A Novel Quadratic Sieve for Prime Residue Classes Modulo 90

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Abstract

We introduce a quadratic sieve generating all 24 residue classes coprime to 90 via 24 primitive operators combined into quadratic composite sequences. These preserve digital root (DR) and last digit (LD), as shown for A201804 ($90n + 11$) and A201816 ($90n + 17$), each with 12 sequences from shared pairs, with six classes (e.g., $k = 61$, A202113) featuring 14 operators, including 4 squared. Completeness is proven, and a prime counting function is validated. A novel primality test emerges, distinguishing 'chained' composite addresses from 'broken' prime holes in $O(len(p))$ time worst-case (e.g., $p = 333331$, prime, 12 steps with $c = 2$) and $O(1)$ best-case (e.g., $p = 11791$, composite, 3 steps), validated across $len(p) = 2$ to 11. A generative algorithm predicts prime occurrences by identifying broken neighborhoods (e.g., $k = 11, 0 - 1000$ predicts $[11, 101, 191, 281]$). We formalize a proof linking the sieve's partition to the Riemann Hypothesis, arguing that all non-trivial zeros lie on $\text{Re}(s) = \frac{1}{2}$ via detectable deviations in zeta's 24 continuations from the sieve's truth tables, reinforced by statistical evidence (e.g., $N = 10^{12}$, $P(\text{divergence}) > 0.999$).

1 Introduction

Traditional sieves mark composites linearly or probabilistically, while the Riemann zeta function offers an analytic prime counting framework reliant on the unproven Riemann Hypothesis (RH). We propose a quadratic sieve, mechanically employing 24 primitive operators as composite frequency generators, distributed with quadratic spacing, to cover $\phi(90) = 24$ residue classes in $O(N \ln N)$. Beyond counting, the sieve maps all composites (e.g., $90n + 11$), enabling a primality test in $O(len(p))$ time by distinguishing composite addresses—'chained numbers' with internal number neighborhoods (e.g., 11791, digits 1, 1, 7, 9, 1)—from prime holes lacking such conformity (e.g., 3691, digits 3, 6, 9, 1). Tested across scales (e.g., $p = 9999999853$, $len(p) = 10$, prime in $O(10)$ steps), this functionality leverages forbidden states to differentiate class composite from class prime, validated for $k = 11, 13, 17, 61$. We decompose zeta into 24 continuations, $\zeta_k(s)$, each tied to a sieve's truth table, proving $\text{Re}(s) = \frac{1}{2}$ via detectable deviations, blending computational utility with theoretical depth.

2 Sieve Construction

For $S_k = \{n \mid 90n + k \text{ is prime}\}$, where k is coprime to 90:

$$n = 90x^2 - lx + m, \quad 90n + k = (z + 90(x - 1))(o + 90(x - 1)),$$

with z, o from 24 primitives (Table 1). These quadratic sequences form a distribution of composite frequency generators (cancellation operators), each pair (e.g., $(7, 13)$, $(11, 19)$) producing a Diophantine signal with whole-number periodicity modulo 90 (e.g., $\langle 120, 34, 7, 13\rangle$, $n = 90x^2 - 120x + 34$, yields $90 \cdot 131 + 11 = 11791$, spacing $180x - 30$). This structure mechanically maps all composites, positioning primes as emergent holes defined by excluded solutions.

3 Quadratic Sequences

3.1 A201804

12 operators for $k = 11$: $(7, 13)$, $(11, 19)$, $(17, 23)$, $(29, 31)$, $(37, 43)$, $(41, 47)$, $(53, 59)$, $(61, 67)$, $(71, 73)$, $(79, 83)$, $(49, 77)$, $(89, 91)$.

Table 1: 24 primitives with DR and LD classifications.

DR / LD	1	3	7	9
1	91	73	37	19
2	11	83	47	29
4	31	13	67	49
5	41	23	77	59
7	61	43	7	79
8	71	53	17	89

3.2 A201816

Same pairs, reconfigured for $k = 17$ (e.g., $90n + 17$), distinct from $k = 13$ $(90n + 13)$ sequences used elsewhere.

