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Session 2019-20
Odd Semester

PUBLISHED BY : Apram Singh
Quantum Page Pvt. Ltd.
Plot No. 59/2/7, Site - 4, Industrial Area,
Sahibabad, Ghaziabad-201 010

Phone : 0120 - 4160479
Email : pagequantum@gmail.com Website: www.quantumpage.co.in
Delhi Office : 1/6590, East Rohtas Nagar, Sahadara, Delhi-110032

MATHEMATICS-IV (CC : Sem-3 & 4)
1st Edition : *2019-20*

Price: Rs. 90/- only

Printed at : Mayank Enterprises, Delhi-110093.

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KAS 302/402 : MATHEMATICS - IV

UNIT 1

Partial Differential Equations

CONTENTS

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Form partial differential equations of the equations by eliminating the arbitrary constants :**
$z = ax + by + ab$

Answer

Differentiating z partially w.r.t. x and y,

$$p = \frac{\partial z}{\partial x} = a,\ q = \frac{\partial z}{\partial y} = b$$

Substituting for a and b in the given equation, we get

$$z = px + qy + pq$$

which is a partial differential equation.

Que 1.2. **Form partial differential equations of the equations by eliminating the arbitrary constants :**
$az + b = a^2x + y$

Answer

Differentiating the given relation partially w.r.t. x, we get

$$a\frac{\partial z}{\partial x} = a^2$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = p = a \quad \text{...(1.2.1)}$$

Again differentiating the given relation partially w.r.t. y, we get

$$a\frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = q = \frac{1}{a} \quad \text{...(1.2.2)}$$

Multiplying eq. (1.2.1) and (1.2.2), we get $pq = 1$

which is a partial differential equation.

Que 1.3. **Form the partial differential equation by eliminating the arbitrary function(s) from the following :**

i. $z = f(x^2 - y^2)$

ii. $z = \phi(x) . \psi(y)$

iii. $z = f(x + it) + g(x - it)$

Answer

i. Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2) . 2x \quad \text{...(1.3.1)}$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = f'(x^2 - y^2) . (-2y) \quad \text{...(1.3.2)}$$

Dividing eq. (1.3.1) by eq. (1.3.2), we get

$$\frac{p}{q} = \frac{x}{(-y)} \Rightarrow py + qx = 0$$

which is a partial differential equation.

ii. Differentiating z w.r.t. x, partially, we get

$$\frac{\partial z}{\partial x} = p = \phi'(x) \psi(y) \quad \text{...(1.3.3)}$$

Differentiating z w.r.t. y, partially, we get

$$\frac{\partial z}{\partial y} = q = \phi(x) \psi'(y) \quad \text{...(1.3.4)}$$

Differentiating eq. (1.3.3) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial y \partial x} = s = \phi'(x) \psi'(y) \quad \text{...(1.3.5)}$$

Multiplying eq. (1.3.3) and (1.3.4), we get

$$pq = \phi'(x) \psi(y) \phi(x) \psi'(y) = zs \quad \text{[Using (1.3.5)]}$$

$$\Rightarrow \quad pq - zs = 0$$

which is a partial differential equation.

iii. Given $z = f(x + it) + g(x - it)$

Differentiating z twice partially w.r.t. x and t, we have

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it) \qquad \text{...(1.3.6)}$$

$$\frac{\partial z}{\partial t} = if'(x+it) - ig'(x-it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x+it) + i^2 g''(x-it)$$

or
$$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it) \qquad \text{...(1.3.7)}$$

Adding eq. (1.3.6) and eq. (1.3.7), we obtain $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

which is a partial differential equation of second order.

Que 1.4. **Solve the following differential equations :**
$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Answer

Here Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\therefore \quad \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

Taking the first two members, we have $\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$

which on integration gives

$$\log(x-y) = \log(y-z) + \log a$$

or
$$\log\left(\frac{x-y}{y-z}\right) = \log a \quad \text{or} \quad \frac{x-y}{y-z} = a \qquad \text{...(1.4.1)}$$

Similarly, taking the last two members, we obtain

$$\frac{y-z}{z-x} = b \qquad \text{...(1.4.2)}$$

From eq. (1.4.1) and eq. (1.4.2), the general solution is

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0.$$

Que 1.5. **Solve** $\frac{y^2 z}{x} p + xzq = y^2$.

Answer

Rewriting the given equation as

$$y^2zp + x^2zq = y^2x$$

The subsidiary equations are

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$$

The first two fractions give $x^2dx = y^2dy$. ...(1.5.1)

Integrating we get $x^3 - y^3 = a$

Again the first and third fractions give $xdx = zdz$ ...(1.5.2)

Integrating, we get $x^2 - z^2 = b$

Hence from eq. (1.5.1) and eq. (1.5.2), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2)$$

Que 1.6. **Solve the partial differential equation $x(y^2 + z)\,p - y(x^2 + z)\,q = z(x^2 - y^2)$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.**

Answer

Lagrange's subsidiary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{...(1.6.1)}$$

Using $x, y, -1$ as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

$$\therefore \quad x\,dx + y\,dy - dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}$$

$$\Rightarrow \quad x^2 + y^2 - 2z = c_1 \quad \text{...(1.6.2)}$$

Again, using $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y} + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dz + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow \quad xyz = c_2 \quad \text{...(1.6.3)}$$

Hence the general solution is

$$\phi(x^2 + y^2 - 2z, xyz) = 0$$

Que 1.7. **Solve $x^2(y - z)\,p + y^2(z - x)\,q = z^2(x - y)$.**

Answer

Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \quad \text{...(1.7.1)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \quad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \quad \text{...(1.7.2)}$$

Hence from eq. (1.7.1) and eq. (1.7.2), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

Que 1.8. **Solve : $\sqrt{p} + \sqrt{q} = 1$.**

Answer

The equation is of the form, $f(p, q) = 0$

The complete solution is $z = ax + by + c$ ...(1.8.1)

where $\sqrt{a} + \sqrt{b} = 1$ or $b = (1 - \sqrt{a})^2$

$\therefore$ From eq. (1.8.1), the complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c$$

Que 1.9. **Solve : $pq = p + q$.**

Answer

The equation is of the form $f(p, q) = 0$

The complete solution is $z = ax + by + c$ ...(1.9.1)

where $ab = a + b$ or $b = \dfrac{a}{a-1}$

$\therefore$ From eq. (1.9.1), the complete solution is $z = ax + \dfrac{a}{a-1} y + c.$

Que 1.10. **Solve : $4xyz = pq + 2px^2y + 2qxy^2$.**

Answer

Given : $4xyz = pq + 2px^2y + 2qxy^2$ ...(1.10.1)

Let $x^2 = X$ and $y^2 = Y$

so that $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = 2x \dfrac{\partial z}{\partial X}$

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = 2y \dfrac{\partial z}{\partial Y}$

$\therefore$ After putting the values of p and q in eq. (1.10.1), we get

$$4xyz = 4xy \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} + 4x^3 y \frac{\partial z}{\partial X} + 4xy^3 \frac{\partial z}{\partial Y}$$

or $$z = x^2 \frac{\partial z}{\partial X} + y^2 \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

$$= X \frac{\partial z}{\partial X} + Y \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

or $$z = PX + QY + PQ,$$

where $P = \dfrac{\partial z}{\partial X}$ and $Q = \dfrac{\partial z}{\partial Y}$

It is of the form $z = PX + QY + f(P, Q)$

Answer

The equation is of the form, $f(p, q) = 0$ ...(1.8.1)

The complete solution is $z = ax + by + c$

where $\sqrt{a} + \sqrt{b} = 1$ or $b = (1 - \sqrt{a})^2$

$\therefore$ From eq. (1.8.1), the complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c$$

Que 1.9. **Solve : $pq = p + q$.**

Answer

The equation is of the form $f(p, q) = 0$ ...(1.9.1)

The complete solution is $z = ax + by + c$

where $ab = a + b$ or $b = \dfrac{a}{a-1}$

$\therefore$ From eq. (1.9.1), the complete solution is $z = ax + \dfrac{a}{a-1}y + c$.

Que 1.10. **Solve : $4xyz = pq + 2px^2y + 2qxy^2$.**

Answer

Given : $4xyz = pq + 2px^2y + 2qxy^2$ ...(1.10.1)

Let $x^2 = X$ and $y^2 = Y$

so that $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = 2x \dfrac{\partial z}{\partial X}$

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = 2y \dfrac{\partial z}{\partial Y}$

$\therefore$ After putting the values of p and q in eq. (1.10.1), we get

$$4xyz = 4xy \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} + 4x^3y \frac{\partial z}{\partial X} + 4xy^3 \frac{\partial z}{\partial Y}$$

or $$z = x^2 \frac{\partial z}{\partial X} + y^2 \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

$$= X \frac{\partial z}{\partial X} + Y \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

or $$z = PX + QY + PQ,$$

where $P = \dfrac{\partial z}{\partial X}$ and $Q = \dfrac{\partial z}{\partial Y}$

It is of the form $z = PX + QY + f(P, Q)$

Its complete solution is $z = aX + bY + ab$ or $z = ax^2 + by^2 + ab$.

Que 1.11. **Solve : $z^2 (p^2 + q^2 + 1) = a^2$.**

Answer

The given equation is of the form $f(z, p, q) = 0$

Let $u = x + by$ (note the use of b instead of a, since a is a given constant)

so that $$p = \frac{dz}{du} \text{ and } q = b\frac{dz}{du}.$$

Substituting these values of p and q in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + b^2\left(\frac{dz}{du}\right)^2 + 1\right] = a^2$$

or $$z^2(1 + b^2)\left(\frac{dz}{du}\right)^2 = a^2 - z^2$$

or $$z\sqrt{1+b^2}\,\frac{dz}{du} = \pm\sqrt{a^2 - z^2}$$

or $$\pm\sqrt{1+b^2}\cdot\frac{z}{\sqrt{a^2-z^2}}\,dz = du$$

Integrating, we have

$$\pm\sqrt{1+b^2}\sqrt{a^2-z^2} = u + c$$

or $$(1 + b^2)(a^2 - z^2) = (x + by + c)^2$$

PART-2

Charpit's Method, Cauchy's Method of Characteristics.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.12. **Solve $(p^2 + q^2)\, y = qz$.**

Answer

Let $$f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0 \quad \text{...(1.12.}$$

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating $\quad p^2 + q^2 = c^2$ ...(1.12.2)

Now solving eq. (1.12.1) and eq. (1.12.2),

put $p^2 + q^2 = c^2$ in eq. (1.12.1), so that $q = c^2y/z$

Substituting this value of q in eq. (1.12.2), we get $p = \dfrac{c\sqrt{(z^2 - c^2y^2)}}{z}$

Hence $$dz = pdx + qdy = \frac{c\sqrt{(z^2 - c^2y^2)}}{z}dx + \frac{c^2y}{z}dy$$

or $$z\,dz - c^2y\,dy = c\sqrt{(z^2 - c^2y^2)}\,dx \text{ or } \frac{\frac{1}{2}d(z^2 - c^2y^2)}{\sqrt{(z^2 - c^2y^2)}} = cdx$$

Integrating, we get $c\sqrt{(z^2 - c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Que 1.13. **Solve $2xz - px^2 - 2qxy + pq = 0$**

Answer

Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(1.13.1)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore \quad dq = 0 \quad \text{or} \quad q = a.$

Putting $q = a$ in eq. (1.13.1), we get

$$p = \frac{2x(z - ay)}{x^2 - a}$$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady$$

or $$\frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$

which is the required complete solution.

Que 1.14. **Solve $2z + p^2 + qy + 2y^2 = 0$.**

Answer

Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$ ...(1.14.1)

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in eq. (1.14.1), we get

$$q = \frac{1}{y}\left[-2z - 2y^2 - (a-x)^2\right]$$

$$\therefore \quad dz = pdx + qdy$$

$$= (a-x)\,dx - \frac{1}{y}\left[2z + 2y^2 + (a-x)^2\right] dy$$

Multiplying both sides by $2y^2$,

$$2y^2\,dz + 4yz\,dy = 2y^2(a-x)\,dx - 4y^3\,dy - 2y(a-x)^2\,dy$$

Integrating $\quad 2zy^2 = -\left[y^2(a-x)^2 + y^4\right] + b$

or $\quad y^2\left[(x-a)^2 + 2z + y^2\right] = b$, which is the desired solution.

Que 1.15. **Solve the PDE $z_x z_y - z = 0$ subject to the condition $z(s, -s) = 1$.**

Answer

Here, we have

$$F(x, y, z, p, q) = pq - z$$

The characteristics system takes the form

$$\frac{dx}{dt} = F_p = q(t), \quad \frac{dy}{dt} = F_q = p(t), \quad \frac{dz}{dt} = pF_p + qF_q = 2p(t)q(t)$$

$$\frac{dp}{dt} = -\left[F_x + p(t)F_z\right] = p(t), \quad \frac{dq}{dt} = -\left[F_y + q(t)F_z\right] = q(t)$$

Note that $\quad \frac{dp}{dt} = p(t) \Rightarrow p(t) = ce^t$ and $\frac{dq}{dt} = q(t) \Rightarrow q(t) = de^t$

where c and d are arbitrary constants. Since we are looking for a characteristics strip (*i.e.*, $F(x, y, z, p, q) = 0$), we set $z(t) = p(t)\,q(t) = cde^{2t}$. The equations for the characteristic system are :

$$x(t) = de^t + d_1,\; y(t) = ce^t + c_1,\; z(t) = cde^{2t},\; p(t) = ce^t,\; q(t) = de^t,$$

where c_1 and d_1 are constants.

The initial condition $z(s, -s) = 1$ is given on the line $y = -x$ traced out by $(s, -s)$, we have $f(s) = s$ and $g(s) = -s$. We must find $h(s)$ and $k(s)$ such that

$$1 = G(s) = h(s)k(s) \quad 0 = G'(s) = h(s) - k(s),$$

$$0 = F_p(\ldots)(-1) - F_q(\ldots)(1) = -k(s) - h(s).$$

Thus, we have two choices $h(s) = 1$ and $k(s) = 1$, or $h(s) = -1$ and $k(s) = -1$.
For the choice $h(s) = 1$ and $k(s) = 1$, we obtain
$x(s, t) = e^t - 1 + s$, $y(s, t) = e^t - 1 - s$, $z(s, t) = e^{2t}$, $p(s, t) = e^t$, $q(s, t) = e^t$.
From the first two equations, we obtain

$$e^t = (x + y + 2)/2.$$

Then the solution is

$$z(x, y) = e^{2t} = \frac{(x + y + 2)^2}{4}$$

If we choose $h(s) = -1$ and $k(s) = -1$, the solution is given by

$$z(x, y) = \frac{(x + y + 2)^2}{4}$$

PART-3

Solution of Linear Partial Differential Equation of Higher Order with Constant Coefficient.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.16. **Solved :** $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

Answer

The given equation is

$$(D^2 - DD' - 6D'^2)z = 0$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Auxiliary equation is

$$m^2 - m - 6 = 0$$

$\Rightarrow$ $(m - 3)(m + 2) = 0 \Rightarrow m = 3, -2$

$\therefore$ $\text{CF} = f_1(y + 3x) + f_2(y - 2x)$

$$\text{PI} = 0$$

Hence the complete solution is

$$z = \text{CF} + \text{PI} = f_1(y + 3x) + f_2(y - 2x)$$

where f_1 and f_2 are arbitrary functions.

Que 1.17. **Solve the linear partial differential equation**

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$$

Answer

The given equation is $(D^4 + D'^4)z = 0$

Auxiliary equation is $m^4 + 1 = 0$

$\Rightarrow$ $m^4 + 1 + 2m^2 = 2m^2$

$\Rightarrow$ $(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$

$\Rightarrow$ $(m^2 + \sqrt{2}m + 1)(m^2 - \sqrt{2}m + 1) = 0$

so that $m^2 + \sqrt{2}m + 1 = 0$ or $m^2 - \sqrt{2}m + 1 = 0$

$\Rightarrow$ $m = \dfrac{-1 \pm i}{\sqrt{2}}, \dfrac{1 \pm i}{\sqrt{2}}$

Let $z_1 = \dfrac{-1 + i}{\sqrt{2}}$ and $z_2 = \dfrac{1 + i}{\sqrt{2}}$,

then, $m = z_1, \overline{z}_1, z_2, \overline{z}_2$

Here $\overline{z}_1$ and $\overline{z}_2$ denote complex conjugate of z_1 and z_2 respectively.

$$\text{CF} = f_1(y + z_1 x) + f_2(y + \overline{z}_1 x) + f_3(y + z_2 x) + f_4(y + \overline{z}_2 x)$$
$$\text{PI} = 0$$

Hence the complete solution is

$$z = \text{CF} + \text{PI} = f_1(y + z_1 x) + f_2(y + \overline{z}_1 x) + f_3(y + z_2 x) + f_4(y + \overline{z}_2 x)$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

Que 1.18. **Solve the linear partial differential equation**

$$\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}.$$

Answer

The given equation is

$(D^3 - 3D^2D' + 4D'^3)\, u = e^{x+2y}$ where $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$

Auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$
$$m^2(m + 1) - 4m(m + 1) + 4(m + 1) = 0$$
$$(m + 1)(m^2 - 4m + 4) = 0$$
$$(m - 2)^2 (m + 1) = 0$$

$$m = 2, 2, -1$$

$$\text{CF} = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y},$$

$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3} \iiint e^u \, du\, du\, du$$

where $x + 2y = u$

$$= \frac{1}{27} e^{x+2y}$$

Hence the complete solution is

$$u = \text{CF} + \text{PI}$$

$$= f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x) + \frac{1}{27} e^{x+2y}$$

where f_1, f_2 and f_3 are arbitrary functions.

Que 1.19. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$.

Answer

Given equation in symbolic form is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Auxiliary equation is $m^2 + m - 6 = 0$ where $m = -3, 2$.

$\therefore$ $\text{CF} = f_1(y - 3x) + f_2(y + 2x)$

Since $D^2 + DD' - 6D'^2 = -(2)^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method

$$\text{PI} = \frac{1}{D^2 - DD' - 6D'^2} \cos(2x + y)$$

$$= \frac{1}{(D + 3D')(D - 2D')} \cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})\, dx\right]_{c \to y+2x}$$

$$= \frac{1}{D + 3D'}\left[\int \cos c \, dx\right]_{c \to y+2x}$$

$[\because \; y = c - mx = c - 2x]$

$$= \frac{1}{D + 3D'} x \cos(y + 2x)$$

$$= \left[\int x \cos\left(\overline{c+3x}+2x\right)dx\right]_{c \to y-3x}$$

$$= \left[\int x \cos(5x+c)\, dx\right]_{c \to y-3x}$$

$$= \left[\frac{x \sin(5x+c)}{5} + \frac{\cos(5x+c)}{25}\right]_{c \to y-3x}$$

[Integrating by parts]

$$= \frac{x}{5}\sin\left(5x+\overline{y-3x}\right)+\frac{1}{25}\cos\left(5x+\overline{y-3x}\right)$$

$$= \frac{x}{5}\sin(2x+y)+\frac{1}{25}\cos(2x+y)$$

Hence the complete solution is

$$z = f_1(y-3x)+f_2(y+2x)+\frac{x}{5}\sin(2x+y)+\frac{1}{25}\cos(2x+y)$$

Que 1.20. Solve $r - 4s + 4t = e^{2x+y}$.

Answer

Given equation is $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Symbolic form is $(D^2 - 4DD' + 4D'^2)\, z = e^{2x+y}$.

Its auxiliary equation is $(m-2)^2 = 0$, where $m = 2, 2$.

$\therefore$

$$\text{CF} = f_1(y+2x) + x f_2(y+2x)$$

$$\text{PI} = \frac{1}{(D-2D')^2}\, e^{2x+y}$$

The usual rule fails because $(D-2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ To obtain the PI, we find from $(D-2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\, dx$$

$$= \int e^{2x+(c-2x)}\, dx = xe^c = xe^{2x+y}$$

$[\because \; y = c - mx = c - 2x]$

and from $(D-2D')\, z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)}\, dy = \frac{1}{2}x^2e^c = \frac{1}{2}x^2e^{2x+y}$$

$[\because \; y = c - mx = c - 2x]$

Hence the complete solution is $z = f_1(y+2x) + xf_2(y+2x) + \dfrac{1}{2}x^2e^{2x+y}$.

Que 1.21. Solve : $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x.$

Answer

The given equation is

$$(D^2 - 2DD' + D'^2)z = \sin x$$

Auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$$

$$\therefore \quad \text{CF} = f_1(y + x) + xf_2(y + x)$$

$$\text{PI} = \frac{1}{(D - D')^2} \sin (x + 0.y)$$

$$= \frac{1}{(1-0)^2} \iint \sin u \, du \, du, \qquad \text{where } x = u$$

$$= -\sin u = -\sin x$$

Hence the complete solution is

$$z = \text{CF} + \text{PI} = f_1(y + x) + xf_2(y + x) - \sin x$$

where f_1 and f_2 are arbitrary functions.

Que 1.22. Solve the linear partial differential equation

$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4 \sin (2x + y).$

Answer

The given equation is

$$(D^3 - 4D^2D' + 4DD'^2)z = 4 \sin (2x + y)$$

The auxiliary equation is

$$m^3 - 4m^2 + 4m = 0$$

$$\Rightarrow \quad m(m^2 - 4m + 4) = 0$$

$$\Rightarrow \quad m = 0, 2, 2.$$

$$\therefore \quad \text{CF} = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)$$

$$\text{PI} = \frac{1}{D^2 - 4D^2D' + 4DD'^2} 4 \sin (2x + y)$$

$$= \frac{4}{D}\left[\frac{1}{D^2 - 4DD' + 4D'^2} \sin (2x + y)\right]$$

$$= \frac{4}{D}\left[\frac{1}{(D - 2D')^2} \sin (2x + y)\right]$$

$$= x \cdot \frac{4}{D}\left[\frac{1}{2(D-2D')}\sin(2x+y)\right]$$

$$= 4x^2 \cdot \frac{1}{D}\left[\frac{1}{2}\sin(2x+y)\right]$$

$$= 2x^2 \frac{1}{D}\sin(2x+y)$$

$$= -2x^2 \frac{\cos(2x+y)}{2} = -x^2\cos(2x+y).$$

Hence the complete solution is

$$z = CF + PI$$

$$= f_1(y) + f_2(y+2x) + xf_3(y+2x) - x^2\cos(2x+y)$$

where f_1, f_2 and f_3 are arbitrary functions.

Que 1.23. **Solve :** $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y) + e^{3x+y}$.

Answer

The given equation is

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x+y}$$

Auxiliary equation is

$$m^3 - 7m - 6 = 0$$

$$(m+1)(m^2 - m - 6) = 0$$

$\Rightarrow$ $$m = -1, -2, 3$$

$\therefore$ $$CF = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$$

$$PI = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{3x+y}.$$

PI corresponding to $\sin(x+2y)$

$$= \frac{1}{(1)^3 - 7(1)(2)^2 - 6(2)^3}\iiint \sin u \, du \, du \, du,$$

where $x + 2y = u$

$$= -\frac{1}{75}\cos u = -\frac{1}{75}\cos(x+2y)$$

PI corresponding to e^{3x+y}

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}(e^{3x+y})$$

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}(D^3 - 7DD'^2 - 6D'^3)} e^{3x+y}$$

$$= x \cdot \frac{1}{3D^2 - 7D'^2} e^{3x+y}$$

$$= x \cdot \frac{1}{3(3)^2 - 7(1)^2} e^{3x+y} = x \cdot \frac{1}{20} e^{3x+y}$$

$\therefore$ Required PI $= -\frac{1}{75}\cos(2y + x) + \frac{x}{20} e^{3x+y}$

$\therefore$ Complete solution is

$$z = \text{CF} + \text{PI} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) - \frac{1}{75}\cos(x + 2y) + \frac{x}{20} e^{3x+y}$$

where f_1, f_2 and f_3 are arbitrary function.

Que 1.24. **Solve $(D^2 - DD')z = \cos x \cos 2y$**

Answer

Auxiliary equation is : $m^2 - m = 0$

$$m(m - 1) = 0$$

$$m = 0, 1$$

$$\text{CF} = f_1(y) + f_2(y + x)$$

$$\text{PI} = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$= \frac{1}{2}\frac{1}{D^2 - DD'}\{\cos(x + 2y) + \cos(x - 2y)\}$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x + 2y) + \frac{1}{D^2 - DD'}\cos(x - 2y)\right]$$

[Put $D^2 = -1$, $DD' = -2$ and $D^2 = -1$, $DD' = 2$]

$$= \frac{1}{2}\left[\frac{1}{-1-(-2)}\cos(x + 2y) + \frac{1}{-1-(2)}\cos(x - 2y)\right]$$

$$= \frac{1}{2}\left[\cos(x + 2y) - \frac{1}{3}\cos(x - 2y)\right]$$

Thus, the complete solution is

$$z = f_1(y) + f_2(y + x) + \frac{1}{2}\left\{\cos(x + 2y) - \frac{1}{3}\cos(x - 2y)\right\}$$

Que 1.25. **Solve : $r + s - 2t = \sqrt{2x + y}$.**

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The given equation is $r + s - 2t = \sqrt{2x + y}$.

We know that

$$r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = \sqrt{2x + y}$$

$$(D^2 + DD' - 2D'^2) z = \sqrt{2x + y}$$

Auxiliary equation is

$$m^2 + m - 2 = 0$$

$$(m - 1)(m + 2) = 0$$

$$m = 1, -2$$

$$\text{CF} = f_1(y + x) + f_2(y - 2x)$$

$$\text{PI} = \frac{1}{D^2 + DD' - 2D'^2}\sqrt{2x + y}$$

Put $D = 2$, $D' = 1$, let $2x + y = u$

$$= \frac{1}{(2)^2 + (2)(1) - 2(1)^2}\iint \sqrt{u}\, du\, du$$

$$= \frac{1}{4} \cdot \frac{4}{15} u^{5/2}$$

$$\text{PI} = \frac{1}{15}(2x + y)^{5/2}$$

$\therefore$ Complete solution is

$$z = \text{CF} + \text{PI}$$

$$= f_1(y + x) + f_2(y - 2x) + \frac{1}{15}(2x + y)^{5/2}$$

where f_1 and f_2 are arbitrary functions

Que 1.26. **Solve $r + (a + b)s + abt = xy$.**

Answer

Given equation $r + (a + b)s + ab\,t = xy$

We know that

$$r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} + (a + b)\frac{\partial^2 z}{\partial x \partial y} + ab\frac{\partial^2 z}{\partial y^2} = xy$$

$$(D^2 + (a + b)DD' + ab D'^2) z = xy$$

Put

$$D = m \text{ and } D' = 1$$

Auxiliary equation is :

$$m^2 + (a+b)\,m + ab = 0$$
$$m^2 + am + bm + ab = 0$$
$$m(m+a) + b(m+a) = 0$$
$$(m+a)(m+b) = 0$$
$$m = -a, -b$$
$$\text{CF} = f_1(y-ax) + f_2(y-bx)$$
$$\text{PI} = \frac{1}{(D^2 + (a+b)\,D\,D' + ab\,D'^2)}\,xy$$
$$\text{PI} = \frac{1}{(D^2 + a\,D\,D' + b\,D\,D' + ab\,D'^2)}\,xy$$
$$= \frac{1}{D(D+aD') + bD'(D+aD')}\,xy$$
$$= \frac{1}{(D+aD')(D+bD')}\,xy$$
$$= \frac{1}{D\left[1+\frac{aD'}{D}\right]D\left[1+\frac{bD'}{D}\right]}\,xy$$
$$= D^{-1}\left[1\;\frac{aD'}{D}\right]^{-1} D^{-1}\left[1+\frac{bD'}{D}\right]^{-1} xy$$
$$= \frac{1}{D^2}\left[1+\frac{aD'}{D}+\frac{a^2D'^2}{D^2}\cdots\right]\left[1-\frac{bD'}{D}+\frac{b^2D'^2}{D^2}\cdots\right]xy$$
$$= \frac{1}{D^2}\left[1+\frac{aD'}{D}\right]\left[1-\frac{bD'}{D}\right]xy$$

Neglecting higher terms

$$= \frac{1}{D^2}\left[1-\frac{bD'}{D}+\frac{aD'}{D}-\frac{ab\,D'^2}{D^2}\right]xy$$
$$= \frac{1}{D^2}\left[xy - b\frac{x^2}{2}+\frac{ax^2}{2}-\frac{ab(1)}{D^2}(0)\right]$$
$$= \frac{1}{D^2}\left[xy+\frac{x^2}{2}(a-b)\right]$$

Integrating twice, we get

$$\text{PI} = \frac{x^3y}{6}+\frac{x^4}{24}(a-b)$$
$$\text{PI} = \frac{x^3y}{6}+(a-b)\frac{x^4}{24}$$

Complete solution

$$z = \text{CF} + \text{PI}$$
$$= f_1(y-ax) + f_2(y-bx) + \frac{x^3y}{6} + \frac{(a-b)}{24}x^4$$

Que 1.27. **Solve the following partial differential equation :**

$$(D^2 - DD' - 2D'^2 + 2D + 2D')\, z = \sin(2x + y)$$

where notations have their usual meaning.

