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Supersymmetric Chern-Simons theory and its application to ABJM theory

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ABSTRACT

This thesis is devoted to the study of superconformal Chern-Simons theory, particularly a $1/2$ BPS domain wall version of ABJM theory. The ABJM theory is found in 2008 after a long time quest of the world volume theory of M_2-branes, it is also dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. This theory is proved to be integrable in the planar limit, where the two-loop dilatation operator can be mapped to the Hamiltonian of certain alternating spin chains due to the bifundamental nature of the gauge structure. In the first few chapters, we will explain some basic facts about Chern-Simons theory and symmetries in $2+1$ spacetime dimensions. Then we introduce how to incorporate supersymmetry into the Chern-Simon gauge theory, this will lay out foundations for Lagrangian construction of ABJM theory. Since integrability turns out to be the quite important in the calculation of various physical observables, we will explain how to use the integrability to compute the anomalous dimension of gauge invariant single-trace operator. After this, we will introduce the $1/2$ BPS domain wall defect ABJM theory, the corresponding non-vanishing tree level one-point functions can be mapped to the overlap between certain matrix product states and Bethe states from the spin chain. In the end, we systematically set up the computation for the semi-classical analysis around the domain wall configuration based on a deformed version of fuzzy spherical harmonics.

RESUMÉ

Denne afhandling er viet til studiet af superkonform Chern-Simons-teori, særligt en $1/2$ BPS-domænevægsversion af ABJM-teorien. ABJM-teorien blev opdaget i 2008 efter en lang søgen efter verdensvolumenteorien for M_2-braner, og den er også dual til type IIA strengteori på $AdS_4 \times \mathbb{CP}^3$. Det er blevet bevist, at denne teori er integrerbar i den planare grænse, hvor den to-loop dilatationsoperator kan omskrives som Hamilton-operatoren for visse alternerende spinkæder på grund af gauge-strukturens bifundamentale natur. I de første kapitler forklarer vi nogle grundlæggende fakta om Chern-Simons-teori og symmetrier i $2+1$ rumtidsdimensioner. Derefter introducerer vi, hvordan supersymmetri kan inkorporeres i Chern-Simons gauge-teori; dette danner grundlaget for Lagrange-konstruktionen af ABJM-teorien. Da integrerbarhed viser sig at være ganske vigtig i beregningen af forskellige fysiske observabler, vil vi forklare, hvordan man kan bruge integrerbarhed til at beregne den anomale dimension af gauge-invariante single-trace operatorer. Herefter introducerer vi $1/2$ BPS-domænevæg-defekt-ABJM-teorien. De tilsvarende ikke-nul træ-niveau étpunktsfunktioner kan omskrives som overlappet mellem visse matrixprodukttilstande og Bethe-tilstande fra spinkæden. Til sidst opstiller vi systematisk beregningen for den semi-klassiske analyse omkring domænevægskonfigurationen baseret på en deformeret version af fuzzy sfæriske harmoniske funktioner.

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CONTENTS

Part I

INTRODUCTION

1

INTRODUCTION

It has been a long-standing challenge for us theoretical physicist to construct a theory of quantum gravity, gauge gravity duality might give us hints of it due to holographic principle. As originally proposed by Juan Maldacena in 1997 [1, 2, 3], gauge-gravity duality states that:

Type IIB string theory on $AdS_5 \times S^5$ is dual to 4D $\mathcal{N} = 4$ super Yang–Mills theory with gauge group $SU(N)$.

We now often refer this correspondence as AdS_5/CFT_4. Such duality has been extensively studied since the birth of it. However, there are certain extensions of such duality in lower dimensions, including $ABJM$ theory [4](3D conformal field theory) and M theory on $AdS_4 \times S^7/\mathbb{Z}_k$, which are less studied.

In the mid-2000s, a major open problem in string/M-theory was to identify the low-energy world-volume theory of multiple M_2-branes (super-membranes in 11-dimensional M theory). One would expect a 3D super-conformal field theory (SCFT) describing N coincident M_2-branes in flat space by analogy of D_3-branes. Early efforts led to the Bagger-Lambert-Gustavsson (BLG) model [5, 6, 7, 8, 9], which achieved maximal $\mathcal{N} = 8$ supersymmetry using a novel 3-algebra structure. However, it was limited to $SO(4) \cong SU(2) \times SU(2)$, essentially describing at most two M_2-branes. This set the stage for Aharony, Bergman, Jafferis, and Maldacena, who constructed the seminal ABJM theory, a 3D $\mathcal{N} = 6$ super-conformal Chern-Simons (CS) matter theory that captured the low-energy dynamics of multiple M_2-branes.

ABJM theory (named for Aharony, Bergman, Jafferis, Maldacena) is a 3D $\mathcal{N} = 6$ SCFT with gauge group $U(N)_k \times U(N)_{-k}$. Here k is the CS level, and the two $U(N)$ factors have opposite levels $+k$ and $-k$ to ensure an overall parity-invariant theory. Importantly, ABJM was conjectured to describe the low-energy limit of N coincident M_2-branes placed at a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity. In particular, for $k = 1, 2$, the orbifold is trivial or symmetric, and the theory is believed to enhance to $\mathcal{N} = 8$ supersymmetry, describing N M_2-branes in flat 11D space. For instance, in the special case $N = 2, k = 1$, ABJM theory coincides with the BLG model, providing a consistency check. For general $k > 1$, the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ breaks some supersymmetry, yielding an $\mathcal{N} = 6$ theory. This is still maximally super-conformal for $k > 2$ in 3D (since $\mathcal{N} = 8$ is not possible for generic k).

Most importantly, ABJM theory provided a concrete example of $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, for a review see [10]. If we stay in the large k regime, we will get an analogy duality like the case in Maldacena's original proposal:

Type IIA superstring theory on $AdS_4 \times CP^3$ with RR four-form flux $F^{(4)} \sim N$ through AdS_4 and RR two-form flux $F^{(2)} \sim k$ through a $CP^1 \subset CP^3$ is dual to $\mathcal{N} = 6$ super-conformal Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$ on $\mathbb{R}^{1,2}$ and CS levels k and $-k$.

Both theories are controlled by only two parameters, k and N, which take integer values. One can take a planar limit which is given by

$$k, N \to \infty, \quad \lambda = \frac{N}{k} = \text{fixed}.$$

On the gravity side, the string coupling constant and effective tension are given by

$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N}, \quad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda},$$

where R is the radius of CP^3 and twice the radius of AdS_4.

Originally formulated in the 1970s [11] and popularized by Edward Witten in the 1980s [12], CS theory is a 3-dimensional gauge theory defined by a topological action (CS term) with no local propagating degrees of freedom. It first appeared as a way to compute knot invariants (like the Jones polynomial) via expectation values of Wilson loops in quantum field theory. Moreover CS terms also naturally arise in high energy theory, notably in all 10- and 11-dimensional supergravity theories. This gives us the motivation for unifying supersymmetry with CS theory.

A major early breakthrough came in 1986, when Achúcarro and Townsend showed that 3D supergravity with a negative cosmological constant can be reformulated as a CS gauge theory based on a supergroup [13]. In their construction, the graviton and gravitino fields are packaged into a CS action, demonstrating that a CS action can accommodate local supersymmetry in three dimensions. This was a strong motivation to consider CS terms in supersymmetric gauge theories more generally. Around the same time, particle physicists investigated topologically massive gauge theories with supersymmetry. In 3D, a CS term gives gauge bosons a mass while preserving gauge invariance (albeit breaking parity), and adding supersymmetry required introducing a matching mass term for the gaugino. By the early 1990s, researchers had constructed explicit supersymmetric Chern-Simons gauge theories with various amount of supersymmetry. For example, Nishino and Gates systematically developed CS gauge models with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry [14]. Marrying a topological field theory with supersymmetry could yield richer mathematical structures, i.e. supersymmetric knot invariants, and new quantum phenomena like supersymmetric anyons.

The strongest motivation to combine supersymmetry and Chern-Simons theory would be the quest to describe the dynamics of certain branes in string/M-theory as mentioned in the beginning. It became clear in the early days study of such dynamics that the worldvolume theory of multiple M_2-branes (membranes in M-theory) should be a 3D superconformal field theory with maximal supersymmetry. However, traditional 3D Yang-Mills terms are not conformally invariant, but a CS term is of the correct dimension to yield a conformal theory. The culmination of these efforts was the ABJM model, which provided a concrete example of a supersymmetric Chern-Simons theory that is conformal and exactly solvable in the large N limit. It immediately became a "hydrogen atom" for 3D quantum field theory, a relatively simple model with rich symmetry, used to test ideas in AdS/CFT.

In the planar (large N) limit with fixed 't Hooft coupling $\lambda = N/k$, ABJM theory exhibits integrability (by calculating the anomalous dimensions) much like planar $\mathcal{N} = 4$ super Yang-Mills in 4D. The complicated problem of operator mixing in planar ABJM can be mapped to an one-dimensional spin chain system that is exactly solvable, but now the corresponding spin chain is an "alternating" chain [15, 16]. Initial studies demonstrated integrability at the first non-trivial loop orders (two-loop planar interactions) in certain sectors, and this was later extended to the complete theory at all loops [17, 18, 19]. The anomalous dimensions can also be studied from the string theory side as done eg. in [20]. As in other gauge theories, planar single-trace operators are the basic gauge-invariant states. An example is an operator constructed by tracing a product of matter fields, e.g. $\mathrm{Tr}(Y^1 Y_4^\dagger Y^1 Y_4^\dagger \dots)$. Here Y^A (with $A = 1, \dots, 4$) are the four complex scalar fields of ABJM, and $Y_A^\dagger$ are their conjugates. Gauge invariance forces the fields in the trace to alternate between these two types: one field carries an index from the first gauge group to the second, the next carries an index from the second back to the first, and so on.

