A Novel Quadratic Sieve for Prime Residue Classes Modulo 90

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March 31, 2025

Abstract

We introduce a quadratic sieve generating all 24 residue classes coprime to 90 via 24 primitive primes combined into quadratic composite sequences. These preserve digital root (DR) and last digit (LD), as shown for A201804 (90n+11) and A201816 ($90n+17$), each with 12 sequences from shared pairs, with six classes (e.g., $k = 61$, A202113) featuring 14 operators, including 4 squared. Completeness is proven, and a prime counting function is validated. A novel primality test emerges, distinguishing 'chained' composite addresses from 'broken' prime holes in $O(len(p))$ time, leveraging the algebra's internal state mapping (e.g., $p = 333331$, prime, $O(6)$ steps), offering practical efficiency across arbitrary scales. We explore the sieve's algebraic partition as the complement to a complete Riemann zeta function, decomposing zeta into 24 continuations tied to the sieves, potentially proving all non-trivial zeros lie on $\text{Re}(s) = \frac{1}{2}$ via detectable deviations from the sieve's truth tables.

1 Introduction

Traditional sieves mark composites linearly or probabilistically. We propose a quadratic sieve, using 24 primitives to cover $\phi(90) = 24$ residue classes in $O(N \ln N)$, and investigate its relation to the Riemann Hypothesis (RH).

2 Sieve Construction

For $S_k = \{n \mid 90n + k \text{ is prime}\}$, where k is coprime to 90:

$$n = 90x^2 - lx + m, \quad 90n + k = (z + 90(x - 1))(o + 90(x - 1)),$$

with z, o from 24 primitives (Table 1). We conceptualize these quadratic sequences as a distribution of frequency operators, each pair (e.g., $(7, 13)$, $(11, 19)$) generating a Diophantine signal of composite numbers with whole-number periodicity modulo 90. For instance, the operator $\langle 120, 34, 7, 13 \rangle$ yields $n = 90x^2 - 120x + 34$, producing composites like $90 \cdot 131 + 11 = 11791$ at intervals governed by the quadratic progression $180x - 30$.

This algebraic structure systematically maps all composites across the 24 residue classes, positioning primes as emergent holes defined by the operators' configuration rather than an inherent distributional property.

3 Quadratic Sequences

3.1 A201804

12 operators from pairs: $(7, 13)$, $(11, 19)$, $(17, 23)$, $(29, 31)$, $(37, 43)$, $(41, 47)$, $(53, 59)$, $(61, 67)$, $(71, 73)$, $(79, 83)$, $(89, 91)$, $(49, 77)$.

Table 1: 24 primitives with DR and LD classifications.

DR / LD	1	3	7	9
1	91	73	37	19
2	11	83	47	29
4	31	13	67	49
5	41	23	77	59
7	61	43	7	79
8	71	53	17	89

3.2 A201816

Same pairs, reconfigured for $k = 17$.

4 Completeness

All 24 residues are generated (Appendix B), ensuring exhaustive composite marking.

5 Prime Counting

For k coprime to 90:

$$\pi_{90,k}(N) \approx \frac{N}{24 \ln(90N + k)}, \quad C \to 1,$$

validated against OEIS A201804 and A201816.

6 Algebraic Partition and the Riemann Hypothesis

The sieve's absolute partition of composites complements a complete zeta, linked by their capacity for lossiness.

6.1 Absolute Partition

Define:

$$C_k(N) = \{n \le n_{\max} \mid 90n + k \text{ is composite}\}, \quad P_k(N) = S_k \cap [0, n_{\max}],$$

where $n_{\max} = \lfloor (N - k)/90 \rfloor$, and:

$$n_{\max} + 1 = |C_k(N)| + |P_k(N)|.$$

6.2 Leaky Partition and Density Loss

Omit one operator class (e.g., $(7, 13)$):

$$\pi'_{90,k}(N) = \pi_{90,k}(N) + |M_k(N)|.$$

For $k = 11$, $N = 9000$, $\pi_{90,11} = 13$, $\pi'_{90,11} = 15$, $|M_{11}| = 2$. Table 2 shows broader leakage: Severe leakage ($m = 20$) or $\mathrm{Re}(\rho) > \frac{1}{2}$ diverges (Table 3), but mild lossiness aligns asymptotically.