Answer

The given differential equation is

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$$

That can be written as

$$(D + D')(D - 2D' + 2)z = 0$$

Comparing it with

$$(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$$

$$m_1 = -1,\ a_1 = 0,\ m_2 = 2,\ a_2 = -2$$

Complementary function

$$\text{CF} = f_1(y - x)e^{0x} + f_2(y + 2x)\, e^{-2x}$$

$$= f_1(y - x) + f_2(y + 2x)e^{-2x}$$

$$\text{PI} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x + y)$$

Put, $D^2 = -4,\ D'^2 = -1,\ DD' = -2$

$$= \frac{1}{-4 + 2 + 2 + 2D + 2D'} \sin(2x + y)$$

$$= \frac{1}{2(D + D')} \sin(2x + y)$$

$$= \frac{1}{2} \frac{(D - D')}{(D + D')(D - D')} \sin(2x + y)$$

$$= \frac{1}{2} \frac{(D - D')}{D^2 - D'^2} \sin(2x + y)$$

$$= \frac{1}{2(-4 + 1)} (D - D') \sin(2x + y)$$

$$= -\frac{1}{6} \{2\cos(2x + y) - \cos(2x + y)\}$$

$$= -\frac{1}{6} \cos(2x + y)$$

$$z = \text{CF} + \text{PI}$$

$$z = f_1(y - x) + f_2(y + 2x)e^{-2x} - \frac{1}{6} \cos(2x + y)$$

PART-4

Equations Reducible to Linear Partial Differential Equations with Constant Coefficients.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.28. Solve the linear partial differential equation

$$x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4.$$

Answer

Put $x = e^X, y = e^Y$ so that $X = \log x$ and $Y = \log y$ and let $D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$ and

$DD' = \frac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$[D(D-1) - 4DD' + 4D'(D'-1) + 6D']z = e^{3X+4Y}$

$\Rightarrow$ $[(D^2 - 4DD' + 4D'^2) - (D - 2D')]z = e^{3X+4Y}$

$\Rightarrow$ $(D - 2D')(D - 2D' - 1)z = e^{3X+4Y}$

Its

$$\begin{aligned} \text{CF} &= f_1(Y + 2X) + e^X f_2(Y + 2X) \\ &= f_1(\log y + 2\log x) + x f_2(\log y + 2\log x) \\ &= f_1(\log yx^2) + x f_2(\log yx^2) = g_1(yx^2)) + x g_2(yx)^2 \end{aligned}$$

$$\begin{aligned} \text{PI} &= \frac{1}{D - 2D' - 1}\left[\frac{1}{D - 2D'} e^{3X+4Y}\right] \\ &= \frac{1}{D - 2D' - 1}\left[\frac{1}{3-8}\int e^u \, du\right] \text{ where } 3X + 4Y = u \\ &= \frac{1}{D - 2D' - 1}\left[-\frac{1}{5} e^{3X+4Y}\right] \\ &= -\frac{1}{5}\left[\frac{1}{D - 2D' - 1} e^{3X+4Y}\right] \\ &= -\frac{1}{5}\left[\frac{1}{3 - 8 - 1} e^{3X+4Y}\right] = \frac{1}{30} e^{3X+4Y} = \frac{1}{30} x^3 y^4 \end{aligned}$$

Hence the complete solution is

$$z = \text{CF} + \text{PI} = g_1(yx^2) + xg_2(yx^2) + \frac{1}{30}x^3y^4$$

where g_1 and g_2 are arbitrary functions.

Que 1.29. **Solve : $(x^2D^2 + 2xy\,DD' + y^2D'^2)z = x^my^n$.**

Answer

Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$ and let $D = \dfrac{\partial}{\partial X}$, $D' = \dfrac{\partial}{\partial Y}$ and

$DD' = \dfrac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$[D(D-1) + 2DD' + D'(D'-1)]z = e^{mX+nY}$

$\Rightarrow \quad (D^2 + 2DD' + D'^2 - D - D')z = e^{mX+nY}$

$\Rightarrow \quad [(D+D')^2 - (D+D')]z = e^{mX+nY}$

$\Rightarrow \quad (D+D')(D+D'-1)z = e^{mX+nY}$

$$\therefore \quad \text{CF} = f_1(Y-X) + e^X f_2(Y-X)$$

$$= f_1(\log y - \log x) + xf_2(\log y - \log x)$$

$$= f_1\left(\log\frac{y}{x}\right) + xf_2\left(\log\frac{y}{x}\right) = g_1\left(\frac{y}{x}\right) + xg_2\left(\frac{y}{x}\right)$$

$$\text{PI} = \frac{1}{(D+D')(D+D'-1)}e^{mX+nY}$$

$$= \frac{1}{(m+n)(m+n-1)}e^{mX+nY}$$

$$= \frac{x^my^n}{(m+n)(m+n-1)}$$

Hence complete solution is

$$z = \text{CF} + \text{PI}$$

$$= g_1(y/x) + xg_2(y/x) + \frac{x^my^n}{(m+n)(m+n-1)}$$

where g_1 and g_2 are arbitrary functions.

Que 1.30. **Solve : $x^2r - y^2t + px - qy = \log x$.**

Answer

Let $x = e^X$, $y = e^Y$ so that $X = \log x$ and $Y = \log y$ and let $D = \dfrac{\partial}{\partial X}$ and $D' = \dfrac{\partial}{\partial X}$,

then the given equation reduces to

$$[D(D-1) - D'(D'-1) + D - D']z = X$$

$$\Rightarrow \quad (D^2 - D'^2)z = X \quad \text{...(1.30.1)}$$

which is a homogeneous linear partial differential equation, with constant coefficients.

$$CF = \phi_1(Y + X) + \phi_2(Y - X)$$

and

$$PI = \frac{1}{D^2 - D'^2}(X) = \frac{1}{(1)^2 - (0)^2}\iint u\, du\, du$$

where

$$X = u$$

$$= \int \frac{u^2}{2}\, du = \frac{u^3}{6} = \frac{X^3}{6}$$

Hence solution to eq. (1.30.1) is

$$z = \phi_1(Y + X) + \phi_2(Y - X) + \frac{X^3}{6}$$

$$= \phi_1(\log y + \log x) + \phi_2(\log y - \log x) + \frac{(\log x)^3}{6}$$

Therefore the complete solution to the given differential equation is

$$z = \phi_1(\log xy) + \phi_2\left(\log \frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

$$z = f_1(xy) + f_2\left(\frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

where f_1 and f_2 are arbitrary functions.

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UNIT

Applications of Partial Differential Equations

CONTENTS

PART-1

Classification of Linear Partial Differential Equation of Second Order, Method of Separation of Variable.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. **Classify the following partial differential equation**

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0.$$

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Answer

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0$$

On comparing above equation with ideal form,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

$$A = (1-x^2)$$

$$B = -2xy$$

$$C = (1-y^2)$$

$$\begin{aligned} B^2 - 4AC &= (-2xy)^2 - 4(1-x^2)(1-y^2) \\ &= 4x^2y^2 - 4(1 - y^2 - x^2 + x^2y^2) \\ &= 4x^2y^2 - 4 + 4y^2 + 4x^2 - 4x^2y^2) \\ &= 4(x^2 + y^2) - 4 \end{aligned}$$

For hyperbolic : $B^2 - 4AC > 0$, for $x \geq 1$ or $y \geq 1$ or both $x, y \geq 1$

For elliptical : $B^2 - 4AC < 0$, for x and $y \leq 0$

For parabolic : $B^2 - 4AC = 0$, for any of x and $y = 1$ and 0

Que 2.2. **Apply method of separation of variables to solv**

$\frac{\partial^2 z}{\partial x \partial y} = e^{-x}\cos y$, **given that $z = 0$ when $x = 0$ and $\frac{\partial z}{\partial x} = 0$ wh**
$y = 0$.

Answer

$$\frac{\partial^2 z}{\partial x\,\partial y} = e^{-x}\cos y \quad ...(2.2.1)$$

Let $$z = X(x).\ Y(y)$$

where X is a function of x only and Y is a function of y only.

$$\frac{\partial z}{\partial y} = X\frac{\partial Y}{\partial y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}$$

From given eq. (2.2.1),

$$\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} = e^{-x}\cos y$$

$$e^{x}\frac{\partial X}{\partial x} = \frac{\cos y}{\frac{\partial Y}{\partial y}} = k \text{ (say)}$$

Now $$e^{x}\frac{\partial X}{\partial x} = k$$

$$\partial X = ke^{-x}\partial x$$

$$X = -ke^{-x} + C_1$$

and $$k\frac{\partial Y}{\partial y} = \cos y$$

$$\partial Y = \frac{1}{k}\cos y\,\partial y$$

$$Y = \frac{1}{k}\sin y + C_2$$

Thus $$z = XY$$

$$z = (-ke^{-x} + C_1)\left(\frac{1}{k}\sin y + C_2\right) \quad ...(2.2.2)$$

Putting $$z = 0 \text{ when } x = 0$$

$$0 = -k + C_1$$

$$C_1 = k$$

From eq. (2.2.2),

$$z = (-ke^{-x} + k)\left(\frac{1}{k}\sin y + C_2\right) \quad ...(2.2.3)$$

$$\frac{\partial z}{\partial x} = ke^{-x}\left(\frac{1}{k}\sin y + C_2\right)$$

Putting $$\frac{\partial z}{\partial x} = 0,\ y = 0$$

$$0 = ke^{-x}(0 + C_2)$$

$$C_2 = 0$$

From eq. (2.2.3), $z = k(1-e^{-x})\left(\frac{1}{k}\sin y\right) = (1-e^{-x})\sin y$

Que 2.3. **Solve by separation of variables :**

$$y^3 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0$$

Answer

The given equation is

$$y^3 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0 \quad \text{...(2.3.1)}$$

Let $u = XY$, where X is a function of x and Y is a function of y only.

$$\frac{\partial u}{\partial x} = X'Y$$

and

$$\frac{\partial u}{\partial y} = XY'$$

Put these values in eq. (2.3.1),

$$y^3 Y \frac{dX}{dx} + x^2 X \frac{dY}{dy} = 0$$

Dividing by XY, we get

$$\frac{y^3 dX}{X dx} + \frac{x^2}{Y}\frac{dY}{dy} = 0$$

$$\frac{y^3}{X}\frac{dX}{dx} = -\frac{x^2}{Y}\frac{dY}{dy}$$

or

$$\frac{dX}{x^2 X dx} = -\frac{1}{y^3 Y}\frac{dY}{dy} = k \text{ (say)}$$

Taking

$$\frac{dX}{x^2 X dx} = k$$

$$\Rightarrow \quad \frac{1}{X} dX = kx^2 dx$$

On integrating we get, $\log X = k\frac{x^3}{3} + C_1$

$$X = e^{\left(\frac{kx^3}{3} + C_1\right)}$$

Taking

$$-\frac{1}{y^3 Y}\frac{dY}{dy} = k$$

$$\frac{dY}{Y} = -y^3\, k dy$$

$$\log Y = -k\frac{y^4}{4} + C_2$$

$$Y = e^{\left(-\frac{ky^4}{4} + C_2\right)}$$

Therefore the complete solution is,

$$u = XY$$

$$= e^{\left(k\frac{x^3}{3} + C_1\right)} e^{\left(-k\frac{y^4}{4} + C_2\right)} = e^{k\left(\frac{x^3}{3} - \frac{y^4}{4}\right) + C}$$

$$[C = C_1 + C_2]$$

Que 2.4. **Solve $\frac{\partial^2 u}{\partial x^2} = 2u + \frac{\partial u}{\partial y}$ using method of separation of variables subject to the conditions $u = 0$ and $\frac{\partial u}{\partial x} = e^{-3y}$ when $x = 0$ for all value of y.**

Answer

$$\frac{\partial^2 u}{\partial x^2} = 2u + \frac{\partial u}{\partial y} \quad \text{...(2.4.1)}$$

Let $u = XY$ ...(2.4.2)

$$\Rightarrow \quad \frac{\partial u}{\partial y} = X\frac{\partial Y}{\partial y}, \frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

From eq. (2.4.1),

$$Y \cdot \frac{\partial^2 X}{\partial x^2} = 2XY + X \cdot \frac{\partial Y}{\partial y}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial Y}{\partial y} + 2 = (k^2) \text{ (say)}$$

Now,

$$\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = k^2$$

$$(D^2 - k^2)X = 0$$

$$m^2 - k^2 = 0$$

$$m = \pm k$$

$$X = C_1 e^{kx} + C_2 e^{-kx}$$

and

$$\frac{1}{Y}\frac{\partial Y}{\partial y} = k^2 - 2$$

$$Y = C_3 e^{(k^2-2)y}$$

From eq. (2.4.2),

$$u = C_3 e^{(k^2-2)y}(C_1 e^{kx} + C_2 e^{-kx}) \quad \text{...(2.4.3)}$$

At $x = 0$, $u = 0$

$$0 = C_3 e^{(k^2 - 2)y} (C_1 + C_2)$$

$$C_1 + C_2 = 0$$

$$C_2 = -C_1$$

From eq. (2.4.3),

$$u = C_1 C_3 e^{(k^2 - 2)y} (e^{kx} - e^{-kx})$$

$$u = A_n e^{(k^2 - 2)y} (e^{kx} - e^{-kx}) \quad \text{...(2.4.4)}$$

$$\frac{\partial u}{\partial x} = A_n e^{(k^2 - 2)y} k(e^{kx} + e^{-kx})$$

At $x = 0$,

$$\frac{\partial u}{\partial x} = e^{-3y}$$

$$2 A_n k e^{(k^2 - 2)y} = e^{-3y}$$

$$2A_1 k = 1 \quad \text{and} \quad (k^2 - 2) = -3$$

$$A_1 = \frac{1}{2k} \qquad k = i$$

So,

$$A_1 = \frac{1}{2i}$$

Thus from eq. (2.4.4),

$$u = \frac{1}{2i} e^{-3y} (e^{ix} - e^{-ix})$$

Que 2.5. **Solve by the method of separation of variables :**

$$x \frac{\partial^2 u}{\partial x \partial y} + 2yu = 0.$$

AKTU 2014-15(II), Marks 05

Answer

$$x \frac{\partial^2 u}{\partial x \partial y} + 2yu = 0$$

Let $u = XY$, where X is a function of x and y is a function of Y only.

$$x \frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} + 2y\, XY = 0$$

$$\frac{x \partial X}{\partial x} \frac{\partial Y}{\partial y} = -2yXY$$

Divide by X, we get

$$\frac{x}{X} \frac{\partial X \partial Y}{\partial x \partial y} = -2yY$$

$$\frac{x}{X} \frac{\partial X}{\partial x} = \frac{-2yY}{\left(\frac{\partial Y}{\partial y}\right)} = k \text{ (say)}$$

Now when

$$\frac{x}{X}\frac{\partial X}{\partial x} = k \text{ and } \frac{-2yY}{k} = \frac{\partial Y}{\partial y}$$

put
$$x = e^z$$
$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial z}$$
$$\frac{\partial X}{\partial z} - kX = 0$$
$$m = k$$
$$X = k_1 e^{kz}$$
$$X = C_1 e^{k \log x}$$
$$X = C_1 x^k$$

When,
$$-\frac{2y}{k}\partial y = \frac{\partial Y}{Y}$$
$$-\frac{y^2}{k} = \log Y - \log C_2$$
$$\frac{Y}{C_2} = e^{-y^2/k}$$
$$Y = C_2 e^{-y^2/k}$$

Thus,
$$u = XY$$
$$u = C_1 C_2 x^k e^{-y^2/k}$$
$$u = Ax^k e^{-y^2/k} \qquad [\because \ C_1C_2 = A]$$

Que 2.6. **Solve by method of separation of variable for PDE**

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0, \ u(x, 0) = 4e^{-x}.$$

AKTU 2016-17(II), Marks 07

Answer

Given, $x\dfrac{\partial u}{\partial x} + \dfrac{2\partial u}{\partial y} = 0$ ...(2.6.1)

Assuming $x = 3$ in eq. (2.6.1), we get

$$3\frac{\partial u}{\partial x} + \frac{2\partial u}{\partial y} = 0$$

Let $u = XY$, where X is a function x and Y is a function of y only.

$$\frac{\partial u}{\partial x} = X'Y \qquad ...(2.6.2)$$

and
$$\frac{\partial u}{\partial y} = XY' \qquad ...(2.6.3)$$

Putting $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in eq. (2.6.1), we get

$$3X'Y + 2XY' = 0$$

Dividing by XY, we get

$$3\frac{X'}{X} + 2\frac{Y'}{Y} = 0$$

or $$\frac{3X'}{X} = -\frac{2Y'}{Y} = k \text{ (say)}$$

Taking $$3\frac{X'}{X} = k$$

$$\frac{dX}{Xdx} = \frac{k}{3}$$

$$\frac{dX}{X} = \frac{k}{3}\,dx$$

On integrating, we get

$$\log X = \frac{k}{3}x + \log C_1$$

$$X = C_1 e^{\frac{k}{3}x}$$

Similarly, $$\frac{Y'}{Y} = -\frac{k}{2}$$

$$\frac{dY}{Y} = -\frac{k}{2}\,dy$$

On integrating, we get,

$$\log Y = -\frac{k}{2}y + \log C_2$$

$$Y = C_2 e^{-\frac{k}{2}y}$$

Therefore the complete solution

$$u = XY$$

$$u = C_1C_2 e^{\frac{k}{3}x} . e^{-\frac{k}{2}y}$$

$$u = C_1C_2 e^{\frac{k}{3}x - \frac{k}{2}y} \quad \text{...(2.6.4)}$$

Now, $$u(x, 0) = C_1C_2 e^{\frac{k}{3}x}$$

$$4e^{-x} = C_1C_2 e^{\frac{k}{3}x}$$

On comparing the coefficients, we get

$$C_1C_2 = 4 \text{ and } \frac{k}{3} = -1 \quad \therefore \quad k = -3$$

Putting the value of C_1C_2 and k in eq. (2.6.4), we get

$$u(x, y) = 4e^{-x + \frac{3}{2}y}$$

Que 2.7. **Solve the P.D.E. by separation of variables method,**

$$u_{xx} = u_y + 2u,\ u(0, y) = 0,\ \frac{\partial}{\partial x} u(0, y) = 1 + e^{-3y}.$$

Answer

Let $$u = XY \quad \text{...(2.7.1)}$$
where X is a function of x only and Y is a function of y only.

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} = YX''$$

From the given equation,

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y}$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{Y'}{Y} + 2 = k\text{(say)} \quad \text{...(2.7.2)}$$

Taking $$\frac{X''}{X} = k$$

$$\Rightarrow \quad X'' - kX = 0$$

Auxiliary equation is

$$m^2 - k = 0$$

$$\Rightarrow \quad m = \pm\sqrt{k}$$

$$\therefore \quad \text{C.F.} = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

$$\text{P.I.} = 0$$

$$\therefore \quad X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x} \quad \text{...(2.7.3)}$$

Taking $$\frac{Y'}{Y} + 2 = k$$

$$\Rightarrow \quad \frac{Y'}{Y} = k - 2$$

$$\Rightarrow \quad \frac{dY}{Y} = (k-2)\,dy$$

Integration yields,

$$\log Y = (k-2)y + \log C_3$$

$$\Rightarrow \quad Y = C_3 e^{(k-2)y} \quad \text{...(2.7.4)}$$

Hence from eq. (2.7.1)

$$u(x, y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x})\, C_3 e^{(k-2)y} \quad \text{...(2.7.5)}$$

Applying the condition $u(0, y) = 0$ in eq. (2.7.5), we get

$$u(0, y) = 0 = (C_1 + C_2)\, C_3 e^{(k-2)y}$$

$$\Rightarrow \quad C_1 + C_2 = 0 \Rightarrow C_2 = -C_1 \qquad \text{...(2.7.6)}$$

From eq. (2.7.5), most general solution is

$$u(x, y) = \Sigma C_1 C_3 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right) e^{(k-2)y} \qquad \text{...(2.7.7)}$$

$$\frac{\partial u}{\partial x} = \Sigma C_1 C_3 \sqrt{k} \left(e^{\sqrt{k}x} + e^{-\sqrt{k}x}\right) e^{(k-2)y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 1 + e^{-3y} = \Sigma C_1 C_3 \sqrt{k}(2)\, e^{(k-2)y} = \sum_{n=1}^{\infty} b_n\, e^{(k-2)y}$$

Comparing the coefficients, we get

$$b_1 = 1, \quad k - 2 = 0$$

$$2C_1C_3\sqrt{k} = 1, k = 2$$

$$\therefore \quad C_1C_3 = \frac{1}{2\sqrt{2}}$$

$$b_3 = -1, \quad k - 2 = -3$$

$$2C_1C_3\sqrt{k} = 1, k = -1$$

$$\therefore \quad C_1C_3 = \frac{1}{2i}$$

Hence from eq. (2.7.1), the particular solution is

$$u(x, y) = \frac{1}{2\sqrt{2}}(e^{\sqrt{2}x} - e^{-\sqrt{2}x}) + \frac{1}{2i}(e^{ix} - e^{-ix})\, e^{-3y}$$

$$\Rightarrow \quad u(x, y) = \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, x + e^{-3y} \sin x$$

Que 2.8. **Solve the following equation by the method of separation of variables $\frac{\partial^2 u}{\partial x\, \partial t} = e^{-t} \cos x$ given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$.**

Answer

Let $\quad u = XT \qquad \text{...(2.8.1)}$

where X is a function of x only and T is a function of t only.

Then,
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\, \partial t} = \frac{\partial}{\partial x}\left(X\frac{dT}{dt}\right) = \frac{dT}{dt} \cdot \frac{dX}{dx} \qquad \text{...(2.8.2)}$$

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \qquad \text{...(2.8.3)}$$

Substituting eq. (2.8.2) in eq. (2.8.3), we get

$$\frac{dT}{dt}\,\frac{dX}{dx} = e^{-t} \cos x$$

$$e^t \frac{dT}{dt} = \frac{\cos x}{\left(\frac{dX}{dx}\right)} = -p^2 \text{ (say)} \quad ...(2.8.4)$$

Now, $e^t \dfrac{dT}{dt} = -p^2$

$\Rightarrow$ $dT = -p^2 e^{-t}\, dt$

Integration yields, $T = -p^2 e^{-t} + C_1$ ...(2.8.5)

Also, $\dfrac{dX}{dx} = -\dfrac{1}{p^2}\cos x$

$$dX = -\frac{1}{p^2}\cos x \; dx$$

Integration yields, $X = -\dfrac{1}{p^2}\sin x + C_2$ ...(2.8.6)

Using eq. (2.8.5) and eq. (2.8.6), we get from eq. (2.8.1),

$$u(x, t) = XT = \left(-\frac{1}{p^2}\sin x + C_2\right)(p^2 e^{-t} + C_1) \quad ...(2.8.7)$$

Applying the condition $u = 0$ when $t = 0$ in eq. (2.8.7), we get

$$0 = \left(-\frac{1}{p^2}\sin x + C_2\right)(p^2 + C_1)$$

$\Rightarrow$ $p_2 + C_1 = 0 \Rightarrow C_1 = -p^2$

From eq. (2.8.7), $\dfrac{\partial u}{\partial t} = \left(-\dfrac{1}{p^2}\sin x + C_2\right)(-p^2 e^{-t})$ ...(2.8.8)

Applying the condition

$\dfrac{\partial u}{\partial t} = 0$ when $x = 0$ in eq. (2.8.8), we get

$$0 = C_2(-p^2 e^{-t})$$

$\Rightarrow$ $C_2 = 0$

Substituting the values of C_1 and C_2 in eq. (2.8.7), we get

$$u(x, t) = -\frac{1}{p^2}\sin x\,(p^2 e^{-t} - p^2) = \sin x\,(1 - e^{-t})$$

PART-2

Solution of Wave and Heat Conduction Equation upto Two Dimensions.

Questions-Answers
Long Answer Type and Medium Answer Type Questions

Que 2.9. Write the solution of one dimensional wave equation.

Answer

One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } C^2 = \frac{T}{m} \qquad ...(2.9.1)$$

T = Tension in the string.

m = Mass per unit length of the string.

Solution of one dimensional wave equation is done by method of separation of variables.

Let

$$u = X(x)\, T(t) \qquad ...(2.9.2)$$

Where X is a function of x only and T is a function of t only.

Differentiating eq. (2.9.2) partially w.r.t. x and t respectively and putting the values in eq. (2.9.1),

$$\frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2} \text{ and}$$

$$\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

From one dimensional wave equation,

$$X \frac{\partial^2 T}{\partial t^2} = C^2 T \frac{\partial^2 X}{\partial x^2}$$

Separating the variables,

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2 \text{ or } k^2 \text{ or } 0 \text{ (say)}$$

Case i : When $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k^2$ and $\frac{1}{C^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2$

or $$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0 \text{ and } \frac{\partial^2 T}{\partial t^2} + k^2 C^2 T = 0$$

or $$(D^2 + k^2)X = 0 \text{ and } (D^2 + k^2C^2)T = 0$$

Auxiliary equations are $m^2 + k^2 = 0$ and $m^2 + k^2C^2 = 0$

$$m = \pm ki \text{ and } m = \pm kCi$$

Thus complementary functions are

$$X = C_1 \cos kx + C_2 \sin kx$$

and $$T = C_3 \cos kCt + C_4 \sin kCt$$

$\therefore$ $$u = XT$$

$\Rightarrow$ $$u = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos kCt + C_4 \sin kCt) \qquad ...(2.9.3)$$

Case ii :

When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$ and $\frac{1}{C^2T}\frac{\partial^2 T}{\partial t^2} = k^2$

$$m = k^2 \text{ and } m^2 = k^2C^2$$

$$m = \pm k \text{ and } m \pm kC$$

$\Rightarrow$ $X = C_5 e^{kx} + C_6 e^{-kx}$

and $T = C_7 e^{kCt} + C_8 e^{-kCt}$

Thus $u = (C_5 e^{kx} + C_6 e^{-kx})(C_7 e^{kCt} + C_8 e^{-kCt})$ ...(2.9.4)

Case iii : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$ and $\frac{1}{C^2T}\frac{\partial^2 T}{\partial t^2} = 0$

$$m = 0, 0 \text{ and } m = 0, 0$$

$\Rightarrow$ $X = C_9 + C_{10}x$ and $T = C_{11} + C_{12}t$

Thus $u = (C_9 + C_{10}x)(C_{11} + C_{12}t)$ ...(2.9.5)

The solution given by eq. (2.9.3) satisfies the one dimensional wave equation. Thus the required solution of one dimensional wave is given by eq. (2.9.3). Now to find the values of C_1, C_2, C_3 and C_4, which are obtained by applying boundary and initial conditions.

Boundary conditions : Let in one dimensional wave equation $u(x, t)$ is the deflection of the string stretched between two fixed points (0, 0) and $(l, 0)$. Let $f(x)$ be the initial deflection and $g(x)$ be the initial velocity. The two boundary conditions are

$u(0, t) = 0$ ...(i)

$u(l, t) = 0$ ...(ii)

Initial conditions : The two initial conditions are

$u(x, 0) = f(x)$ ...(iii)

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$ ...(iv)

Initially if the string is at rest, $g(x) = 0$. Now to find the constants of eq. (2.9.3), apply the boundary condition (i) to eq. (2.9.3).

$$0 = C_1 (C_3 \cos kCt + C_4 \sin kCT)$$

$\Rightarrow$ $C_1 = 0$ $[\because \; C_3 \cos kCt + C_4 \sin kCt \neq 0]$

From eq. (2.9.3),

$u = C_2 \sin kx \,(C_3 \cos kCt + C_4 \sin kCt)$ ...(2.9.6)

Now put boundary condition (ii), in eq. (2.9.6),

$$0 = C_2 \sin kl \,(C_3 \cos kCt + C_4 \sin kCt)$$

$\Rightarrow$ $\sin kl = 0$

$$kl = n\pi$$

$$k = \frac{n\pi}{l}$$

From eq. (2.9.5),

$$u = C_2 \sin\frac{n\pi x}{l}\left(C_3 \cos\frac{n\pi Ct}{l} + C_4 \sin\frac{n\pi Ct}{l}\right)$$

$$u = \sin\frac{n\pi x}{l}\sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi Ct}{l} + B_n \sin\frac{n\pi Ct}{l}\right) \quad \text{...(2.9.7)}$$

Now apply initial conditions (iii) and (iv)

$$u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = f(x) \quad \text{...(2.9.8)}$$

and

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{t=0}^{\infty}\frac{n\pi C}{l} B_n \sin\frac{n\pi x}{l} = g(x) \quad \text{...(2.9.9)}$$

The left hand side of the eq. (2.9.8) and eq. (2.9.9) represents Fourier sine expansion of the right hand side. Thus

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

and

$$\frac{n\pi C}{l} B_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Putting the values of A_n and B_n in eq. (2.9.7), we obtain the required solution of one dimensional wave equation.

Que 2.10. **Find the deflection $u(x, t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by**

$$\sin \pi x + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x),\ 0 \le x \le 1$$

Answer

Given wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad \text{...(2.10.1)}$$

Let $y = XT$, where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 y}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2}$$

and

$$\frac{\partial^2 y}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

Substituting these values in eq. (2.10.1)

$$X\frac{\partial^2 T}{\partial t^2} = a^2 T\frac{\partial^2 X}{\partial x^2}$$

Separating the variables,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{a^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \text{(say)}$$

$$(D^2 + k^2)X = 0 \text{ and } (D^2 + a^2k^2)T = 0$$

$$m = \pm ki \text{ and } m = \pm aki$$

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 \cos akt + C_4 + \sin akt$$

Thus $y = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos akt + C_4 \sin akt)$ ...(2.10.2)

The boundary conditions are,

$$y(0, t) = 0$$

$$y(1, t) = 0$$

Initial conditions are,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$y(x, 0) = \sin \pi x + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x),\ 0 \le x \le 1$$

Put $x = 0, y = 0$ in eq. (2.10.2),

$$0 = C_1(C_3 \cos akt + C_4 \sin akt)$$

$$C_1 = 0$$

From eq. (2.10.2),

$$y = C_2 \sin kx \cdot (C_3 \cos akt + C_4 \sin akt) \quad ...(2.10.3)$$

Now

$$\frac{\partial y}{\partial t} = C_2 \sin kx \cdot (-ak\, C_3 \sin akt + akC_4 \cos akt)$$

$$= ak\, C_2 \sin kx \cdot (-C_3 \sin akt + C_4 \cos akt)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$0 = ak\, C_2 \sin kx (C_4)$$

$$C_4 = 0$$

From eq. (2.10.3),

$$y = C_2 \sin kx \cdot C_3 \cos akt$$

$$y = A_n \sin kx \cdot \cos akt \quad [\because C_2C_3 = A_n] \quad ...(2.10.4)$$

Put $x = 1, y = 0$ in eq. (2.10.4)

$$0 = A_n \sin k \cdot \cos akt$$

$$\sin k = 0$$

$$k = n\pi$$

From eq. (2.10.4),

$$y = A_n \sin(n\pi x) \cos(an\pi t) \quad ...(2.10.5)$$

Now, $t = 0$ $\quad y = \sin \pi x + \frac{1}{3} \sin (3\pi x) + \frac{1}{5} \sin (5\pi x)$

$\sum A_n \sin (n\pi x) = \sin \pi x + \frac{1}{3} \sin (3\pi x) + \frac{1}{5} \sin (5\pi x)$

which will be satisfied by taking

$$A_n = \frac{1}{n} \text{ and } n = 1, 3, 5$$

Hence the required solution is from eq. (2.10.5)

$$y = \frac{1}{n} \sin (n\pi x) \cos (an\pi t) \text{ for } n = 1, 3, 5$$

Que 2.11. **Solve completely the equation $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$ representing the vibrations of the string of length l fixed at both ends. Given that $y(0, t) = 0$; $y(l, t) = 0$, $y(x, 0) = f(x)$ and $\frac{\partial y}{\partial t}(x,0) = 0$; $0 < x < l$.**

Answer

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(2.11.1)}$$

Let

$$y = XT \quad \text{...(2.11.2)}$$

Where X is a function of x only and T is a function of t only, be a solution of eq. (2.11.1)

Then, $\quad \frac{\partial^2 y}{\partial t^2} = XT'' \quad$ and $\quad \frac{\partial^2 y}{\partial x^2} = X''T$

Putting these values in eq. (2.11.1), we get

$$\frac{X''}{X} = \frac{1}{C^2} \frac{T''}{T} = k \text{ (say)} \quad \text{...(2.11.3)}$$

$$X'' = kX$$

$$X'' - kX = 0 \quad \text{...(2.11.4)}$$

When k is negative and $k = -p^2$, say

$$X = C_1 \cos px + C_2 \sin px$$

$$T = C_3 \cos Cpt + C_4 \sin Cpt$$

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt) \quad \text{...(2.11.5)}$$

Due to vibrations problem, y must be periodic function of x and t.