Recently, a lot progress has been made in the integrable theory when there is an integrable boundary shown up. In the condensed matter, such integrable boundary is related to the study of non-equilibrium phenomena, i.e. the quantum quench [21, 22], which plays a key role in investigation of thermalization of a closed system. Similarly, a domain wall version $\mathcal{N} = 4$ Super-Yang-Mills theory which separates vacua with $SU(N)$ and $SU(N-k)$ gauge groups also has integrable structure, the integrable defect [23, 24]. When we say integrable boundaries, in the sense of spin chain language, these integrable boundaries can be mapped to certain integrable boundary states [24, 25] which satisfy corresponding algebraic equations. Such domain wall $\mathcal{N} = 4$ Super-Yang-Mills theory has a gravitational dual, the D_3-D_5 probe brane in $AdS_5 \times S_5$ with k units of magnetic flux on its world-volume [26]. The domain wall is defined by a set of Nahm equations [27, 28]. The simplest probes of external heavy objects in a conformal field theory are one-point functions of local operators in the presence of defects. In the above configuration, the one-point functions can be mapped to the overlap between Bethe states of the spin chain and matrix product states [29, 30, 31, 32], where the matrix product states are integrable boundary states satisfy certain algebraic equations [33]. Similarly, there exists a $1/2$ BPS domain wall version of ABJM theory [34], with a string theory dual taking the form of D_2-D_4 probe brane system with flux, which shares many characteristics with the D_3-D_5 domain wall version of $\mathcal{N}=4$ Super-Yang-Mills, and for which the one-point functions has been found in a closed form. The BPS conditions can again be reformulated into a set of Nahm equations [35], which is related to a deformed version fuzzy spherical harmonics [36, 37].

The present thesis is devoted to further elucidating the domain wall defect configuration in ABJM theory. We will do the semi-classical analysis around the $1/2$ BPS domain wall vacua. We also analyze the deformed version fuzzy spherical harmonics. We will start of with the theory foundation, including the introduction of Chern-Simons theory, supersymmetry in $2+1$ dimensions and how to combine these two together. We will further investigate into the supersymmetric Chern-Simon-Matter theories, with particular emphasis in ABJM theory and its defect version. In the appendices, we will layout the conventions of gamma matrices, spinors and super-space in 3D, the details about deform version of fuzzy sphere harmonics and the structure of correlation functions in AdS/CFT. Published papers are appended at the very end.

Part II

THEORETICAL FOUNDATIONS

2

THEORETICAL FOUNDATIONS

In quantum field theory, topological terms are added to the action to capture global properties of fields and spacetime that are invisible to local dynamics but can dramatically affect quantum behavior. One the most famous examples is the θ-term in quantum chromodynamics, the theory of strong interactions [38, 39]:

$$S_\theta = \theta \frac{g^2}{32\pi^2} \int d^4x \, \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{1}$$

where $F_{\mu\nu}$ is the gluon field strength tensor, $\tilde{F}^{\mu\nu}$ is its dual, θ is a real parameter and the integral represents a topological invariant known as the Pontryagin index or instanton number. This term does not affect classical equations of motion because it's a total derivative, but in quantum theory, it influences vacuum structure, CP symmetry, and the path integral. A non-zero θ-term would lead to observable CP violation in QCD, yet experiments show it's extremely small, this is known as the strong CP problem. Topological terms in QFT also appear in: Chern-Simon theory (used in $2+1$ D topological quantum field theories), Wess-Zumino-Witten terms [40, 41] in effective field theories, and in the classification of topological quantum field theories (TQFTs). In this chapter, we mainly focus on the Chern-Simons theory, we will review the basics of Abelian Chern-Simon theory and non-Abelian Chern-Simon theory, for instance the gauge invariance and how it leads to the quantization condition. We will also spit out the supersymmetry in $2+1$ dimensions, in the end, we will combine the supersymmetry with Chern-Simons theory to construct a consistent theory.

2.1 CHERN-SIMONS BASICS

In the following, we will talk about Abelian and non-Abelian Chern-Simon theory along with their various details, for pedagogical introduction, one can check [42, 43, 44].

2.1.1 *Abelian Chern-Simon theory*

2.1.1.1 *Mathematical setup*

We begin with a gauge group on some spacetime M based on an Abelian Lie group $U(1)$. Locally, the gauge field is a real 1-form denoted by $A = A_\mu dx^\mu$. $F = dA$ is a globally well-defined 2-form, and our gauge field is normalized such that $F/2\pi$ has integer period. That is the integrals around any closed 2-cycles (compact and without boundary) $\Sigma_2 \subset M$ are always integers:

$$\int_{\Sigma_2} F \in 2\pi\mathbb{Z}. \tag{2}$$

With the above convention the gauge transformations are such that the covariant derivative $D = d + iA$ transform by $D \mapsto g^{-1}Dg$ where

$$\begin{aligned} g : M &\mapsto U(1) \\ x &\mapsto g(x) \end{aligned} \tag{3}$$

is a gauge transformation in $G = U(1)$. If we can take a logarithm and take $g = e^{i\epsilon}$ for a globally well-defined function mapping spacetime point to a real point

$$\epsilon : M \mapsto R, \tag{4}$$

then

$$A \to A - d\epsilon. \tag{5}$$

We call these small gauge transformations. On the other hand, if $g(x)$ cannot be written as $g(x) = e^{i\epsilon(x)}$ for a globally well-defined function $\epsilon(x)$, we say that is a large gauge transformations. In this case the gauge transformations are better thought as shifts

$$A \to A + \omega, \quad \omega \in \Omega^1_{2\pi\mathbb{Z}}(M) \tag{6}$$

where $\Omega^1_{2\pi\mathbb{Z}}(M)$ is the space of all differentiable closed one-forms whose periods are all in $2\pi\mathbb{Z}$.

2.1.1.2 *The Chern-Simon Action from θ term*

Since $F = dA$, and we observe the θ term for $3+1$ Maxwell theory can be written as

$$F \wedge F = d(AdA) \tag{7}$$

If we consider a $4d$ path integral with $\partial M_4 = M_3$, then we are asked to consider due to Stokes' theorem

$$\int_{M_3} AdA, \tag{8}$$

as a term in the action. At first Eq. (8) might not be gauge invariant, before this let us consider the physics of this term. The vector potential A has a topological current given by

$$J = \frac{1}{2\pi} \star dA, \tag{9}$$

where $\star$ is the Hodge star, such that it is conserved identically in $2+1$ dimensions[45]

$$d^\dagger J = \frac{1}{2\pi} \star d^2 A = 0. \tag{10}$$

Where the adjoint exterior derivative operator $d^\dagger : \Omega^r(M) \mapsto \Omega^{r-1}(M)$ is defined by

$$d^\dagger = (-1)^{mr+m+1} \star d \star \tag{11}$$

with $m = \dim M$. The current defined in Eq. (9) is gauge invariant both locally and globally. Since we are interested in the long-range physics. This means we want to construct our action in terms of *relevant operator* or *marginal operator*. Since A has mass dimension 1 and J has mass dimension

2, the lowest-dimension 3-$form$ we can construct is $A_\mu J^\mu = A \wedge \star J$. So the Chern-Simons action in a $U(1)$ gauge theory on an oriented 3 manifold M with conventional normalization is

$$S_{CS} = \frac{k}{4\pi} \int_M A \wedge dA, \tag{12}$$

with k as the coupling constant.

2.1.1.3 *Gauge invariance*

Now check this term is indeed gauge invariant. First, under a small gauge transformation Eq. (5), we have

$$A \wedge F \to A \wedge F + d(\epsilon F). \tag{13}$$

There is a total derivative, and if we neglect boundary contributions (which generally is not true, called gauge anomaly), then the gauge invariance is assured. Under large gauge transformations Eq. (6), we have

$$A \wedge F \to A \wedge F + \omega \wedge F. \tag{14}$$

Notice that any 1-form with quantized periods must be a closed 1-form, but in general it need not to be exact, so $\omega \wedge F$ is not necessarily exact globally. This is another problem apart from boundary term.

2.1.1.4 *Quantization of k*

For now let us consider the manifold M has no boundary as in Eq. (13). But we still need to consider Eq. (14). The change of this term in the path integral would be

$$e^{2\pi i \frac{k}{4\pi^2} \int_M A \wedge dA} \to e^{2\pi i \frac{k}{4\pi^2} \int_M A \wedge dA} \cdot e^{2\pi i k \int_M \frac{\omega}{2\pi} \wedge \frac{F}{2\pi}}. \tag{15}$$

So we need to impose

$$e^{2\pi i k \int_M \frac{\omega}{2\pi} \wedge \frac{F}{2\pi}} = 1. \tag{16}$$

By assumption we take $\omega \in \Omega^1_{2\pi\mathbb{Z}}(M)$ to be an arbitrary closed 1-form with $2\pi\mathbb{Z}$ periods. Now due to the quantization condition of the magnetic monopole, then the form $F/2\pi$ has arbitrary integer periods. Therefore, the path integral is well defined iff $k \in \mathbb{Z}$, and we call it $level$.