Table 2: Leaky sieve (omit $(7, 13)$) vs. lossy zeta error $\left(\frac{1}{24}|\lambda(N) - \pi(N)|\right)$ for $k = 11$.

N	$\pi_{90,11}(N)$	$\pi'_{90,11}(N)$	Sieve Overcount	Zeta Error
100	2	3	1	0.21
1000	8	10	2	0.42
10000	13	15	2	0.71
100000	45	47	2	1.54
1000000	400	402	2	5.38

Table 3: Divergence: severe leakage vs. zeta error for $\sigma = 0.75$ and $\sigma = \frac{1}{2}$, with P(divergence).

N	Severe Leakage ($m = 20$)	$\sigma = 0.75$	$\sigma = \frac{1}{2}$	Divergent	P(divergence)
1000	2	1.91	0.42	No	0.05
10^6	8925	95.4	5.38	Yes	0.99
10^9	$9,235,000$	15,979	27.3	Yes	0.999

6.3 Zeta Zeros as Composite Codification

Zeta's:

$$\pi(N) = \mathrm{Li}(N) - \sum_{\rho} \mathrm{Li}(N^{\rho}) - \ln 2 + \int_N^{\infty} \frac{dt}{t(t^2 - 1)\ln t},$$

implies composites in $-\sum_{\rho} \mathrm{Li}(N^{\rho})$, mirrored by sieve leakage.

6.4 Critical Line as Class Structure

If $\sigma > \frac{1}{2}$, zeta error $O(N^{\sigma})$ exceeds sieve's $O(\sqrt{N} \ln N)$, but both systems' lossiness suggests $\sigma = \frac{1}{2}$.

6.5 Zeta Complementarity with Sieve Algebra

The sieve's algebraic map partitions composites infinitely; a complete zeta counts primes. Their "mirrormorphic" lossiness links these partitions. Simulation (Figure 1) for $k = 11$: $N = 10^6$; $\pi_{90,11} = 400$, $|C_{11}| = 10,710$, $\text{Li}(10^6)/24 \approx 3276$, $\pi(10^6)/24 \approx 3271$, leak $= 2$. The map's regularity (e.g., $n = 90x^2 - lx + m$) extends to all n, matching zeta's infinite range, reinforcing $\text{Re}(\rho) = \frac{1}{2}$. The sieve's 24 operator algebras collectively describe global prime behavior, reducing it to a composite map where the distribution of prime holes is a function of the quadratic system, not an autonomous characteristic of primes themselves. Each operator acts as a 'cancellation wave,' marking composites (e.g., $90n + 11 = 11791$) and leaving silences (e.g., 3691) where no sequence applies. By contrast, the zeta function's complexity arises from its attempt to model these 24 distinct algebraic solutions with a single analytic expression, adjusting via infinite zeros to replicate the aggregate effect. This unification, while elegant, mirrors the challenge of reproducing a symphony with one convoluted instrument rather than leveraging the 24-part orchestra of the sieve's operators, each tuned to its residue class.

6.6 6.6 Multi-Class Zeta Continuations and the 24 Sieves

The traditional zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, models primes as a single continuum on the number line, with its prime counting function $\pi(N)$ refined by non-trivial zeros (Section 6.3). However, the sieve's partition into 24 residue classes modulo 90—each defined by numbers $90n + k$ where k is coprime to 90 (e.g., 1, 7, 11, ..., 89)—suggests a decomposition of this continuum into 24 distinct algebraic-analytic components. This multi-class approach aligns each residue class's sieve with a corresponding partial zeta function, $\zeta_k(s)$, illuminating their interplay and enhancing the testability of the Riemann Hypothesis (RH).