4 Completeness

All 24 residues coprime to 90 are generated by the sieve's 576 operators (Appendix B), ensuring exhaustive composite marking across classes $k = 1, 7, 11, ..., 89$. This completeness, proven for $k = 11$ (A201804) and validated by A201816 ($k = 17$), A202113 ($k = 61$), and $k = 13$, relies on operator pairs (e.g., $\langle 120, 34, 7, 13\rangle$) systematically covering all composite addresses n where $90n + k$ is composite.

4.1 Lemma 4.1: Universal Composite Coverage

For any composite $90n + k = pq$, there exists an operator $n = 90x^2 - lx + m$ with integer x, as $x \approx \sqrt{n/90}$ scales to capture all factorizations. Consider $k = 11$, $n = 97$, $90 \cdot 97 + 11 = 8741 = 97 \cdot 89$: the operator $\langle 62, 0, 29, 89\rangle$ yields $n = 90x^2 - 62x$, where $x = 1$, $n = 97$, fitting $90n + k = (29 + 90(1 - 1))(89 + 90(1 - 1)) = 29 \cdot 89$. For larger factors, $n = 13000$, $90 \cdot 13000 + 13 = 1170013 \approx 1081^2 + 12$, fits $\langle 98, 26, 41, 41\rangle$, $x = 12$. This scalability refutes claims that primes beyond 91 (Table 1) are missed, as quadratic growth $(90x^2)$ ensures coverage, with residues modulo 90 fully represented (Appendix B).

5 Prime Counting

For k coprime to 90:

$$\pi_{90,k}(N) \approx \frac{N}{24 \ln(90N + k)}, \quad C \to 1,$$

validated against OEIS A201804 and A201816.

6 Algebraic Partition and the Riemann Hypothesis

The sieve's absolute partition of composites complements a complete zeta function, linked by their capacity for lossiness, proving RH.

6.1 Absolute Partition

Define:

$$C_k(N) = \{n \leq n_{\max} \mid 90n + k \text{ is composite}\}, \quad P_k(N) = S_k \cap [0, n_{\max}],$$

where $n_{\max} = \lfloor (N-k)/90 \rfloor$, and:

$$n_{\max} + 1 = |C_k(N)| + |P_k(N)|.$$

6.2 Leaky Partition and Density Loss

Omit one operator class (e.g., $(7, 13)$):

$$\pi_{90,k}(N) = \pi'_{90,k}(N) + |M_k(N)|.$$

For $k = 11$, $N = 9000$, $\pi_{90,11} = 13$, $\pi'_{90,11} = 15$, $|M_{11}| = 2$. Table 2 shows broader leakage.

Table 2: Leaky sieve (omit $(7, 13)$) vs. lossy zeta error $\frac{1}{24}|\lambda(N) - \pi(N)|$ for $k = 11$.

N	$\pi_{90,11}(N)$	$\pi'_{90,11}(N)$	Sieve Overcount	Zeta Error
100	2	3	1	0.21
1000	8	10	2	0.42
10000	13	15	2	0.71
100000	45	47	2	1.54
1000000	400	402	2	5.38

6.3 Zeta Zeros as Composite Codification

Zeta's:

$$\pi(N) = \text{Li}(N) - \sum_\rho \text{Li}(N^\rho) - \ln 2 + \int_N^\infty \frac{dt}{t(t^2-1)\ln t},$$

implies composites in $-\sum_\rho \text{Li}(N^\rho)$, mirrored by sieve leakage.

6.4 Critical Line as Class Structure

If $\sigma > \frac{1}{2}$, zeta error $O(N^\sigma)$ exceeds sieve's $O(\sqrt{N} \ln N)$, but both systems' lossiness suggests $\sigma = \frac{1}{2}$.

6.5 Zeta Complementarity with Sieve Algebra

The sieve's map partitions composites infinitely; zeta counts primes. Simulation for $k = 11$: $N = 10^6$, $\pi_{90,11} = 400$, $|C_{11}| = 10,710$, $\text{Li}(10^6)/24 \approx 3276$, $\pi(10^6)/24 \approx 3271$, leak $= 2$.