2.1.1.5 *Wilson loops and linking number*

Now let consider the partition function in the presence of of a source of J [46]:

$$\mathcal{Z}[J]=\frac{1}{\mathcal{Z}}\int\mathcal{D}A\exp[i\int_M d^3x\Big(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+A_\mu J^\mu\Big)]=\frac{1}{\mathcal{Z}}\int\mathcal{D}A\, e^{i\tilde{S}_{CS}[A,J]},\qquad(17)$$

consider two particles moving in the closed loops γ_1 and γ_2 around each other in the plane. Model them by the source $J=J_1+J_2$, with

$$J_a^\mu=\oint dx_a^\mu\delta^3(x-x_a(t)),\quad a=1,2.\qquad(18)$$

Then one can get the source term as

$$\int_M d^3xA_\mu J^\mu=\oint_{\gamma_1}dx_1^\mu A_\mu(x_1)+\oint_{\gamma_2}dx_2^\mu A_\mu(x_2).\qquad(19)$$

This is just nonlocal observable Wilson loops $W_a=\exp[\oint_{\gamma_a}dx_a^\mu A_\mu(x_a)]$. In terms of W_1 and W_2, the path integral Eq. (17) will becomes

$$\mathcal{Z}[J]=\frac{1}{\mathcal{Z}}\int\mathcal{D}AW_1W_2e^{iS_{CS}}=\langle W_1W_2\rangle.\qquad(20)$$

Because $\tilde{S}_{CS}$ is quadratic in A, the path integral Eq. (20) is Gaussian. Thus it can be evaluated exactly (up to a normalization constant) by substituting the classical A_μ^{cl} into $\tilde{S}_{CS}$. From Eq. (17), we can get that the EoMs are

$$\begin{aligned}\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}F_{\nu\rho}&=\frac{k}{2\pi}\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho=J^\mu,\\ \Leftrightarrow\partial_\alpha A_\beta(x)&=\frac{2\pi}{k}\varepsilon_{\mu\alpha\beta}J^\mu(x).\end{aligned}\qquad(21)$$

Then one can solve the EoM using Lorentz gauge

$$\begin{aligned}A_\beta(x)&=\frac{1}{k}\varepsilon_{\mu\alpha\beta}\int_{\mathcal{M}}d^3y(\partial^\alpha_{x-y}\frac{1}{|x-y|})J^\mu(y)\\&=-\frac{1}{k}\varepsilon_{\mu\alpha\beta}\int_{\mathcal{M}}d^3y\frac{x^\alpha-y^\alpha}{|x-y|^3}J^\mu(y)\\&=-\frac{1}{k}\varepsilon_{\mu\alpha\beta}\sum_{a=1}^{2}\oint_{\gamma_a}dx_a^\mu\frac{(x-x_a)^\alpha}{|x-y|^3},\end{aligned}\qquad(22)$$

where we have used the Green's function

$$(\partial_t^2-\nabla_{2D}^2)\frac{1}{2\pi|x-y|}=\delta^3(x-y),\qquad(23)$$

with $|x-y|$ measure the Minkowski distance between point x and y. After substituting

$$\begin{aligned}
\mathcal{Z}[J] = \langle W_1 W_2 \rangle &= \mathcal{N} \exp i\tilde{S}[A^{\text{cl}}, 0] \\
&= \mathcal{N} \exp[\frac{ik}{4\pi} \int d^3x \varepsilon^{\mu\nu\rho} A^{\text{cl}}_\mu \partial_\nu A^{\text{cl}}_\rho] \\
&= \mathcal{N} \exp[\frac{i}{4\pi k} \int d^3x \oint_{\gamma_a} dx^\alpha_a \oint_{\gamma_b} dx^\theta_b (\delta^\nu_\alpha \varepsilon_{\beta\theta\omega} - \delta^\nu_\beta \varepsilon_{\alpha\theta\omega}) \\
&\quad \times \frac{(x-x_a)^\beta}{|x-x_a|^3} \partial_\nu \frac{(x-x_b)^\omega}{|x-x_b|^3}],
\end{aligned} \tag{24}$$

where I have used $\varepsilon^{\mu\nu\rho}\varepsilon_{\mu\alpha\beta} = \delta^\nu_\alpha \delta^\rho_\beta - \delta^\nu_\beta \delta^\rho_\alpha$, and the sum over a, b is implicit. Use the trick Eq. (23) again

$$\begin{aligned}
\mathcal{Z}[J] &= \mathcal{N} \exp[\frac{i}{4\pi k} \int d^3x \oint_{\gamma_a} dx^\alpha_a \oint_{\gamma_b} dx^\theta_b (\delta^\nu_\alpha \varepsilon_{\beta\theta\omega} - \delta^\nu_\beta \varepsilon_{\alpha\theta\omega}) \\
&\quad \times \frac{(x-x_a)^\beta}{|x-x_a|^3} \partial_\nu \partial^\omega \frac{-1}{|x-x_b|}] \\
&= \mathcal{N} \exp[\frac{i}{4\pi k} \int d^3x \oint_{\gamma_a} dx^\alpha_a \oint_{\gamma_b} dx^\theta_b (\delta^\nu_\alpha \varepsilon_{\beta\theta\omega} - \delta^\nu_\beta \varepsilon_{\alpha\theta\omega}) \\
&\quad \times \frac{(x-x_a)^\beta}{|x-x_a|^3} \delta^\omega_\nu \times 2\pi \delta^3(x-x_b)] \\
&= \mathcal{N} \exp[-\frac{i}{k} \varepsilon_{\beta\theta\omega} \oint_{\gamma_a} dx^\beta_a \oint_{\gamma_b} dx^\theta_b \frac{(x_a-x_b)^\omega}{|x_a-x_b|^3}],
\end{aligned} \tag{25}$$

where the sum over a, b is implicit. Actually, when $x_a = x_b$, the above integral diverges. So to keep our theory finite, we will absorb them into the normalization, then

$$\begin{aligned}
\mathcal{Z}[J] = \bar{\mathcal{N}} \exp[-\frac{2i}{k} \varepsilon_{\beta\theta\omega} \oint_{\gamma_1} dx^\beta_1 \oint_{\gamma_2} dx^\theta_2 \frac{(x_1-x_2)^\omega}{|x_1-x_2|^3}] \\
= \tilde{N} \exp[\frac{i}{2k} \varepsilon_{\beta\theta\omega} \oint_{\gamma_1} dx^\beta_1 \oint_{\gamma_2} dx^\theta_2 \frac{(x_1-x_2)^\omega}{|x_1-x_2|^3}] \\
= \exp\left(\frac{2i\pi}{k} \Phi[\gamma_1, \gamma_2]\right),
\end{aligned} \tag{26}$$

where in the last step I have set $\tilde{N} = 1$, and the *Guass linking integral*:

$$\Phi[\gamma_1, \gamma_2] = \frac{1}{4\pi} \varepsilon_{\beta\theta\omega} \oint_{\gamma_1} dx^\beta_1 \oint_{\gamma_2} dx^\theta_2 \frac{(x_1-x_2)^\omega}{|x_1-x_2|^3} \tag{27}$$

is a topological invariant called the *linking number* of γ_1 and γ_2, it is an integer counts the number of times that one curves winds around the other.

2.1.1.6 *Canonical quantization*

The Chern-Simons action can be decomposed into its A_0 and A_i component

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} d^3x (\varepsilon^{ij} A_i \dot{A}_j + 2A_0 \varepsilon^{ij} \partial_i A_j), \tag{28}$$

there is no kinetic term for A_0, which means it can be serve as an Lagrangian multiplier. Its equation of motion is

$$\frac{k}{2\pi} F_{12} = 0. \tag{29}$$

If we imposed it at the action, meaning we are choosing the gauge $A_0 = 0$, and Eq. (29) serves as a constraint. Then Eq. (28) turns to

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} d^3x (\varepsilon^{ij} A_i \dot{A}_j). \tag{30}$$

One can easily see that the conjugate momentum of A_i would be $\Pi^i = \frac{k}{4\pi} \varepsilon^{ij} A_j$. And the Hamiltonian $H = \Pi^\mu \dot{x}_\mu - L$ vanishes as we expected from

$$T^{\mu\nu} \propto \frac{\delta S_{CS}}{\delta g_{\mu\nu}} = 0, \tag{31}$$

Indicate the conformal nature of Chern-Simons theory from metric independence. The Poisson brackets can be read off from Eq. (30) as

$$\{A_i, A_j\} = \frac{2\pi}{k} \varepsilon_{ij} \delta^2(\mathbf{x} - \mathbf{y}), \tag{32}$$

promote to operators would be

$$[A_i, A_j] = \frac{2\pi i}{k} \varepsilon_{ij} \delta^2(\mathbf{x} - \mathbf{y}). \tag{33}$$

2.1.2 *Non-Abelian Chern-Simons theory*

To construct a non-Abelian Chern-Simons theory, we consider the vector potential takes the values in general Lie algebra. In the following, we will use fundamental $\mathfrak{su}_2$ as example, so our Lie algebra valued gauge field is then

$$A_\mu(x) = A^a_\mu(x)(\frac{\sigma_a}{2i}), \tag{34}$$

where σ_a are the Pauli matrices. And the field strength reads

$$F = dA + A \wedge A, \tag{35}$$

or in indices

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{36}$$

As in the Yang-Mills theory, we need to consider the Wilson loop operators (like a holonomy) as observables (like the Aharonov-Bohm phase)

$$W_L = \mathrm{Tr}\left[P\exp(\oint_L dl^\mu A_\mu)\right], \tag{37}$$

where we have consider the integral follows some closed path L, P indicates path ordering. Now let's consider the gauge transformation in the case of a non-Abelian gauge field

$$A_\mu \to U^{-1}A_\mu U + U^{-1}\partial_\mu U(x), \tag{38}$$

where U is the map

$$\begin{aligned} U : M &\mapsto SU(2), \\ x &\mapsto U(x). \end{aligned} \tag{39}$$

Or in differential form

$$A \to g^{-1}Ag + g^{-1}dg. \tag{40}$$

It can be shown that W_L is invariant under the gauge transformation.