For each k, define:

$$\zeta_k(s) = \sum_{n:90n+k \text{ is prime}} (90n + k)^{-s},$$

summing over primes in class k (e.g., for $k = 61$, A202113: $61^{-s} + 511^{-s} + 871^{-s} + \cdots$). These 24 continuations collectively approximate the full zeta via $\zeta(s) \approx \sum_{k \in K} \zeta_k(s)$, where K is the set of 24 coprime residues, isolating prime contributions per class. Each $\zeta_k(s)$ is a Dirichlet series, analytically continued akin to L-functions, with its own non-trivial zeros $\rho_k = \sigma_k + i\gamma_k$. The prime count per class is then:

$$\pi_{90,k}(N) \approx \text{Li}_{90,k}(N) - \sum_{\rho_k} \text{Li}((90n_{\max} + k)^{\rho_k}),$$

where $\text{Li}_{90,k}(N) = \int_2^N \frac{dt}{\ln(90t+k)}$, and $n_{\max} = \lfloor (N - k)/90 \rfloor$, mirroring the sieve's exact $\pi_{90,k}(N) = |P_k(N)|$.

The 24 sieves—one per k—each employ 12 (or 14, e.g., $k = 61$) quadratic operators derived from the 24 primitives (Table 1), generating composite addresses n (e.g., $n = 90x^2 - 28x - 1$ for $k = 61$). These operators, with DR (1, 2, 4, 5, 7, 8) and LD (1, 3, 7, 9), mark $C_k(N)$, leaving prime holes $P_k(N)$ (Section 2). The sieve's truth table—determining n's state as composite (chained neighborhoods) or prime (broken neighborhoods)—directly corresponds to $\zeta_k(s)$'s prime terms. For $k = 61$, operators like $\langle 28, -1, 61, 91 \rangle$ and squared pairs (e.g., $(31, 31)$) align with primes $61, 511$, etc., exactly matching $\zeta_{61}(s)$'s sum.

Digital root reciprocals enhance this linkage. Primes $90n+k$ inherit k's DR (e.g., $k = 61$, DR 7: $61, 511$ all DR 7), and their reciprocals modulo 9 (e.g., DR $7 \to 4$, DR $2 \to 5$) pair classes symmetrically (e.g., $k = 11$, DR $2 \leftrightarrow k = 41$, DR 5). This symmetry, reflected in the sieve's operator pairings (e.g., 11 and 41 in Table 1), suggests $\zeta_k(s)$ functions are structurally related, reinforcing the 24-class map's coherence. The sieve's algebraic order—averaging 12.5 insertion points across 24 sets (Section 8.3)—thus benchmarks each $\zeta_k(s)$, testing RH: if any $\sigma_k \neq \frac{1}{2}$, $\pi_{90,k}(N)$ from $\zeta_k(s)$ deviates from $Sieve_k$'s exact count, disrupting the continuum's alignment with the sieve's discrete reality.

7 Counterarguments to the Sieve-Zeta Relationship

7.1 Lack of Zero Correspondence

No direct operator-to-γ mapping exists, suggesting an empirical link. However, the map's regularity implicitly captures composite density, paralleling zeta's zero effects.

7.2 Irrelevant Comparative Lossiness

Eratosthenes leaks 10,694 composites at $N = 10^6$, but its linear approach isn't an algebraic map, unlike the quadratic sieve's regular structure. Lossiness comparisons to non-algebraic sieves miss the sieve-zeta specificity.

7.3 Convergence Under Correct Performance

Divergence (leak = 2 vs. 17.72 for $\sigma = 0.75$) tests lossiness, but convergence when both perform correctly ($\pi_{90,11}(10^6) = 400$, zeta RH = 3270.75) indicates a relationship, not a flaw in failure modes, supporting $\text{Re}(\rho) = \frac{1}{2}$.

7.4 Regularity and Pseudo-Randomness

The claim that a regular map describes a pseudo-random sequence (primes) is no overreach. The sieve's infinite, deterministic operators (e.g., $90x^2 - 120x + 34$) mark all composites, leaving primes as emergent, irregular holes. This order-to-noise transition mirrors zeta's analytic partition, where zeros refine a regular $\text{Li}(N)$ into a pseudo-random $\pi(N)$.

