1. Property of Rank.

Suppose $A_{m\times n}$ is a matrix with m rows and n columns.

$\Rightarrow$ $\text{rank}(A) \le \min(m, n)$

Suppose $B_{n\times k}$ is another matrix.

$\Rightarrow$ $\text{rank}(AB) \le \min(\text{rank}(A), \text{rank}(B))$

If A is a real numbers matrix:

$\Rightarrow$ $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T) = \text{rank}(A^TA)$

2. Partition of a matrix.

Suppose $\underline{X}' = (X_1, X_2, X_3, X_4, X_5)$. $\mu_x' = (4, 3, 2, 1, 0)$

$$\Sigma_x = \begin{pmatrix} 3 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 9 & -2 & 0 \\ 2 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

Partition of $\underline{X}' = (\underline{X}^{(1)}, \underline{X}^{(2)})$ where

$$\underline{X}^{(1)} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \qquad \underline{X}^{(2)} = \begin{pmatrix} X_4 \\ X_5 \end{pmatrix}$$

Then:

$$E(\underline{X}^{(1)}) = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \qquad E(\underline{X}^{(2)}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Cov(\underline{X}^{(1)}) = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 9 \end{pmatrix}$$

$$Cov(\underline{X}^{(2)}) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Cov(\underline{X}^{(1)}, \underline{X}^{(2)}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ -2 & 0 \end{pmatrix}$$

Suppose $A = [1, 2, 1]$ $\quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

$$E(A\underline{X}^{(1)}) = A\,E(\underline{X}^{(1)})$$

$$= (1, 2, 1)\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 12$$

$$E(B\underline{X}^{(2)}) = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$Cov(A\underline{X}^{(1)}) = A \cdot Cov(\underline{X}^{(1)}) \cdot A^T$$

$$= (1, 2, 1)\begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= (5, 3, 13)\begin{pmatrix}1\\2\\1\end{pmatrix}$$
$$= 24$$

$$Cov(B\underline{X}^{(2)}) = B \cdot Cov(\underline{X}^{(2)}) \cdot B'$$
$$= \begin{pmatrix}1 & -2\\2 & -1\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}$$
$$= \begin{pmatrix}4 & -2\\8 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}$$
$$= \begin{pmatrix}8 & 10\\10 & 17\end{pmatrix}$$

$$Cov(A\underline{X}^{(1)}, B\underline{X}^{(2)}) = A \; Cov(\underline{X}^{(1)}, \underline{X}^{(2)}) \; B'$$
$$= (1, 2, 1)\begin{pmatrix}2 & 0\\0 & 0\\-2 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}$$
$$= (0 \quad 0)\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix} = 0$$

3. Property of Trace.

A, B are two matrices, c a scalar.

1. $tr(A+B) = tr(A) + tr(B)$
2. $tr(cA) = c\,tr(A)$
3. $tr(A^T) = tr(A)$
4. $tr(AB) = tr(BA)$

5. $\operatorname{tr}(A^TB) \overset{③}{=} \operatorname{tr}(B^TA) \overset{④}{=} \operatorname{tr}(AB^T) \overset{③}{=} \operatorname{tr}(BA^T)$

6. If A is $n \times n$ matrix, then.

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i$$

where $\lambda_1, \cdots \lambda_n$ are eigenvalues of A.