2.1.2.1 *Chern-Simons action from θ term*

Recall that in the dim $=4$ Yang-Mills theory, we have the topological θ term action as

$$\begin{aligned} S &= \frac{1}{8\pi^2}\int_{\mathcal{N}} \text{tr } F\wedge F \\ &= \frac{1}{16\pi^2}\int_{\mathcal{N}} d^4x \text{ tr } (F_{\mu\nu}{}^*F^{\mu\nu}) \end{aligned} \tag{41}$$

One can show that this term can written as in terms of an exact form

$$\begin{aligned} F\wedge F &= (dA+A\wedge A)\wedge(dA+A\wedge A) \\ &= dA\wedge dA + dA\wedge A\wedge A + A\wedge A\wedge dA + A\wedge A\wedge A\wedge A \\ &= d(A\wedge dA+\frac{2}{3}A\wedge A\wedge A) + A\wedge A\wedge A\wedge A \end{aligned} \tag{42}$$

where the second term is quite unusual:

$$\begin{aligned} d(\overset{\alpha\,\beta}{A}\wedge\overset{\gamma}{A}\wedge\overset{\theta}{A}) &= \overset{\alpha\,\beta}{dA}\wedge\overset{\gamma}{A}\wedge\overset{\theta}{A} - \overset{\beta}{A}\wedge\overset{\alpha\,\gamma}{dA}\wedge\overset{\theta}{A} + \overset{\beta}{A}\wedge\overset{\gamma}{A}\wedge\overset{\alpha\,\theta}{dA} \\ &= \overset{\alpha\,\beta}{dA}\wedge\overset{\gamma}{A}\wedge\overset{\theta}{A} + \overset{\alpha\,\gamma}{dA}\wedge\overset{\theta}{A}\wedge\overset{\beta}{A} + \overset{\alpha\,\theta}{dA}\wedge\overset{\beta}{A}\wedge\overset{\gamma}{A} \\ &= 3\times dA\wedge A\wedge A. \end{aligned} \tag{43}$$

And one can recognize the third term just gives us the volume of $\mathcal{N}$, which will give us $2n\pi$ in the path integral in the end, hence integrated out. So one can conclude

$$S_{CS} = \frac{2\pi k}{8\pi^2}\int_{\mathcal{M}} \text{tr}(AdA+\frac{2}{3}A\wedge A\wedge A) \tag{44}$$

is a good topological action candidate for dim $=3$. If one works in indices by plugging Eq. (36) into Eq. (41), one would get

$$\begin{aligned} tr[F_{\mu\nu}{}^*F^{\mu\nu}] &= \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\text{tr}[(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])(\partial_\lambda A_\rho - \partial_\rho A_\lambda + [A_\lambda, A_\rho])] \\ &= \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\text{tr}\{\Big(\partial_\mu A_\nu\partial_\lambda A_\rho - \partial_\mu A_\nu\partial_\rho A_\lambda - \partial_\nu A_\mu\partial_\lambda A_\rho + \partial_\nu A_\mu\partial_\rho A_\lambda\Big) \\ &+ \Big(\partial_\mu A_\nu[A_\lambda, A_\rho] - \partial_\nu A_\mu[A_\lambda, A_\rho] + [A_\mu, A_\nu]\partial_\lambda A_\rho - [A_\mu, A_\nu]\partial_\rho A_\lambda\Big) \\ &+ [A_\mu, A_\nu][A_\lambda, A_\rho]\} \\ &= 2\epsilon^{\mu\nu\lambda\rho}\text{tr}\{\partial_\mu\Big(A_\nu\partial_\lambda A_\rho + \frac{2}{3}A_\nu A_\lambda A_\rho\Big) + A_\mu A_\nu A_\lambda A_\rho\}, \end{aligned} \tag{45}$$

so Eq. (41) is equivalent to

$$S = \frac{1}{8\pi^2}\int_{\mathcal{N}} d^4x\, \partial_\mu G^\mu, \tag{46}$$

with

$$G^{\mu} = \epsilon^{\mu\nu\lambda\rho}\text{tr}\Big(A_{\nu}\partial_{\lambda}A_{\rho} + \frac{2}{3}A_{\nu}A_{\lambda}A_{\rho}\Big). \tag{47}$$

Use Stokes' theorem, one can reduce this to dim $= 3$ [44].

2.1.2.2 *Quantization condition*

Now let's check action Eq. (44) is gauge invariant under transformation Eq. (40)

$$\mathcal{L}' = \epsilon^{\mu\nu\rho}\ \text{tr}\{\overset{(A)}{(A^g_{\mu})\partial_{\nu}(A^g_{\rho})} + \frac{2}{3}\overset{(B)}{A^g_{\mu}A^g_{\nu}A^g_{\rho}}\}. \tag{48}$$

One can expand it our (use $dgg^{-1} = -gdg^{-1}$):

$$\begin{aligned}(A) =& A_{\mu}\partial_{\nu}A_{\rho} - A_{\mu}\partial_{\nu}gg^{-1}A_{\rho} + g^{-1}A_{\mu}A_{\rho}\partial_{\nu}g - g^{-1}A_{\mu}\partial_{\nu}gg^{-1}\partial_{\rho}g - \partial_{\mu}gg^{-1}\partial_{\nu}gg^{-1}A_{\rho} \\ &+\partial_{\mu}gg^{-1}\partial_{\nu}A_{\rho} + g^{-1}\partial_{\mu}gg^{-1}A_{\rho}\partial_{\nu}g - (g^{-1}\partial_{\mu}g)(g^{-1}\partial_{\nu}g)(g^{-1}\partial_{\rho}g)\end{aligned}, \tag{49}$$

and

$$\begin{aligned}(B) =& A_{\mu}A_{\nu}A_{\rho} + g^{-1}A_{\mu}A_{\nu}\partial_{\rho}g + A_{\mu}\partial_{\nu}gg^{-1}A_{\rho} + \partial_{\mu}gg^{-1}A_{\nu}A_{\rho} \\ &+g^{-1}A_{\mu}\partial_{\nu}gg^{-1}\partial_{\rho}g + g^{-1}\partial_{\mu}gg^{-1}A_{\nu}\partial_{\rho}g + \partial_{\mu}gg^{-1}\partial_{\nu}gg^{-1}A_{\rho}, \\ &+(g^{-1}\partial_{\mu}g)(g^{-1}\partial_{\nu}g)(g^{-1}\partial_{\rho}g)\end{aligned} \tag{50}$$

combine together, one can get

$$\mathcal{L}' = \mathcal{L} - \epsilon^{\mu\nu\rho}\partial_{\mu}\text{tr}\ (\partial_{\nu}gg^{-1}A_{\rho}) - \frac{1}{3}\epsilon^{\mu\nu\rho}\text{tr}\ (g^{-1}\partial_{\mu}gg^{-1}\partial_{\nu}gg^{-1}\partial_{\rho}g) \tag{51}$$

with $\mathcal{L} = \epsilon^{\mu\nu\rho}\ \text{tr}\{(A_{\mu})\partial_{\nu}(A_{\rho}) + \frac{2}{3}A_{\mu}A_{\nu}A_{\rho}\}$. If we assume that our manifold $\mathcal{M}$ doesn't have boundary. Then the only no-trivial term would be the winding number density term, put it into integral

$$-\frac{1}{24\pi^2}\int_{\mathcal{M}}\text{tr}(g^{-1}dg)^3 = \text{Integer}, \tag{52}$$

then our action is indeed gauge invariant given

$$k \in \mathbb{Z}, \tag{53}$$

and it is called the level of Chern-Simons theory. Or in terms of differential form, one would get

$$\mathcal{L}' - \mathcal{L} = -d\big(\text{tr}\ (dgg^{-1}A)\big) - \frac{1}{3}\text{tr}\ (g^{-1}dg)^3. \tag{54}$$

