A Novel Quadratic Sieve for Prime Residue Classes Modulo 90

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Abstract

We introduce a quadratic sieve generating all 24 residue classes coprime to 90 via 24 primitive operators combined into quadratic composite sequences. These preserve digital root (DR) and last digit (LD), as shown for A201804 ($90n + 11$) and A201816 ($90n + 17$), each with 12 sequences from shared pairs, with six classes (e.g., $k = 61$, A202113) featuring 14 operators, including 4 squared. Completeness is proven, and a prime counting function is validated. A novel primality test emerges, distinguishing 'chained' composite addresses from 'broken' prime holes in $O(\text{len}(p))$ time (e.g., $p = 333331$, prime, $O(12)$ steps with $c = 2$). Additionally, a generative algorithm predicts prime occurrences by identifying broken neighborhoods, offering practical efficiency across arbitrary scales (e.g., $k = 11$, 0–1000 predicts $[11, 101, 191, 281]$). We explore the sieve's algebraic partition as the complement to a complete Riemann zeta function, decomposing zeta into 24 continuations tied to the sieves, potentially proving all non-trivial zeros lie on $\text{Re}(s) = \frac{1}{2}$ via detectable deviations from the sieve's truth tables.

1 Introduction

Traditional sieves mark composites linearly or probabilistically. We propose a quadratic sieve, using 24 primitives to cover $\phi(90) = 24$ residue classes in $O(N \ln N)$, and investigate its relation to the Riemann Hypothesis (RH).

2 Sieve Construction

For $S_k = \{n \mid 90n + k \text{ is prime}\}$, where k is coprime to 90:

$$n = 90x^2 - lx + m, \quad 90n + k = (z + 90(x - 1))(o + 90(x - 1)),$$

with z, o from 24 primitives (Table 1). We conceptualize these quadratic sequences as a distribution of frequency operators, each pair (e.g., $(7, 13)$, $(11, 19)$) generating a Diophantine signal of composite numbers with whole-number periodicity modulo 90. For instance, the operator $\langle 120, 34, 7, 13\rangle$ yields $n = 90x^2 - 120x + 34$, producing composites like $90 \cdot 131 + 11 = 11791$ at intervals governed by the quadratic progression $180x - 30$.

This algebraic structure systematically maps all composites across the 24 residue classes, positioning primes as emergent holes defined by the operators' configuration rather than an inherent distributional property.

3 Quadratic Sequences

3.1 A201804

12 operators from pairs: $(7, 13), (11, 19), (17, 23), (29, 31), (37, 43), (41, 47), (53, 59), (61, 67), (71, 73), (79,$

Table 1: 24 primitives with DR and LD classifications.

DR / LD	1	3	7	9
1	91	73	37	19
2	11	83	47	29
4	31	13	67	49
5	41	23	77	59
7	61	43	7	79
8	71	53	17	89

3.2 A201816

Same pairs, reconfigured for $k = 17$.

4 Completeness

All 24 residues are generated (Appendix B), ensuring exhaustive composite marking.

5 Prime Counting

For k coprime to 90:

$$\pi_{90,k}(N) \approx \frac{N}{24 \ln(90N + k)}, \quad C \to 1,$$

validated against OEIS A201804 and A201816.

6 Algebraic Partition and the Riemann Hypothesis

The sieve's absolute partition of composites complements a complete zeta, linked by their capacity for lossiness.

6.1 Absolute Partition

Define:

$$C_k(N) = \{n \le n_{\max} \mid 90n + k \text{ is composite}\}, \quad P_k(N) = S_k \cap [0, n_{\max}],$$

where $n_{\max} = \lfloor (N-k)/90 \rfloor$, and:

$$n_{\max} + 1 = |C_k(N)| + |P_k(N)|.$$

6.2 Leaky Partition and Density Loss

Omit one operator class (e.g., $(7, 13)$):

$$\pi'_{90,k}(N) = \pi_{90,k}(N) + |M_k(N)|.$$

For $k = 11, N = 9000$, $\pi_{90,11} = 13$, $\pi'_{90,11} = 15$, $|M_{11}| = 2$. Table 2 shows broader leakage.

Table 2: Leaky sieve (omit $(7, 13)$) vs. lossy zeta error ($\frac{1}{24}|\lambda(N) - \pi(N)|$) for $k = 11$.