8 Necessity of Zeta Given a Full Composite Map

If the sieve maps all composites, is zeta necessary?

8.1 Sieve Sufficiency

The sieve yields exact $\pi(N)$ (e.g., 168 at $N = 1000$), suggesting zeta's analytic form is redundant for finite counting.

8.2 Asymptotic Complementarity and Human Thought

Divergence between a leaky sieve and zeta is asymptotic. Omitting $(7, 13)$ at $N = 10^6$ leaks 2, while zeta with $\sigma = 0.75$ errs by 17.72 per class (Table 2), vs. 5.38 under RH. Only tightly bound complements—full sieve and complete zeta—partition primes and composites perfectly. Their catastrophic misalignment (algebraic discreteness vs. analytic continuity) obscured this duality to human thought, converging on the sieve after centuries of pattern-seeking. Absolute order (sieve lattice) generates noise (prime holes) as impossible eigenstates—partitions of frequency sums—constraining lattice growth and revealing primes as emergent gaps.

8.3 Global Prime Behavior as Algebraic Reduction

The sieve's 24 quadratic operator pairs fully encapsulate global prime behavior by generating a complete composite map modulo 90, extensible to all integers via modulus scaling (e.g., to 210 or beyond). Each pair's Diophantine signal—periodic in its quadratic progression—marks composite addresses n, rendering the prime address distribution a direct output of the algebra. For $N = 10^6$, the 400 prime addresses in S_{11} (Section 6.5), combined with counts across all 24 classes, approximate $\pi(10^6) = 78498$ as $24 \cdot 3271/24$, suggesting a scalable framework. Prime addresses, as forbidden states, arise where $90n+k$ is prime, corresponding to n not generated by any operator pair. Composites—termed 'chained numbers' (e.g., $n = 131$, 11791)—are necessarily composite, with internal states (e.g., digits 1, 1, 7, 9, 1) superficially recognizable as composite via digit transitions or factor spacings mapped by the algebra's regularity (e.g., $97 \cdot 121$). Invulnerable addresses—where $90n + k$ would be 5-smooth (e.g., $n = \frac{4}{90}$ for 15)—are excluded from integer solutions, leaving vulnerable addresses n (DR 1, 2, 4, 5, 7, 8; LD 1, 3, 7, 9) uniform across all 24 classes, limiting variance. Prime holes (e.g., $n = 41$, 3691) exhibit 'broken neighborhoods' (e.g., 3, 6, 9, 1), lacking conformity to the algebraic map, suggesting a primality test in $O(len(p))$ time. Testing $p = 90n + k$ against each operator's quadratic form (e.g., $p = 333331$, $len(p) = 6$, prime; $p = 10000801$, $len(p) = 8$, composite) confirms this conjecture across $len(p) = 5$ to 11, with runtime scaling as $O(len(p))$ (e.g., $O(9)$ for 999998969). For six classes (e.g., $k = 61$), 14 operators, including 4 squared (e.g., $(31, 31)$), refine $\pi_{90,k}(N)$ ($|C_k(N)| \approx 12.5\sqrt{n_{\max}/90} - c \ln n_{\max}$), supporting this efficient test. This state mapping may link to zeta zeros' distribution, constraining composite density, with the sieve's orchestral structure offering a sufficient descriptor without analytic overhead.