2.1.2.3 *Equation of motion and Lax connection*

Let us proceed to the Equation of motion. Take a variation of the Chern-Simons action,

$$\begin{aligned}\mathcal{L}' &= \text{tr}\left[(A+\delta A)\wedge d(A+\delta A)+\frac{2}{3}(A+\delta A)^3\right] \\ &\simeq \text{tr}[A\wedge dA + A\wedge d(\delta A)+\delta A\wedge dA+\frac{2}{3}A^3 \\ &\quad +\frac{2}{3}(\delta A\wedge A\wedge A + A\wedge\delta A\wedge A + A\wedge A\wedge \delta A)]\end{aligned} \tag{55}$$

then

$$\begin{aligned}\delta\mathcal{L} = \mathcal{L}'-\mathcal{L} &= \text{tr}[d(\delta A\wedge A)+2\delta A\wedge A+2\delta A\wedge A\wedge A] \\ &= d\ \text{tr}[\delta A\wedge A]+2\text{tr}\ [\delta A\wedge F],\end{aligned} \tag{56}$$

where one has to use the same method as in Eq. (45). So the EoM is just flatness condition or zero-curvature condition

$$F = dA + A\wedge A = 0, \tag{57}$$

where again we assume no boundary contribution. For a one-from satisfied above condition, one typically calls it the *Lax connection*, which serves as the building block of a monodromy matrix. Given a one-form, one is tempted to construct a parallel transport operator (path-order integral, Wilson line) U_γ which translates between the vector spaces attached to the endpoints of path $\gamma : (x, y)$

$$U_\gamma(u; y, x) := P\exp\int_x^y A(u), \tag{58}$$

with $x = \{\tau_0, \sigma_0\}, y = \{\tau_1, \sigma_1\}$, u as the spectral parameter. This operator is the main ingredient for construction of a Lax pair for our system. Due to the flatness condition, operator U is invariant under homotopic curves with same endpoint. And the typically we want $U_\gamma(u|x; x) = 1$ and

$$U_\gamma(u|x+\delta x; x) = 1 + A_\mu(x)\delta x^\mu. \tag{59}$$

Or in term of differential equations

$$\begin{aligned}\partial_\mu U &= A_\mu U, \text{ at point } y \\ \partial_\mu U &= -UA_\mu, \text{ at point } x.\end{aligned} \tag{60}$$

The Lax pair is constructed from the parallel transport operator, but we have to take the boundary condition into account. Let assume the periodic boundaries $\sigma = \sigma + L$, then the *Lax pair* is defined

$$\begin{gathered} T(u|\tau) = T(u|\tau, L; \tau, 0) = P \exp \int_0^L d\sigma A_\sigma(u), \\ M(u) = A_\tau(u)\Big|_{\sigma=0}. \end{gathered} \tag{61}$$

Due to the flatness condition, the monodromy matrix is well defined, otherwise it will depends on the path it is defined. Hence Eq. (61) indicate the Lax equations

$$\dot{T}(u|\tau) = A_\tau\Big|_{\sigma=L} T - TA_\tau\Big|_{\sigma=0} = [M(u), T(u)]. \tag{62}$$

The traces of powers of T are conserved

$$F_k(u) = \text{tr}\ T(u)^k, \tag{63}$$

because of

$$\frac{d}{d\tau} F_k = k\ \text{tr}\ T^{k-1}[M, T] = 0. \tag{64}$$

2.1.2.4 *Wilson loops and knot invariant*

For the case of non-Abelian case, our previous insight of Wilson loops will still give us topological invariant terms

$$\text{Knot invariant} = \frac{\mathcal{Z}(S^3, \gamma_1, \gamma_2)}{\mathcal{Z}(S^3)} = \frac{\int_{S^3} \mathcal{D}A_\mu(x) W_{\gamma_1} W_{\gamma_2} e^{iS_{CS}}}{\int_{S^3} \mathcal{D}A_\mu(x) e^{iS_{CS}}}. \tag{65}$$

This has been dealt by E. Witten in his famous paper [12]. We will not show how to derive it here.

2.1.2.5 *Canonical quantization*

The canonical quantization is kind of the same as the Abelian one, We will not drill on it further.

2.1.2.6 *A short comments*

Before we turn to supersymmetry in $2+1$ dimensions, it is worth briefly mentioning theories of matter fields coupled to Chern-Simons gauge field. Interestingly, $SU(N)_k$ (k is the Chern-Simons level) Chern-Simons theories coupled to fundamental matter turn out to be effectively solvable in the 't Hooft large N limit, a particular analysis of the partition function, see e.g. [47].

2.2 SUPERSYMMETRY IN $2+1$ DIMENSIONS

In this section, we introduce supersymmetry in three dimensions. We will restrict ourselves to theories with $\mathcal{N}=2$ supersymmetry algebra in three dimensions and supersymmetric actions.

2.2.1 *Spinors in three dimensions*

In three dimensions, spinors have two components and transform in the two-dimensional representation of $SO(2,1) \simeq SL(2,\mathbb{R})$ or $SU(2)$ according to whether the signature is Minkowskian or Euclidean, respectively. To see this in Minkowskian signature, take a 3 vector X and a corresponding 2×2 matrix $\tilde{x}$,

$$X = x_\mu e^\mu = (x_0, x_1, x_2), \quad \tilde{x} = x_\mu \sigma^\mu = \begin{pmatrix} x_0 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 \end{pmatrix}, \tag{66}$$

where σ^μ is the three vector of Pauli matrix $(\sigma_0, \sigma_1, \sigma_2)$

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}. \tag{67}$$

The choice of σ^μ is not unique (notice in this way we have chosen x_3 is invariant under $SL(2,\mathbb{C}) \simeq SO(3,1)$ when do the dimension reduction). Transformation $X \mapsto \Lambda X$ under $SO(2,1)$ leaves the square

$$|X|^2 = x_0^2 - x_1^2 - x_2^2 \tag{68}$$

invariant, whereas the action of $SL(2,\mathbb{R})$ mapping $\tilde{x} \mapsto N\tilde{x}N^{\dagger}$ with $N \in SL(2,\mathbb{R})$ preserves the determinant. A typical element in $SL(2,\mathbb{R})$ would be [48]

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \text{with} \quad \alpha\bar{\alpha} - \beta\bar{\beta} = 1. \tag{69}$$

The map between $SL(2,\mathbb{R})$ and $SO(2,1)$ is 2-1, since $N = \pm 1$ both correspond to $\Lambda = \mathbb{I}$. The basic representation of $SL(2,\mathbb{R})$ are:

- The fundamental representation:

$$\psi'_{\alpha} = N_{\alpha}{}^{\beta}\psi_{\beta}, \quad \alpha, \beta = 1, 2. \tag{70}$$

- The contravariant representation

$$\psi'^{\alpha} = \psi^{\beta} N_{\beta}^{\alpha}, \quad \alpha, \beta = 1, 2. \tag{71}$$

We will see next these two representations are not independent, to see this we will consider the ways to raise and lower the indices. $\epsilon_{\alpha\beta}$ or $\epsilon^{\alpha\beta}$ are the invariant symbols (like metric) in $SL(2,\mathbb{R})$, we use the following convention

$$\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{72}$$

since

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} N_{\rho}{}^{\alpha} N_{\sigma}{}^{\beta} = \epsilon^{\alpha\beta} \cdot \det N = \epsilon^{\alpha\beta}. \tag{73}$$

We can define

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}, \quad \psi_{\alpha} = \epsilon_{\alpha\beta}\psi^{\beta}, \tag{74}$$

without violating the transformation rules. And we define the product of spinors as

$$\psi_1\psi_2 = \psi_1{}^{\alpha}\psi_{2\alpha} = \epsilon^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}. \tag{75}$$

In $SL(2,\mathbb{C})$ one would also have a conjugate representation, that is an inequivalent representation of the fundamental representation [49]. We need to specify the rule for elements, i.e. $(\sigma^{\mu})_{\alpha\dot{\alpha}}$, transform in both fundamental representation and conjugate representation. However, in our case things get simplified since we don't have the conjugate representation, that is to say we can safely replace $\dot{\alpha}$ with β for the transformation rules. For example

$$\sigma^{\mu}_{\alpha\beta} \mapsto N_{\alpha}{}^{\gamma}(\sigma^{\nu})_{\gamma\theta}(\Lambda^{-1})^{\mu}{}_{\nu}N_{\beta}{}^{\theta}, \tag{76}$$

with Λ denotes the Lorentz transformation. For a better understanding, we will talk about the spinors from Dirac equations. The most obvious difference is that the irreducible set of Dirac matrices consists of 2×2 matrices in 3D, rather than 4×4 matrices in 4D. Correspondingly, the irreducible fermion fields are 2-component spinors. The Dirac equations from Dirac Lagrangian with a $U(1)$ gauge coupling $\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - e\bar{\psi}\gamma^{\mu}A_{\mu}\psi$ is

$$(i\gamma^{\mu}\partial_{\mu} - m - e\gamma^{\mu}A_{\mu})\psi = 0, \quad \text{or} \quad i\partial_t\psi = \big(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta\big)\psi, \tag{77}$$

where $\vec{\alpha} = \gamma^0\vec{\gamma}$ and $\beta = \gamma^0$. The Dirac gamma matrices satisfy the anticommutation relations:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}, \tag{78}$$

where we use the Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1)$. One would have a Dirac representation (ψ is complex from Eq. (77))

$$\begin{aligned} \gamma^0 = \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma^1 = i\sigma^1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ \gamma^2 = i\sigma^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{79}$$

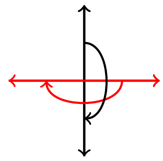

Figure 1: parity transformation in 3 D

And a Majorana representation is

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\gamma^1 = i\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (80)$$
$$\gamma^2 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Eq. (79) and Eq. (80) satisfy the identities:

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu}\mathbb{I} - i\epsilon^{\mu\nu\rho}\gamma_\rho, \mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho) = -2i\epsilon^{\mu\nu\rho}. \quad (81)$$

Above the second identity is different from 4D where the trace of odd number of gamma matrices will be zero. Another important point of $2+1$ dimensions is that there is no γ^5 matrix that anti-commutes with all Dirac matrices, note that $i\gamma^0\gamma^1\gamma^2 = \mathbb{I}$. Thus there is no notation of chirality, which agrees with our degree of freedom. We also need to pay attentions to discrete symmetries if we consider the full Poincare symmetry. These symmetries, namely parity, change conjugation and time reversal, act very differently in $2+1$ dimension. Our usual notion of a parity transformation is a reflection $\vec{x} \to -\vec{x}$ of the spatial coordinates. However, in the plane, such a transformation is equivalent to a rotation as following (since $\det \Lambda = 1$).