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt)$$

Now applying boundary conditions that

$y = 0$ when $x = 0$ and $y = 0$ when $x = l$, we get

$$0 = C_1 (C_3 \cos cpt + C_4 \sin Cpt) \quad \text{...(2.11.6)}$$

$$0 = (C_1 \cos pl + C_2 \sin pl)(C_3 \cos Cpt + C_4 \sin Cpt) \quad ...(2.11.7)$$

From eq. (2.11.6), we have $C_1 = 0$ and eq. (2.11.7) reduces to

$$C_2 \sin pl\,(C_3 \cos Cpt + C_4 \sin Cpt) = 0$$

$$\sin pl = 0$$

$$pl = n\pi \text{ or } p = \frac{n\pi}{l} \text{ where } n = 1,2,3,....$$

A solution of wave equation

$$y = C_2\left(C_3 \cos\frac{n\pi Ct}{l} + C_4 \sin\frac{n\pi Ct}{l}\right)\sin\frac{n\pi x}{l}$$

$$= \left(a_n \cos\frac{n\pi Ct}{l} + b_n \sin\frac{n\pi Ct}{l}\right)\sin\frac{n\pi x}{l}$$

where $C_2 C_3 = a_n$ and $C_2 C_4 = b_n$

$$y = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi Ct}{l} + b_n \sin\frac{n\pi Ct}{l}\right)\sin\frac{n\pi x}{l} \quad ...(2.11.8)$$

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$, where $t = 0$ ...(2.11.9)

we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad ...(2.11.10)$$

and

$$0 = \sum_{n=1}^{\infty} \frac{n\pi C}{l} b_n \sin\frac{n\pi x}{l} \quad ...(2.11.11)$$

Since eq. (2.11.10) represents Fourier series for $f(x)$, we have

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \quad ...(2.11.12)$$

From eq. (2.11.11), $b_n = 0$, for all n

Hence eq. (2.11.8) reduces to

$$y = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi Ct}{l}\sin\frac{n\pi x}{l} \quad ...(2.11.13)$$

where a_n is given by eq. (2.11.12) when $f(x)$ *i.e.*, $y(x, 0)$ is known.

Que 2.12. **Find the displacement of a finite string of length L that is fixed at both ends and is released from rest with an initial displacement $f(x)$.**

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Answer

Let the equation of the string be

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = XT$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$$

It will satisfy the given differential equation

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 \cos kCt + C_4 \sin kCt$$

$$u = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos kct + C_4 \sin kCt) \quad ...(2.12.1)$$

According to given conditions,

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = f(x)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$$

Applying $u(0, t) = 0$ in eq. (2.12.1), $C_1 = 0$

$$u = \sin kx\,(A_n \cos kCt + B_n \sin kCt) \quad ...(2.12.2)$$

where,

$$C_2 C_3 = A_n$$

$$C_2 C_4 = B_n$$

Applying $u(L, t) = 0$

$$k = \frac{n\pi}{L}$$

$$u = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi Ct}{L}\right) + B_n \sin\left(\frac{n\pi Ct}{L}\right)\right] \quad ...(2.12.3)$$

$$\left(\frac{\partial u}{\partial t}\right) = \frac{n\pi C}{L}\sin\left(\frac{n\pi x}{L}\right)\left[-A_n \sin\frac{n\pi Ct}{L} + B_n \cos\frac{n\pi Ct}{L}\right]$$

Applying $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0,\ B_n = 0$

From eq. (2.12.3)

$$u = \sum A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi Ct}{L}\right) \quad ...(2.12.4)$$

Applying $u(x, 0) = f(x)$ in eq. (2.12.4)

$$f(x) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad ...(2.12.5)$$

Thus complete solution is given by eq. (2.12.4) where A_n is given by eq. (2.12.5).

Que 2.13. **Write the solution of two dimensional wave equation.**

Answer

Equation of two dimensional wave is given by

$$\frac{\partial^2 u}{\partial t^2} = C^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad ...(2.13.1)$$

The boundary conditions are $u(0, y, t) = 0$ ...(i)

$u(a, y, t) = 0$ ...(ii)

$u(x, 0, t) = 0$ ...(iii)

$u(x, b, t) = 0$ ...(iv)

The initial conditions are

$u(x, y, 0) = f(x, y)$ ...(v)

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x,y)$ ...(iv)

Let $u = X(x)\,Y(y)\,T(t)$ is a solution of eq. (2.13.1). Differentiating partially w.r.t. x, y, and t and putting the values in eq. (2.13.1),

$$XY\frac{\partial^2 T}{\partial t^2} = C^2\left(YT\frac{\partial^2 X}{\partial x^2} + XT\frac{\partial^2 Y}{\partial y^2}\right)$$

Dividing both sides by XYT,

$$\frac{1}{C^2T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case i : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_1^2, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_2^2, \frac{1}{C^2T}\frac{\partial^2 T}{\partial t^2} = -k^2$

Where $k^2 = k^2_1 + k^2_2$

$X = C_1 \cos k_1x + C_2 \sin k_1x,$

$Y = C_3 \cos k_2y + C_4 \sin k_2y$

and $T = C_5 \cos kCt + C_6 \sin kCt$

Thus $u = (C_1 \cos k_1x + C_2 \sin k_1x)(C_3 \cos k_2y + C_4 \sin k_2y)$

$(C_5 \cos kCt + C_6 \sin kCt)$ ...(2.13.2)

Case ii : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k_1^2, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k_2^2$ and $\frac{1}{C^2T}\frac{\partial^2 T}{\partial t^2} = k^2$

Where, $$k^2 = k_1^2 + k_2^2$$

Its solution is given by

$$u = (C_1 e^{k_1 x} + C_2 e^{-k_1 x})(C_3 e^{k_2 y} + C_4 e^{-k_2 y})(C_5 e^{kCt} + C_6 e^{-kCt}) \quad ...(2.13.3)$$

Case iii : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$ and $\frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

Its solution is

$$u = (C_1 x + C_2)(C_3 y + C_4)(C_5 t + C_6) \quad ...(2.13.4)$$

Solution of two dimensional wave equation is given by eq. (2.13.2). Apply boundary condition (i) to eq. (2.13.2),

$$0 = C_1$$

$\Rightarrow$ From eq. (2.13.2),

$$u = C_2 \sin k_1 x (C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt) \quad ...(2.13.5)$$

At $$x = a, u = 0$$

$$0 = C_2 \sin k_1 a (C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt)$$

$$\sin k_1 a = 0 = \sin m\pi$$

$$k_1 = \frac{m\pi}{a}$$

From eq. (2.13.5),

$$u = C_2 \sin \frac{m\pi x}{a} (C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt) \quad ...(2.13.6)$$

Now at $$y = 0, u = 0$$

From eq. (2.13.6),

$\Rightarrow$ $0 = C_3$

$$u = C_2 \sin \frac{m\pi x}{a} C_4 \sin k_2 y (C_5 \cos kCt + C_6 \sin kCt) \quad ...(2.13.7)$$

At $y = b, u = 0$

$$\sin k_2 b = 0 = \sin n\pi$$

$$k_2 = \frac{n\pi}{b}$$

From eq. (2.13.7)

$$u = C_2 C_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (C_5 \cos kCt + C_6 \sin kCt)$$

$$\text{or } u = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\left(A_{mn} \cos kCt + B_{mn} \sin kCt\right) \quad ...(2.13.8)$$

Now apply initial condition (v),

$$f(x, y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) A_{mn}$$

$$A_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \qquad ...(2.13.9)$$

Differentiate eq. (2.13.8) w.r.t. t,

$$\frac{\partial u}{\partial t} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)[-kCA_{mn}\sin kCt + kCB_{mn}\cos kCt]$$

At $\quad t = 0, \dfrac{\partial u}{\partial t} = g(x,y)$

$$g(x, y) = kC\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)B_{mn}$$

$$kCB_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \qquad ...(2.13.10)$$

Solution of two dimensional wave equation is given by eq. (2.13.8) and the values of A_{mn} and B_{mn} are given by eq. (2.13.9) and eq. (2.13.10).

Que 2.14. **State and solve one dimensional heat flow equation.**

Answer

One dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2} \qquad ...(2.14.1)$$

Let $\quad u = X(x)\,T(t)$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2 T}\frac{\partial T}{\partial t} = -k^2, k^2, 0$$

The three possible solutions are :

i. $\quad X = C_1\cos kx + C_2\sin kx,\ T = C_3 e^{-k^2C^2t}$

i.e., $\quad u = (C_1\cos kx + C_2\sin kx)C_3 e^{-k^2C^2t}$

ii. $\quad X = C_1 e^{kx} + C_2 e^{-kx},\ T = C_3 e^{k^2C^2t}$

i.e., $\quad u = C_3 e^{k^2C^2t}(C_1 e^{kx} + C_2 e^{-kx})$

iii. $\quad X = C_1 x + C_2,\ T = C_3$

i.e., $\quad u = (C_1 x + C_2)C_3$

The only solution which satisfies the condition of one dimensional heat equation is given by solution (i).

i.e., $\quad u = (C_1\cos kx + C_2\sin kx)C_3 e^{-k^2C^2t} \qquad ...(2.14.2)$

Now the boundary and initial conditions depend on the nature of the problem.

Case i : When ends of a homogeneous bar at zero temperature, the boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$ and the initial condition is $u(x, 0) = f(x)$, $0 < x < l$.

Case ii : One face at temperature u_0.

The boundary conditions are $u(0, t) = 0$

$$u(l, t) = u_0$$

and the initial condition is $u(x, 0) = v_0$

Case iii : In case of infinite bar,

There is no boundary condition but only the initial condition is

$$u(x, 0) = f(x)$$

Case iv : Insulated faces

Boundary conditions are $u_x(0, t) = 0$

$$u_x(l, t) = 0$$

and the initial condition is $u_x(x, 0) = f(x)$, $0 < x < l$.

Que 2.15. **Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$.**

Answer

Equation of heat in one dimension is given by

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = X(x)\, T(t)$$

$$X\frac{\partial T}{\partial t} = C^2 T \frac{\partial^2 X}{\partial x^2} \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2 T}\frac{\partial T}{\partial t} = -k^2 \text{(let)}$$

$$X = C_1 \cos kx + C_2 \sin kx$$

and

$$T = C_3 e^{-k^2 C^2 t}$$

Thus,

$$u = (C_1 \cos kx + C_2 \sin kx)\, C_3 e^{-k^2 C^2 t} \quad \text{...(2.15.1)}$$

Given boundary conditions are $u(0, t) = u(2, t) = 0$

Put $u(0, t) = 0$ in eq. (2.15.1),

$$0 = C_1 C_3 e^{-k^2 C^2 t}$$

$$C_1 = 0$$

From eq. (2.15.1),

$$u = C_2 C_3 \sin kx\; e^{-k^2 C^2 t}$$

$$u = A_n \sin kx \, e^{-k^2c^2t} \quad \ldots(2.15.2)$$

Apply $u(2, t) = 0$

$$0 = A_n \sin 2k \, e^{-k^2c^2t}$$

$$\sin 2k = 0$$

$$k = \frac{n\pi}{2}$$

From eq. (2.15.2),

$$u = \sum A_n \sin \frac{n\pi x}{2} e^{-n^2\pi^2c^2t/4} \quad \ldots(2.15.3)$$

Apply initial condition $u(x, 0) = 100\,x$

$$100\,x = \sum A_n \sin \frac{n\pi x}{2}$$

$$A_n = \frac{2}{2}\int_0^2 100x \cdot \sin \frac{n\pi x}{2} dx$$

$$A_n = 100\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= 100\left[-\frac{4}{n\pi}\cos n\pi\right] = -\frac{400}{n\pi}(-1)^n = \frac{400}{n\pi}(-1)^{n+1}$$

Thus from eq. (2.15.3), $u = \sum_{n=1}^{\infty} \frac{400}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{2} e^{-n^2\pi^2c^2t/4}$

Que 2.16. **Find the temperature distribution in a rod of length 'a' which is perfectly insulated including the ends and the initial temperature distribution is $x(a - x)$, $0 < x < a$.**

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Answer

Let the equation of the temperature distribution is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \ldots(2.16.1)$$

$$u = X(x)\, T(t)$$

$$X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial T}{\partial t} = -k^2 \text{ (let)}$$

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 e^{-k^2c^2t}$$

Thus
$$u = (A_n \cos kx + B_n \sin kx)\, e^{-k^2c^2t} \quad \text{...(2.}$$

Given boundary and initial conditions are

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$$

$$u(x, 0) = x(a - x),\ 0 < x < a$$

$$\frac{\partial u}{\partial x} = (-k A_n \sin kx + B_n k \cos kx)\, e^{-k^2c^2t}$$

$$0 = B_n k e^{-k^2c^2t}$$

$$B_n = 0$$

From eq. (2.16.2)

$$u = A_n \cos kx\, e^{-k^2c^2t} \quad \text{...(2.16.}$$

$$\frac{\partial u}{\partial x} = k A_n \sin kx\, e^{-k^2c^2t}$$

$$0 = k A_n \sin ka\, e^{-k^2c^2t}$$

$$k = \frac{n\pi}{a}$$

$$u = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2c^2t/a^2} \quad \text{...(2.16.4)}$$

Now at $t = 0$,

$$u(x, 0) = \sum A_n \cos\left(\frac{n\pi x}{a}\right)$$

$$A_n = \frac{2}{a}\int_0^a u(x, 0) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$A_n = \frac{2}{a}\int_0^a x(a - x) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a}\left[(ax - x^2)\left(\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right) - (a - 2x)\left(\frac{-a^2}{n^2\pi^2}\cos\frac{n\pi x}{a}\right) + (-2)\left(\frac{-a^3}{n^3\pi^3}\sin\frac{n\pi x}{a}\right)\right]_0^a$$

$$A_n = \frac{2}{a}\left[-(-a)\left(\frac{-a^2}{n^2\pi^2}\cos n\pi\right) + a\left(\frac{-a^2}{n^2\pi^2}\right)\right]$$

$$= \frac{-2a^2}{n^2\pi^2}[1 + \cos n\pi]$$

Thus

$$u(x, t) = \sum_0^{\infty} \frac{-2a^2}{n^2\pi^2}(1 + \cos n\pi) \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2c^2t/a^2}$$

Thus
$$u = (A_n \cos kx + B_n \sin kx)\, e^{-k^2 C^2 t} \qquad \text{...(2.16.2)}$$

Given boundary and initial conditions are

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$$

$$u(x, 0) = x(a - x),\ 0 < x < a$$

$$\frac{\partial u}{\partial x} = (-k A_n \sin kx + B_n k \cos kx)\, e^{-k^2 C^2 t}$$

$$0 = B_n k e^{-k^2 C^2 t}$$

$$B_n = 0$$

From eq. (2.16.2)

$$u = A_n \cos kx\, e^{-k^2 C^2 t} \qquad \text{...(2.16.3)}$$

$$\frac{\partial u}{\partial x} = k A_n \sin kx\, e^{-k^2 C^2 t}$$

$$0 = k A_n \sin ka\, e^{-k^2 C^2 t}$$

$$k = \frac{n\pi}{a}$$

$$u = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-n^2 \pi^2 C^2 t/a^2} \qquad \text{...(2.16.4)}$$

Now at $t = 0$,

$$u(x, 0) = \sum A_n \cos\left(\frac{n\pi x}{a}\right)$$

$$A_n = \frac{2}{a}\int_0^a u(x, 0) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$A_n = \frac{2}{a}\int_0^a x(a - x) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a}\left[(ax - x^2)\left(\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right) - (a - 2x)\left(\frac{-a^2}{n^2\pi^2}\cos\frac{n\pi x}{a}\right) + (-2)\left(\frac{-a^3}{n^3\pi^3}\sin\frac{n\pi x}{a}\right)\right]_0^a$$

$$A_n = \frac{2}{a}\left[-(-a)\left(\frac{-a^2}{n^2\pi^2}\cos n\pi\right) + a\left(\frac{-a^2}{n^2\pi^2}\right)\right]$$

$$= \frac{-2a^2}{n^2\pi^2}[1 + \cos n\pi]$$

Thus

$$u(x, t) = \sum_0^{\infty} \frac{-2a^2}{n^2\pi^2}(1 + \cos n\pi) \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2 C^2 t/a^2}$$

$\Rightarrow \quad u = (C_1 e^{k_1 x} + C_2 e^{-k_1 x})(C_3 e^{k_2 y} + C_4 e^{-k_2 y}) C_5 e^{C^2 k^2 t}$

Case iii :

$$X = C_1 \cos k_1 x + C_2 \sin k_1 x,$$

$$Y = C_3 \cos k_2 y + C_4 \sin k_2 y, \; T = C_5 e^{-C^2 k^2 t}$$

$$u = (C_1 \cos k_1 x + C_2 \sin k_1 x)(C_3 \cos k_2 y + C_4 \sin k_2 y) C_5 e^{-C^2 k^2 t}$$

Out of these three solutions, we have to choose that solution which satisfies the heat equation. Accordingly, case (iii) is accepted here.

$$u = (C_1 \cos k_1 x + C_2 \sin k_1 x)(C_3 \cos k_2 y + C_4 \sin k_2 y) C_5 e^{-C^2 k^2 t} \quad ...(2.17.2)$$

Now we apply boundary conditions on putting $u = 0$ and $x = 0$ in eq. (2.17.2), we get

$$0 = C_1 (C_3 \cos k_2 y + C_4 \sin k_2 y) C_5 e^{-C^2 k^2 t}$$

$\Rightarrow \quad C_1 = 0$

$\therefore$ The eq. (2.17.2) reduces to

$$u = (C_2 \sin k_1 x)(C_3 \cos k_2 y + C_4 \sin k_2 y) C_5 e^{-C^2 k^2 t}$$

or

$$u = \sin k_1 x (A_{mn} \cos k_2 y + B_{mn} \sin k_2 y) e^{-k^2 C^2 t} \quad ...(2.17.3)$$

Now at $\quad x = a, u = 0$

$$0 = \sin k_1 a (A_{mn} \cos k_2 y + B_{mn} \sin k_2 y) e^{-k^2 C^2 t}$$

$$\sin k_1 a = 0 = \sin m\pi$$

$$k_1 = \frac{m\pi}{a}$$

From eq. (2.17.3),

$$u = \sin\left(\frac{m\pi x}{a}\right)[A_{mn} \cos k_2 y + B_{mn} \sin k_2 y] \, e^{-k^2 C^2 t}$$

Now at $\quad y = 0, u = 0 \quad ...(2.17.4)$

$\Rightarrow \quad A_{mn} = 0$

and at $\quad y = b, u = 0$

$$\sin k_2 b = 0 = \sin n\pi$$

$$k_2 = \frac{n\pi}{b}$$

Thus from eq. (2.17.4),

$$u = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-k^2 C^2 t} \quad ...(2.17.5)$$

where

$$k^2 = k_1^2 + k_2^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$k = \pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$$

Now the initial condition $u(x, y, 0) = f(x, y)$

From eq. (2.17.3),

$$f(x, y) = B_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

where
$$B_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x, y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

Solution is given by eq. (2.17.3) and value of B_{mn} is given by the above equation.

Que 2.18. **A square plate is bounded by lines $x = 0$, $y = 0$, $x = 20$, $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while the upper three edges are kept at 0 °C. Find the steady state temperature.**

AKTU 2016-17, Marks 10

Answer

The two dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

Its solution is

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

$$u(0, y) = 0$$

$$0 = C_1(C_3 e^{py} + C_4 e^{-py})$$

$$C_1 = 0$$

Then,
$$u(x, y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py})$$

$$u(20, y) = 0$$

$$0 = C_2 \sin 20p\,(C_3 e^{py} + C_4 e^{-py})$$

$$\sin 20p = \sin n\pi = 0$$

$$p = \frac{n\pi}{20}$$

$$u(x, y) = \sin\frac{n\pi x}{20}\left(C_2C_3 e^{\frac{n\pi}{20}y} + C_2C_4 e^{-\frac{n\pi}{20}y}\right)$$

$$= \sin\frac{n\pi x}{20}\left(Ae^{\frac{n\pi}{20}y} + Be^{-\frac{n\pi}{20}y}\right)$$

where,
$$A = C_2C_3 \text{ and } B = C_2C_4$$

$$u(x, 0) = 0$$

$$0 = \sin \frac{n\pi x}{20}(A + B)$$

$$A = -B$$

$$\therefore \quad u(x, y) = A \sin \frac{n\pi}{20} x \left[e^{\frac{n\pi}{20}y} - e^{-\frac{n\pi}{20}y} \right]$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{20} x \sinh \frac{n\pi}{20} y$$

$$u(x, 20) = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin \frac{n\pi x}{20}$$

$$= x(20 - x)$$

where, $$b_n = \frac{2}{20 \times \sinh n\pi} \int_0^{20} x(20 - x) \sin \frac{n\pi}{20} x \, dx$$

$$= \frac{2}{20 \times \sinh n\pi} \left[\left\{ (20x - x^2) \left(\frac{-\cos \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right) \right\}_0^{20} - \int_0^{20} (20 - 2x) \left(\frac{\frac{-\cos n\pi}{20} x}{\frac{n\pi}{20}} \right) dx \right]$$

$$= \frac{1}{10 \sinh n\pi} \times \frac{20}{n\pi} \int_0^{20} (20 - 2x) \cos \frac{n\pi}{20} x \, dx$$

$$= \frac{2}{n\pi \sinh n\pi} \left[\left\{ (20 - 2x) \frac{\sin \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right\}_0^{20} - \int_0^{20} (-2) \left(\frac{\sin \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right) dx \right]$$

$$= \frac{4}{n\pi \sinh n\pi} \times \frac{20}{n\pi} \left(\frac{-\cos \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right) = \frac{4 \times 20^2}{n^3 \pi^3 \sinh n\pi} \left(1 - \cos \frac{n\pi}{20} \right)$$

$$= \begin{cases} \dfrac{3200}{n^3 \pi^3 \sinh n\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$\therefore \quad u(x, y) = \frac{3200}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{\sin \frac{n\pi}{20} x \sinh \frac{n\pi}{20} y}{n^3 \sinh n\pi}$$

ii. When k is negative and $k = -p^2$

$$X = C_1 \cos px + C_2 \sin px,\ Y = C_3 e^{py} + C_4 e^{-py}$$

iii. When $k = 0$

$$X = C_1 x + C_2,\ Y = C_3 y + C_4$$

Thus, the various possible solutions of Laplace equation (2.19.2) are

$$u = (C_1 e^{px} + C_1 e^{-px})(C_3 \cos py + C_4 \sin py) \quad \text{...(2.19.5)}$$

$$u = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{...(2.19.6)}$$

$$u = (C_1 x + C_2)(C_3 y + C_4) \quad \text{...(2.19.7)}$$

From these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions.

Que 2.20. **Solve :** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$**, subject to boundary conditions** $u(0, y) = 0 = u(\pi, y)$**, and** $u(x, 0) = u_0$**,** $\lim\limits_{y \to \infty} u(x, y) = 0,\ 0 < x < \pi$**.**

Answer

Given, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Let, $u = XY$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 \cos kx + C_2 \sin kx,\ Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$u = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \text{...(2.20.1)}$$

$$u(0, y) = 0$$

From eq. (2.20.1),

$$0 = C_1 (C_3 e^{ky} + C_4 e^{-ky})$$

$$C_1 = 0$$

$$u = \sin kx\,(A_n e^{ky} + B_n e^{-ky})$$

$$u(\pi, y) = 0$$

$$\sin k\pi = 0 \Rightarrow k = n$$

$$u = \sin nx\,(A_n e^{ny} + B_n e^{-ny}) \quad \text{...(2.20.2)}$$

From eq. (2.20.3), $\lim\limits_{y \to \infty} u(x, y) = 0$, it satisfies only when $A_n = 0$. ...(2.20.3)

$$u = \sum B_n e^{-ny} \sin nx$$

ii. When k is negative and $k = -p^2$

$$X = C_1 \cos px + C_2 \sin px,\ Y = C_3 e^{py} + C_4 e^{-py}$$

iii. When $k = 0$

$$X = C_1 x + C_2,\ Y = C_3 y + C_4$$

Thus, the various possible solutions of Laplace equation (2.19.2) are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad ...(2.19.5)$$

$$u = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(2.19.6)$$

$$u = (C_1 x + C_2)(C_3 y + C_4) \quad ...(2.19.7)$$

From these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions.

Que 2.20. Solve : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, **subject to boundary conditions** $u(0, y) = 0 = u(\pi, y)$, **and** $u(x, 0) = u_0$, $\lim\limits_{y \to \infty} u(x, y) = 0,\ 0 < x < \pi$.

Answer

Given,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let,
$$u = XY$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 \cos kx + C_2 \sin kx,\ Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$u = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad ...(2.20.1)$$

$$u(0, y) = 0$$

From eq. (2.20.1),

$$0 = C_1 (C_3 e^{ky} + C_4 e^{-ky})$$

$$C_1 = 0$$

$$u = \sin kx\,(A_n e^{ky} + B_n e^{-ky})$$

$$u(\pi, y) = 0$$

$$\sin k\pi = 0 \Rightarrow k = n$$

$$u = \sin nx\,(A_n e^{ny} + B_n e^{-ny}) \quad ...(2.20.2)$$

$\lim\limits_{y \to \infty} u(x, y) = 0$, it satisfies only when $A_n = 0$.

From eq. (2.20.3),

$$u = \sum B_n e^{-ny} \sin nx \quad ...(2.20.3)$$

Now $u(x, 0) = u_0$

$$u_0 = \sum B_n \sin nx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\,dx = \frac{-2u_0}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-2u_0}{\pi}\left[\frac{(-1)^n - 1}{n}\right]$$

Thus from eq. (2.20.4),

$$u = \sum \frac{-2u_0}{\pi n}[(-1)^n - 1]e^{-ny}\sin nx$$

Que 2.21. **Solve : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions $u(x, 0) = 0$, $u(x, 1) = 0$, $u(\infty, y) = 0$ and $u(0, y) = u_0$.** **AKTU 2014-15(II), Marks 10**

Answer

Given, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $$u = XY$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \text{ (let)}$$

$$X = C_1 e^{kx} + C_2 e^{-kx}$$

$$Y = C_3 \cos kCy + C_4 \sin kCy$$

$$u = (C_1 e^{kx} + C_2 e^{-kx})(C_3 \cos kCy + C_4 \sin kCy) \quad ...(2.21.1)$$

This solution will satisfy given differential equation.