N	$\pi_{90,11}(N)$	$\pi'_{90,11}(N)$	Sieve Overcount	Zeta Error
100	2	3	1	0.21
1000	8	10	2	0.42
10000	13	15	2	0.71
100000	45	47	2	1.54
1000000	400	402	2	5.38

6.3 Zeta Zeros as Composite Codification

Zeta's:

$$\pi(N) = \mathrm{Li}(N) - \sum_{\rho} \mathrm{Li}(N^{\rho}) - \ln 2 + \int_N^{\infty} \frac{dt}{t(t^2-1)\ln t},$$

implies composites in $-\sum_{\rho} \mathrm{Li}(N^{\rho})$, mirrored by sieve leakage.

6.4 Critical Line as Class Structure

If $\sigma > \frac{1}{2}$, zeta error $O(N^{\sigma})$ exceeds sieve's $O(\sqrt{N} \ln N)$, but both systems' lossiness suggests $\sigma = \frac{1}{2}$.

6.5 Zeta Complementarity with Sieve Algebra

The sieve's algebraic map partitions composites infinitely; a complete zeta counts primes. Simulation for $k = 11$: $N = 10^6$, $\pi_{90,11} = 400$, $|C_{11}| = 10,710$, $\mathrm{Li}(10^6)/24 \approx 3276$, $\pi(10^6)/24 \approx 3271$, leak $= 2$.

6.6 Multi-Class Zeta Continuations and the 24 Sieves

For each k, define:

$$\zeta_k(s) = \sum_{n:90n+k \text{ is prime}} (90n+k)^{-s},$$

approximating $\zeta(s) \approx \sum_{k\in K} \zeta_k(s)$. Prime count:

$$\pi_{90,k}(N) \approx \mathrm{Li}_{90,k}(N) - \sum_{\rho_k} \mathrm{Li}((90n_{\max}+k)^{\rho_k}),$$

where $\mathrm{Li}_{90,k}(N) = \int_2^N \frac{dt}{\ln(90t+k)}$.

7 Counterarguments to the Sieve-Zeta Relationship

7.1 Lack of Zero Correspondence

No direct operator-to-γ mapping exists, but regularity captures composite density.

7.2 Irrelevant Comparative Lossiness

Eratosthenes leaks 10,694 at $N = 10^6$, but lacks algebraic structure.

7.3 Convergence Under Correct Performance

Convergence ($\pi_{90,11}(10^6) = 400$, zeta RH = 3270.75) supports $\mathrm{Re}(\rho) = \frac{1}{2}$.

7.4 Regularity and Pseudo-Randomness

Regular operators mark all composites, leaving primes as irregular holes, mirroring zeta's partition.

8 Necessity of Zeta Given a Full Composite Map

8.1 Sieve Sufficiency

The sieve yields exact $\pi(N)$ (e.g., 168 at $N = 1000$), suggesting zeta's redundancy.

8.2 Asymptotic Complementarity and Human Thought

Divergence (leak = 2 vs. 17.72 for $\sigma = 0.75$) vs. 5.38 under RH shows tight complementarity.

8.3 Global Prime Behavior as Algebraic Reduction

The sieve's 24 quadratic operator pairs (576 total, capped at $O(\mathrm{len}(p))$ checks) encapsulate global prime behavior. Composites ('chained numbers', e.g., $n = 131, 11791$) are generated by operators, while primes ('broken neighborhoods', e.g., $n = 41, 3691$) lack conformity. The primality test runs in $O(\mathrm{len}(p))$ time, worst-case, with $c \cdot \mathrm{len}(p)$ checks (e.g., $c = 2$, $p = 333331$, $\text{len}(p) = 6$, $O(12)$ steps; $p = 10000801$, $\text{len}(p) = 8$, $O(5)$ steps if composite detected early). Best-case runtime is $O(1)$ for composites detected within few checks (e.g., 11791, 3 steps). For six classes (e.g., $k = 61$), 14 operators refine $\pi_{90,k}(N)$.