So the improper discrete parity transformation should be taken as reflection of just one of the spatial axes [50]

$$\mathcal{P} : x^1 \to -x^1, \quad x^2 \to x^2, \quad x^0 \to x^0. \quad (82)$$

The above transformation rules is set to vector of $SO(2,1)$, we want to know how the spinors transform. From $i\bar{\psi}\gamma^\mu\partial_\mu\psi$ is a scalar under parity transformation we know that

$$\begin{aligned} i\bar{\psi}\gamma^\mu\partial_\mu\psi &\to i(\psi')^\dagger\gamma^0\gamma^1(-1)\partial_1\psi + i(\psi')^\dagger\gamma^0\gamma^2\partial_2\psi + i(\psi')^\dagger\gamma^0\gamma^0\partial_1\psi \\ &= i\bar{\psi}\gamma^\mu\partial_\mu\psi, \end{aligned} \tag{83}$$

a proper solution would be

$$\psi \;\to\; \gamma^1\psi. \tag{84}$$

In this way, a fermion mass term breaks parity

$$\bar{\psi}\psi \to -\bar{\psi}\psi. \tag{85}$$

Under $\mathcal{P}$, the gauge field A^μ transform like a vector, hence the Chern-Simons term changes sign under $\mathcal{P}$. Charge conjugation converts the charge of all particle, so the Dirac equation would look like

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi_c = 0, \tag{86}$$

This can be achieved by the definition $\psi_c = \mathcal{C}\bar{\psi}^T$, where the charge conjugation matrix $\mathcal{C}$ should satisfy

$$(\gamma^\mu)^T = -\mathcal{C}^{-1}\gamma^\mu\mathcal{C}. \tag{87}$$

In Eq. (87), we can choose $\mathcal{C} = \gamma^2$. Then one can easy see that $\bar{\psi}\psi$ is invariant under $\mathcal{C}$. For the gauge field

$$C : A_\mu \to -A_\mu. \tag{88}$$

Time reversal operator $\mathcal{T}$ is an anti-unitary operator ($\mathcal{T} : i \to -i$) in order to implement $x^0 \to -x^0$ without change the energy $P^0 \to -P^0$ with P^μ is the three momentum operator. The action on the spinor and gauge field would be

$$\mathcal{T} : \psi \to \gamma^2\psi, \quad \vec{A} \to -\vec{A}, \quad A^0 \to A^0. \tag{89}$$

Note that $\bar{\psi}\psi$ changes sign under this transformation as we expected from CPT invariance.

2.2.2 *From 4d $\bar{\sigma}^m$, σ^n to 3d γ^μ*

Normally, if we choose $\sigma^m_{\alpha\dot{\beta}} = (\mathbb{I}, \vec{\sigma})$ and $(\bar{\sigma}^m)^{\dot{\alpha}\beta} = (-\mathbb{I}, \vec{\sigma})$, the Dirac algebra yields

$$(\sigma^m)_{\alpha\dot{\alpha}}(\bar{\sigma}^n)^{\dot{\alpha}\beta} + (\sigma^n)_{\alpha\dot{\alpha}}(\bar{\sigma}^m)^{\dot{\alpha}\beta} = 2\eta^{mn}\delta^\beta_\alpha \tag{90}$$

where $\eta^{mn} = \mathrm{diag}(-1,1,1,1)$. (If we choose $\sigma^m_{\alpha\dot{\beta}} = (\mathbb{I}, \vec{\sigma})$ and $(\bar{\sigma}^m)^{\dot{\alpha}\beta} = (\mathbb{I}, -\vec{\sigma})$, then the above algebra still hold, but $\eta^{mn} = \mathrm{diag}(1,-1,-1,-1)$.) This is the typical choose for σ^m matrix, but if we follow the convention by [51], we wish to find $(\sigma^m)_\alpha^{\ \dot{\beta}}$ and $(\bar{\sigma}^m)_{\dot{\alpha}}^{\ \beta}$, which still satisfy the Dirac algebra. (The reason why we doing this is in 3D case, there is no chirality, which means if we set $\dot{\alpha} \sim \alpha$, we can sort of get 3d out of 4d.) Let's lower our index using

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{91}$$

then

$$\begin{aligned}(\sigma^m)_\alpha^{\ \dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^m)_{\alpha\dot{\beta}} = (-i\sigma_2, \sigma_3, -i\sigma_0, -\sigma_1), \\ (\bar{\sigma}^m)_{\dot{\alpha}}^{\ \alpha} &= \epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^m)^{\dot{\beta}\alpha} = (i\sigma_2, -\sigma_3, -i\sigma_0, \sigma_1).\end{aligned} \tag{92}$$

One can find that

$$(\sigma^m)_\alpha^{\ \dot{\alpha}}(\bar{\sigma}^n)_{\dot{\alpha}}^{\ \beta} + (\sigma^n)_\alpha^{\ \dot{\alpha}}(\bar{\sigma}^m)_{\dot{\alpha}}^{\ \beta} = 2\tilde{\eta}^{mn}\delta^\beta_\alpha \tag{93}$$

with $\tilde{\eta}^{mn} = \mathrm{diag}(1,1,-1,1)$. So we might as well rearrange the index (we do dimension reduction in space direction 2 to get real γ matrix), and set

$$\begin{aligned}(\sigma^m)_\alpha^{\ \dot{\beta}} &= (-i\sigma_0, -i\sigma_2, \sigma_3, -\sigma_1) = (-i\sigma_0, (\gamma^\mu)_\alpha^{\ \beta}), \\ (\bar{\sigma}^m)_{\dot{\alpha}}^{\ \beta} &= (-i\sigma_0, i\sigma_2, -\sigma_3, \sigma_1) = (-i\sigma_0, -(\gamma^\mu)_\alpha^{\ \beta})\end{aligned} \tag{94}$$

one can easily check that the corresponding γ^μ satisfy the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{95}$$

with $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1)$. In this case, we can safely say that one can just replace σ^m and $\bar{\sigma}^m$ with γ^μ and $i\mathbb{I}$ (What a miracle coincident!). And this explains why we seeking $\gamma_{\alpha\beta}$ to be symmetric under exchange of α, β due to the symmetric properties of σ^m and $\bar{\sigma}^m$.

2.2.3 *Superspace conventions and superfields*

We start by formally defining superspace, parametrized by bosonic coordinates x^μ and fermionic coordinates $\theta^\alpha, \bar{\theta}^\alpha$. In product like $\theta^\alpha\theta_\alpha = \theta^2$, $\theta^\alpha\bar{\theta}_\alpha = \theta\bar{\theta}$ and $\bar{\theta}^\alpha\bar{\theta}_\alpha = \bar{\theta}\bar{\theta}$, $\theta^\alpha\gamma_{\mu\nu}\theta^\beta = \theta\gamma^\mu\bar{\theta}$ we suppress the indices. We have

$$\theta^\alpha\theta^\beta = \frac{1}{2}\epsilon^{\alpha\beta}\theta^2, \quad \bar{\theta}_\alpha\bar{\theta}_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\bar{\theta}^2. \tag{96}$$

We identify a point in superspace with the operator:

$$e^{ix^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}^\alpha \bar{Q}_a}. \tag{97}$$

Supercovariant derivatives and SUSY generators are

$$\begin{aligned} D_a = \partial_\alpha + i(\gamma^\mu\bar{\theta})_\alpha\partial_\mu, \quad Q_a = \partial_\alpha - i(\gamma^\mu\bar{\theta})_\alpha\partial_\mu, \\ \bar{D}_a = -\bar{\partial}_\alpha - i(\theta\gamma^\mu)_\alpha\partial_\mu, \quad \bar{Q}_a = -\bar{\partial}_\alpha + i(\theta\gamma^\mu)_\alpha\partial_\mu, \end{aligned} \tag{98}$$

which give

$$\{D_a, \bar{D}_\beta\} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu, \qquad \{Q_a, \bar{Q}_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu \tag{99}$$

The convention for integration also follows [51],

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta_\alpha, \quad d^2\bar{\theta} = -\frac{1}{4}d\bar{\theta}^\alpha d\bar{\theta}_\alpha, \quad d^4\theta = d^2\theta d^2\bar{\theta}, \tag{100}$$

such that

$$\int d^2\theta\theta^2 = 1, \quad \int d^2\bar{\theta}\bar{\theta}^2 = 1, \quad \int d^4\theta\theta^2\bar{\theta}^2 = 1 \tag{101}$$

Then we can define superfield in our superspace, the typical superfield (irreducible representation of SUSY) would be

- Chiral superfield

$$D_\alpha\Phi = 0 \tag{102}$$

- Anti-chiral superfield

$$\bar{D}_\alpha\Phi = 0 \tag{103}$$

- Vector superfield

$$V = V^\dagger \tag{104}$$

- Linear superfield

$$DDL = 0 \quad \text{with} \quad L^\dagger = L \tag{105}$$

These have been well presented in [52, 53].