Applying $u(x, 0) = 0$ in eq. (2.21.1), we get

$$C_3 = 0$$

$$u = (A_n e^{kx} + B_n e^{-kx}) \sin ky \quad ...(2.21.2)$$

where $$C_1 C_4 = A_n,\ C_2 C_4 = B_n$$

Applying $u(x, 1) = 0$ in eq. (2.21.2)

$$u = 0 = \sin k$$

$$k = n\pi$$

$$u = (A_n e^{kn\pi x} + B_n e^{-kn\pi x}) \sin n\pi y \quad ...(2.21.3)$$

Applying $$u(\infty, y) = 0$$

As $x \to \infty$, $u \to 0$, this is only possible if

$$A_n = 0$$

Thus from eq. (2.21.3)

$$u = \sum B_n e^{-kn\pi x} \sin n\pi y \quad \text{...(2.21.4)}$$

Applying $u(0, y) = u_0$

$$u_0 = \sum B_n \sin n\pi y$$

where,

$$B_n = 2\int_0^1 u_0 \sin n\pi y \, dy$$

$$= 2u_0 \left| -\cos n\pi y \right|_0^1 = -2u_0 (\cos n\pi - 1)$$

$$B_n = 2u_0 (1 - \cos n\pi)$$

From eq. (2.21.4)

$$u = \sum 2u_0 (1 - \cos n\pi) e^{-kn\pi x} \sin n\pi y$$

Que 2.22. **Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the xy-plane, $0 \le x \le a$ and $0 \le y \le b$ satisfying the following boundary conditions $u(x, 0) = 0$, $u(x, b) = 0$ and $u(0, y) = 0$, $u(a, y) = f(y)$.**

AKTU 2015-16(II), Marks 10

Answer

Given Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{...(2.22.1)}$$

Let $u = XY$, where X is a function of x only and Y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$$

From eq. (2.22.1),

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case i :

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \text{ (say)}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$$

and

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$X = C_1 x + C_2,\; Y = C_3 y + C_4$$

At $\quad y = 0,\ Y = 0 \Rightarrow C_4 = 0$

Also, $\quad y = b,\ Y = 0 \Rightarrow C_3 = 0$

$\therefore \quad Y = 0$

Thus, $\quad u = XY = X(0)$

$u = 0$ (not possible)

Case ii :
$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \text{ (say)}$$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

and
$$\frac{\partial^2 Y}{\partial y^2} - k^2 Y = 0$$

$$X = C_1 \cos kx + C_2 \sin kx,\ Y = C_3 e^{ky} + C_4 e^{-ky}$$

If $\quad y = 0,\ Y = 0$

$$C_3 + C_4 = 0$$
$$C_4 = -C_3$$

and $\quad Y = 0$ at $y = b$

$$0 = C_3 e^{kb} - C_3 e^{-kb}$$
$$C_3 (e^{kb} - e^{-kb}) = 0$$
$$C_3 = 0,\ C_4 = 0,\ Y = 0$$

(Not possible)

Case iii :
$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 e^{kx} + C_2 e^{-kx},$$
$$Y = C_3 \cos ky + C_4 \sin ky$$

At $\quad y = 0,\ Y = 0,\ C_3 = 0$

$$Y = C_4 \sin ky$$

At $\quad y = b,\ Y = 0$

$$0 = C_4 \sin kb$$
$$\sin kb = 0$$
$$kb = n\pi$$
$$k = \frac{n\pi}{b}$$

Thus,
$$u = (C_1 e^{kx} + C_2 e^{-kx}) C_4 \sin \frac{n\pi y}{b} \quad \text{...(2.22.2)}$$

At $\quad x = 0,\ u = 0$

$$0 = (C_1 + C_2) C_4 \sin \frac{n\pi y}{b}$$
$$C_1 + C_2 = 0$$
$$C_2 = -C_1$$

From eq. (2.22.2),

$$u = \frac{2}{2} C_4 C_1 \left(e^{kx} - e^{-kx}\right) \sin \frac{n\pi y}{b}$$

$$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}}{2} \right) \sin\left(\frac{n\pi y}{b}\right) \quad ...(2.22.3)$$

Let $\quad b_n = 2C_1 C_4$

At $\quad x = a, \; u = f(y)$

From eq. (2.22.3),

$$f(y) = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi a}{b}} - e^{\frac{-n\pi a}{b}}}{2} \right) \sin\left(\frac{n\pi y}{b}\right)$$

$$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

$$b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy \quad ...(2.22.4)$$

Thus, $$u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

where b_n is given by eq. (2.22.4)

Que 2.23. **In a telephone of wire of length l, a steady voltage distribution of 20 volts at the source end and 12 volts at the terminal end is maintained. At time $t = 0$, the terminal is grounded. Assuming $L = 0$, $G = 0$, determine the voltage and current where symbols have their usual meanings.**

Answer

The telegraph line equation is

$$\frac{\partial^2 V}{\partial x^2} = RC \frac{\partial V}{\partial t}$$

$$\frac{\partial V}{\partial t} = \frac{1}{RC} \frac{\partial^2 V}{\partial x^2} \quad ...(2.23.1)$$

Here, V_s = Initial steady voltage satisfying

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$$V_s = 20 + \frac{(12-20)}{l}x$$

$$V_s = 20 - \frac{8}{l}x = V(x, 0)$$

$$V(x, 0) = 20 - \frac{8}{l}x \quad \text{...(2.23.2)}$$

And let $V_s' =$ steady voltage after grounding the terminal end (terminal voltage = 0)

$$V_s' = 20 - \frac{20x}{l} \quad \text{...(2.23.3)}$$

$$V(x, t) = V_s' + V_t(x, t)$$

$$V(x, t) = 20 - \frac{20x}{l} + \sum b_n e^{-n^2\pi^2 t/l^2 RC} \sin\left(\frac{n\pi x}{l}\right) \quad \text{...(2.23.4)}$$

Putting $t = 0$, $V(x, 0)$ is given by eq. (2.23.2)

From eq. (2.23.4),

$$20 - \frac{8}{l}x = 20 - \frac{20x}{l} + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{12x}{l} = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l \frac{12}{l} x \sin\frac{n\pi x}{l} dx$$

$$b_n = \frac{24}{l^2}\left[x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{24}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi\right] = -\frac{24}{n\pi}(-1)^n$$

$$b_n = \frac{24}{n\pi}(-1)^{n+1}$$

Thus from eq. (2.23.4),

$$V(x, t) = 20 - \frac{20x}{l} + \sum \frac{24}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t/l^2 RC} \sin\left(\frac{n\pi x}{l}\right)$$

Also,

$$\frac{\partial V}{\partial x} = -\frac{20}{l} + \sum \frac{24}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t/l^2 RC}\left(\frac{n\pi}{l}\right)\cos\frac{n\pi x}{l} = -L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{20}{lL} - \sum \frac{24}{Ll}(-1)^{n+1} e^{-n^2\pi^2 t/l^2 RC} \cos\left(\frac{n\pi x}{l}\right)$$

On integrating

$$i = \frac{20\,t}{lL} + \frac{24\,l^2\,RC}{n^2\,\pi^2 Ll} \sum (-1)^{n+1} e^{\frac{-n^2\pi^2 t}{l^2\,RC}} \cos\left(\frac{n\pi x}{l}\right) + A$$

At $t = 0$, $i = 0$, $A = 0$

$$i = \frac{20\,t}{lL} + \frac{24l\,RC}{n^2\pi^2\,L} \sum (-1)^{n+1} e^{\frac{-n^2\,\pi^2 t}{l^2\,RC}} \cos\left(\frac{n\pi x}{l}\right)$$

Que 2.24. **Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that**

$$i(x, 0) = i_0,\ e(x, 0) = e_0 \sin\frac{\pi x}{l}$$

Also R and G are negligible.

Answer

Since R and G are negligible, transmission line equations becomes

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t} \quad \text{....(2.24.1)}$$

and
$$\frac{\partial i}{\partial x} = -C\frac{\partial e}{\partial t} \quad \text{....(2.24.2)}$$

For elimination of i, differentiating eq. (2.24.1) partially w.r.t x and eq. (2.24.2) partially w.r.t t, we have

$$\frac{\partial^2 e}{\partial x^2} = -L\frac{\partial^2 i}{\partial x \partial t} \text{ and } \frac{\partial^2 i}{\partial t \partial x} = C\frac{\partial^2 e}{\partial t^2}$$

Hence,
$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} \quad \text{....(2.24.3)}$$

The initial conditions are $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin\dfrac{\pi x}{l}$ ....(2.24.4)

Since, the ends are suddenly grounded, the boundary conditions are

$$e(0, t) = e(l, t) = 0$$

Also $i = i_0$ (constant) when $t = 0$ ....(2.24.5)

$$\therefore \quad \frac{\partial i}{\partial x} = 0 \text{ which gives } \frac{\partial e}{\partial t} = 0 \text{ when } t = 0 \quad \text{....(2.24.6)}$$

Now let $e = XT$ be a solution of eq. (2.24.3) where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 e}{\partial x^2} = X''T \text{ and } \frac{\partial^2 e}{\partial t^2} = XT''$$

$\therefore$ From eq. (2.24.3) $X''T = LCXT''$

Separating the variables $\dfrac{X''}{X} = LC\dfrac{T''}{T} = -p^2$ (say)

This leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} + p^2X = 0 \text{ and } \frac{d^2T}{dt^2} + \frac{p^2}{LC}T = 0$$

$\therefore$
$$X = C_1 \cos px + C_2 \sin px$$
$$T = C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}$$

$\Rightarrow$
$$e = XT = (C_1 \cos px + C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right) \quad ...(2.24.7)$$

Applying the boundary conditions eq. (2.24.5) in eq. (2.24.7), we get

$$C_1 = 0 \text{ and } p = \frac{n\pi}{l}, n \text{ being an integer}$$

$\therefore$ Eq. (2.24.7) becomes

$$e = C_2 \sin \frac{n\pi x}{l}\left(C_3 \cos \frac{n\pi t}{l\sqrt{LC}} + C_4 \sin \frac{n\pi t}{l\sqrt{LC}}\right)$$

or
$$e = \sin \frac{n\pi x}{l}\left(A \cos \frac{n\pi t}{l\sqrt{LC}} + B \sin \frac{n\pi t}{l\sqrt{LC}}\right) \quad ...(2.24.8)$$

where
$$A = C_2C_3 \text{ and } B = C_2C_4$$

$$\frac{\partial e}{\partial t} = \sin \frac{n\pi x}{l}\left(-\frac{An\pi}{l\sqrt{LC}} \sin \frac{n\pi t}{l\sqrt{LC}} + \frac{Bn\pi}{l\sqrt{LC}} \cos \frac{n\pi t}{l\sqrt{LC}}\right)$$

Since $\frac{\partial e}{\partial t} = 0$ when $t = 0$, we get

$$B = 0$$

$\therefore$ From eq. (2.24.8)
$$e = A \sin \frac{n\pi x}{l} \cos \frac{n\pi x}{l\sqrt{LC}}$$

By superposition,
$$e = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi x}{l\sqrt{LC}} \text{ is also a solution}$$

But
$$e = e_0 \sin \frac{\pi x}{l} \text{ when } t = 0$$

$\therefore$
$$e_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$\Rightarrow$
$$A_1 = e_0 \quad \text{and } A_2 = A_3 = ... = 0$$

Hence,
$$e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Now,
$$-L\frac{\partial i}{\partial t} = \frac{\partial e}{\partial x}$$

$\therefore$
$$\frac{\partial i}{\partial t} = -\frac{1}{L} \cdot \frac{e_0\pi}{l} \cos \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Integrating w.r.t. t, regarding x as constant

$$i = -\frac{e_0 \pi}{Ll} \cos\frac{\pi x}{l} \cdot \frac{l\sqrt{LC}}{\pi} \sin\frac{\pi t}{l\sqrt{LC}} + f(x) \quad \ldots(2.24.9)$$

where $f(x)$ is an arbitrary constant function.
Since $i = i_0$ when $t = 0$, we have $i_0 = 0 + f(x)$ or $f(x) = i_0$
$\therefore$ From eq. (2.24.9), we have

$$i = i_0 - e_0 \sqrt{\frac{C}{L}} \cos\frac{\pi x}{l} \sin\frac{\pi t}{l\sqrt{LC}}$$

Que 2.25. **Solve $\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$ assuming that the initial voltage is $V_0 \sin\frac{\pi x}{l}$; $V_t(x_0) = 0$ and $V = 0$ at the ends $x = 0$ and $x = l$ for all t.**

Answer

Let
$$V = XT \quad \ldots(2.25.1)$$
where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 V}{\partial x^2} = TX'' \text{ and } \frac{\partial^2 V}{\partial t^2} = TX''$$

Substituting in the given equations, we get

$$TX'' = LCTX''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -p^2 \text{ (say)}$$

i.
$$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$$

$\therefore$
$$X = C_1 \cos px + C_2 \sin px$$

ii.
$$LC\frac{T''}{T} = -p^2 \Rightarrow \quad T'' + \frac{p^2}{LC}T = 0$$

$\therefore$
$$T = C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}$$

Hence,
$$V = XT = (C_1 \cos px + C_2 \sin px)\left(C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}\right) \quad \ldots(2.25.2)$$

Boundary conditions are

$$V(0, t) = 0 = V(l, t) \text{ and } \frac{\partial V}{\partial t} = 0 \text{ when } t = 0$$

Applying conditions on eq. (2.25.2), we get

$$0 = C_1\left(C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}\right)$$

$\Rightarrow$
$$C_1 = 0$$

From eq. (2.25.2), $$V = C_2 \sin px \left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}} \right) \quad ...(2.25.3)$$

$$V(l, t) = 0 = C_2 \sin pl \left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}} \right)$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \; (n \in I)$$

$\therefore$ $$p = \frac{n\pi}{l}$$

Hence, from eq. (2.25.3)

$$V = C_2 \sin \left[C_3 \cos \frac{n\pi t}{l\sqrt{LC}} + C_4 \sin \frac{n\pi t}{l\sqrt{LC}} \right] \quad ...(2.25.4)$$

$$\frac{\partial V}{\partial t} = C_2 \frac{n\pi}{l\sqrt{LC}} \sin \frac{n\pi x}{l} \left[-C_3 \sin \frac{n\pi t}{l\sqrt{LC}} + C_4 \sin \frac{n\pi t}{l\sqrt{LC}} \right]$$

At $t = 0$,

$$0 = C_2 \frac{n\pi}{l\sqrt{LC}} \sin \frac{n\pi x}{l} . C_4$$

$\Rightarrow$ $$C_4 = 0$$

Hence from eq. (2.25.4),

$$V = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi x}{l\sqrt{LC}} \quad ...(2.25.5)$$

Now, $$V(x, 0) = V_0 \sin \frac{\pi x}{l} = C_2 C_3 \sin \frac{n\pi x}{l}$$

Comparing, we get

$$C_2 C_3 = V_0 \text{ and } n = 1$$

Hence, the required solution is

$$V(x, t) = V_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

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UNIT

Statistical Techniques I

CONTENTS

PART-1

Introduction : Measures of Central Tendency, Moments, Moment Generating Function (MGF), Skewness, Kurtosis.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **The first four moments of a distribution about the value 4 of the variable are – 1.5, 17, – 30 and 108. Find the moments about the origin.**

Answer

Given : $\mu_1' = -1.5, \mu_2' = 17, \mu_3' = -30, \mu_4' = 108$
$A = 4.$

Moments about the mean,

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$= 17 - (-1.5)^2 = 14.75$$
$$\mu_3 = \mu_3' - 3\,\mu_1'\mu_2' + 2(\mu_1')^3$$
$$= (-30) - 3\,(-1.5)(17) + 2\,(-1.5)^3 = 39.75$$
$$\mu_4 = \mu_4' - 4\,\mu_1'\mu_3' + 6\,(\mu_1')^2\,(\mu_2') - 3(\mu_1')^4$$
$$= 108 - 4(-1.5)(-30) + 6(-1.5)^2(17) - 3(-1.5)^4$$
$$= 142.3125$$

Moments about origin,

$$\upsilon_1 = A + \mu_1' = 4 - 1.5 = 2.5$$
$$\upsilon_2 = \mu_2 + (\upsilon_1)^2 = 14.75 + (2.5)^2 = 21$$
$$\upsilon_3 = \mu_3 + 3\upsilon_1\upsilon_2 - 2\upsilon_1^{\,3}$$
$$= 39.75 + 3 \times 2.5 \times 21 - 2 \times (2.5)^3 = 166$$
$$\upsilon_4 = \mu_4 + 4\upsilon_1\upsilon_3 - 6\upsilon_1^{\,2}\upsilon_2 + 3\upsilon_1^{\,4}$$
$$= 142.3125 + 4 \times 2.5 \times 166 - 6 \times (2.5)^2 \times 21 + 3 \times (2.5)^4$$
$$= 1132$$

Que 3.2. **Define skewness and kurtosis of a distribution. The first four moments of a distribution are 0, 2.5, 0.7, and 18.71. Find the coefficient of skewness and kurtosis.**

Answer

Skewness : The term skewness means lack of symmetry *i.e.*, when a distribution is not symmetric then it is called a skewed distribution and this distribution may be positively skewed or negatively skewed.

Kurtosis : It tells whether the distribution, if plotted on a graph would give us a normal curve, a curve more flat than the normal curve, or more peaked than the normal curve.

Numerical :

$$\mu_1 = 0,\ \mu_2 = 2.5,\ \mu_3 = 0.7,\ \mu_4 = 18.71$$

$$\text{Coefficient of skewness } (\beta_1) = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{(0.7)^2}{(2.5)^3} = 0.03136\ (+\text{ ve})$$

The distribution is positively skewed.

$$\text{Kurtosis } (\beta_2) = \frac{\mu_4}{{\mu_2}^2} = \frac{18.71}{(2.5)^2} = 2.9936 < 3$$

The distribution is platykurtic.

Que 3.3. **Find the M.G.F. of the random variable X having the following probability density function**

$$F(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Also find mean and variance of X.

Answer

Moment generating function is given by

$$M_x(t) = E(e^{tx})$$

$$= \int_0^1 xe^{tx}\,dx + \int_1^2 (2-x)e^{tx}\,dx + \int_2^\infty 0.e^{tx}\,dx$$

$$= \left[\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2}\,(e^{2t} + 1 - 2e^t)$$

$$= \frac{(e^t - 1)^2}{t^2} = \left(\frac{e^t - 1}{t}\right)^2$$

$$= \frac{1}{t^2}\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right]^2$$

$$M_x(t) = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots$$

Now

$$V_1 = \frac{d}{dt}\,M_x(t) = \frac{1}{2!} + 0 = \frac{1}{2}$$

$$V_2 = \frac{d^2}{dt^2} M_x(t) = \frac{d}{dt}\left(\frac{1}{2!} + \frac{2t}{3!} + \ldots\right) = \frac{1}{3}$$

$$\text{Mean} = \bar{x} = V_1 = \frac{1}{2}$$

$$\text{Variance} = u_2 = V_2 - \bar{x}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Que 3.4. The first four moments of a distribution about $x = 4$ are 1, 4, 10 and 45. Calculate the moments about the mean and comment upon the skewness and kurtosis of the distribution.

Answer

Given : $\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$ and $\mu_4' = 45$, $x = 4$

Moments about mean :

$$\mu_1 = 0,\ \mu_2 = \mu_2' - (\mu_1')^2 = 4 - (1)^2 = 4 - 1 = 3$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= 10 - 3\,(1)\,(4) + 2\,(1)^3 = 10 - 12 + 2 = 0$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\,(\mu_1')^2(\mu_2') - 3\,(\mu_1')^4$$

$$= 45 - 4(1)\,(10) + 6\,(1)^2\,(4) - 3\,(1)^4$$

$$= 45 - 40 + 24 - 3 = 26$$

Coefficient of skewness :

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

Coefficient of kurtosis :

$$\frac{\mu_4}{\mu_2^2} = \frac{26}{3^2} = 2.88 < 3,\ i.e., \text{ curve is platykurtic.}$$

Que 3.5. Find all four central moments and discuss skewness and kurtosis and also Karl Pearson skewness for the frequency distribution given below :

Range of Expend in ₹ (100)/month	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12
No. of Families	38	292	389	212	69

Answer

Moments about mean are given by

Range	f	x	$(x-A)$	$f(x-A)$	$(x-A)^2$	$f(x-A)^2$	$(x-A)^3$	$f(x-A)^3$	$(x-A)^4$	$f(x-A)^4$	Cumulative Frequency
2 – 4	38	3	–4	–152	16	608	–64	–2432	256	9728	38
4 – 6	292	5	–2	–584	4	1168	–8	–2336	16	4672	330
6 – 8	389	7(A)	0	0	0	0	0	0	0	0	719
8 – 10	212	9	2	424	4	848	8	1696	16	3392	931
10 – 12	69	11	4	276	16	1104	64	4416	256	17664	1000
	$\Sigma f = 1000$			$\Sigma f(x-A) = -36$		$\Sigma f(x-A)^2 = 3728$		$\Sigma f(x-A)^3 = 1344$		$\Sigma f(x-A)^4 = 35,456$	

$$\mu'_1 = \frac{\Sigma f(x-A)}{\Sigma f} = \frac{-36}{1000} = -0.036$$

$$\mu'_2 = \frac{\Sigma f(x-A)^2}{\Sigma f} = \frac{3728}{1000} = 3.728$$

$$\mu'_3 = \frac{\Sigma f(x-A)^3}{\Sigma f} = \frac{1344}{1000} = 1.344,\ \mu_4' = \frac{\Sigma f(x-A)^4}{\Sigma f} = \frac{35456}{1000} = 35.456$$

Central moments are given by

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 3.728 - (-0.036)^2 = 3.7267$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$$

$$= 1.344 - 3(3.728)(-0.036) + 2(-0.036)^3$$

$$= 1.344 + 0.402624 - 0.000093312$$

$$\mu_3 = 1.7465$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$

$$= 35.456 - 4(1.344)(-0.036) + 6(3.728)(-0.036)^2 - 3(-0.036)^4$$

$$= 35.456 + 0.193536 + 0.028988 - 5.0388 \times 10^{-6}$$

$$\mu_4 = 35.6785$$

Coefficient of skewness, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

$$\beta_1 = \frac{(1.7465)^2}{(3.7267)^3}$$

$$\beta_1 = 0.0589 \text{ (positive)}$$

The curve is positively skewed.

Coefficient of kurtosis, $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{35.6785}{(3.7267)^2}$

$\beta_2 = 2.569\ (< 3)$, *i.e.*, curve is platykurtic.

Measure of Karl Pearson's skewness is given by

$$\text{Mean} = A + \frac{\Sigma f(x-A)}{\Sigma f}$$

$$= 7 + \frac{(-36)}{1000} = 6.964$$

$$\text{Median} = l + \frac{\frac{N}{2} - c.f}{f} . i = 6 + \frac{\frac{1000}{2} - 330}{389} \times 2$$

$$= 6 + 0.437 \times 2 = 6.874$$

$$\text{Standard Deviation (S.D.)} = \sqrt{\frac{\Sigma f(x-A)^2}{\Sigma f} - \left(\frac{\Sigma f(x-A)}{\Sigma f}\right)^2}$$

$$= \sqrt{\frac{3728}{1000} - \left(\frac{-36}{1000}\right)^2}$$

$$= \sqrt{3.728 - 0.001296} = \sqrt{3.726} = 1.930$$

$$\text{Karl Pearson's coefficient of skewness} = \frac{3(\text{Mean} - \text{Median})}{S.D}$$

$$S_k = \frac{3(6.964 - 6.874)}{1.930}$$

$$S_k = \frac{0.27}{1.930} = 0.1398$$

Since $S_k > 0$

$\therefore$ Distribution is positively skewed.

Que 3.6. **The following table represents the height of a batch of 100 students. Calculate skewness and kurtosis :**

Height (in cm)	59	61	63	65	67	69	71	73	75
No. of students	0	2	6	20	40	20	8	2	2

Answer

Height (cm) x	No. of student f	$u = \frac{x-67}{2}$	fu	fu^2	fu^3	fu^4
59	0	−4	0	0	0	0
61	2	−3	−6	18	−54	162
63	6	−2	−12	24	−48	96
65	20	−1	−20	20	−20	20
67	40	0	0	0	0	0
69	20	1	20	20	20	20
71	8	2	16	32	64	128
73	2	3	6	18	54	162
75	2	4	8	32	128	512
	$N = \Sigma f = 100$		$\Sigma fu = 12$	$\Sigma fu^2 = 164$	$\Sigma fu^3 = 144$	$\Sigma fu^4 = 1100$

Moments about 67 :

$$\mu_1' = \left(\frac{\sum fu}{N}\right) h = \left(\frac{12}{100}\right)(2) = 0.24$$

$$\mu_2' = \left(\frac{\sum fu^2}{N}\right) h^2 = \left(\frac{164}{100}\right)(4) = 6.56$$

$$\mu_3' = \left(\frac{\sum fu^3}{N}\right) h^3 = \frac{144}{100} \times 8 = 11.52$$

$$\mu_4' = \left(\frac{\sum fu^4}{N}\right) h^4 = \frac{1100}{100} \times 16 = 176$$

Moments about mean :

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - \mu_1'^2 = 6.56 - (.24)2 = 6.5024$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= 11.52 - 3(6.56)(0.24) + 2(0.24)^3 = 6.824448$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= 176 - 4(11.52)(0.24) + 6(6.56)(0.24)^2 - 3(0.24)^4$$
$$= 167.19798$$

Coefficient of skewness, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

$$= \frac{(6.824448)^2}{(6.5024)^3} = 0.1694 \text{ (positive)}$$

Hence, the curve is positively skewed.

Coefficient of kurtosis, $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 3.9544 > 3$

Hence the distribution is leptokurtic.

PART-2

Curve Fitting, Method of Least Square, Fitting of Straight Lines, Fitting of Second Degree Parabola, Exponential Curve.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.7. **Use the method of least squares to obtain the normal equations and fit the curve for $y = \frac{c_0}{x} + c_1\sqrt{x}$ to the following table of values :**

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

Answer

Normal equations to the curve $y = \frac{c_0}{x} + c_1\sqrt{x}$ are

$$\sum \frac{y}{x} = c_0 \sum \frac{1}{x^2} + c_1 \sum \frac{1}{\sqrt{x}}$$

$$\sum y\sqrt{x} = c_0 \sum \frac{1}{\sqrt{x}} + c_1 \sum x$$

Table of values is :

x	y	y/x	$y\sqrt{x}$	$\frac{1}{\sqrt{x}}$	$\frac{1}{x^2}$
0.1	21	210	6.64078	3.16228	100
0.2	11	55	4.91935	2.23607	25
0.4	7	17.5	4.42719	1.58114	6.25
0.5	6	12	4.24264	1.41421	4
1	5	5	5	1	1
2	6	3	8.48528	0.70711	0.25
$\Sigma x = 4.2$		$\Sigma(y/x) = 302.5$	$\Sigma y\sqrt{x} = 33.71524$	$\Sigma \frac{1}{\sqrt{x}} = 10.10081$	$\Sigma \frac{1}{x^2} = 136.5$

Substituting the values in normal equations, we get

$$302.5 = 136.5\, c_0 + 10.10081\, c_1 \qquad ...(3.7.1)$$

and

$$33.71524 = 10.10081\, c_0 + 4.2\, c_1 \qquad ...(3.7.2)$$

Solving eq. (3.7.1) and eq. (3.7.2), we get

$$c_0 = 1.97327 \text{ and } c_1 = 3.28182$$

Hence the required equation of curve is

$$y = \frac{1.97327}{x} + 3.28182\sqrt{x}$$

Que 3.8. **Using method of least squares, derive the normal equation to fit a parabola $y = a + bx + cx^2$ from the following data :**

x	2	3	4	5	6
y	14	17	20	24	29

Answer

Equation of parabola, $y = a + bx + cx^2$

Normal equations for the parabola are :

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

x	y	xy	x^2	x^3	x^4	x^2y
2	14	28	4	8	16	56
3	17	51	9	27	81	153
4	20	80	16	64	256	320
5	24	120	25	125	625	600
6	29	174	36	216	1296	1044
$\Sigma x = 20$	$\Sigma y = 104$	$\Sigma xy = 453$	$\Sigma x^2 = 90$	$\Sigma x^3 = 440$	$\Sigma x^4 = 2274$	$\Sigma x^2 y = 2173$

Substituting the values in normal equations, we have

$$104 = 5a + 20b + 90c \quad (\because n = 5)$$

$$453 = 20a + 90b + 440c$$

$$2173 = 90a + 440b + 2274c$$

On solving, we get

$$a = 11, b = 0.843 \text{ and } c = 0.357$$

So the required equation of parabola is given by

$$y = 11 + 0.843x + 0.357\,x^2$$

Que 3.9. **Using the least square method, fit a second degree polynomial from the following data :**

x	0	1	2	3	4	5	6	7	8
y	12.0	10.5	10.0	8.0	7.0	8.0	7.5	8.5	9.0

Also, estimate y at $x = 6.5$.