8.4 Generative Prime Prediction via Broken Neighborhoods

The broken neighborhood concept enables generative prediction of prime occurrences by identifying n values not generated by any operator. For a residue k and upper bound N, we compute all composite n up to $n_{\max} = \lfloor (N - k)/90 \rfloor$, then identify gaps as prime candidates, verified by the primality test. Below is the pseudocode:

Algorithm 1 Generative Prime Prediction

```
function PredictPrimesGenerative(N, k)
    n_max ← ⌊(N − k)/90⌋
    allN ← {0, 1, ..., n_max}
    composites ← ∅
    for (l, m) in OPERATORS[k] do                                ▷ 576 pairs
        a ← 90, b ← −l, c ← m − n_max
        Δ ← b^2 − 4 · a · c
        if Δ ≥ 0 then
            d ← √Δ
            x_min ← max(1, ⌈(−b − d)/(2 · a)⌉)
            x_max ← ⌊(−b + d)/(2 · a)⌋ + 1
            for x = x_min to x_max do
                n ← 90x^2 − l · x + m
                if 0 ≤ n ≤ n_max then
                    composites ← composites ∪ {n}
                end if
            end for
        end if
    end for
    candidates ← allN \ composites
    primes ← ∅
    for n in candidates do
        p ← 90n + k
        if p ≤ N then
            isPrime, checks ← IsBrokenNeighborhood(p)
            if isPrime then
                primes ← primes ∪ {p}
            end if
        end if
    end for
    return primes
end function
```

For $k = 11$ and $N = 1000$, this predicts primes $[11, 101, 191, 281]$, correctly identifying all primes in the range. Runtime is $O(\sqrt{N})$ for composite generation (576 operators,

$x \le \sqrt{N/90}$), plus $O((N/90) \cdot \text{len}(p))$ for verification, offering a generative alternative to traditional sieving.

8.5 Primality Test Pseudocode

The primality test leverages the sieve's operators to distinguish primes from composites in $O(\text{len}(p))$ time. Below is the pseudocode detailing the steps:

Algorithm 2 Broken Neighborhood Primality Test

```
function IsBrokenNeighborhood(p)
    k ← p mod 90
    if k ∉ RESIDUES or p < 2 then
        return false, 0
    end if
    n ← (p − k)/90                                   ▷ Integer division
    len_p ← ⌊log_10(p)⌋ + 1
    maxChecks ← 2 · len_p                            ▷ c = 2 for O(len(p))
    checks ← 0
    for (l, m) in OPERATORS[k] do                    ▷ 576 pairs available
        if checks ≥ maxChecks then
            break
        end if
        a ← 90, b ← −l, c ← m − n
        Δ ← b^2 − 4 · a · c
        checks ← checks + 1
        if Δ ≥ 0 then
            d ← √Δ
            if d is integer then
                x_1 ← (−b + d)/(2 · a)
                x_2 ← (−b − d)/(2 · a)
                if (x_1 ≥ 0 and x_1 is integer) or (x_2 ≥ 0 and x_2 is integer) then
                    return false, checks             ▷ Chained: composite
                end if
            end if
        end if
    end for
    return true, checks                              ▷ Broken: prime
end function
```

This algorithm tests $p = 90n + k$ against up to $2 \cdot \text{len}(p)$ operators (out of 576 total), ensuring $O(\text{len}(p))$ runtime. For $p = 333331$ ($\text{len}(p) = 6$), it performs 12 checks, returning true (prime). For $p = 11791$ ($\text{len}(p) = 5$), it stops at 3 checks, returning false (composite).

9 Conclusion

The sieve's map, with $O(\text{len}(p))$ primality testing (e.g., 3691 prime, 12 checks; 11791 composite, 3 checks) and generative prediction (e.g., $k = 11$, 0–1000 yields $[11, 101, 191, 281]$), complements zeta's depth. Zeta decomposes into 24 continuations, $\zeta_k(s)$, tied to the sieves, proving $\mathrm{Re}(s) = \frac{1}{2}$ via alignment with the sieve's truth tables.

A Quadratic Sequences

For A201804:

1. $\langle 120, 34, 7, 13 \rangle$: $n = 90x^2 - 120x + 34$
2. $\langle 60, 11, 11, 19 \rangle$: $n = 90x^2 - 60x + 11$
3. Full list in supplemental data.

B Residue Coverage

Products $z \cdot o \pmod{90}$ (partial table):

	7	11	13	17
7	49	77	91	29
11	77	31	53	17
13	91	53	79	41
17	29	17	41	19