STA 135 (Sample Exam Sol)

P1. Proof:

Left inverse: $B^{-1}A^{-1}AB = B^{-1}B = I$

Right inverse: $(AB)(B^{-1}A^{-1}) = AA^{-1} = I$

$\Rightarrow$ $(AB)^{-1} = B^{-1}A^{-1}$ due to the uniqueness of inverse

P2. See HW1 3.16.

P3. (a) $\Sigma^{-1/2}(\underline{X}_1 - \underline{\mu}) \sim N_4(\underline{0}, I_4)$

$$\Rightarrow \left(\Sigma^{-1/2}(\underline{X}_1 - \underline{\mu})\right)'\left(\Sigma^{-1/2}(\underline{X}_1 - \underline{\mu})\right) \sim \chi^2_4$$

$$\Leftrightarrow (\underline{X}_1 - \underline{\mu})'\Sigma^{-1}(\underline{X}_1 - \underline{\mu}) \sim \chi^2_4$$

(b) $\bar{X} \sim N_4(\mu, \Sigma/60)$

$$\Rightarrow \sqrt{60}\,\Sigma^{-1/2}(\bar{X} - \underline{\mu}) \sim N_4(\underline{0}, I_4)$$

$$\Rightarrow 60(\bar{X} - \underline{\mu})'\Sigma^{-1}(\bar{X} - \underline{\mu}) \sim \chi^2_4$$

P4. Assume $S = \boxed{\frac{1}{n}} \sum_{j=1}^{n} (\underline{x}_j - \bar{x})(\underline{x}_j - \bar{x})'$

$$\Lambda = \left(\frac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|} \right)^{n/2}$$

$$= \left(\frac{\left| \begin{matrix} 10 & 14 \\ 14 & 21 \end{matrix} \right|}{\left| \begin{matrix} 11 & 15 \\ 15 & 22 \end{matrix} \right|} \right)^{61/2}$$

$$= \left(\frac{14}{17} \right)^{\frac{61}{2}}$$

$$= 3.6 \times 10^{-6}$$

$$\cancel{T^2 = \frac{(61-1)|\hat{\Sigma}|}{|\hat{\Sigma}_0|}}$$

$$T^2 = \frac{(n-1)|\hat{\Sigma}_0|}{|\hat{\Sigma}|} - (n-1)$$

$$= \frac{60 \times 17}{14} - 60$$

$$= 12.86$$

$$C_\alpha = \frac{(n-1)p}{n-p} F_{p,n-p}(\alpha) = \frac{(61-1)\times 2}{61-2} \times F_{2,59}(0.05) = 6.41 < T^2$$

$\Rightarrow$ Reject H_0

b. $p = 2$, $\alpha = 0.05$

$\Rightarrow$ (5-29)

$$\bar{x}_L - t_{60}\left(\frac{0.05}{2\times 2}\right)\sqrt{\frac{10}{61}} \le \mu_L \le \bar{x}_L + t_{60}\left(\frac{0.05}{2\times 2}\right)\sqrt{\frac{10}{61}}$$

$$\bar{x}_W - t_{60}\left(\frac{0.05}{2\times 2}\right)\sqrt{\frac{21}{61}} \le \mu_W \le \bar{x}_W + t_{60}\left(\frac{0.05}{2\times 2}\right)\sqrt{\frac{21}{61}}$$

$\Rightarrow$

$$17.07 \le \mu_L \le 18.93$$

$$29.65 \le \mu_W \le 32.35$$

P5.

$$\begin{array}{ccc} & x_1 & x_2 \\ \end{array}$$
$$\left(\begin{array}{c|cc} 5 & 2 & 3 \\ \hline 2 & 2 & 0 \\ 3 & 0 & 3 \end{array}\right)$$

$$E(Y \mid X_1, X_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2)$$

$$= 2 + (2 \quad 3)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix}$$

$$= 2 + (1 \quad 1)\begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix}$$

$$= x_1 + x_2 - 2$$

when $(x_1, x_2) = (2, 2) \Rightarrow E(Y \mid x_1 = 2, x_2 = 2) = 2$

$$Var(Y \mid X_1, X_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{+}\Sigma_{21}$$

$$= 5 - (2\ 3)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= 0$$