Answer

Let a second degree polynomial, $y = ax^2 + bx + c$

The normal equations for the given polynomial are given as follows :

$$\Sigma x^2 y = c\Sigma x^2 + b\Sigma x^3 + a\Sigma x^4$$

$$\Sigma y = nc + b\Sigma x + a\Sigma x^2$$
$$\Sigma yx = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

x	y	x^2	x^3	x^4	xy	x^2y
0	12.0	0	0	0	0	0
1	10.5	1	1	1	10.5	10.5
2	10.0	4	8	16	20.0	40.0
3	8.0	9	27	81	24.0	72.0
4	7.0	16	64	256	28.0	112.0
5	8	25	125	625	40.0	200.0
6	7.5	36	216	1296	45.0	270
7	8.5	49	343	2401	59.5	416.5
8	9.0	64	512	4096	72.0	576
$\Sigma x = 36$	$\Sigma y = 80.5$	$\Sigma x^2 = 204$	$\Sigma x^3 = 1296$	$\Sigma x^4 = 8772$	$\Sigma xy = 299$	$\Sigma x^2y = 1697$

$$n = 9$$

Putting value in normal equations, we have

$$204a + 36b + 9c = 80.5 \quad ...(3.9.1)$$
$$1296a + 204b + 36c = 299 \quad ...(3.9.2)$$
$$8772a + 1296b + 204c = 1697 \quad ...(3.9.3)$$

On solving eq. (3.9.1), eq. (3.9.2) and eq. (3.9.3),

$$a = 0.18, \quad b = -1.85, \quad c = 12.18$$

Then, $\quad y = 0.18x^2 - 1.85x + 12.18$

At $\quad x = 6.5$

$$y = 7.76$$

Que 3.10. **Fit the curve $pv^{\gamma} = K$ to the following data :**

p (kg / cm²)	0.5	1	1.5	2	2.5	3
v (litres)	1620	1000	750	650	520	460

Answer

$$pv^{\gamma} = K$$

$$v = \left(\frac{K}{p}\right)^{1/\gamma} = K^{1/\gamma}\, p^{-1/\gamma}$$

Taking log, $\quad \log v = \frac{1}{\gamma}\log K - \frac{1}{\gamma}\log p$

which is the form $\quad Y = A + BX$

Where $\; Y = \log v, \;\; X = \log p, \;\; A = \frac{1}{\gamma}\log K \;$ and $\; B = -\frac{1}{\gamma}$

p	v	X	Y	XY	X^2
0.5	1620	– 0.30103	3.20952	– 0.96616	0.09062
1	1000	0	3	0	0
1.5	750	0.17609	2.87506	0.50627	0.03101
2	620	0.30103	2.79239	0.84059	0.09062
2.5	520	0.39794	2.716	1.08080	0.15836
3	460	0.47712	2.66276	1.27046	0.22764
Total		ΣX = 1.05115	ΣY = 17.25573	ΣXY = 2.73196	ΣX^2 = 0.59825

Here, $m = 6$

Substituting the values in normal equations, we get

$$17.25573 = 6A + 1.05115\,B$$

and

$$2.73196 = 1.05115\,A + 0.598825\,B$$

On solving, we get

$$A = 2.99911 \quad \text{and} \quad B = -0.70298$$

$$\therefore \qquad \gamma = -\frac{1}{B} = \frac{1}{0.70298} = 1.42252$$

Again,

$$\log K = \gamma A = 4.26629$$

$$\therefore \qquad K = \text{antilog}\,(4.26629) = 18462.48$$

Hence required curve is

$$pv^{1.42252} = 18462.48.$$

Que 3.11. **Determine the least square approximation of the type $ax^2 + bx + c$ to the function 2^x, at points $x_i = 0, 1, 2, 3, 4$.**

Answer

Here

$$y = 2^x = ax^2 + bx + c$$

Normal equations for the given curve are,

$$\Sigma yx^2 = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$$

$$\Sigma yx = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

$$\Sigma y = a\Sigma x^2 + b\Sigma x + mc$$

(Here $m = 5$)

Table of values is,

x	y	xy	x^2	yx^2	x^3	x^4
0	1	0	0	0	0	0
1	2	2	1	2	1	1
2	4	8	4	16	8	16
3	8	24	9	72	27	81
4	16	64	16	256	64	256
Σx = 10	Σy = 31	Σxy = 98	Σx^2 = 30	Σyx^2 = 346	Σx^3 = 100	Σx^4 = 354

Substituting the values in normal equations, we get

$$346 = 354a + 100b + 30c$$

$$98 = 100a + 30b + 10c$$

$$31 = 30a + 10b + 5c$$

On solving,

$$a = 1.143,\ b = -0.971 \text{ and } c = 1.286$$
$$y = 1.143x^2 - 0.971x + 1.286$$

Que 3.12. **Determine the normal equations if the curve $y = ax + bx^2$ is fitted to the data (x_i, y_i), $i = 1, 2, ..., m$. Hence fit this curve to the data :**

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Answer

Normal equations are,

$$\Sigma xy = a\Sigma x^2 + b\Sigma x^3 \quad ...(3.12.1)$$

and

$$\Sigma x^2y = a\Sigma x^3 + b\Sigma x^4 \quad ...(3.12.2)$$

Table of values is

x	y	x^2	x^3	x^4	xy	x^2y
1	1.8	1	1	1	1.8	1.8
2	5.1	4	8	16	10.2	20.4
3	8.9	9	27	81	26.7	80.1
4	14.1	16	64	256	56.4	225.6
5	19.8	25	125	625	99	495
Total		$\Sigma x^2 = 55$	$\Sigma x^3 = 225$	$\Sigma x^4 = 979$	$\Sigma xy = 194.1$	$\Sigma x^2y = 822.9$

Substituting the values in eq. (3.12.1) and eq. (3.12.2), we get

$$194.1 = 55\,a + 225\,b$$

and

$$822.9 = 225\,a + 979\,b$$

On solving, we get $a = \dfrac{83.85}{55} = 1.52$ and $b = \dfrac{317.4}{664} = 0.49$

Hence required parabolic curve is $y = 1.52x + 0.49\,x^2$

Que 3.13. **Using the method of least square fit a curve of the form $y = ab^x$ to the following data :**

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

Answer

$$y = ab^x$$
$$\log y = \log a + x \log b$$

Let

$$Y = \log_e y,\ A = \log_e a,\ B = \log_e b$$

$$Y = A + Bx$$

Normal equations are,

$$\sum_{i=1}^{5} Y_i = n\,A + B\sum_{i=1}^{5} x$$

$$\sum_{i=1}^{5} x_i\,Y_i = A\sum_{i=1}^{5} x_i + B\sum_{i=1}^{5} x_i^2 \qquad \text{(Here } n = 5\text{)}$$

x	y	$Y = \log y$	x^2	$x_i\,Y_i$
2	8.3	2.1163	4	4.2326
3	15.4	2.7344	9	8.2032
4	33.1	3.4995	16	13.998
5	65.2	4.1775	25	20.8875
6	127.4	4.8473	36	29.0838
$\sum_{i=1}^{5} x = 20$		$\sum_{i=1}^{5} Y = 17.3750$	$\sum_{i=1}^{5} x^2 = 90$	$\sum_{i=1}^{5} x_i Y_i = 76.4051$

Substituting the values in normal equations, we get

$$17.375 = 5\,A + 20\,B$$

$$76.4051 = 20\,A + 90\,B$$

On solving, we get

$$B = 0.69051$$

and $\quad A = 0.71296$

$\therefore \quad 0.71296 = \log_e a$

$\Rightarrow \quad a = e^{0.71296} = 2.04$

and $\quad 0.69051 = \log_e b$

$\Rightarrow \quad b = e^{0.69051} = 1.99$

$\therefore \quad y = ab^x \Rightarrow y = 2.04(1.99)^x$

Que 3.14. **Fit a second degree parabola to the following data :**

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Answer

We shift the origin to (2.5, 0) and take 0.5 as the new unit. This changes the variable x to X, by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. Normal equations are :

$$\Sigma y = am + b\Sigma X + c\Sigma X^2$$

$$\Sigma Xy = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$

$$\Sigma X^2 y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

x	X	y	Xy	X^2	X^2y	X^3	X^4
1.0	– 3	1.1	– 3.3	9	9.9	– 27	81
1.5	– 2	1.3	– 2.6	4	5.2	– 8	16
2.1	– 1	1.6	– 1.6	1	1.6	– 1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total =	$\Sigma X = 0$	$\Sigma y = 16.2$	$\Sigma Xy = 14.3$	$\Sigma X^2 = 28$	$\Sigma X^2y = 69.9$	$\Sigma X^3 = 0$	$\Sigma X^4 = 196$

Substituting the values in normal equations, we get

$$7a + 28c = 16.2; \qquad 28b = 14.3; \qquad 28a + 196c = 69.9$$

On solving, we get

$$a = 2.07, \quad b = 0.511, \quad c = 0.061$$

$$y = 2.07 + 0.511X + 0.061\,X^2$$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511\,(2x - 5) + 0.061\,(2x - 5)^2$$

which simplifies to $\quad y = 1.04 - 0.198\,x + 0.244\,x^2.$

This is the required parabola of best fit.

PART-3

Correction and Rank Correlation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.15. **Calculate the rank coefficient from the sales and expenses of 10 firms as given below :**

Sales X	45	56	39	54	45	40	56	60	30	36
Expenses Y	40	36	30	44	36	32	45	42	20	36

Answer

X	Y	R_1	R_2	$d = R_1 - R_2$	d^2
45	40	5.5	4	1.5	2.25
56	36	2.5	6	– 3.5	12.25
39	30	8	9	– 1	1
54	44	4	2	2	4
45	36	5.5	6	– 0.5	0.25
40	32	7	8	– 1	1
56	45	2.5	1	1.5	2.25
60	42	1	3	– 2	4
30	20	10	10	0	0
36	36	9	6	3	9
					$\Sigma d^2 = 36$

Repeated Rank of X column :

$$45 = 2 \text{ times} = m_1$$
$$56 = 2 \text{ times} = m_2$$

Repeated Rank of Y column :

$$36 = 3 \text{ times} = m_3$$

Rank correlation coefficient,

$$r = 1 - \frac{6\left[\Sigma d^2 + \frac{1}{12}\Sigma(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right]}{N(N^2 - 1)}$$

$$= 1 - \frac{6\left[36 + \frac{1}{12}(2^3 - 2 + 2^3 - 2 + 3^3 - 3)\right]}{10(99)}$$

$$= 1 - \frac{6[36 + 3]}{990} = 1 - \frac{234}{990}$$

$$r = 0.7636$$

Que 3.16. **Calculate the coefficient of correlation between the following ages of husband (x) and wife (y) by taking 30 and 28 as assumed mean in case of x and y respectively :**

x	24	27	28	28	29	30	32	33	35	35	40
y	18	20	22	25	22	28	28	30	27	30	32

Answer

Given : $\bar{x} = 30$ and $\bar{y} = 28$

x	y	$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
24	18	−6	−10	36	100	60
27	20	−3	−8	9	64	24
28	22	−2	−6	4	36	12
28	25	−2	−3	4	9	6
29	22	−1	−6	1	36	6
30	28	0	0	0	0	0
32	28	2	0	4	0	0
33	30	3	2	9	4	6
35	27	5	−1	25	1	−5
35	30	5	2	25	4	10
40	32	10	4	100	16	40
				$\Sigma(x-\bar{x})^2 = 217$	$\Sigma(y-\bar{y})^2 = 270$	$\Sigma(x-\bar{x})(y-\bar{y}) = 159$

Now we know that the coefficient of correlation is given as,

$$r_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \times \Sigma(y-\bar{y})^2}} = \frac{159}{\sqrt{217 \times 270}}$$

$$= \frac{159}{242.0537}$$

$$r_{xy} = 0.657$$

Que 3.17. **Ten students got the following percentage of marks in principles of Economics and Statistics :**

Roll Nos.	1	2	3	4	5	6	7	8	9	10
Marks in Economics	78	36	98	25	75	82	90	62	65	39
Marks in Statistics	84	51	91	60	68	62	86	58	53	47

Calculate the coefficient of correlation.

Answer

Let the marks in the two subjects be denoted by x and y respectively.

x	y	$u = x - 65$	$v = y - 66$	u^2	v^2	uv
78	84	13	18	169	324	234
36	51	– 29	– 15	841	225	435
98	91	33	25	1089	625	825
25	60	– 40	– 6	1600	36	240
75	68	10	2	100	4	20
82	62	17	– 4	289	16	– 68
90	86	25	20	625	400	500
62	58	– 3	– 8	9	64	24
65	53	0	– 13	0	169	0
39	47	– 26	– 19	676	361	494
Total		$\Sigma u = 0$	$\Sigma v = 0$	$\Sigma u^2 = 5398$	$\Sigma v^2 = 2224$	$\Sigma uv = 2704$

Here, $$n = 10,\ \bar{u} = \frac{1}{n}\Sigma u_i = 0,\ \bar{v} = \frac{1}{n}\Sigma v_i = 0$$

$$r_{uv} = \frac{n\Sigma uv - \Sigma u \Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$$

$$= \frac{(10 \times 2704) - (0 \times 0)}{\sqrt{(10 \times 5398) - (0)^2}\sqrt{(10 \times 2224) - (0)^2}} = 0.780$$

Hence, $$r_{xy} = r_{uv} = 0.780.$$

PART-4

Regression Analysis : Regression Lines of y on x and x on y, Regression Coefficients, Properties of Regressions Coefficient and Non-Linear Regression.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.18. If the θ is the acute angle between the two regression lines in the case of two variables x and y, show that $\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$, where r, σ_x, σ_y have their usual meanings. Explain the significance of the formula when $r = 0$ and $r = \pm 1$.

Answer

Coefficient of correlation is given by,

$$r = \frac{\eta \Sigma\, dxdy - \Sigma\, dx\, \Sigma\, dy}{\sqrt{\eta \Sigma\, dx^2 - (\Sigma\, dx)^2}\, \sqrt{\eta \Sigma\, dy^2 - (\Sigma\, dy)^2}}$$

where,

$$dx = x - \bar{x}$$
$$dy = y - \bar{y}$$

Coefficients of regression are given by,

$$b_{xy} = \frac{\eta \Sigma\, dxdy - \Sigma\, dx\, \Sigma\, dy}{\eta \Sigma\, dy^2 - (\Sigma\, dy)^2}$$

$$b_{yx} = \frac{\eta \Sigma\, dxdy - \Sigma\, dx\, \Sigma\, dy}{\eta \Sigma\, dx^2 - (\Sigma\, dx)^2}$$

Equations to the lines of regression of y on x and x on y are

$$y - \bar{y} = \frac{r\sigma_y}{\sigma_x}(x - \bar{x}) \text{ and } x - \bar{x} = \frac{r\sigma_x}{\sigma_y}(y - \bar{y})$$

Their slopes are $m_1 = \dfrac{r\sigma_y}{\sigma_x}$ and $m_2 = \dfrac{\sigma_y}{r\sigma_x}$

$$\therefore \quad \tan\theta = \pm \frac{m_2 - m_1}{1 + m_2 m_1} = \pm \frac{\dfrac{\sigma_y}{r\sigma_x} - \dfrac{r\sigma_y}{\sigma_x}}{1 + \dfrac{\sigma_y^2}{\sigma_x^2}}$$

$$= \pm \frac{1 - r^2}{r} \cdot \frac{\sigma_y}{\sigma_x} \cdot \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2} = \pm \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Since $r^2 \le 1$ and σ_x, σ_y are positive.

$\therefore$ +ve sign gives the acute angle between the lines.

Hence,

$$\tan\theta = \frac{1 - r^2}{r} \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When $r = 0$, $\theta = \dfrac{\pi}{2}$ $\therefore$ The two lines of regression are perpendicular to each other.

Hence the estimated value of y is the same for all values of x and vice-versa.

When $r = \pm 1$, $\tan\theta = 0$ so that $\theta = 0$ or π.

The lines of regression will coincide and there is a perfect correlation between the two variables x and y.

Que 3.19. **In a partially destroyed laboratory record of an analysis of correlation data, the following results are legible variance of $x = 9$. Regression equations are**

$$8x - 10y = -66$$
$$40x - 18y = 214$$

Find

I. Mean values of x and y

II. Standard deviation of y

III. Correlation coefficient between x & y.

Answer

Since both the lines of regression pass through the point $(\bar{x}, \bar{y})$ there we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \quad ...(3.$$

$$40\bar{x} - 18\bar{y} - 214 = 0 \quad ...(3.$$

Multiplying eq. (3.19.1) by 5, we get

$$40\bar{x} - 50\bar{y} + 330 = 0 \quad ...(3.$$

Subtracting eq. (3.19.3) from eq. (3.19.2), we get

$$32\bar{y} - 544 = 0 \therefore \bar{y} = 17$$

$\therefore$ From eq. (3.19.1), $8\bar{x} - 170 + 66 = 0$

$$8\bar{x} = 104$$

$$\bar{x} = 13$$

Hence $\bar{x} = 13,\ \bar{y} = 17$

Now, variance of $x = \sigma_x^2 = 9$

$$\sigma_x = 3$$

The equations of lines of regression can be written as

$$y = 0.8x + 6.6 \text{ and } x = 0.45y + 5.35$$

$\therefore$ The regression coefficient of y on x is $\frac{r\sigma_y}{\sigma_x} = 0.8$ ...(3.

The regression coefficient of x on y is $\frac{r\sigma_x}{\sigma_y} = 0.45$ ...(3.

Multiplying eq. (3.19.4) and eq. (3.19.5), we get

$$r^2 = 0.8 \times 0.45 = 0.36$$

$$r = 0.6$$

From eq. (3.19.4), we get standard deviation of y,

$$\sigma_y = \frac{0.8\sigma_x}{r} = \frac{0.8 \times 3}{0.6} = 4.$$

Que 3.20. **Find the coefficient of correlation (r) and obtain equation to the lines of regression for the following data :**

x	6	2	10	4	8
y	9	11	5	8	7

Answer

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	XY	X^2	Y^2
6	9	0	1	0	0	1
2	11	-4	3	-12	16	9
10	5	4	-3	-12	16	9
4	8	-2	0	0	4	0
8	7	2	-1	-2	4	1
$\Sigma x = 30$	$\Sigma y = 40$	$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma XY = -26$	$\Sigma X^2 = 40$	$\Sigma Y^2 = 20$

$$\bar{x} = \frac{\Sigma x}{N} = \frac{30}{5} = 6$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{40}{5} = 8$$

Regression coefficient of y on x,

$$b_{yx} = \frac{\Sigma XY}{\Sigma X^2} = \frac{-26}{40} = -0.65$$

Regression coefficient of x on y,

$$b_{xy} = \frac{\Sigma XY}{\Sigma Y^2} = \frac{-26}{20} = -1.3$$

Equation of regression line (y on x) is

$$y - \bar{y} = \frac{\Sigma XY}{\Sigma X^2}(x - \bar{x})$$

$$y - 8 = \frac{-26}{40}(x - 6)$$

or, $$y - 8 = -0.65\,(x - 6)$$

or, $$y = -0.65x + 11.9$$

Regression equation (x on y) is

$$x - \bar{x} = \frac{\Sigma XY}{\Sigma Y^2}(y - \bar{y})$$

$$x - 6 = -1.3\,(y - 8)$$

$$x = -1.3y + 16.4$$

Correlation coefficient,

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$r^2 = -0.65 \times (-1.3) = 0.845$$

$$r = -\sqrt{0.845}$$

$$r = -0.919$$ (As both b_{yx}, b_{xy} are negativ

Que 3.21. **If for two random variables, x and y with same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$, then show that**

$$\frac{b}{\beta} = \frac{1-a}{1-\alpha}$$

Also find the common mean.

Answer

Here, $b_{yx} = a, b_{xy} = \alpha$

Let the common mean be m, then regression lines are

$$y - m = a(x - m)$$

$$y = ax + m(1 - a) \quad ...(3.21.1)$$

and

$$x - m = \alpha (y - m)$$

$$x = \alpha y + m (1 - \alpha) \quad ...(3.21.2)$$

Comparing eq. (3.21.1) and eq. (3.21.2) with the given equations,

$$b = m (1 - a), \beta = m (1 - \alpha)$$

$$\therefore \quad \frac{b}{\beta} = \frac{1-a}{1-\alpha}$$

Que 3.22. **For 10 observations on price (x) and supply (y) the following data were obtained**

$$\Sigma x = 130, \Sigma y = 220, \Sigma x^2 = 2288$$

$$\Sigma y^2 = 5506 \text{ and } \Sigma xy = 3467$$

Obtain the two lines of regression.

Answer

$$\bar{x} = \frac{\Sigma x}{N}$$

$$\bar{x} = \frac{130}{10} = 13$$

$$\bar{y} = \frac{\Sigma y}{N}$$

$$\bar{y} = \frac{220}{10} = 22$$

Regression coefficient of y on x, $b_{yx} = \frac{\Sigma XY}{\Sigma X^2}$

$$b_{yx} = \frac{3467}{2288} = 1.52$$

Regression coefficient of x on y, $b_{xy} = \dfrac{\Sigma XY}{\Sigma Y^2} = \dfrac{3467}{5506} = 0.63$

Equation of regression line (y on x) is,

$$y - \bar{y} = \frac{\Sigma XY}{\Sigma X^2}(x - \bar{x})$$

$$y - 22 = 1.52(x - 13)$$

$$y = 1.52x + 2.24$$

Regression equation (x on y) is,

$$x - \bar{x} = \frac{\Sigma XY}{\Sigma Y^2}(y - \bar{y})$$

$$x - 13 = 0.63\,(y - 22)$$

$$x = 0.63y - 0.86$$

☺☺☺

4

UNIT

Statistical Techniques II

CONTENTS

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. **From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction, (ii) two particular engineers must be included, (iii) one particular architect must be excluded.**

Answer

i. Number of committees $= {}^6C_3 \times {}^5C_2 = \dfrac{6.5.4}{3.2.1} \times \dfrac{5.4}{2.1} = 200.$

ii. Here we have to choose one engineer from the remaining four engineers.

$\therefore$ Number of committees $= {}^4C_1 \times {}^5C_2 = 4 \times \dfrac{5.4}{2.1} = 40.$

iii. Here we have to choose two architects from the remaining four architects.

$\therefore$ Number of committees $= {}^6C_3 \times {}^4C_2 = \dfrac{6.5.4}{3.2.1} \times \dfrac{4.3}{2.1} = 120.$

Que 4.2. **A five figure number is formed by the digit 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.**

Answer

The five digits can be arranged in 5! ways, out of which 4! will begin with zero.

$\therefore$ Total number of 5-figure number formed $= 5! - 4! = 96.$

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, *i.e.*, number ending in 04, 12, 20, 24, 32, 40.

Number ending in $04 = 3! = 6$, Number ending in $12 = 3! - 2! = 4.$

Number ending in $20 = 3! = 6$, Number ending in $24 = 3! - 2! = 4.$

Number ending in $32 = 3! = 2! = 4$, and Number ending in $40 = 3! = 6.$

The number having 12, 24, 32 in the extreme right are (3! – 2!) since the number having zero on the extreme left are excluded.

Que 4.3. **A has one share in a lottery in which there is 1 prize and 2 blanks ; B has three shares in a lottery in which there are 3 prizes and 6 blanks. Compare the probability of A's success to that of B's success.**

Answer

A can draw a ticket in $^3C_1 = 3$ ways.

The number of cases in which A can get a prize is 1.

$\therefore$ The probability of A's success $= \dfrac{1}{3}$.

Again B can draw a ticket in $^9C_3 = \dfrac{9.8.7}{3.2.1} = 84$ ways.

The number of ways in which B gets all blanks $= {}^6C_3 = \dfrac{6.5.4}{3.2.1} = 20$

$\therefore$ The number of ways of getting a prize $= 84 - 20 = 64$

Thus the probability of B's success $= 64/84 = 16/21$

Hence A's probability of success : B's probability of success $= \dfrac{1}{3} : \dfrac{16}{21}$

$$= 7 : 16$$

Que 4.4. **A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.**

Answer

Two balls out of 14 can be drawn in $^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in $^8C_2$ ways. Thus the probability of drawing 2 white balls

$$= \frac{^8C_2}{^{14}C_2} = \frac{28}{91}$$

Similarly, 2 red balls out of 6 can be drawn in $^6C_2$ ways. Thus the probability of drawing 2 red balls

$$= \frac{^6C_2}{^{14}C_2} = \frac{15}{91}$$

Hence, the probability of drawing 2 balls of the same colour (either both white or both red).

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}$$

Que 4.5. **A bag contains four white and two black balls and a second bag contains three of each colour. A bag is selected at random, and a ball is then drawn at random from the bag chosen. What is the probability that tne ball drawn is white ?**

Answer

There are two mutually exclusive cases,

i. When the first bag is chosen.

ii. When the second bag is chosen.

Now the chance of choosing the first bag is $\frac{1}{2}$ and if this bag is chosen, the probability of drawing a white ball is 4/6. Hence the probability of drawing a white ball from first bag is

$$\frac{1}{2} \times \frac{4}{6} = \frac{1}{3}$$

Similarly the probability of drawing a white ball from second bag is

$$\frac{1}{2} \times \frac{3}{6} = \frac{1}{4}$$

Since the events are mutually exclusive the required probability

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Que 4.6. **Three machines I, II and III are manufacture respectively 0.4, 0.5 and 0.1 of the total production. The percentage of defective items produced by I, II and III is 2, 4 and 1 per cent respectively. For an item chosen at random, what is the probability it is defective ?**

Answer

The defective item produced by machine I $= \frac{0.4 \times 2}{100} = \frac{0.8}{100}$

The defective item produced by machine II $= \frac{0.5 \times 4}{100} = \frac{2}{100}$

The defective item produced by machine III $= \frac{0.1 \times 1}{100} = \frac{0.1}{100}$

The total defective items produced by machines, I, II or III

$$= \frac{0.8}{100} + \frac{2}{100} + \frac{0.1}{100} = \frac{2.9}{100} = 0.029$$

Que 4.7. **Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king and the second a queen if the first card is (i) Replaced, (ii) Not replaced.**

Answer

i. The probability of drawing a king $= \frac{4}{52} = \frac{1}{13}$

If the card is replaced, the pack will again have 52 cards so that the probability of drawing a queen in 1/13.

The two events being independent, the probability of drawing both cards in succession $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$

ii. The probability of drawing a king $= \frac{1}{13}$

If the card is not replaced, the pack will have 51 cards only so that the chance of drawing a queen is 4/51.

Hence, the probability of drawing both cards $= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$

Que 4.8. **There are two groups of objects : one of which consists of 5 science and 3 engineering subjects, and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If the number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately.**

Answer

Probability of turning up 3 or 5 $= \frac{2}{6} = \frac{1}{3}$

Probability of selecting an engineering subject from first group $= \frac{3}{8}$

$\therefore$ Probability of selecting an engineering subject from first group on turning up 3 or 5

$$= \frac{1}{3} \times \frac{3}{8} = \frac{1}{8}$$

Now, probability of not turning 3 or 5 $= 1 - \frac{1}{3} = \frac{2}{3}$

Probability of selecting an engineering subject from second group $= \frac{5}{8}$

∴ Probability of selecting an engineering subject from second group & turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12}$$

Thus, the probability of selecting an engineering subject

$$= \frac{1}{8} + \frac{5}{12} = \frac{13}{24}$$

Que 4.9. **A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd ?**

Answer

Let p be the probability of getting a head and q the probability of getting a tail in a single toss, so that $p + q = 1$.

Then probability of getting head on an odd toss

= Probability of getting head in the 1st toss

+ Probability of getting head in the 3rd toss

+ Probability of getting head in the 5th toss + ... ∞

$$= p + qqp + qqqqp + ... \infty$$

$$= p(1 + q^2 + q^4 + ...) = p\frac{1}{1-q^2} (q < 1)$$

$$= p\frac{1}{(1-q)(1+q)} = p\frac{1}{(1+q)} = \frac{1}{1+q}$$

Que 4.10. **A can hit a target 3 times in 5 shots, B can hit a target 2 times in 5 shots and C can hit a target 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) at least two shots hit ?**

Answer

Probability of A hitting the target = 3/5

Probability of B hitting the target = 2/5

Probability of C hitting the target = 3/4

i. In order that two shots may hit the target, the following cases must be considered :

p_1 = Chance that A and B hit and C fails to hit $= \frac{3}{5} \times \frac{2}{3} \times \left(1 - \frac{3}{4}\right) = \frac{6}{100}$

p_2 = Chance that B and C hit and A fails to hit $= \frac{2}{5} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{12}{100}$

p_3 = Chance that C and A hit and B fails to hit $= \frac{3}{4} \times \frac{3}{5} \times \left(1 - \frac{2}{5}\right) = \frac{27}{100}$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = 0.45$$

iii. In order that at least two shots may hit the target, we must also consider the cases of all A, B, C hitting the target in addition to the three cases of (i) for which

p_4 = Chance that A, B, C all hit $= \frac{3}{5} \times \frac{2}{5} \times \frac{3}{4} = \frac{18}{100}$

Since all these are mutually exclusive events, the probability of at least two shots hit.

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = 0.45$$

Que 4.11. **Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd, if**

i. The two cards are drawn together.

ii. The two cards are drawn one after the other without replacement.

iii. The two cards are drawn one after the other with replacement.

Answer

i. Two cards out of 10 can be selected in $^{10}C_2 = 45$ ways. The sum is odd if one number is odd and the other number is even. There being five odd numbers cards (1, 3, 5, 7, 9) and five even number cards (2, 4, 6, 8, 10), an odd and an even number cards are chosen in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25}{45} = \frac{5}{9}$$

ii. Two cards out of 10 can be selected on after the other without replacement in $10 \times 9 = 90$ ways. Odd number cards are selected in $5 \times 5 = 25$ ways and even number cards are selected in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25 + 25}{90} = \frac{5}{9}$$

iii. Two cards can be selected one after the other with replace... $10 \times 10 = 100$ ways. Odd number cards are selected in $5 \times 5 = ...$ and even number cards are selected in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25 + 25}{100} = \frac{1}{2}$$

Que 4.12. **The students in a class are selected at random, one ... the other, for an examination. Find the probability p that the ... and girls in the class alternate if**

i. The class consists of 4 boys and 3 girls.

ii. The class consists of 3 boys and 3 girls.

Answer

i. As there are 7 student in the class, the first examined must be ...

$\therefore$ Probability that first is a boy $= \frac{4}{7}$

Then the probability that the second is a girl $= \frac{3}{6}$

$\therefore$ Probability of the next boy $= \frac{3}{5}$

Similarly the probability that the fourth is a girl $= \frac{3}{4}$,

The probability that the fifth is a boy $= \frac{2}{3}$,

The probability that the sixth is a girl $= \frac{1}{2}$

and the last is a boy $= \frac{1}{1}$

Thus $$p = \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{35}$$

ii. The first student is a boy and the first student is a girl, are two m... exclusive cases. If the first student is a boy, then the probability ... the students alternate is

$$p_1 = \frac{3}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{20}$$

If the first student is a girl, then the probability p_2 that the s... alternate is

$$P_2 = \frac{3}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{20}$$

Thus the required prob. $p = P_1 + P_2 = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$

PART-2

Conditional Probability, Baye's Theorem, Random Variable (Discrete, Continuous Random Variable) Probability Mass Function, Probability Density Function, Expectation and Variance.

Questions-Answers
Long Answer Type and Medium Answer Type Questions

Que 4.13. **There are three bags : first containing 1 white, 2 red, 3 green balls : second containing 2 white, 3 red, 1 green balls and third containing 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.**

Answer

Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two bags are white and red.

Now $$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$P(A/B_1) = P$ (a white and a red ball are down from first bag)

$$= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15}$$

Similarly $$P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \frac{2}{5}$$

$$P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \frac{1}{5}$$

By Baye's theorem,

$$P(B_2/A) = \frac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}$$

Que 4.14. **Three urns contain 6 red, 4 black; 4 red, 6 black; 5 red, black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red find the probability that it is drawn from the first urn.**

Answer

Let

U_1 : the ball is drawn from U_1.
U_2 : the ball is drawn from U_2.
U_3 : the ball is drawn from U_3.
R : the ball is red.

We have to find $P(U_1/R)$.