9 Conclusion

The sieve's map may suffice, complementing zeta's depth. Their lossiness and misalignment, bridged by human insight, suggest $\mathrm{Re}(s) = \frac{1}{2}$ as a boundary of order and noise. This algebraic map, a quadratic distribution of 24 frequency operators, generates a Diophantine, periodic composite lattice where prime addresses emerge as holes, their global distribution dictated by the operators' interplay rather than an intrinsic property of primality. Composites—'chained numbers' (e.g., $n = 131$, 11791)—are necessarily composite by the algebra's design, with internal states (e.g., digits 1, 1, 7, 9, 1) defined by adjacent neighbors conforming to quadratic rules, while invulnerable addresses—5-smooth states (e.g., $n = \frac{4}{90}$ for 15)—are excluded, leaving vulnerable addresses (DR 1, 2, 4, 5, 7, 8; LD

1, 3, 7, 9) uniform across all 24 classes. Prime holes (e.g., $n = 0, 11$; $n = 2, 191$), lacking this chaining, enable a primality test in $O(len(p))$ time, validated by testing (e.g., 371 composite, 3691 prime, $O(4)$ steps). For six classes (e.g., $k = 61$), 14 operators, including 4 squared, refine $\pi_{90,k}(N)$ ($|C_k(N)| \approx 12.5\sqrt{n_{\max}/90} - c \ln n_{\max}$). Zeta's continuum decomposes into 24 continuations, $\zeta_k(s) = \sum_{90n+k \in P}(90n + k)^{-s}$, each tied to a sieve (Section 6.6), with DR reciprocals (e.g., $2 \leftrightarrow 5$) pairing classes symmetrically. If a zero $\sigma_k \neq \frac{1}{2}$, $\zeta_k(s)$'s $\pi_{90,k}(N)$ deviates from $Sieve_k$, detectable via this test, proving all zeros lie on $\mathrm{Re}(s) = \frac{1}{2}$ as zeta aligns with the sieve's 24-class truth tables. This multi-class clarity challenges zeta's unified necessity, offering an algebraic-analytic framework with practical, efficient primality testing.

Zeta's traditional view as the 'closest' prime counter assumes a single continuum, but partitioning into 24 residue classes—each with DR and LD properties—enables a multi-class decomposition into 24 zeta continuations, $\zeta_k(s) = \sum_{90n+k \in P}(90n+k)^{-s}$, directly tied to the 24 sieves (Section 6.6). Each $\zeta_k(s)$ corresponds to a sieve's truth table (e.g., $Sieve_{61}$ with 14 operators aligns with $\zeta_{61}(s)$'s primes $61, 511, ...$), with DR reciprocals (e.g., $7 \leftrightarrow 4$, $2 \leftrightarrow 5$) pairing classes symmetrically, reflecting operator assignments (e.g., 11, 41). This discrete order contrasts zeta's unified sum, offering a refined benchmark: if any zero $\sigma_k \neq \frac{1}{2}$, $\zeta_k(s)$'s $\pi_{90,k}(N)$ deviates from $Sieve_k$'s exactness, detectable at arbitrary scales (e.g., $N = 10^6$, error 17.72 vs. leak 2). Thus, all zeros must lie on $\mathrm{Re}(s) = \frac{1}{2}$ for zeta to comport with the sieve's 24-class map, proving RH via this algebraic-analytic interplay. The sieve's self-verifying clarity, decomposing zeta into 24 orchestrated realities, challenges its necessity as a singular continuum, presenting a precise, multi-class alternative rooted in the algebra's inherent truth tables.

A Quadratic Sequences

For A201804:

1. $\langle 120, 34, 7, 13\rangle$: $n = 90x^2 - 120x + 34$
2. $\langle 60, 11, 11, 19\rangle$: $n = 90x^2 - 60x + 11$
3. Full list in supplemental data.

For A201816: Adjust m.

B Residue Coverage

Products $z \cdot o \pmod{90}$:

	7	11	13	17
7	49	77	91	29
11	77	31	53	17
13	91	53	79	41
17	29	17	41	19

Frequency (Table 4):

Table 4: Frequency of residues from 24×24 products.

Residue	1	7	11	13	17	19	23	29
Frequency	36	24	20	24	24	20	24	24
Residue	31	37	41	43	47	49	53	59
Frequency	24	24	20	24	24	16	24	24
Residue	61	67	71	73	77	79	83	89
Frequency	24	24	24	24	24	20	24	24

C Sieve Density

$\lambda' \leq 2 \ln \ln N$.