2.3 COMBINING SUSY AND CS TERMS

Here we will discuss the $\mathcal{N} = 2$ superspace formulation of Abelian super-conformal Chern-Simons theories and then extend it to non-Abelian case.

2.3.1 *Notation*

We will use the convention from [51], our previous convention of gamma matrices also work in this case. There are two important properties that we need to use

$$\text{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho) = 2\epsilon^{\mu\nu\rho}, \tag{106}$$

and

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma, \tag{107}$$

due to our choice of γ matrices and ϵ convention. $\mathcal{N} = 2$ in three dimension is closely related to $\mathcal{N} = 1$ in four dimension (similarly based on spinor components, one can link 3d $\mathcal{N} = 2$ with 4d $\mathcal{N} = 1$ superfields), the Chern-Simons part of the action is constructed out of a vector supermultiplet that can be obtained by dimensional reduction of a four-dimensional $\mathcal{N} = 1$ supermultiplet. In 4 dim, the offshell massless vector multiplet contains a gauge field v_M, a four-component Majorana spinor, and a real scalar D. On reduction to three dimensions, the gauge field gives $v_M = (A_\mu, \sigma)$, where σ

is along the imaginary direction when we do the dimension reduction. Also the spinor can be recast as a two-component Dirac spinor χ, and we still have the scalar D. Note that once the reduction is completed, there is no chirality, which means

$$(\sigma^M)_{\beta\dot{\alpha}} \sim ((\gamma^\mu)_{\beta\alpha}, i\delta_{\beta\alpha}), \quad (\bar{\sigma}^M)^{\dot{\alpha}\beta} \sim (-(\gamma^\mu)^{\alpha\beta}, i\delta^{\alpha\beta}), \tag{108}$$

if we choose the metric $\eta_{MN} = \mathrm{diag}\{-1,1,1,1\}$ for 4 dim. Off shell there are four bosonic and four fermionic degrees of freedom. Note that the dimension of A_μ and σ is 1, the dimension of χ is $3/2$, and the dimension of D is 2. We choose the vector multiplet as

$$\mathcal{V}(x,\bar{\theta},\theta) = 2i\theta\bar{\theta}\sigma(x) + 2\theta\gamma^\mu\bar{\theta}A_\mu(x) + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi}(x) - \sqrt{2}i\bar{\theta}^2\theta\chi(x) + \theta^2\bar{\theta}^2 D(x). \tag{109}$$

The supersymmetry transformation sort of like the case in 4 dim, we will not show it here. (One can compare [54] with i.e. [55], and keep Eq. (108) in mind)

Note that unlike vector multiplet, the chiral supermultiplet is also charged under flavor symmetry $R = U(2)$. We will omit the gauge group index, and keep the flavor group index as A. If the matter representation R is complex, let us use the notation $\bar{\Phi}^A$, Φ_A to distinguish holomorphic fields and their antiholomorphic conjugates. The multiplet contains a complex scalar ϕ^A of dimension $1/2$, a Dirac two-component spinor ψ^A of dimension 1, and a complex auxiliary scalar F of dimension $3/2$. We also have the following R charge assignments: ϕ^A has R charge $1/2$, ψ^A has R charge $-1/2$, and F^A has R charge $-3/2$ (the conjugate will take the opposite values.). These conventions correspond to the holomorphic superspace coordinate θ have R charge 1, and the supersymmetry parameter ε having R charge 1. We choose the chiral superfield as

$$\Phi = \phi(x_L) + \sqrt{2}\theta\psi(x_L) + \theta^2 F(x_L), \tag{110}$$

also anti-chiral superfield

$$\bar{\Phi} = \bar{\phi}(x_R) + \sqrt{2}\bar{\theta}\bar{\psi}(x_R) + \bar{\theta}^2\bar{F}(x_R). \tag{111}$$

where

$$x_L^\mu = x^\mu + i\theta\gamma^\mu\bar{\theta}, \quad x_R^\mu = x^\mu - i\theta\gamma^\mu\bar{\theta}. \tag{112}$$

One can do the same kind of expansion as SYM, we will not show it here. The supersymmetric transformation again takes the similar form as in 4 dim SYM.

2.3.2 *Abelian superconformal Chern-Simons theories*

The superspace Lagrangian for abelian $\mathcal{N}=2$ Chern-Simons theory, coupled to matter chiral super field Φ, is given by

$$S=\int d^3x\int d^4\theta\Big(\frac{k}{4\pi}V\Sigma+\bar{\Phi}e^V\Phi\Big)+\int d^3x\big(\int d^2\theta W(\Phi)+\int d^2\bar{\theta}\bar{W}(\bar{\Phi})\big), \tag{113}$$

where $\Sigma=\bar{D}^\alpha D_\alpha V$, can be viewed as the supersymmetric field strength in this case. Under gauge transformation:

$$V\mapsto V+i\Lambda-i\bar{\Lambda},\quad \Phi\mapsto e^{-i\Lambda}\Phi,\quad \bar{\Phi}\mapsto\bar{\Phi}e^{i\bar{\Lambda}} \tag{114}$$

where Λ and $\bar{\Lambda}$ are chiral and anti-chiral superfields respectively to keep V in Wess-Zumino gauge. So

$$\begin{aligned}\Sigma\mapsto&\bar{D}^\alpha D_\alpha V+i\bar{D}^\alpha D_\alpha\Lambda-i\bar{D}^\alpha D_\alpha\bar{\Lambda}\\ &=\Sigma+i\epsilon^{\alpha\beta}\{\bar{D}_\beta,D_\alpha\}\Lambda+0\\ &=\Sigma+i\epsilon^{\alpha\beta}\big(2i(\gamma^\mu)_{\beta\alpha}\partial_\mu\big)\Lambda\\ &=\Sigma\end{aligned} \tag{115}$$

where the second line used the properties of chiral and anti-chiral superfields. The third line used the anti-commutation relation between two super derivatives, the last line is due to the symmetric properties of gamma matrices. So in order to keep the gauge invariant, k must be quantized. The matter part is obviously gauge invariant under Eq. (114). Let's now complete the superspace integration. First, as in the SYM, we will switch to the coordinate

$$y^\mu=x^\mu+i\theta\gamma^\mu\bar{\theta}. \tag{116}$$

In this language, the super derivatives yields

$$\bar{\mathcal{D}}_\alpha=-\bar{\partial}_\alpha,\quad D_\alpha=\partial_\alpha+2i(\gamma^\mu\bar{\theta})_\alpha\partial_\mu, \tag{117}$$

the first term is easily derived by acting with arbitrary superfields, the second term is just to keep the anticommutator holds, and ∂_μ is with respect to y. Then one can get:

$$\begin{aligned}\theta\gamma^\mu\bar{\theta}A_\mu(x) = \theta\gamma^\mu\bar{\theta}A_\mu(y) - \frac{i}{2}\theta^2\bar{\theta}^2\partial_\mu A^\mu(y) \\ \theta\bar{\theta}\sigma(y) = \theta\bar{\theta}\sigma(x) + \theta\bar{\theta}\theta\gamma^\alpha\bar{\theta}\partial_\alpha\sigma(x) = \theta\bar{\theta}\sigma(x)\end{aligned} \tag{118}$$

where I have used Eq. (A.3) in 0806.1519. Then one can rewrite Eq. (109) in the form

$$\mathcal{V}(x,\bar{\theta},\theta) = 2i\theta\bar{\theta}\sigma(x) + 2\theta\gamma^\mu\bar{\theta}A_\mu(x) + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi}(x) - \sqrt{2}i\bar{\theta}^2\theta\chi(x) + \theta^2\bar{\theta}^2\Big(D(x) - i\partial_\mu A^\mu(y)\Big) \tag{119}$$

2.3.2.1 *Supersymmetric field strength in components*

Let's now calculate the supersymmetric field strength in components,

$$\begin{aligned}\mathcal{D}_\alpha\mathcal{V} =& 2\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta A_\mu(y) + 2i\bar{\theta}_\alpha\sigma(y) + 2\sqrt{2}i\theta_\alpha\bar{\theta}\bar{\chi}(y) - \sqrt{2}i\bar{\theta}^2\chi_\alpha(y) \\ &+ 2\theta_\alpha\bar{\theta}^2\big(D(y) - iA_\mu\big) \\ &+ 4i(\gamma^\mu\bar{\theta})_\alpha\theta\gamma^\nu\bar{\theta}\partial_\mu A_\nu(y) + (2i)*(2i)(\gamma^\mu\bar{\theta})_\alpha\theta\bar{\theta}\partial_\mu\sigma(y) \\ &+ (\sqrt{2}i)*(2i)(\gamma^\mu\bar{\theta})_\alpha\theta^2\bar{\theta}\partial_\mu\bar{\chi}(y) + 0.\end{aligned} \tag{120}$$

Then

$$\Sigma = \bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha\mathcal{V} = \epsilon^{\alpha\beta}\bar{\mathcal{D}}_\beta\mathcal{D}_\alpha\mathcal{V} = \epsilon^{\alpha\beta}\bar{\partial}_\beta(\mathcal{D}_\alpha\mathcal{V}). \tag{121}$$