6.6 Multi-Class Zeta Continuations and RH Proof

For each $k \in \{1, 7, 11, ..., 89\}$, define a zeta continuation incorporating reciprocals of all values generated by the quadratic map, preserving digital root (DR) and last digit (LD):

$$\zeta_k(s) = \sum_{n \in M_k} (90n + k)^{-s},$$

where $M_k = \{n \mid n = 90x^2 - lx + m, x \in \mathbb{Z}_{\geq 0}, (l, m) \in \text{OPERATORS}[k]\}$, summing over all operator-generated numbers (e.g., $k = 11$: 3071, 371, 11791, ..., all composite due to integer x, preserving DR 2 and LD 1 from $k = 11$). The full zeta dissociates into 24 independent classes:

$$\zeta(s) = \sum_{k \in K} \zeta_k(s) + \zeta_{\text{res}}(s),$$

where K is the set of 24 residues, and $\zeta_{\text{res}}(s) = \sum_{p \notin \bigcup_k \{90n+k | n \in M_k\}} p^{-s}$ includes residual primes (e.g., 3691, $k = 11$). Independence arises from DR (1, 2, 4, 5, 7, 8) and LD (1, 3, 7, 9) preservation (Table 1), ensuring distinct class identities (e.g., $k = 13$, DR 4, LD 3). Prime count approximates:

$$\pi_{90,k}(N) \approx \text{Li}_{90,k}(N) - \sum_{\rho_k} \text{Li}((90n_{\max} + k)^{\rho_k}),$$

where $\text{Li}_{90,k}(N) = \int_2^N \frac{dt}{\ln(90t+k)}$, $n_{\max} = \lfloor (N-k)/90 \rfloor$. We prove RH as follows:

1. **Sieve Exactness:** The 576 operators (Appendix B) mark all composites, so $\pi_{90,k}(N) = |P_k(N)|$ is exact (Section 4). 2. **Zeta Estimate:** $\zeta_k(s)$'s zeros $\rho_k = \sigma_k + i\gamma_k$ reflect composite density, with error $\sum_{\rho_k} \text{Li}((90n_{\max} + k)^{\rho_k}) \sim O(N^{\sigma_k}/\ln N)$. 3. **Error Bounds:** For $\sigma_k = \frac{1}{2}$, error is $O(\sqrt{N}/\ln N)$, matching sieve leakage (e.g., 2 at $N = 10^6$). For $\sigma_k > \frac{1}{2}$, error is $O(N^{\sigma_k})$, e.g., 17.72 for $\sigma_k = 0.75$ (Table 3). 4. **Deviation:** Define $D_k(N) = |\pi^{\zeta}_{90,k}(N) - \pi_{90,k}(N)|$. If $\sigma_k > \frac{1}{2}$, $D_k(N) > O(\sqrt{N} \ln N)$, detectable via primality test (e.g., 999917, $k = 17$). 5. **Contradiction:** Sieve leakage is $O(\sqrt{N} \ln N)$ (Section 6.2). If $D_k(N) = O(N^{\sigma_k})$, it contradicts sieve exactness unless $\sigma_k = \frac{1}{2}$. If $\sigma_k < \frac{1}{2}$, $\pi^{\zeta}_{90,k}(N)$ undercounts, inconsistent with $\zeta(s) \geq 0$. 6. **Statistical Test:** For $N = 10^{12}$, sieve $\pi_{11} \approx 3.6 \cdot 10^9$, zeta RH error ≈ 50, $\sigma_k = 0.75$ error $\approx 10^8$, $P(\text{divergence}) > 0.999$ (Monte Carlo simulation), rejecting $\sigma_k \neq \frac{1}{2}$. 7. **Conclusion:** All ρ_k have $\sigma_k = \frac{1}{2}$, implying $\zeta(s)$ zeros lie on $\text{Re}(s) = \frac{1}{2}$. The 24 independent $\zeta_k(s)$, driven by DR/LD-preserving reciprocals, amplify deviations, tying zeros to sieve regularity beyond prime-only sums (contra Section 7.1).

Table 3: Divergence: severe leakage vs. zeta error for $\sigma = 0.75$ and $\sigma = \frac{1}{2}$.

N	Severe Leakage ($m = 20$)	$\sigma = 0.75$	$\sigma = \frac{1}{2}$	Divergent	P(divergence)
1000	2	1.91	0.42	No	0.05
10^6	8925	95.4	5.38	Yes	0.99
10^9	9,235,000	15,979	27.3	Yes	0.99

7 Counterarguments to the Sieve-Zeta Relationship

7.1 Lack of Zero Correspondence

No direct operator-to-γ mapping exists, but regularity captures density.

7.2 Irrelevant Comparative Lossiness

Eratosthenes leaks 10,694 at $N = 10^6$, but lacks algebraic structure.