By Baye's theorem,

$$P(U_1/R) = \frac{P(U_1)P(R/U_1)}{P(U_1)P(R/U_1) + P(U_2)P(R/U_2) + P(U_3)P(R/U_3)} \quad \text{...(4.14.1)}$$

Since the three urns are equally likely to be selected

$$P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$$

Alos $P(R/U_1) = P$(a red ball is drawn from urn I) $= \frac{6}{10}$

$P(R/U_2) = P$(a red ball is drawn from urn II) $= \frac{4}{10}$

$P(R/U_3) = P$(a red ball is drawn from urn III) $= \frac{5}{10}$

$\therefore$ From eq. (4.14.1), we have $P(U_1/R) = \dfrac{\frac{1}{3} \times \frac{6}{10}}{\frac{1}{3} \times \frac{6}{10} + \frac{1}{4} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}} = \frac{2}{5}$

Que 4.15. **The probability density function of a variate X is**

X	0	1	2	3	4	5	6
$p(X)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

i. **Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$**
ii. **What will be the minimum value of k so that $P(X \leq 2) > 3$.**

Answer

i. If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1$$

$$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$k = 1/49.$$

$$P(X < 4) = k + 3k + 5k + 7k = 16k = 16/19$$

$$P(X \geq 5) = 11k + 13k = 24k = 24/49$$

$$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = 33/49.$$

ii. $$P(X \leq 2) = k + 3k + 5k = 9k = 0.3 \text{ or } k > 1/30$$

Thus, minimum value of $k = \dfrac{1}{30}$

Que 4.16. **A random variable X has the following probability function :**

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$3k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

i. Find the value of k
ii. Evaluate $P(X < 6)$, $(P \geq 6)$
iii. $P(0 < X < 5)$

Answer

i. If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1$$

$$0 + k + 2k + 2k + 3k^2 + 2k^2 + 7k^2 + k = 1$$

i.e., $$7k^2 + 9k - 1 = 0$$

$$(10 - k)(k + 1) = 0 \text{ i.e., } k = \frac{1}{10}$$

ii. $P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k^2 + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

iii. $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= k + 2k + 2k + 3k = 8k = \frac{8}{10} = \frac{4}{5}$$

Que 4.17.

i. **Is the function defined as follows a density function ?**

$$f(x) = e^{-x}, \quad x \geq 0$$
$$= 0, \quad x < 0,$$

ii. **If so, determine the probability that the variate having this density will fall in the interval (1, 2) ?**

iii. **Also find the cumulative probability function $F(2)$?**

Answer

i. $f(x) \geq 0$ for every x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

ii. Required probability $= P(1 \leq x \leq 2) = \int_{1}^{2} e^{-x}\,dx = e^{-1} - e^{-2}$

$$= 0.368 - 0.135 = 0.233$$

This probability is equal to the shaded area in Fig. 4.17.1(a).

iii. Cumulative probability function $P(2)$:

$$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} e^{-x}\,dx = 1 - e^{-2}$$
$$= 1 - 0.135 = 0.865$$

which is shown in Fig. 4.17.1(b).

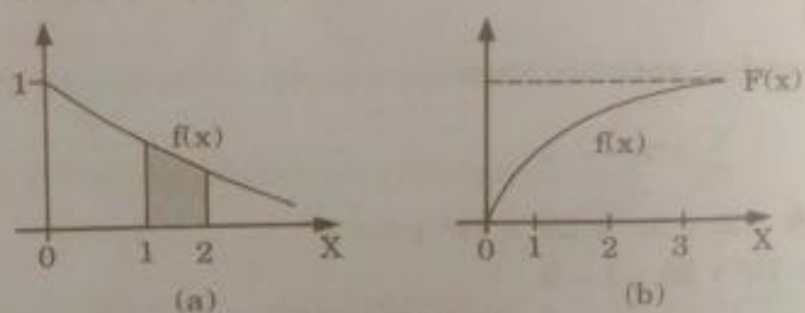

Fig. 4.17.1.

Que 4.18. **A variate X has the probability distribution**

x	–3	6	9
$P(X = x)$	1/6	1/2	1/3

Find $E(X)$ and $E(X^2)$. Hence evaluate $E\,(2X + 1)^2$.

$$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = 11/2$$

$$E(X)^2 = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = 93/2$$

$$\therefore \quad E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$

$$= 4(93/2) + 4(11/2) + 1 = 209$$

Que 4.19. **The frequency distribution of a measurable characteristic varying between 0 and 2 is as under**

$$f(x) = x^3, \quad 0 \le x \le 1$$

$$= (2-x)^3, \; 1 \le x \le 2$$

Calculate the standard deviation and also the mean deviation about the mean.

Answer

Total frequency $\quad N = \int_0^1 x^3\, dx + \int_1^2 (2-x)^3\, dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\therefore \quad \mu_1'$ (about the origin) $= \frac{1}{N}\left[\int_0^1 x\, x^3\, dx + \int_1^2 x(2-x)^3\, dx\right]$

$$= 2\left\{\left[\frac{x^5}{5}\right]_0^1 + \left[-x\frac{(2-x)^4}{4}\right]_1^2 - \left[\frac{(2-x)^5}{20}\right]_1^2\right\} = 2\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{20}\right) = 1$$

μ_2' (about the origin) $= \frac{1}{N}\left[\int_0^1 x^2\, x^3\, dx + \int_1^2 x^2(2-x)^3\, dx\right]$

$$= 2\left\{\left[\frac{x^6}{6}\right]_0^1 + \left[-x^2\frac{(2-x)^4}{4}\right]_1^2 + \frac{1}{2}\int_1^2 x(2-x)^4\, dx\right\}$$

$$= 2\left\{\frac{1}{6} + \frac{1}{4} + \frac{1}{2}\left[\frac{1}{5} + \frac{1}{30}\right]\right\} = \frac{16}{15}$$

Hence $\quad \sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2 = \frac{16}{15} - 1 = \frac{1}{15}$

i.e., Standard deviation $\sigma = \frac{1}{\sqrt{15}}$

Mean deviation about the mean :

$$= \frac{1}{N}\left\{\int |x-1|\, x^3\, dx + \int_1^2 |x-1|(2-x)^3\, dx\right\}$$

$$= 2\left[\int_0^1 (1-x)x^3\,dx + \int_1^2 (x-1)(2-x)^3\,dx\right]$$

$$= 2\left[\left(\frac{1}{4}-\frac{1}{5}\right)+\left(0+\frac{1}{20}\right)\right]=\frac{1}{5}$$

PART-3

Discrete and Continuous Probability Distribution : Binomial, Poisson and Normal Distribution.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.20. **Define a binomial distribution. Prove that the Poisson distribution is the limiting case of binomial distribution.**

Answer

A. **Binomial distribution :** A random variable x is said to have a binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$p(x=r)=\begin{cases} {}^nC_r\, p^r q^{n-r}, & r=0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

Also $\quad p+q=1$

B. **Poisson distribution as a limiting case of binomial distribution :**

If the parameters n and p of a binomial distribution are known, we can find the distribution. But in situations where n is very large and p is very small, application of binomial distribution is very labourious. However, if we assume that as $n \to \infty$ and $p \to 0$ such that np always remains finite, say λ, we get the Poisson approximation to the binomial distribution.

Now, for a binomial distribution

$$P(X=r)={}^nC_r\, q^{n-r}p^r=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\times(1-p)^{n-r}\times p^r$$

$$=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\times\left(1-\frac{\lambda}{n}\right)^{n-r}\times\left(\frac{\lambda}{n}\right)^r$$

$$\left(\text{Since } np=\lambda \;\therefore\; p=\frac{\lambda}{n}\right)$$

$$= \frac{\lambda^r}{r!} \times \frac{n(n-1)(n-2)\ldots(n-r+1)}{n^r} \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\ldots\left(\frac{n-r+1}{n}\right) \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) \times \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

As $n \to \infty$, each of the $(r-1)$ factors,

$\left(1-\frac{1}{n}\right), \left(1-\frac{2}{n}\right), \ldots \left(1-\frac{r-1}{n}\right)$ tends to 1. Also $\left(1-\frac{\lambda}{n}\right)^r$ tends to 1.

Since $\underset{x \to \pm\infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$, the Naperian base.

$$\therefore \quad \left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda} \to e^{-\lambda} \text{ as } n \to \infty$$

Hence in the limiting case when $n \to \infty$, we have

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad (r = 0, 1, 2, 3, \ldots)$$

Here P is called the Poisson probability distribution.

Que 4.21. **Find the mean and variance of binomial distribution.**

Answer

Mean of binomial distribution :

For the binomial distribution, $P(r) = {}^nC_r q^{n-r} p^r$

Mean $\qquad \mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r\,{}^nC_r q^{n-r} p^r$

$$= 0 + 1.\,{}^nC_1 q^{n-1}p + 2.\,{}^nC_2 q^{n-2}p^2 + 3.\,{}^nC_3 q^{n-3}p^3 + \ldots + n.\,{}^nC_n p^n$$

$$= nq^{n-1}p + 2.\frac{n(n-1)}{2.1}q^{n-2}p^2 + 3.\frac{n(n-1)(n-2)}{3.2.1}q^{n-3}p^3 + \ldots + np^n$$

$$= np^{n-1}p + n(n-1)q^{n-2}p^2 + \frac{n(n-1)(n-2)}{2.1}q^{n-3}p^3 + \ldots + np^n$$

$= np[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2}p + {}^{n-1}C_2 q^{n-3}p^2 + \ldots + {}^{n-1}C_{n-1}p^{n-1}]$

$= np(q+p)^{n-1} = np$

Hence, the mean of binomial distribution is np.

Variance of binomial distribution :

$$\sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2 = \sum_{r=0}^{n} [r + r(r-1)] P(r) - \mu^2$$

$$= \sum_{r=0}^{n} rP(r) + \sum_{r=0}^{n} r(r-1) P(r) - \mu^2$$

$$= \mu + \sum_{r=2}^{n} r(r-1)\, {}^nC_r q^{n-r} p^r - \mu^2$$

(Since the contribution due to $r = 0$ and $r = 1$ is zero)

$$= \mu + [2.1 \,.\, {}^nC_2 q^{n-2}p^2 + 3.2 \,.\, {}^nC_3 q^{n-3}p^3 + \ldots + n(n-1)\, {}^nC_n p^n] - \mu^2$$

$$= \mu + \left[2.1.\frac{n(n-1)}{2.1} q^{n-2}p^2 + 3.2 \frac{n(n-1)(n-2)}{3.2.1} q^{n-3}p^3 + \ldots + n(n-1)p^n\right] - \mu^2$$

$$= \mu + [n(n-1)q^{n-2}p^2 + n(n-1)(n-2)q^{n-3}p^3 + \ldots + n(n-1)p^n] - \mu^2$$

$$= \mu + n(n-1)p^2[q^{n-2} + (n-2)q^{n-3}p + \ldots + p^{n-2}] - \mu^2$$

$$= \mu + n(n-1)p^2[{}^{n-2}C_0 q^{n-2} + {}^{n-2}C_1 q^{n-3}p + \ldots + {}^{n-2}C_{n-2}p^{n-2}] - \mu^2$$

$$= \mu + n(n-1)p^2(q+p)^{n-2} - \mu^2 = \mu + n(n-1)p^2 - \mu^2$$

$$= np + n(n-1)p^2 - n^2p^2 = np[1-p] = npq$$

Hence, the variance of binomial distribution is npq.

Que 4.22. **Out of 800 families with four children each, how many families would be expected to have (i) 2 boys and 2 girls, (ii) at least one boy, (iii) no girl, (iv) at most two girls. Assume equal probabilities for boys and girls.**

Answer

Probability of having boy $= P = \frac{1}{2}$

Probability of having girl $= Q = \frac{1}{2}$

Number of children $= n$

i. Probability of getting 2 boy and 2 girl $= {}^nC_r P^r Q^{n-r}$

$$= {}^4C_2 P^2 Q^{4-2}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2! \times 2!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$$

$$= 6 \times \frac{1}{4} \times \frac{1}{4} = 0.375$$

ii. Probability of getting at least one boy $= 1 - {}^4C_0 P^0 Q^4$

$$= 1 - \frac{4!}{0! \times 4!} \times \left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16} = 0.937$$

iii. Probability of getting no girl $= {}^4C_4 P^4 Q^0$

$$= {}^4C_4\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^0$$

$$= \frac{4!}{0!4!}\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625$$

iv. Probability of getting at most two girl

$$= {}^4C_4 P^4 Q^0 + {}^4C_3 P^3 Q^1 + {}^4C_2 P^2 Q^2$$

$$= 0.0625 + \frac{4 \times 3!}{1 \times 3!} \times \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 0.375$$

$$= 0.0625 + \frac{1}{4} + 0.375$$

$$= 0.625 + 0.250 + 0.375$$

$$= 0.6875$$

Que 4.23. **Find the mean and variance of Poisson distribution.**

Answer

For the Poisson distribution, $P(r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$

Mean of Poisson distribution :

$$\text{Mean, } \mu = \sum_{r=0}^{\infty} rP(r) = \sum_{r=0}^{\infty} r \cdot \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= e^{-\lambda}\sum_{r=0}^{\infty}\frac{r\lambda^r}{r!} = e^{-\lambda}\left(0 + \frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \ldots\ldots\right)$$

$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\ldots\right) = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Thus, the mean of the Poisson distribution is equal to the parameter λ.

Variance of Poisson distribution :

$$\text{Variance,} \qquad \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2$$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{\lambda^r e^{-\lambda}}{r!} - \lambda^2 = e^{-\lambda}\sum_{r=0}^{\infty}\frac{r^2\lambda^r}{r!} - \lambda^2$$

$$= e^{-\lambda}\left[\frac{1^2\lambda}{1!} + \frac{2^2\lambda^2}{2!} + \frac{3^2\lambda^3}{3!} + \frac{4^2\lambda^4}{4!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{2\lambda^2}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{(1+1)\lambda}{1!} + \frac{(1+2)\lambda^2}{2!} + \frac{(1+3)\lambda^3}{3!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\ldots\right) + \left(\frac{\lambda}{1!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \ldots\ldots\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\ldots\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}[e^{\lambda} + \lambda e^{\lambda}] - \lambda^2$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda}(1 + \lambda) - \lambda^2$$

$$= \lambda(1 + \lambda) - \lambda^2 = \lambda.$$

Hence, the variance of the Poisson distribution is also λ.

Que 4.24. The distribution of the number of road accidents per day in a city is Poisson with mean 4. Find the number of days out of 100 days when there will be

i. no accident

ii. at least 2 accidents

iii. at most 3 accidents

iv. between 2 and 5 accidents

Answer

Mean, $\lambda = 4$, Number of days, $N = 100$

i. $$P(r = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-4} = 0.01831$$

$\therefore$ Required number of days $= N \,.\, P(r = 0)$
$= 100 \times 0.01831 = 1.831 \approx 2$

ii. $P(r \geq 2) = 1 - P(r < 2) = 1 - [P(r = 0) + P(r = 1)]$
$$= 1 - \left[e^{-4} + \frac{e^{-4}(4)}{1!}\right] = 1 - 5e^{-4} = 0.90842$$

$\therefore$ Required number of days $= N \,.\, P(r \geq 2)$
$= 100 \times 0.90842 = 90.842 \approx 91$

iii. $P(r \leq 3) = P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$
$$= \frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} + \frac{e^{-4}(4)^2}{2!} + \frac{e^{-4}(4)^3}{3!}$$
$$= e^{-4} + 4e^{-4} + 8e^{-4} + \frac{64}{6}e^{-4} = 0.43347$$

$\therefore$ Required number of days $= N.\, P(r \leq 3)$
$= 100 \times 0.43347 = 43.347 \approx 43$

iv. $$P(2 < r < 5) = P(r = 3) + P(r = 4) = \frac{e^{-4}(4)^3}{3!} + \frac{e^{-4}(4)^4}{4!}$$
$$= \left(\frac{64}{6} + \frac{256}{24}\right)e^{-4} = 0.3907$$

$\therefore$ Required number of days
$= N \,.\, P(2 < r < 5) = 100 \times 0.3907 = 39.07 \approx 39$

Que 4.25. **Assuming that half the population of a town consumes chocolates and 100 investigators each take 10 individuals to see whether they are consumers. How many investigators would be needed to report that 3 people or less were consumers ?**

Answer

The chance for an individual to be consumer $= p = \frac{1}{2}$

The chance of not being a consumer $= q = 1 - \frac{1}{2} = \frac{1}{2}$

$$P(r \leq 3) = P(0) + P(1) + P(2) + P(3)$$
$$= q^{10} + {}^{10}C_1\, q^9 p^1 + {}^{10}C_2\, q^8 p^2 + {}^{10}C_3\, q^7 p^3$$
$$= \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + 45\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + 120\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3$$
$$= \frac{176}{1024}$$

Number of investigators to report that three or less people were consumers of chocolates

$$= \frac{176}{1024} \times 100 = 17.2.$$

Hence, 17 investigators would be needed to report that 3 or less people were consumers.

Que 4.26. **In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.**

Answer

Mean number of defectives = 2 = np = $20p$

$\therefore$ The probability of a defective part is $p = 2/20 = 0.1$

The probability of a non-defective part = 0.9

$\therefore$ The probability of at least three defectives in a sample of 20

= 1 – (Probability that either none, or one, or two are non-defective parts)

$= 1 - [{}^{20}C_0(0.9)^{20} + {}^{20}C_1(0.1)(0.9)^{19} + {}^{20}C_2(0.1)^2(0.9)^{18}]$

$= 1 - (0.9)^{18} \times 4.51 = 0.323$

Thus the number of samples having at least three defective parts out of 1000 samples

$= 1000 \times 0.323 = 323$

Que 4.27. **Fit a Poisson distribution to the set of observation :**

x	0	1	2	3	4
f	122	60	15	2	1

Answer

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{60 + 36 + 6 + 4}{200} = 0.5$$

$\therefore$ Mean of Poisson distribution *i.e.*, $m = 0.5$

Hence, the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

The theoretical frequencies are

x	0	1	2	3	4
f	121	61	15	2	0

$[\because \quad e^{-0.5} = 0.61]$

Que 4.28. **If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction.**

Answer

It follows a Poisson distribution as the probability of occurrence is very small.

Mean $(m) = np = 2000(0.001) = 2$

Probability that more than 2 will get a bad reaction

= 1 – [Probability that no one gets a bad reaction + Probability that one gets a bad reaction + Probability that two gets bad reaction]

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] \qquad [\because \quad m = 2]$$

$$= 1 - \frac{5}{e^2} = 0.32 \qquad [\because \quad e = 2.718]$$

Que 4.29. **In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and SD of 60 hours. Estimate the number of bulbs likely to burn for**

a. More than 2150 hours

b. Less than 1950 hours and

c. More than 1920 hours and but less than 2160 hours.

Answer

Here $\mu = 2040$ hours and $\sigma = 60$ hours

a. For $x = 2150$, $\qquad z = \dfrac{x - \mu}{\sigma} = 1.833$

$\therefore$ Area against $z = 1.83$ is 0.4664

(According to normal distribution table)

The area required in this case is to the right of the ordinate at $z = 1.83$.

i.e., $\qquad$ Area $= 0.5 - 0.4664 = 0.0336$

Thus the number of bulbs to burns for more than 2150 hours

$= 0.0336 \times 2000 = 67$ approximately

b. For $x = 1950$, $z = \frac{x-\mu}{\sigma} = -1.5$

The area required in this case is to the left of $z = -1.33$

i.e., Area $= 0.5 - 0.4082$(table value for $z = 1.33) = 0.0918$

$\therefore$ The number of bulbs expected to burn for less than 1950 hours.

$= 0.0918 \times 2000 = 184$ approximately.

c. When $x = 1920$, $z = \frac{1920 - 2040}{60} = -2$

When $x = 2160$, $z = \frac{2160 - 2040}{60} = 2$

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area for $z = 2$, *i.e.*, $= 2 \times 0.4772 = 0.9544$.

Thus, the required number of bulbs $= 0.9544 \times 2000 = 1909$ nearly.

Que 4.30. **In a normal distribution, 31 % of the items are under 45 and 8 % are over 64. Find the mean and standard deviation of the distribution. It is given that if $f(t) = \frac{1}{\sqrt{2\pi}}\int_0^t e^{-\frac{x^2}{2}}dx$, then $f(0.5) = 0.19$ and $f(1.4) = 0.42$.**

Answer

Let μ and σ be the mean and standard deviation respectively.

31 % of the items are under 45.

Area to the left of the ordinate $x = 45$ is 0.31.

When $x = 45$, let $z = z_1$

$$P(z_1 < z < 0) = 0.5 - 0.31 = 0.19$$

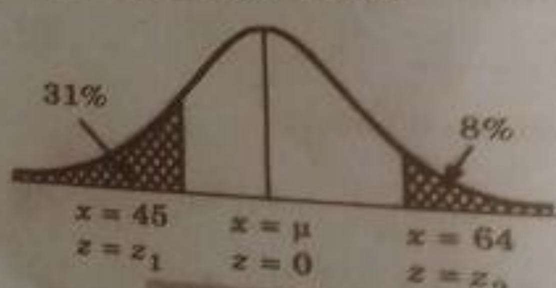

Fig. 4.30.1.

From the normal tables, the value of z corresponding to this area is 0.5.

$\therefore$ $z_1 = -0.5$ $[z_1 < 0]$

When $x = 64$, let $z = z_2$

$$P(0 < z < z_2) = 0.5 - 0.08 = 0.42$$

From normal tables, the value of z corresponding to this area is 1.4.

$$z_2 = 1.4$$

Since

$$z = \frac{x - \mu}{\sigma}$$

$$-0.5 = \frac{45 - \mu}{\sigma} \quad \text{and} \quad 1.4 = \frac{64 - \mu}{\sigma}$$

$\Rightarrow$

$$45 - \mu = -0.5\sigma \quad \text{...(4.30.1)}$$

and

$$64 - \mu = 1.4\sigma \quad \text{...(4.30.2)}$$

Subtracting eq. (4.30.2) from eq. (4.30.1)

$$-19 = -1.9\sigma$$

$$\sigma = 10$$

From eq. (4.30.1), $45 - \mu = -0.5 \times 10 = -5$

$$\mu = 50$$

☺☺☺

5

UNIT

Statistical Techniques III

CONTENTS

PART-1

Sampling, Testing of Hypothesis and Statistical Quality Control : Introduction, Sampling Theory (Small and Large), Hypothesis, Null Hypothesis, Alternative Hypothesis, Testing a Hypothesis, Level of Significance, Confidence Limits.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5 % level of significance.

Answer

Suppose the coin is unbaised.

Then the probability of getting the head in a toss = 1/2

∴ Expected number of successes = 1/2 × 400 = 200

The observed value of successes = 216

Thus the excess of observed value over expected value = 216 – 200 = 16

Also SD of simple sampling $= \sqrt{npq} = \sqrt{\left(400 \times \frac{1}{2} \times \frac{1}{2}\right)} = 10$

Hence $$z = \frac{x - np}{\sqrt{(npq)}} = \frac{16}{10} = 1.6$$

As $z < 1.96$, the hypothesis is accepted at 5 % level of significance *i.e.*, we conclude that the coin is unbaised at 5 % level of significance.

Que 5.2. In a city A, 20 % of a random sample of 900 school boys had a certain slight physical defect. In another city *B*, 18.5 % of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant ?

Answer

We have $$n_1 = 900,\ n_2 = 1600$$

and $$p_1 = \frac{20}{100} = \frac{1}{5},\ p_2 = \frac{18.5}{100}$$

and $$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.19$$

$$q = 1 - 0.19 = 0.81$$

$$e^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

Thus

$$= 0.19 \times 0.81\left(\frac{1}{900} + \frac{1}{1600}\right) = 0.0017$$

giving

$$e = 0.04 \text{ (approx.)}$$

Also

$$p_1 - p_2 = \frac{1.5}{100} = 0.015 \quad \therefore z = \frac{p_1 - p_2}{e} = \frac{0.015}{0.04} = 0.37$$

As $z < 1$, the difference between the proportions is not significant.

Que 5.3. **The mean of a certain normal population is equal to the Standard Error (SE) of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative ?**

Answer

If μ be the mean and σ the SD of the distribution, then

$$\mu = \text{S.E. of the sample means} = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$$

Also for a sample of size 25, we have

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{25}} = \frac{\bar{x} - \sigma/10}{\sigma/5}$$

$$= \frac{10\bar{x} - \sigma}{10} \times \frac{5}{\sigma} = \frac{10\bar{x} - \sigma}{2\sigma} = \frac{5\bar{x}}{\sigma} - \frac{1}{2}$$

Since $\bar{x}$ is negative, $z < -\frac{1}{2}$

$\therefore$ The probability that a normal variate $z < -\frac{1}{2}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{1}{2}} e^{-\frac{1}{2}z^2}\, dz = \frac{1}{\sqrt{2\pi}}\int_{1/2}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$

$= 0.5 - 0.915 = 0.3085$, from normal table.

Que 5.4. **An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of n that will ensure this result with 90% confidence.**

Answer

SE of the proportion of heads $= \sqrt{\frac{pq}{n}} = \sqrt{\frac{1}{2}\frac{1}{2}\frac{1}{n}} = \frac{1}{2\sqrt{n}}$

90 % of confidence = 45 % of the total area under the normal curve on each side of the mean.

$\therefore$ The corresponding value of $z = 1.645$, from normal table.

Thus $p \mp 1.645\,\sigma = 0.49$ or 0.51

i.e., $0.5 - 1.645\dfrac{1}{2\sqrt{n}} = 0.49$ and $0.5 + 1.645\dfrac{1}{2\sqrt{n}} = 0.51$

Hence $\dfrac{1.645}{2\sqrt{n}} = 0.01$ or $\sqrt{n} = \dfrac{329}{4}$ or $n = 6765$ approximately

PART-2

Test of Significance of Difference of Means.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.5. **Explain the test of significance of difference of means.**

Answer

Given two independent samples $x_1, x_2, x_3, \ldots, x_{n_1}$ and $y_1, y_2, y_3, \ldots, y_{n_2}$, with means $\bar{x}$ and $\bar{y}$ and standard deviations σ_x and σ_y from a normal populations with the same variance, we have to test the hypothesis that the population mean μ_1 and μ_2 are the same.

For this, we calculate $$t = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \ldots(5.5.1)$$

where $$\bar{x} = \frac{1}{n_1}\sum_1^{n_1} x_i,\ \bar{y} = \frac{1}{n_2}\sum_1^{n_2} y_i$$

and

$$\sigma^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)\sigma_x^2 + (n_2 - 1)\sigma_y^2] = \frac{1}{n_1 + n_2 - 2}\left\{\sum_1^{n_1}(x_i - \bar{x})^2 + \sum_1^{n_2}(y_i - \bar{y})^2\right\}$$

It can be shown that the variate t defined by eq. (5.5.1) follows the t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

If the calculated value of $t > t_{0.05}$, the difference between the sample means is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1 % level of significant.

If $t < t_{0.05}$, the data is said to be consistent with hypothesis, that $\mu_1 = \mu_2$.

Que 5.6. **Eleven students were given a test in statistics. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence that the students have benefitted by extra coaching ?**

Boys	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Marks I test	**23**	**20**	**19**	**21**	**18**	**20**	**18**	**17**	**23**	**16**	**19**
Marks II test	**24**	**19**	**22**	**18**	**20**	**22**	**20**	**20**	**23**	**20**	**17**

Answer

We compute the mean and the S.D. of the difference between the marks of the two tests as under :

$$\bar{d} = \text{mean of } d\text{'s} = \frac{11}{11} = 1;$$

$$\sigma_s^2 = \frac{\Sigma(d-\bar{d})^2}{n-1} = \frac{50}{10} = 5 \qquad i.e., \sigma_s = 2.24$$

Assuming that the students have not been benefited by extra coaching it implies that the mean of the difference between the marks of the two test is zero $i.e., \mu = 0$.

Then
$$t = \frac{\bar{d}-\mu}{\sigma_s}\sqrt{n} = \frac{1-0}{2.24}\sqrt{11} = 1.48 \text{ nearly and}$$

$$df\ \nu = 11 - 1 = 10.$$

Students	x_1	x_2	$d = x_2 - x_1$	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	–1	–2	4
3	19	22	3	2	4
4	21	18	–3	–4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	–	–1	1
10	16	20	4	3	9
11	19	17	–2	–3	9
			$\Sigma d = 11$		$\Sigma(d-\bar{d})^2 = 50$

We know that $t_{0.05}$ (for $v = 10$) = 2.228. As the calculated value of $t < t_{0.05}$, the value of t is not significant at 5 % level of significance *i.e.*, the test provides no evidence that the students have benefited by extra coaching.

Que 3.7. Samples of sizes 10 and 14 were taken from two normal populations with SD 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5 % level.

Answer

We have, $\bar{x}_1 = 20.3$, $\bar{x}_2 = 18.6$, $n_1 = 10$, $n_2 = 14$, $s_1 = 3.5$, $s_2 = 5.2$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = 22.775$$

$$S = 4.772$$

Null hypothesis, $H_0 : \mu_1 = \mu_2$, *i.e.*, the means of the two populations are the same.

Alternative hypothesis, $H_1 : \mu_1 \neq \mu_2$

Test statistic : Under H_0, the test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{20.3 - 18.6}{4.772\sqrt{\dfrac{1}{10} + \dfrac{1}{14}}} = 0.8604$$

The tabulated value of t at 5 % level of significance for 22 *df* is $t_{0.05} = 2.0739$

Conclusion :

Since $t = 0.8604 < t_{0.05}$, the null hypothesis H_0 is accepted; *i.e.*, there is no significant difference between their means.

Que 3.8. The heights of 6 randomly chosen sailors in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.

Answer

Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

$$n_1 = 6, n_2 = 9$$

Null hypothesis, $H_0 : \mu_1 = \mu_2$. *i.e.*, the mean of both the population are the same.

Alternative hypothesis, $H_1 : \mu_1 > \mu_2$

Calculation of two sample means :

X_1	63	65	68	69	71	72
$X_1 - \bar{X}_1$	−5	−3	0	1	3	4
$(X_1 - \bar{X}_1)^2$	25	9	0	1	9	16

$$\bar{X}_1 = \frac{\Sigma X_1}{n_1} = 68 ; \Sigma(X_1 - \bar{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \bar{X}_2$	−6.66	−5.66	−2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \bar{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\bar{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66 ; \Sigma(X_2 - \bar{X}_2)^2 = 152.0002$$

$$S^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \bar{X}_1)^2 + \Sigma(X_2 - \bar{X}_2)^2]$$

$$= 16.3077$$

$$S = 4.038$$

Test statistic :

Under H_0,

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{68 - 67.666}{4.038\sqrt{\frac{1}{6} + \frac{1}{9}}} = 0.1569$$

The value of t at 5 % level of significance for 13 df is 1.77 $(df = n_1 + n_2 - 2)$

Conclusion : Since $t_{calculated} < t_{0.05} = 1.77$, the null hypothesis H_0 is accepted.

i.e., there is no significant difference between their averages.

i.e., the sailors are not on the average taller than the soldiers.

