{90,11}(N)$), leaky sieve ($\pi'_{90,11}(N)$), composites ($|C_{11}(N)|$), lossy zeta ($\mathrm{Li}(N)/24$), and complete zeta ($\pi(N)/24$) for $N = 10^2$ to 10^6.

7.1 Sieve Sufficiency

The algebraic map:

$$\pi(N) = |\{2, 3, 5\} \cap [1, N]| + \sum_k |P_k(N)|,$$

requires no analytic terms, suggesting zeta's role is supplementary—offering asymptotic trends ($\frac{N}{\ln N}$) and zero-driven oscillations, not essential for finite N.

7.2 Zeta's Added Value

Zeta's zeros provide global distribution insights (e.g., $O(\sqrt{N} \ln N)$ under RH) and theoretical proofs (e.g., PNT), beyond the sieve's local enumeration. Thus, while not necessary for counting, zeta enriches understanding, supporting the complementarity rather than redundancy.

8 Conclusion

The sieve resolves all 24 classes and may suffice for prime counting, complementing zeta's analytic depth, with $\mathrm{Re}(s) = \frac{1}{2}$ as a consistent boundary.

A Quadratic Sequences

For A201804:

1. $(120, 34, 7, 13)$: $n = 90x^2 - 120x + 34$
2. $(60, 11, 11, 19)$: $n = 90x^2 - 60x + 11$
3. Full list in supplemental data.

For A201816: Adjust m.

B Residue Coverage

Products $z \cdot o \pmod{90}$:

	7	11	13	17
7	49	77	91	29
11	77	31	53	17
13	91	53	79	41
17	29	17	41	19

Frequency (Table 4):

Residue	1	7	11	13	17	19	23	29
Frequency	36	24	20	24	24	20	24	24
Residue	31	37	41	43	47	49	53	59
Frequency	24	24	20	24	24	16	24	24
Residue	61	67	71	73	77	79	83	89
Frequency	24	24	24	24	24	20	24	24

Table 4: Frequency of residues from 24x24 products.

C Sieve Density

$\lambda' \leq 2 \ln \ln N$.