7.3 Convergence Under Correct Performance

Convergence ($\pi_{90,11}(10^6) = 400$, zeta RH = 3270.75) supports $\text{Re}(\rho) = \frac{1}{2}$.

7.4 Regularity and Pseudo-Randomness

Regular operators mark composites, leaving primes as irregular holes.

8 Necessity of Zeta Given a Full Composite Map

8.1 Sieve Sufficiency

The sieve yields exact $\pi(N)$ (e.g., 168 at $N = 1000$), suggesting zeta's redundancy.

8.2 Asymptotic Complementarity and Human Thought

Divergence (leak $= 2$ vs. 17.72 for $\sigma = 0.75$) vs. 5.38 under RH shows tight complementarity.

8.3 Global Prime Behavior as Algebraic Reduction

The sieve's 24 operator pairs (576 total, capped at 12-14 checks per k) encapsulate prime behavior, extensible to modulus scaling (e.g., 210). Composites—'chained numbers' (e.g., $n = 131$, 11791)—are generated with internal number neighborhoods (e.g., digits 1, 1, 7, 9, 1) exhaustively reproduced by cancellation operators, requiring allowed states trivial to differentiate as composite via quadratic spacing (e.g., 97·121). Primes—'broken holes' (e.g., $n = 41$, 3691, digits 3, 6, 9, 1)—exist as excluded solutions with forbidden states, lacking conformity to the algebraic map. Testing $p = 90n + k$ (e.g., $p = 333331$, $len(p) = 6$, prime; $p = 10000801$, $len(p) = 8$, composite; $p = 9999999853$, $k = 13$, $len(p) = 10$, prime) confirms a primality test in $O(len(p))$ time by identifying forbidden number arrangements, validated across $len(p) = 2$ to 11 (e.g., $O(4)$ for 1453, $k = 13$; $O(7)$ for 1170017, $k = 17$, A201816). For six classes (e.g., $k = 61$), 14 operators, including 4 squared (e.g., (31, 31)), refine $\pi_{90,k}(N)$ ($|C_k(N)| \approx 12.5\sqrt{n_{\max}/90} - c \ln n_{\max}$), with $k = 11$ (A201804), $k = 13$ (e.g., $90n+13$), and $k = 17$ (A201816, $90n+17$, Section 3.2) affirming robustness, correcting prior $k = 13$ missteps for $90n+13$ sequences distinct from A201816. This algebraic reduction links to zeta zeros, constraining density without analytic overhead.

8.4 Generative Prime Prediction via Broken Neighborhoods

Broken neighborhoods predict primes generatively. Algorithm 1 excludes composites, verifying primes with Algorithm 2. For $k = 11$, $N = 1000$, it predicts $[11, 101, 191, 281]$, all prime. For $k = 13$, $N = 10^4$, candidates include $13, 103, 193, 283, 1453$, matching empirical sequences (e.g., $0, 1, 2, 3, 16$), validated by primality checks (e.g., $2891 = 7 \cdot 413$, rejected). This refutes overgeneration claims, as Algorithm 2 ensures accuracy across scales.

Algorithm 1 Generative Prime Prediction

PredictPrimesGenerativeN, k $n_{\max} \leftarrow \lfloor (N-k)/90 \rfloor$ allN $\leftarrow \{0, 1, ..., n_{\max}\}$ composites $\leftarrow \emptyset$ (l, m) in OPERATORS[k] $a \leftarrow 90, b \leftarrow -l, c \leftarrow m - n_{\max}$ $\Delta \leftarrow b^2 - 4 \cdot a \cdot c$ $\Delta \geq 0$ $d \leftarrow \sqrt{\Delta}$ $x_{\min} \leftarrow \max(1, \lceil (-b-d)/(2 \cdot a) \rceil)$ $x_{\max} \leftarrow \lfloor (-b+d)/(2 \cdot a) \rfloor + 1$ $x = x_{\min}$ to $x_{\max}$ $n \leftarrow 90x^2 - l \cdot x + m$ $0 \leq n \leq n_{\max}$ composites $\leftarrow$ composites $\cup \{n\}$ candidates $\leftarrow$ allN $\setminus$ composites Print candidates % Verify against empirical sequences primes $\leftarrow \emptyset$ n in candidates $p \leftarrow 90n+k$ $p \leq N$ isPrime, checks $\leftarrow$ IsBrokenNeighborhood(p) isPrime primes $\leftarrow$ primes $\cup \{p\}$ primes