PART-3

T-Test, F-Test, and Chi-Square Test, One Way Analysis of Variance (ANOVA).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

ue 5.9. **The annual rainfall at a certain place is normally stributed with mean 45 cm. The rainfall during the last five years e 48 cm, 42 cm, 40 cm, 44 cm and 43 cm. Can we conclude that the erage rainfall during the last five years is less than the normal infall ? Test at 5% level of significance.**

nswer

e have the mean and standard deviation of the small sample as

$$\bar{x} = \frac{1}{n}\sum_i x_i = \frac{1}{5}\ (48 + 42 + 40 + 44 + 43) = 43.4$$

$$s^2 = \left(\frac{1}{n}\sum_i x_i^2\right) - \bar{x}^2 = \frac{1}{5}\ (9453) - 43.4)^2 = 7.04$$

ull hypothesis, $H_0 : \bar{x} = \mu$ (there is no significant difference in the rain all).

Alternate hypothesis, $H_1 : \bar{x} < \mu$.

We use the left tailed test with 5 % level of significance. Now t (0.05) for one tailed test = t(0.1) for two tailed test with $n = 5$. The value of t for $P = 0.05$ and $\upsilon = 4$ is 2.132. The test statistic is given by

$$t = \frac{\bar{x} - \mu}{s / \sqrt{n = 1}} = \frac{43.4 - 45}{1.3266} = -1.206$$

Since, $|t| = 1.206 < 2.132$, we accept the null hypothesis. There is no significance difference in the rainfall.

Que 5.10. **The height of 8 males participating in an athletic championship are found to be 175 cm, 168 cm, 165 cm, 167 cm, 160 cm, 173 cm and 168 cm. Can we conclude that the average height is greater than 165 cm ? Test at 5 % level of significance.**

Answer

Null hypothesis, $H_0 : \mu = 165$ cm.

Alternate hypothesis, $H_1 : \mu < 165$ cm

We use the right tailed test with 5 % level of significance. We have $n = 8$. Since, t (0.05) for one tailed test = t (0.1) for two tailed test, we have for 7 degrees of freedom and $P = 0.05$, $t = 1.895$. We compute the sample mean and standard deviation. We have

$$\bar{x} = \frac{1}{8}(175 + 168 + 170 + 167 + 160 + 173 + 168)$$
$$= 168.25$$

$$s^2 = \frac{1}{8}\left(\frac{1}{n}\sum_i x_i^2\right) - \bar{x}^2 = \frac{1}{8}(226616) - (168.25)^2$$

$$= 18.9375.$$

The test statistic is given by

$$t = \frac{\bar{x} - \mu}{s / \sqrt{n-1}} = \frac{168.25 - 165}{4.3517 / 2.6458} = 1.976$$

Since, $|t| = 1.976 < 1.895$, we reject the null hypothesis and accept the alternative hypothesis. The average height is greater than 165 cm.

Que 5.11. **The scores of 10 candidates obtained in tests before and after attending some coaching classes are given below :**

Before	54	76	92	65	75	78	66	82	80	78
After	60	80	86	72	80	72	66	88	82	73

Is the coaching for the test effective ? Test at 5 % level of significance.

Answer

The data relates to the marks obtained by the same set of students. Hence, we can regard that the marks are correlated.

If x_i, y_i denote the marks obtained in the two tests, we obtain the values of $d_i = x_i - y_i$ as –6, –4, 6, –7, –5, 6, 0, –6, –2, 5.

We find
$$\bar{d} = \frac{1}{n}\Sigma d_i = -\frac{13}{10} = -1.3,$$

We define
$$s_d^2 = \frac{1}{n}\Sigma d_i^2 - \bar{d}^2 = \frac{1}{10}(263) - 1.69 = 24.61.$$

Null hypothesis, H_0 : $\bar{d} = 0$ (the students have not benefited from coaching).

Alternate hypothesis, H_1 : $\bar{d} < 0$ (the students have benefited from coaching).

We shall use the one tailed test. Now, t (0.05) for one tailed test = $t(0.1)$ for two tailed test, with the degrees of freedom = $n - 1 = 9$. The value of t for $P = 0.05$ and $\upsilon = 9$ is 1.833.

$$t = \frac{\bar{d}}{s_d / \sqrt{n-1}} = \frac{-1.3}{4.9608 / 3} = -0.786.$$

We find $|t| = 0.786 < 1.833$. Hence, we accept the null hypothesis that the students have not benefited from coaching.

Que 5.12. **Two random samples of sizes 9 and 7 gave the sum of square of deviations from their respective means as 175 and 95 respectively. Can they be regarded as drawn from normal populations with the same variance ?**

Answer

We have

$$n_1 = 9,\ \Sigma (x_i - \bar{x})^2 = n_1 s_1^2 = 175,$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{175}{8} = 21.875.$$

$$n_2 = 7,\ \Sigma (y_i - \bar{y})^2 = n_2 s_2^2 = 95,$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{95}{6} = 15.8333.$$

Now, $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$. Hence, we take $v_1 = n_1 - 1 = 8$, and $v_2 = n_2 - 1 = 6$. We define

Null hypothesis, H_0 : $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$.

Alternate hypothesis, H_1 : $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

At 5 % level of significance, we have, $F_{0.05}(8, 6) = 4.15$

Now, the F-statistic is given by

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{21.875}{15.8333} = 1.381 < 4.15$$

Therefore, we accept the null hypothesis H_0. The two random samples might have come from two normal populations with the same variance.

Que 5.13. **Two random samples of sizes 9 and 6 gave the following values of the variable**

Sample 1	15	22	28	26	18	17	29	21	24
Sample 2	8	12	9	16	15	10			

Test the difference of the estimates of the population variance at 5 % level of significance.

Answer

We have

$$n_1 = 9,\ \bar{x} = 22.222,\ s_1^2 = 21.7294,$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = 24.4456.$$

$$n_2 = 6,\ \bar{y} = 11.6667,\ s_2^2 = 8.8881,$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = 10.6657.$$

Now, $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$. We take $v_1 = n_1 - 1 = 8$, and $v_2 = n_2 - 1 = 5$.

We define

Null hypothesis, $H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$.

Alternate hypothesis, $H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

At 5 % level of significance, we have, $F_{0.05}(8, 5) = 4.82$

Now, the F-statistic is given by

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{24.4456}{10.6657} = 1.381 < 4.82$$

Therefore, we accept the null hypothesis. There is no significant difference between the population variance.

Que 5.14. By using χ^2-test, find out whether there is any association between income level and type of schooling :

Social Status Health	Poor	Rich	Total
Below Normal	130	20	150
Normal	102	108	210
Above Normal	24	96	120
Total	256	224	480

Answer

Let us suppose the null hypothesis is that there is no association between income level and type of schooling.

The expected frequencies are

$$E(130) = \frac{256 \times 150}{480} = 80$$

$$E(20) = \frac{224 \times 150}{480} = 70$$

$$E(102) = \frac{256 \times 210}{480} = 112$$

$$E(108) = \frac{224 \times 210}{480} = 98$$

$$E(24) = \frac{256 \times 120}{480} = 64$$

$$E(96) = \frac{224 \times 120}{480} = 56$$

f_o	f_e	$(f_o - f_e)$	$(f_o - f_e)^2$	$(f_o - f_e)^2/f_e$
130	80	50	2500	31.25
20	70	−50	2500	35.71
102	112	−10	100	0.89
108	98	10	100	1.02
24	64	−40	1600	25.00
96	56	40	1600	28.57
				$\Sigma\frac{(f_o - f_e)^2}{f_e} = 122.44$

$$\chi^2 = \Sigma\frac{(f_o - f_e)^2}{f_e}$$
$$= 122.44$$

Table value of χ^2 at 5 % level of significance for 5 degree of freedom is 11.07. Since the calculated value is greater than table value therefore null hypothesis is rejected. Thus, there is an association between income level and type of schooling.

Que 5.15. **A survey of 240 families with 4 children shows the following distribution :**

Number of boys	4	3	2	1	0
Number of families	10	55	105	58	12

Test the hypothesis that male and female births are equal probable. (Given : $\chi^2_{0.05}$ = 9.49 and 11.1 for 4 *d.f.* and 5 *d.f.* respectively)

Answer

Null hypothesis, H_0 : Male and female are equally probable.

Number of boys	4	3	2	1	0
Number of girls	0	1	2	3	4
Number of families	10	55	105	58	12

Alternate hypothesis, H_1 : Male and female birth are not equally probable.
Calculation of expected frequencies $(q + p)^n$,

Answer

Null hypothesis, H_0 : There is no difference in the number of deaths per months among three hospitals.

Alternate hypothesis, H_1: There is a significant difference in the number of deaths per months among three hospitals.

Level of significance : We use 5 % level of significance.

Test statistic : To find the variance ratio, F, we set up an ANOVA table and find the sample totals as :

$$\Sigma y_A = 3 + 4 + 3 + 5 + 0 = 15$$
$$\Sigma y_B = 6 + 3 + 3 + 4 + 4 = 20$$
$$\Sigma y_C = 7 + 3 + 4 + 6 + 5 = 25$$

Grand Total (G.T.) $= \Sigma y_A + \Sigma y_B + \Sigma y_C = 60$

Correction Factor (C.F.)

$$= \frac{(G.T)^2}{n} = \frac{(60)^2}{15} = 240$$

Sum of squares of samples :

$$\Sigma y_A^2 = 3^2 + 4^2 + 3^2 + 5^2 + 0^2 = 59$$
$$\Sigma y_B^2 = 6^2 + 3^2 + 3^2 + 4^2 + 4^2 = 86$$
$$\Sigma y_C^2 = 7^2 + 3^2 + 4^2 + 6^2 + 5^2 = 135$$

Total sum of squares $= \Sigma y_A^2 + \Sigma y_B^2 + \Sigma y_C^2 - \text{C.F.}$

$$= 59 + 86 + 135 - 240 = 40$$

Sum of squares between samples

$$= \frac{(\Sigma y_A)^2}{n_1} + \frac{(\Sigma y_B)^2}{n_2} + \frac{(\Sigma y_C)^2}{n_3} - \text{C.F.}$$

$$= \frac{(15)^2}{5} + \frac{(20)^2}{5} + \frac{(25)^2}{5} - 240 = 10$$

Sum of squares within samples

= Total sum of squares – Sum of squares between sample = 40 – 10 = 30

Degree of freedom for total sum squares $= n - 1 = 15 - 1 = 14$

Degree of freedom for hospital $= k - 1 = 3 - 1 = 2$

Degree of freedom for error $= n - k = 15 - 3 = 12$

Answer

Null hypothesis, H_0 : There is no difference in the number of deaths per months among three hospitals.

Alternate hypothesis, H_1: There is a significant difference in the number of deaths per months among three hospitals.

Level of significance : We use 5 % level of significance.

Test statistic : To find the variance ratio, F, we set up an ANOVA table and find the sample totals as :

$$\Sigma y_A = 3 + 4 + 3 + 5 + 0 = 15$$
$$\Sigma y_B = 6 + 3 + 3 + 4 + 4 = 20$$
$$\Sigma y_C = 7 + 3 + 4 + 6 + 5 = 25$$

Grand Total (G.T.) = $\Sigma y_A + \Sigma y_B + \Sigma y_C = 60$

Correction Factor (C.F.)

$$= \frac{(G.T)^2}{n} = \frac{(60)^2}{15} = 240$$

Sum of squares of samples :

$$\Sigma y_A^2 = 3^2 + 4^2 + 3^2 + 5^2 + 0^2 = 59$$
$$\Sigma y_B^2 = 6^2 + 3^2 + 3^2 + 4^2 + 4^2 = 86$$
$$\Sigma y_C^2 = 7^2 + 3^2 + 4^2 + 6^2 + 5^2 = 135$$

Total sum of squares $= \Sigma y_A^2 + \Sigma y_B^2 + \Sigma y_C^2 - \text{C.F.}$

$$= 59 + 86 + 135 - 240 = 40$$

Sum of squares between samples

$$= \frac{(\Sigma y_A)^2}{n_1} + \frac{(\Sigma y_B)^2}{n_2} + \frac{(\Sigma y_C)^2}{n_3} - \text{C.F.}$$

$$= \frac{(15)^2}{5} + \frac{(20)^2}{5} + \frac{(25)^2}{5} - 240 = 10$$

Sum of squares within samples

= Total sum of squares – Sum of squares between sample = 40 – 10 = 30

Degree of freedom for total sum squares = $n - 1 = 15 - 1 = 14$

Degree of freedom for hospital = $k - 1 = 3 - 1 = 2$

Degree of freedom for error = $n - k = 15 - 3 = 12$

ANOVA Table

Source of variation	Sum of squares	Degree of freedom	Mean sum of squares	Variance ratio or F
Between samples	10	2	5	$F_{2,12} = \frac{5}{2.5} = 2$
Within samples	30	12	2.5	
Total	40	14	–	–

The tabular value of F at 5 % level of significance with $\upsilon_1 = 2, \upsilon_2 = 12$ is 3.89

Conclusion : Since $F_{cal} < F_{tab}$, the difference is insignificant and we conclude that data do not suggest a difference in the number of deaths per month among the three hospitals.

Que 5.17. **A manufacturing company purchased three new machines of different makes and wishes to determine whether one of them is faster than the others in producing a certain output. Five hourly production figures are observed at random from each machine and results are given below :**

Observations	A_1	A_2	A_3
1	25	31	24
2	30	39	30
3	36	38	28
4	38	42	25
5	31	35	28

Use ANOVA and determine whether the machines are significantly different in their mean speed. (Given: at 5 % level, $F_{2, 12} = 3.89$)

Answer

Null hypothesis, H_0 : Machines are not significantly different in their mean speeds *i.e.*, $\mu_1 = \mu_2 = \mu_3$

Alternate hypothesis, H_1 : Machines are significantly different in their mean speed.

Level of significance : We use 5% level of significance.

Test statistic : To find the variance ratio F, we set up an ANOVA table as follows :

Let us shift the origin at 30 *i.e.*, reduce each observation by 30.

Now,

Observations	A_1	A_2	A_3
1	–5	1	–6
2	0	9	0
3	6	8	–2
4	8	12	–5
5	1	5	–2

Machine totals :

$$\Sigma A_1 = -5 + 0 + 6 + 8 + 1 = 10$$
$$\Sigma A_2 = 1 + 9 + 8 + 12 + 5 = 35$$
$$\Sigma A_3 = -6 + 0 - 2 - 5 - 2 = -15$$

Grand Total (G.T.) $= \Sigma A_1 + \Sigma A_2 + \Sigma A_1 = 30$

Correction Factor (C.F.) $= \dfrac{(30)^2}{15} = 60$

Sum of squares of samples :

$$\Sigma A_1^2 = (-5)^2 + 0^2 + 6^2 + 8^2 + 1^2 = 126$$

Similarly, $\Sigma A_2^2 = 315,\ \Sigma A_3^2 = 69$

Total sum of squares $= 126 + 315 + 69 - 60 = 510 - 60 = 450$

Machine sum of squares (between samples)

$$= \frac{(10)^2}{5} + \frac{(35)^2}{5} + \frac{(-15)^2}{5} - \text{C.F.}$$
$$= 310 - 60 = 250$$

Error sum of squares (within samples)

= Total sum of squares – Machine sum of squa

$= 450 - 250 = 200$

Degree of freedom for total sum of squares $= n - 1 = 15 - 1 = 14$

Degree of freedom for machines $= k - 1 = 3 - 1 = 2$

Degree of freedom for error $= n - k = 15 - 3 = 12$

ANOVA Table

Source of variation	Sum of squares	Degree of freedom	Mean sum of squares	Variance ratio of F
Machine (between samples)	250	2	$\frac{250}{2} = 125$	$F_{2,12} = \frac{125}{16.67}$
Error (within samples)	200	12	$\frac{200}{12} = 16.67$	$= 7.498$
Total	450	14	–	–

The tabular value of F at 5 % level of significance with $\upsilon_1 = 2, \upsilon_2 = 12$ is 3.89 (given).

Conclusion : Since $F_{\text{calculated}} > F_{\text{tabulated}}$, the null hypothesis is rejected and the difference is significant and we conclude that there is significant difference in the mean speed of machines.

PART-4

Statistical Quality Control (SQC), Control Charts, Control Charts for Variables ($\overline{X}$, R Charts).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.18. **If number of samples = 20, size of each sample = 5, $\bar{R} = 2.32\,\bar{\sigma}$, $\bar{\bar{x}} = 99.6$, $\bar{R} = 7.0$. Find the values of control limit for drawing a mean chart. [$n = 5$, mean range = 2.32 (population S.D.)]**

Answer

Here, we have

$$\bar{\bar{x}} = 99.6$$

$$\bar{R} = 7.0$$

$$\bar{R} = 2.32\,\bar{\sigma} \quad \Rightarrow \quad \bar{\sigma} = \frac{\bar{R}}{232} = \frac{7}{2.32} = 3.0172$$

$n = 5$

UCL $= \bar{X} + 3\left(\dfrac{\bar{\sigma}}{\sqrt{n}}\right) = 99.6 + \left(3 \times \dfrac{3.0172}{\sqrt{5}}\right)$

$= 99.6 + \dfrac{9.0516}{2.2361} = 99.6 + 4.0479 = 103.6479$

LCL $= \bar{X} - 3\left(\dfrac{\bar{\sigma}}{\sqrt{n}}\right) = 99.6 - 4.0479 = 95.5521$

CL $= 99.6$

$\bar{X}$ - Chart :

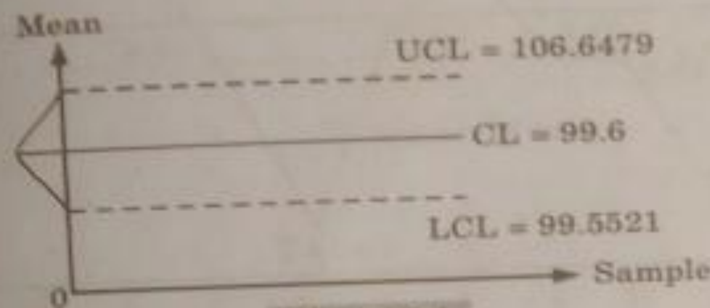

Fig. 5.18.1.

Que 5.19. **The following table gives the sample means and ranges for *I.Q.* samples each of size 6, in the production of certain component. Construct the central chart for mean and range comment on the nature of control.**

Sample Number	1	2	3	4	5	6	7	8	9	10
Mean $\bar{X}$	37.5	49.8	51.5	59.2	54.7	34.7	51.4	61.4	70.7	75.3
Range R	9.5	12.8	10.8	9.1	7.8	5.8	14.5	2.8	3.7	8.0

Answer

Mean of means $\bar{\bar{X}} = \dfrac{\Sigma \bar{X}}{n}$

$= \dfrac{37.5 + 49.8 + 51.5 + 59.2 + 54.7 + 34.7 + 61.4 + 70.7 + 75.3 + 51.4}{10}$

$= \dfrac{546.0}{10}$

From the table of control chart, for sample size of 6

$A_2 = 0.483, D_3 = 0, D_4 = 2.004$

Control limits of $\bar{X}$ – Chart

ULC $= \bar{X} + A_2\bar{R}$

$$= 54.6 + (0.483)(8.4) = 58.657$$

$$\text{LCL} = \bar{X} - A_2\bar{R}$$
$$= 54.6 - (0.483)(8.4) = 50.543$$

$\bar{X}$ **- Chart :**

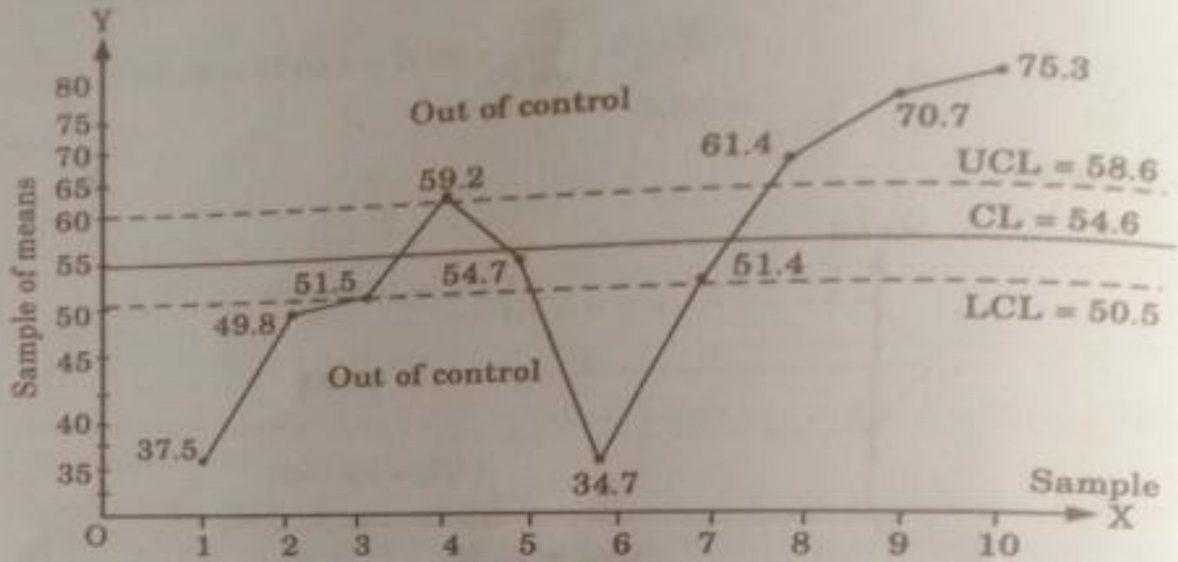

Fig. 5.19.1.

Here we have the means of the sample number 4, 8, 9, 10 which are greater than the Upper Control Limit (UCL).

Also the mean corresponding to the sample number 1, 2, 6 are less than Lower Control Limit (LCL) value. Seven of ten sample points fall outside the control limits.

Hence, the process is very much out of control.

Que 5.20. **Control on measurements of pitch diameter of thread in aircraft fittings is checked with 5 samples each containing 5 times at equal intervals of time. The measurements are given below. Constant $\bar{X}$ and R charts and state your inference from the charts.**

Sample No.	1	2	3	4	5
Measurements x	46	41	40	42	43
	45	41	40	43	44
	44	44	42	43	47
	43	42	40	42	47
	42	40	42	45	45

Answer

Here, we have

Sample No.	1	2	3	4	5
Measurements x	46 45 44 43 42	41 41 44 42 40	40 40 42 40 42	42 43 43 42 45	43 44 47 47 45
Total	$\Sigma x = 220$	$\Sigma x = 208$	$\Sigma x = 204$	$\Sigma x = 215$	$\Sigma x = 226$
$\overline{X} = \frac{\Sigma x}{n}$	$\frac{220}{5} = 44$	$\frac{208}{5} = 41.6$	$\frac{204}{5} = 40.8$	$\frac{215}{5} = 43$	$\frac{226}{5} = 43$
R	$46 - 42 = 4$	$44 - 40 = 4$	$42 - 40 = 2$	$45 - 42 = 3$	$47 - 43 = 4$

$$\overline{\overline{X}} = \frac{\Sigma \overline{X}}{5} = \frac{44 + 41.6 + 40.8 + 43.0 + 45.2}{5}$$

From the table of control chart, for sample size of 5 items, $A_2 = 0.577$.

Limits for $\overline{X}$-Chart :

$$\text{UCL}_{\overline{X}} = \overline{\overline{X}} + A_2\overline{R} = 42.92 + 0.577 \times 3.4 = 44.88$$

$$\text{LCL}_{\overline{X}} = \overline{\overline{X}} - A_2\,\overline{R} = 42.92 - 0.577 \times 3.4 = 40.96$$

$$\overline{X}_3 = 40.8 < \text{LCL} = 40.96$$

$$\overline{X}_5 = 45.2 > \text{UCL} = 44.88$$

All sample points do not lie between control limits. Hence, the process is out of control.

$\bar{X}$ -Chart :

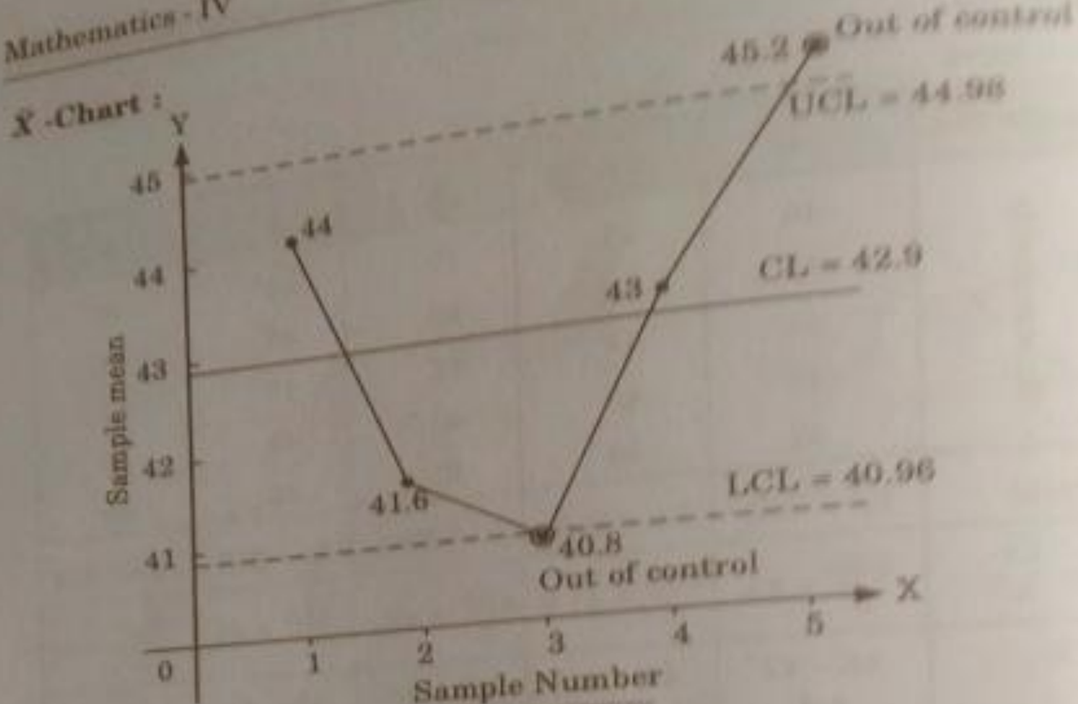

Fig. 5.20.1.

From the control chart table, $D_3 = 0, D_4 = 2.115$

R-chart :

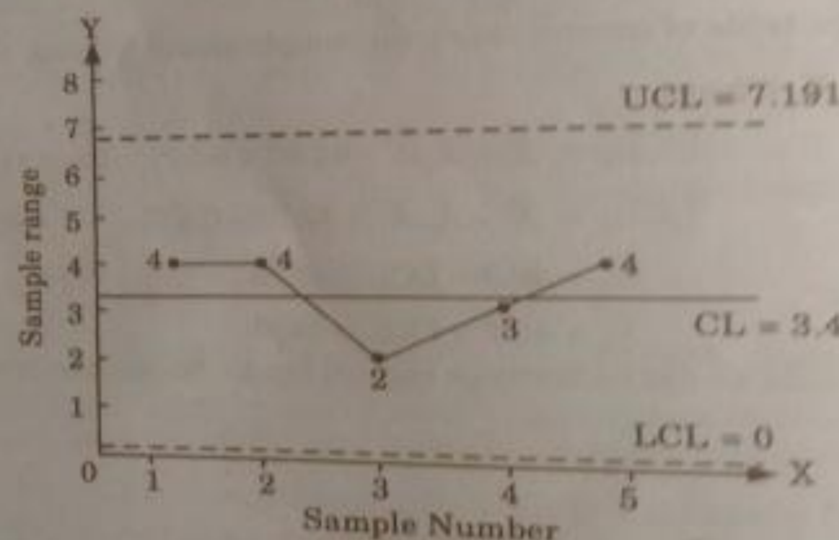

Fig. 5.20.2.

Limits for R-Chart :

$$\begin{aligned} UCL_R &= D_4 \bar{R} \\ &= 2.115 \times 3.4 = 7.191 \\ LCL_R &= D_3 \bar{R} = 0 \\ CL_R &= \bar{R} = 3.4 \end{aligned}$$

All sample points lie between control limits. Hence, the variability is under control. But process is out of control due to $\bar{X}$ - chart.

Que 5.21. **The following data give the measurements of 10 samples each of size 5 in the production process taken in an interval of 2 hours. Calculate the sample means and ranges and draw the control charts for mean and range.**

Sample Number	1	2	3	4	5	6	7	8	9	10
Observed measurement x	49	50	50	48	47	52	49	55	53	54
	55	51	53	53	49	55	49	55	50	54
	54	53	48	51	50	47	49	50	54	52
	49	46	52	50	44	56	53	53	47	54
	53	50	47	53	45	50	45	57	51	56

Answer

Here, we have

Sample Number	1	2	3	4	5	6	7	8	9	10
x Observed measurement	49	50	50	48	47	52	49	55	53	54
	55	51	53	53	49	55	49	55	50	54
	54	53	48	51	50	47	49	50	54	52
	49	46	52	50	44	56	53	53	47	54
	53	50	47	53	45	50	45	57	51	56
Σx	260	250	250	255	235	260	245	270	225	270
$\overline{X} = \frac{\Sigma x}{5}$	52	50	50	51	47	52	49	54	51	54
R	55–49 = 6	53–46 = 7	53–47 = 6	53–48 = 5	50–44 = 6	56–47 = 9	53–45 = 8	57–50 = 7	54–47 = 7	56–52 = 4

Mean of the means $\overline{\overline{X}} = \frac{\Sigma \overline{X}}{n}$

From the control chart table for sample of 5 item, $A_2 = 0.577$

$$= \frac{52 + 50 + 50 + 51 + 47 + 52 + 49 + 54 + 51 + 54}{10}$$

$$= \frac{510}{10} = 51$$

Control Limits for $\overline{X}$ - chart :

$$\text{UCL} = \overline{\overline{X}} + A_2 \overline{R} = 51.0 + (0.577)\,6.5 = 54.7505$$

$$\text{LCL} = \overline{\overline{X}} - A_2 \overline{R} = 51.0 - (0.577)\,6.5 = 47.2495$$

$$CL = \bar{\bar{X}} = 51.0$$

Construction of control chart for mean :

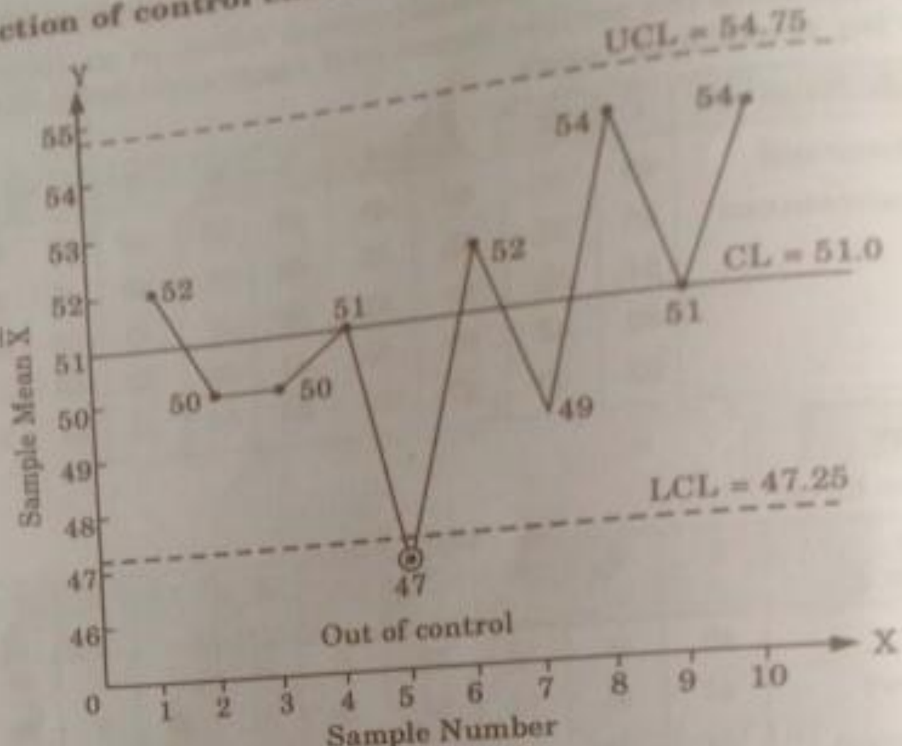

Fig. 5.21.1.

Since $\bar{X}_5 = 47 <$ LCL and this value lies outside the control limits and hence the process is out of control.

R-chart :

From the table of control chart $D_3 = 0, D_4 = 2.115$

$$\bar{R} = \frac{\Sigma R}{n} = \frac{6+7+6+5+6+9+8+7+7+4}{10} = 6.5$$

From the control chart table, for sample size of 5, $A_2 = 0.577$.

Control limits for *R*-chart :

$$\text{UCL} = D_4 \bar{R} = 2.115 \times 6.5 = 13.7475$$

$$\text{LCL} = D_3 \bar{R} = 0$$

$$\text{CL} = \bar{R} = 6.5$$

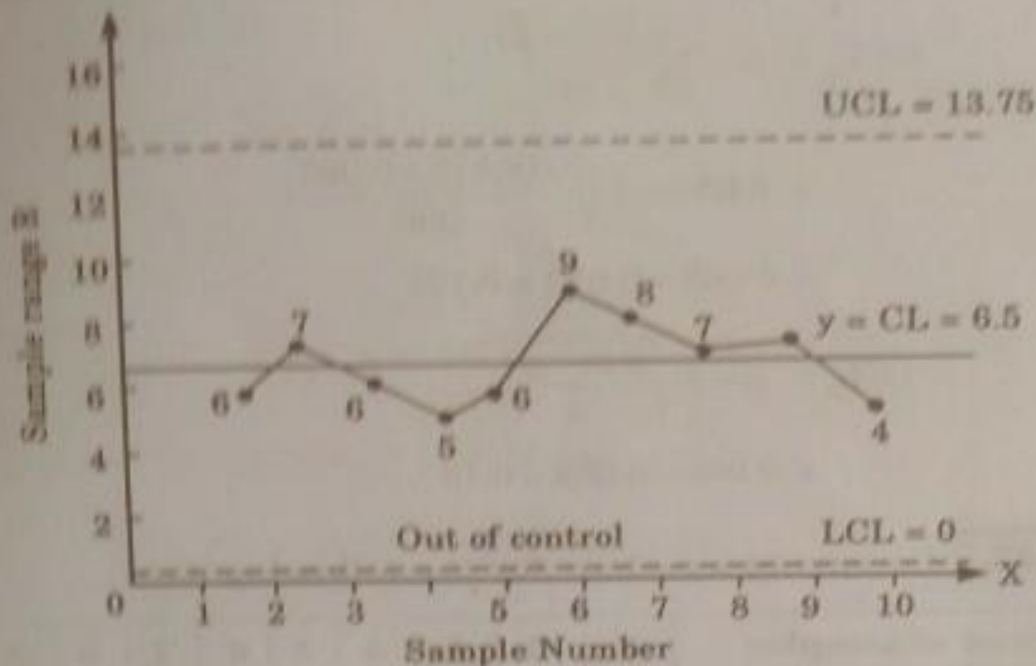

Fig. 5.21.2.

All values of R lie between the control limits 13.7475 and 0.

Hence, the variability is under control. Still the process is out of control due to $\bar{X}$ - chart.

PART-5

Control Charts for Variables (p, np and C-Charts).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.22. **15 sample with size 200 each taken at an interval of 45 minutes from a manufacturing process the average fraction defective was 0.068. Calculate the values of central line, upper and lower control line.**

Answer

Here, we have

Average fraction defective = 0.068

∴ Central line CL = $\bar{p}$ = 0.068

We know that

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$= 0.068 + 3\sqrt{\frac{0.068(1-0.068)}{200}}$$

$$= 0.068 + 0.053 = 0.121$$

$$LCL_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$= 0.068 - 0.053 = 0.15$$

Que 5.23. Construct a p-chart for the following data :

Number of samples (each of 100 items)	1	2	3	4	5	6	7	8	9	10
Number of defectives	12	10	6	8	9	9	7	10	11	8

Answer

Number of Sample	Number of units in a sample (n)	Number of defectives d	Fraction defective $p = d/n$
1	100	12	0.12
2	100	10	0.10
3	100	6	0.06
4	100	8	0.08
5	100	9	0.09
6	100	9	0.09
7	100	7	0.07
8	100	10	0.10
9	100	11	0.11
10	100	8	0.08
Total	$N = 100$	$\Sigma d = 90$	

Average fraction defective $= \bar{p}$

$$= \frac{\text{Total no. of defective in all samples combined}}{\text{Total no. of items in all samples}}$$

$$= \frac{\Sigma d}{N} = \frac{90}{1000} = 0.09$$

Standard limits :

Upper Control Limit UCL $= \bar{p} + 3\sqrt{\dfrac{\bar{p}(1-\bar{p})}{n}}$

$$= 0.09 + 3\sqrt{\frac{0.09\,(1-0.09)}{100}}$$

$$= 0.09 + 3\sqrt{0.000819}$$

$$= 0.09 + 3 \times 0.0286 = 0.09 + 0.0858 = 0.1758$$

Lower Control Limit LCL $= \bar{p} - 3\sqrt{\dfrac{\bar{p}(1-\bar{p})}{n}}$

$$= 0.09 - 0.0858 = 0.0042$$

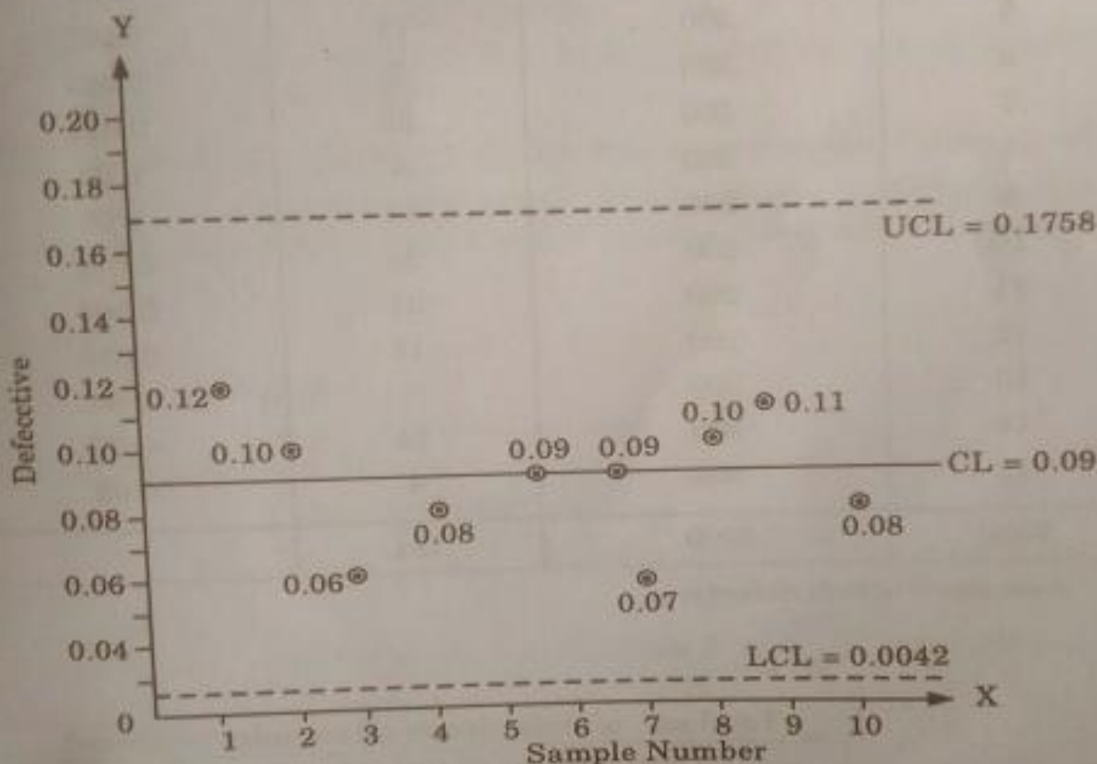

Fig. 5.23.1

All the values lie between control limits.

Hence, the variability is under control.

Que 5.24. **The following set of data covering 15 consecutive production days on the number of defectives found in daily production from a sample of 200 units. Draw a *p*-chart and test whether the production process was in control.**

Production day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of defectives	10	5	10	12	11	9	22	4	12	24	21	15	8	14	4

Number of Sample	Number of units in a sample (n)	Number of defectives d	Fraction defective $p = d/n$
1	200	10	0.05
2	200	5	0.025
3	200	10	0.05
4	200	12	0.06
5	200	11	0.055
6	200	9	0.045
7	200	22	0.011
8	200	4	0.02
9	200	12	0.06
10	200	24	0.012
11	200	21	0.105
12	200	15	0.075
13	200	8	0.04
14	200	14	0.07
15	200	4	0.02
Total	3000	181	

i. Average fraction defective

$$= \bar{p} = \frac{\Sigma d}{N}$$

$$= \frac{\text{Total no. of defective in all samples combined}}{\text{Total number of items in all samples}}$$

$$= \frac{181}{30{,}00} = 0.0603$$

ii. Standard limits

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$= 0.0603 + 3\sqrt{\frac{0.0603\,(1-0.0603)}{200}}$$

$$= 0.0603 + 0.0505 = 0.1108$$

$$LCL_p = 0.0603 - 0.0505 = 0.0098$$

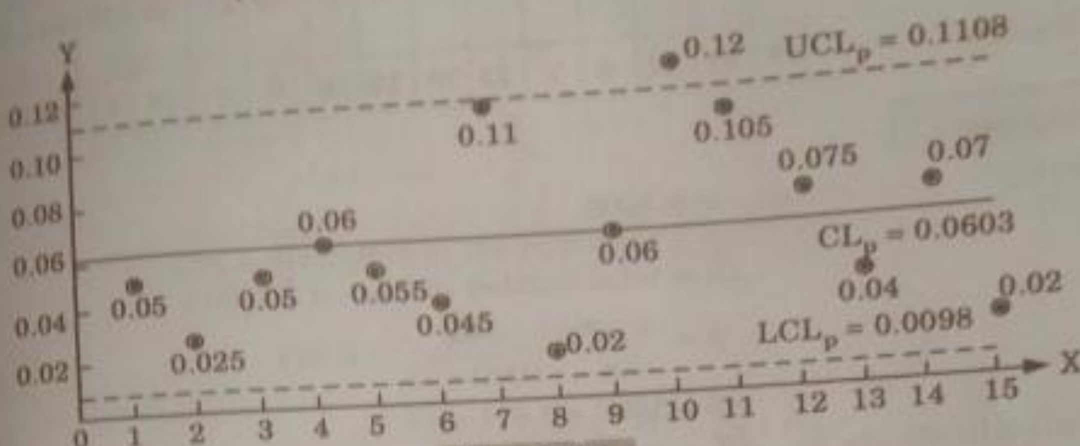

Fig. 5.24.1.

One sample point is above the UCL line, so the production process is to be corrected to make it under control.

Que 5.25. **A drilling machine bores holes with a mean diameter of 0.5230 cm and a standard deviation of 0.0032 cm. Calculate the 2-sigma and 3-sigma upper and lower control limits for means of sample of 4.**

Answer

Mean diameter $\bar{\bar{x}} = 0.5230$ cm

S.D. $\sigma = 0.0032$ cm, $n = 4$

i. 2-sigma limits are as follows :

$$CL = \bar{\bar{x}} = 0.5230 \text{ cm}$$

$$UCL = \bar{\bar{x}} + 2\frac{\sigma}{\sqrt{n}} = 0.5230 + 2 \times \frac{0.0032}{\sqrt{4}} = 0.5262 \text{ cm}$$

$$LCL = \bar{\bar{x}} - 2\frac{\sigma}{\sqrt{n}} = 0.5230 - 2 \times \frac{0.0032}{\sqrt{4}} = 0.5198 \text{ cm.}$$

ii. 3-sigma limits are as follows :

$$CL = \bar{\bar{x}} = 0.5230 \text{ cm}$$

$$UCL = \bar{\bar{x}} + 3\frac{\sigma}{\sqrt{n}} = 0.5230 + 3 \times \frac{0.0032}{\sqrt{4}} = 0.5278 \text{ cm}$$

$$LCL = \bar{\bar{x}} - 3\frac{\sigma}{\sqrt{n}} = 0.5230 - 3 \times \frac{0.0032}{\sqrt{4}} = 0.5182 \text{ cm.}$$

Que 5.26. **In a blade manufacturing factory, 1000 blades are examined daily. Draw the *np*-chart for the following table and examine whether the process is under control ?**

Date number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
of defective Blades	9	10	12	8	7	15	10	12	10	8	7	13	14	15	16

Answer

Here,

$$n = 1000$$

Σnp = total number of defectives = 166

Σn = total number inspected = 1000×15

$$\therefore \quad \bar{p} = \frac{\Sigma np}{\Sigma n} = \frac{166}{1000 \times 15} = 0.011$$

$$\therefore \quad n\bar{p} = 1000 \times 0.011 = 11$$

Control limits are

$$\text{CL} = n\bar{p} = 11$$

$$\text{UCL}_{np} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} = 11 + 3\sqrt{11(1-0.011)} = 20.894$$

$$\text{LCL}_{np} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} = 11 - 3\sqrt{11(1-0.011)} = 1.106$$

***np*-Chart :**

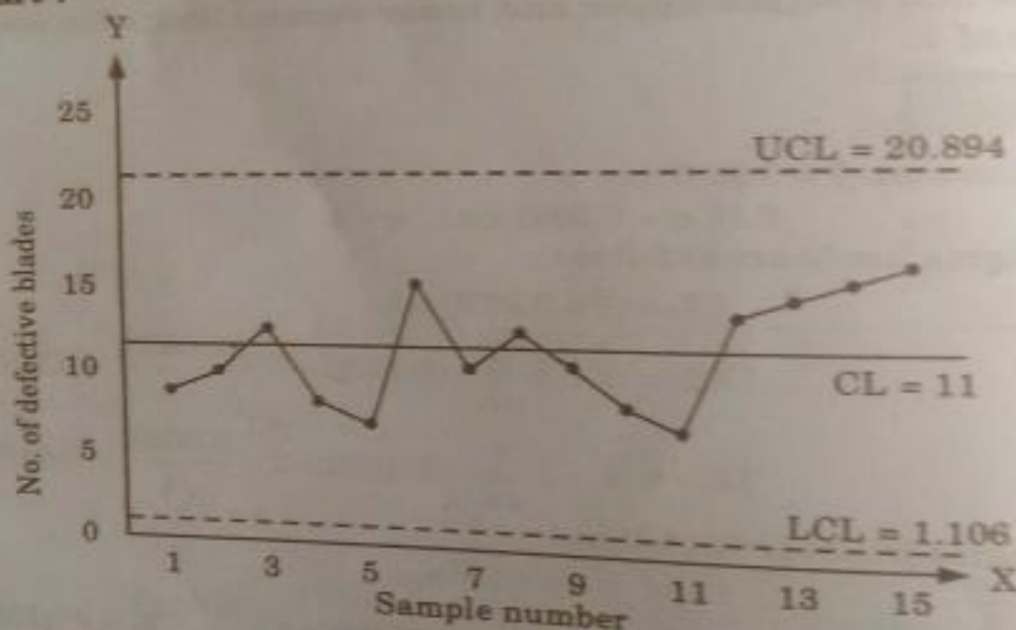

Fig. 5.26.1.

Since all the points lie within the control limits, the process is under control.

Que 5.27. **The following of defective of 10 samples of size 100 each. Construct *np*-chart.**

Sample number	1	2	3	4	5	6	7	8	9	10
Number of defectives	4	8	11	3	11	7	7	16	12	6

Answer

Here, we have

Number of samples = 10

Size of each samples = 100

Number of defectives = $\Sigma d = 4 + 8 + 11 + 3 + 11 + 7 + 7 + 16 + 12 + 6$

$= 85$

$$\therefore \bar{p} = \frac{\text{Total number of defective in all samples combined}}{\text{Total number of items of all the samples combined}}$$

$$= \frac{85}{10 \times 100} = 0.085$$

Central control line,

$$\text{CL} = n\bar{p} = 100 \times 0.085 = 8.5$$

$$\text{UCL} = n\bar{p} + 3\sqrt{n\,\bar{p}\bar{q}}$$

$$= 100\,(0.85) + 3\sqrt{100 \times 0.085 \times (1 - 0.085)}$$

$$= 8.5 + 3\sqrt{100 \times 0.085 \times 0.915}$$

$$= 8.5 + 3 \times 2.79 = 8.5 + 8.37 = 16.87$$

$$\text{LCL} = 8.5 - 8.37 = 0.13$$

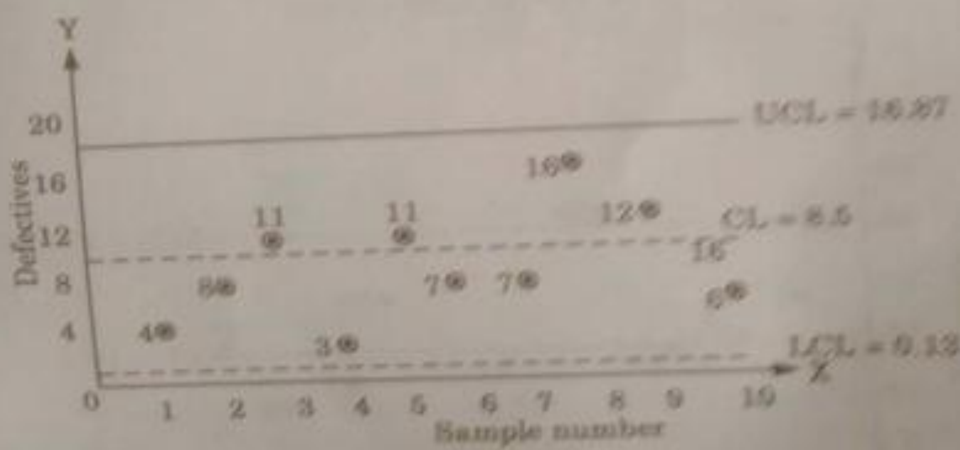

Fig. 5.27.1.

Since all the sample points are inside the control limits, the process is in a state of statistical control.

Que 5.28. **An inspection of 10 samples of size 400 each from 10 lots revealed the following number of defective units 17, 15, 14, 26, 9, 4, 19, 12, 9, 15.**

Calculate control limits for the number of defective units and state whether the process is under control or not.

Answer

Here, we have

Total number of items in 10 samples $= N = 10 \times 400 = 4000$

Total number of defectives in 10 samples $= \Sigma d$

$= 17 + 15 + 14 + 26 + 9 + 4 + 19 + 12 + 9 + 15$

$= 140$

Average fraction defective $= \bar{p} = \dfrac{\Sigma d}{N}$

$$= \frac{140}{4000} = 0.035$$

$$n\bar{p} = 400 \times 0.035 = 14$$

$$CL = n\bar{p} = 14$$

$$UCL = n\,\bar{p} + 3\sqrt{n\,\bar{p}\,(1-\bar{p})} = 14 + 3\sqrt{14\,(1-0.035)}$$

$$= 14 + 3\sqrt{14 \times 0.965} = 14 + 3\sqrt{13.51}$$

$$= 14 + 3 \times 3.676 = 14 + 11.028 = 25.028$$

$$LCL = n\,\bar{p} - 3\sqrt{n\,\bar{p}\,(1-\bar{p})}$$

$$= 14 - 11.0288 = 2.972$$

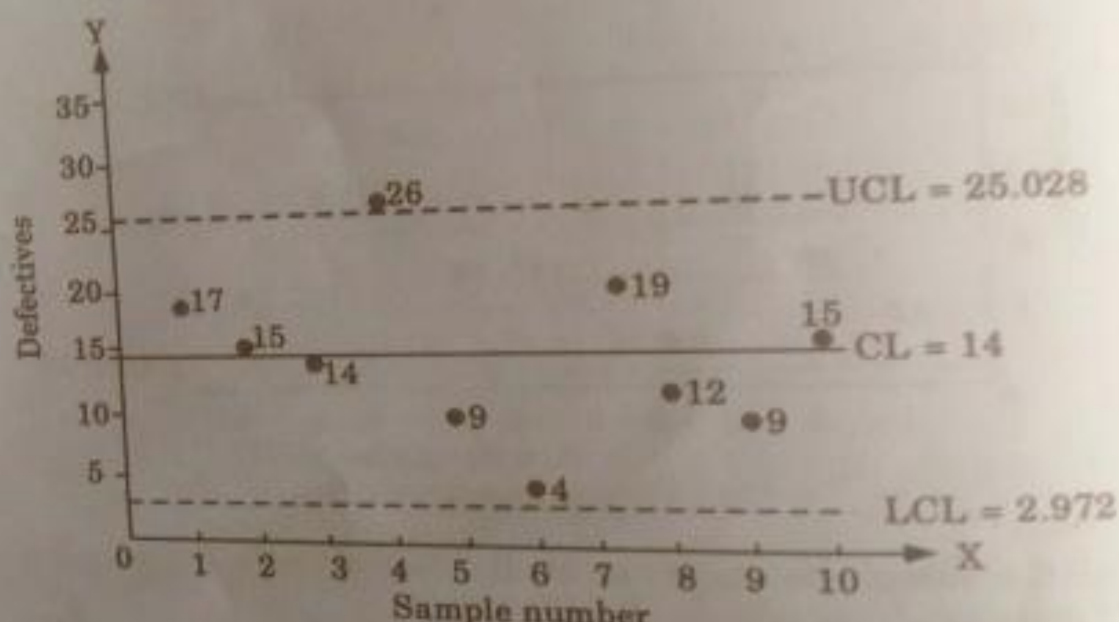

Fig. 5.28.1.

The number of defective units 26 in the 4th sample is greater than the value of the upper control limits of 25. Hence, the sample point falls outside the upper control limit, so, the process is not under control.

Que 5.29. **Draw a C-chart for the following data pertaining to the number of foreign coloured threads (considered as defects) in 15 pieces of cloth of 2 m × 2 m in a certain make of synthetic fibre and state your conclusions.**

7, 12, 13, 20, 21, 5, 4, 3, 10, 8, 0, 9, 6, 7, 20.

Answer

Here, we have number of cloth pieces = 15

i. The total number of defects (C)

$$= 7 + 12 + 3 + 20 + 21 + 5 + 4 + 3 + 10 + 8 + 0 + 9 + 6 + 7 + 20 = 135$$

ii. The average number of defects $(\bar{C})$:

$$\bar{C} = \frac{\text{Total number of defects}}{\text{Total number of samples}} = \frac{\Sigma C}{n} = \frac{135}{15} = 9.$$

iii. The 3-σ control limits for C- chart are given by

Central limit line = $\bar{C} = 9$

$$\text{UCL} = \bar{C} + 3\sqrt{\bar{C}} = (9 + 3\sqrt{9}) = 9 + 9 = 18$$

$$\text{LCL} = \bar{C} - 3\sqrt{\bar{C}} = 9 - 3\sqrt{9} = 9 - 9 = 0$$

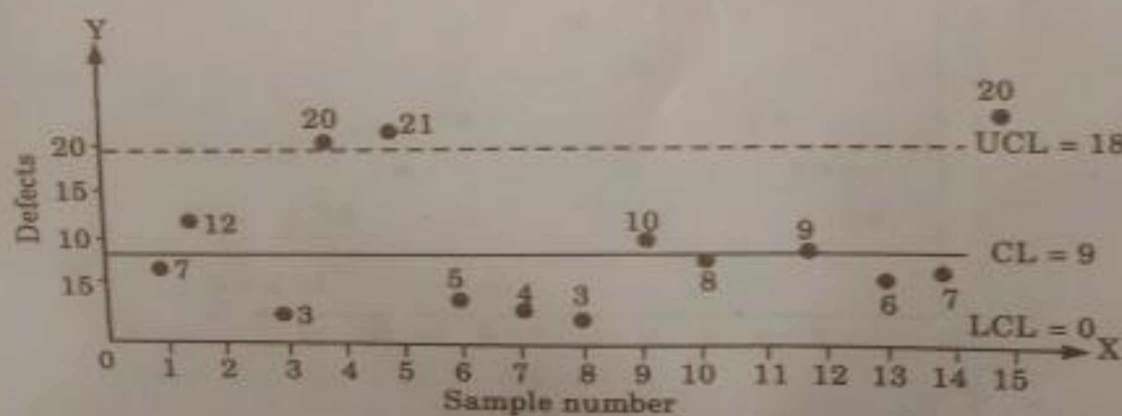

Fig. 5.29.1.

Since three sample points are outside the limits, the process is not under statistical control.

Que 5.30. **During an examination of equal length of cloth the following are the number of defects observed :**

2, 3, 4, 0, 5, 6, 7, 4, 3, 2

Draw a control chart for the number of defects and comment whether the process is under control or not.

Answer

Let the number of defects per unit (equal length of cloth) be denoted by C. Then the average number of defects in the 10 sample units is given by

$$\bar{C} = \frac{\Sigma C}{10} = \frac{2+3+4+0+5+6+7+4+3+2}{10} = \frac{36}{10}$$

$$= 3.6$$

The 3-σ control limits for C-chart are given by

$$UCL = \bar{C} + 3 \times \sqrt{\bar{C}} = 3.6 + 3 \times \sqrt{3.6} = 9.2921$$

$$LCL = \bar{C} - 3 \times \sqrt{\bar{C}} = 3.6 - 3 \times \sqrt{3.6} = 3.6 - 5.2922$$

$$= -2.0902 = 0$$

Since, the number of defects per unit cannot be negative so LCL is zero.

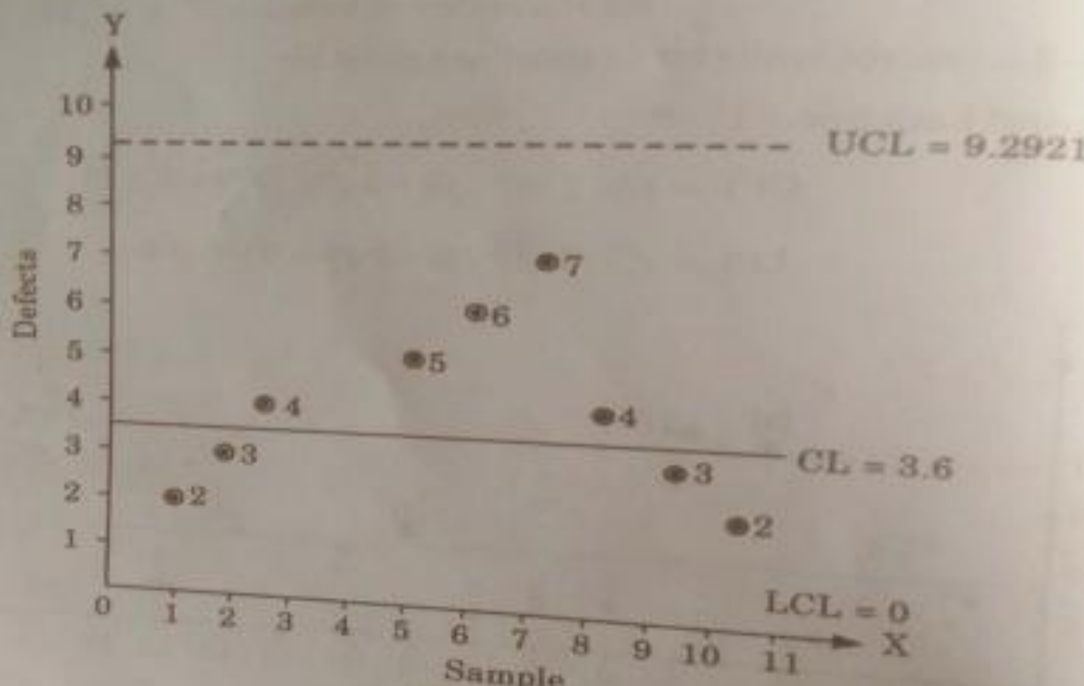

Fig. 5.30.1.

Since all the points are within the control limits, the process is in a state of statistical quality control.

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Plot No. 59/2/7, Site-4, Industrial Area, Sahibabad,
Ghaziabad, 201010, (U.P.) Phone: 0120-4160479
E-mail: pagequantum@gmail.com Web: www.quantumpage.co.in
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