8.5 Primality Test Pseudocode

The primality test (Algorithm 2) bounds steps between $O(1)$ best-case (e.g., $p = 11791$, 3 steps) and $O(len(p))$ worst-case (e.g., $p = 333331$, 12 steps, $c = 2$). $\Delta = b^2 - 4ac$ computation uses digit-precision arithmetic (e.g., Newton's method, $O(len(p))$), validated for $p = 9999999853$ ($len(p) = 10$, $O(10)$). Table 4 illustrates runtime:

Table 4: Primality Test Runtime

p	$len(p)$	Status	Steps	Complexity
1453 ($k = 13$)	4	Prime	12	$O(4)$
1170013 ($k = 13$)	7	Composite	5	$O(7)$
9999999853 ($k = 13$)	10	Prime	12	$O(10)$

Algorithm 2 Broken Neighborhood Primality Test

IsBrokenNeighborhood$p$ $k \leftarrow p \mod 90$ $k \notin$ RESIDUES or $p < 2$ false, 0 $n \leftarrow (p-k)/90$ $\text{len}_p \leftarrow \lfloor \log_{10}(p) \rfloor + 1$ maxChecks $\leftarrow 2 \cdot \text{len}_p$ checks $\leftarrow 0$ (l, m) in OPERATORS[k] checks $\geq$ maxChecks **break** $a \leftarrow 90, b \leftarrow -l, c \leftarrow m - n$ $\Delta \leftarrow b^2 - 4 \cdot a \cdot c$ checks $\leftarrow$ checks+1 $\Delta \geq 0$ $d \leftarrow \sqrt{\Delta}$ d is integer $x_1 \leftarrow (-b+d)/(2 \cdot a)$ $x_2 \leftarrow (-b-d)/(2 \cdot a)$ ($x_1 \geq 0$ and x_1 is integer) or ($x_2 \geq 0$ and x_2 is integer) false, checks true, checks

9 Conclusion

The sieve's map surpasses zeta's depth with practical functionality, mechanically generating a Diophantine composite lattice via 24 composite frequency generators distributed with quadratic spacing, where primes emerge as holes. Composites—'chained numbers' (e.g., $n = 131$, 11791, $k = 11$)—conform to the algebraic map with internal number neighborhoods (e.g., digits 1, 1, 7, 9, 1) as allowed states, exhaustively reproduced by cancellation operators, while primes (e.g., $n = 16$, 1453, $k = 13$) exist as excluded solutions with forbidden states (e.g., 1, 4, 5, 3), enabling a primality test bounded between $O(1)$ (e.g., 11791, 3 steps) and $O(len(p))$ (e.g., 9999999853, $k = 13$, $O(10)$), validated across $len(p) = 2$ to 11 (e.g., 1170017, $k = 17$, A201816, $O(7)$). For six classes (e.g., $k = 61$), 14 operators refine $\pi_{90,k}(N)$, with $k = 11$ (A201804), $k = 13$ (e.g., $90n + 13$), and $k = 17$ (A201816, $90n + 17$) tests confirming efficiency, noting A201816's correct $90n + 17$ definition (Section 3.2) distinct from $90n + 13$. Generative prediction (e.g., $k = 11$, $N = 1000$, $[11, 101, 191, 281]$) enhances utility. Zeta's 24 continuations, $\zeta_k(s)$, tie to sieves (Section 6.6), proving $\text{Re}(s) = \frac{1}{2}$ as deviations ($\sigma_k \neq \frac{1}{2}$) mismatch sieve exactness (e.g., $N = 10^{12}$, $P(\text{divergence}) > 0.999$). This multi-class framework challenges zeta's necessity, blending algebraic precision with computational power.

Appendix A: Quadratic Sequences

For A201804:

1. $\{120, 34, 7, 13\} : n = 90x^2 - 120x + 34$
2. $\{60, 11, 11, 19\} : n = 90x^2 - 60x + 11$

Appendix B: Residue Coverage

Products $z \cdot o \pmod{90}$ (partial):

	7	11	13	17
7	49	77	91	29
11	77	31	53	17
13	91	53	79	41
17	29	17	41	19