Expand it all in components

$$\begin{aligned}\Sigma = \epsilon^{\alpha\beta}\{&2\gamma^\mu_{\alpha\omega}\delta^\omega_\beta A_\mu(y) + 2i\epsilon_{\alpha\omega}\delta^\omega_\beta\sigma(y) + 2\sqrt{2}i\theta_\alpha\bar{\chi}_\beta(y) \\ &- 2\sqrt{2}i\bar{\theta}_\beta\chi_\alpha(y) - 2\theta_\alpha 2\bar{\theta}_\beta\big(D(y) - i\partial_\mu A^\mu(y)\big) \\ &+ 4i[\gamma^\mu_{\alpha\omega}\delta^\omega_\beta\theta\gamma^\nu\bar{\theta}\partial_\mu A_\nu(y) + (\gamma^\mu\bar{\theta})_\alpha\theta^\epsilon\gamma^\nu_{\epsilon\omega}\delta^\omega_\beta\partial_\mu A_\nu(y)] \\ &+ (-4)[\gamma^\mu_{\alpha\omega}\delta^\omega_\beta\theta\bar{\theta}\partial_\mu\sigma(y) + (\gamma^\mu\bar{\theta})_\alpha(-\theta_\beta)\partial_\mu\sigma(y)] \\ &+ (-2\sqrt{2})[\gamma^\mu_{\alpha\omega}\delta^\omega_\beta\theta^2\bar{\theta}\partial_\mu\bar{\chi}(y) + (\gamma^\mu\bar{\theta})_\alpha\theta^2(-1)\partial_\mu\bar{\chi}_\beta(y)]\},\end{aligned} \tag{122}$$

where the blue terms give us zero individually. Collect all the term

$$\begin{aligned}\Sigma =& -4i\sigma(y) + (-2\sqrt{2}i)\theta\bar{\chi}(y) - 2\sqrt{2}i\chi\bar{\theta} + 4\theta\bar{\theta}[D(y) - i\partial_\mu A^\mu(y)] \\ & + 4i\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha(\theta\gamma^\nu)_\beta\partial_\mu A_\nu(y) + 4\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha\theta_\beta\partial_\mu\sigma(y) \\ & + 2\sqrt{2}\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha\theta^2\partial_\mu\bar{\chi}_\beta(y).\end{aligned} \tag{123}$$

One can combine the green term using the property $\gamma^\mu\gamma^\nu - \gamma^{\mu\nu} = \eta^{\mu\nu}$,

$$\begin{aligned}& -4i\theta\bar{\theta}\partial_\mu A^\mu(y) + 4i\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha(\theta\gamma^\nu)_\beta\partial_\mu A_\nu(y) \\ & = -4i\theta\bar{\theta}\partial_\mu A^\mu(y) + 4i\bar{\theta}^a\theta^b(\gamma^\nu)_b{}^\alpha(\gamma^\mu)_\alpha{}^\beta\epsilon_{a\beta}\partial_\mu A_\nu \\ & = -4i\theta\bar{\theta}\partial_\mu A^\mu(y) + 4i\theta\gamma^\nu\gamma^\mu\bar{\theta}\partial_\mu A_\nu \\ & = 4i\theta^b(\gamma^{\nu\mu})_b{}^\beta\bar{\theta}_\beta\partial_\mu A_\nu = -2i\theta\gamma^{\mu\nu}\bar{\theta}F_{\mu\nu}\end{aligned} \tag{124}$$

So one can write the supersymmetric field strength in a more compact form

$$\begin{aligned}\Sigma =& -4i\sigma(y) - 2\sqrt{2}i\theta\bar{\chi}(y) - 2\sqrt{2}i\chi\bar{\theta} + 4\theta\bar{\theta}D(y) \\ & + 4\epsilon^{\alpha\beta}(\gamma^\mu)_{\alpha\omega}\bar{\theta}^\omega\theta_\beta\partial_\mu\sigma(y) + 2\sqrt{2}\epsilon^{\alpha\beta}(\gamma^\mu)_{\alpha\omega}\bar{\theta}^\omega\theta^2\partial_\mu\bar{\chi}_\beta \\ & - 2i\theta\gamma^{\mu\nu}\bar{\theta}F_{\mu\nu} \\ =& -4i\sigma(y) - 2\sqrt{2}i\theta\bar{\chi}(y) - 2\sqrt{2}i\chi\bar{\theta} + 4\theta\bar{\theta}D(y) \\ & - 4\theta\gamma^\mu\bar{\theta}\partial_\mu\sigma - 2\sqrt{2}(\partial_\mu\bar{\chi}\gamma^\mu\bar{\theta})\theta^2 - 2i\theta\gamma^{\mu\nu}\bar{\theta}F_{\mu\nu}\end{aligned} \tag{125}$$

2.3.2.2 *Superspace integration*

First let's get the integrand ($\theta^2\bar{\theta}^2$ term), we will use the Σ in Eq. (123)

$$\begin{aligned}V\Sigma =& \theta^2\bar{\theta}^2(D(y) - i\partial_\mu A^\mu)(-4i\sigma(y)) + (-2\sqrt{2}i)\theta\bar{\chi}(y)(-\sqrt{2}i\bar{\theta}^2)(\theta\chi(y)) \\ & - 2\sqrt{2}i\chi(y)\bar{\theta}(\sqrt{2}i\theta^2\bar{\theta}\bar{\chi}) + 4\theta\bar{\theta}[D(y) - i\partial_\mu A^\mu]\big(2i\theta\bar{\theta}\sigma(y) + 2\theta\gamma^\nu\bar{\theta}A_\nu\big) \\ & + 4i\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha(\theta\gamma^\nu)_\beta\partial_\mu A_\nu\big(2i\theta\bar{\theta}\sigma(y) + 2\theta\gamma^\omega\bar{\theta}A_\omega\big) \\ & + 4\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha\theta_\beta\partial_\mu\sigma(y)\big(2i\theta\bar{\theta}\sigma(y) + 2\theta\gamma^\omega\bar{\theta}A_\omega(y)\big) \\ & + 0\end{aligned} \tag{126}$$

up to a total derivative, one can replace the superspace integration with derivative:

$$\int d^4\theta\cdots = \frac{1}{16}(\mathcal{D}^2\bar{\mathcal{D}}^2\ldots)\big|_{\theta=\bar{\theta}=0}. \tag{127}$$

I will show one of the derivative the red one, the rest will be the same:

$$
\begin{aligned}
\bar{\mathcal{D}}^a\bar{\mathcal{D}}_a(red) =&\#\bar{D}^a\bar{\mathcal{D}}_a[\epsilon^{\alpha\beta}(\gamma^\mu\bar{\theta})_\alpha(\theta\gamma^\nu)_\beta\partial_\mu A_\nu(y)\big(\theta\gamma^\omega\bar{\theta}A_\omega(y)\big)] \\
=&\bar{D}^a\bar{\mathcal{D}}_a[\epsilon^{\alpha\beta}\gamma^\mu_{\alpha b}\bar{\theta}^b\theta^c\gamma^\nu_{c\beta}\partial_\mu A_\nu(y)\theta^d\gamma^\omega_{de}\bar{\theta}^e A_\omega(y)] \\
=&\epsilon^{\alpha\beta}\gamma^\mu_{\alpha b}\delta^b_a\theta^c\gamma^\nu_{c\beta}\partial_\mu A_\nu(y)\theta^d\gamma^\omega_{de}\epsilon^{ef}(-\delta^a_f)A_\omega(y) \\
&-\epsilon^{\alpha\beta}\gamma^\mu_{\alpha b}\epsilon^{af}\delta^b_f\theta^c\gamma^\nu_{c\beta}\partial_\mu A_\nu(y)\theta^d\gamma^\omega_{de}(\delta^e_a)A_\omega(y) \\
=&-2\epsilon^{\alpha\beta}\epsilon^{ab}\gamma^\mu_{\alpha b}\gamma^\nu_{c\beta}\gamma^\omega_{da}\theta^c\theta^d\partial_\mu A_\nu(y)A_\omega(y)
\end{aligned}
\tag{128}
$$

then

$$
\begin{aligned}
\mathcal{D}^e\mathcal{D}_e\bar{\mathcal{D}}^a\bar{\mathcal{D}}_a(red) =&-2\epsilon^{\alpha\beta}\epsilon^{ab}\gamma^\mu_{\alpha b}\gamma^\nu_{c\beta}\gamma^\omega_{da}\delta^c_e\epsilon^{ef}\delta^d_f\partial_\mu A_\nu(y)A_\omega(y) \\
&+2\epsilon^{\alpha\beta}\epsilon^{ab}\gamma^\mu_{\alpha b}\gamma^\nu_{c\beta}\gamma^\omega_{da}\epsilon^{ef}\delta^c_f\delta^d_e\partial_\mu A_\nu(y)A_\omega(y) \\
=&-4\epsilon^{\alpha\beta}\epsilon^{ab}\gamma^\mu_{\alpha b}\gamma^\nu_{c\beta}\gamma^\omega_{da}\epsilon^{cd}\partial_\mu A_\nu A_\omega \\
=&-4\ \mathrm{tr}\ (\gamma^\nu\gamma^\mu\gamma^\omega)\partial_\mu A_\nu A_\omega \\
=&8\epsilon^{\mu\nu\omega}\partial_\mu A_\nu A_\omega
\end{aligned}
\tag{129}
$$

The rest can be derived in a similar way, notice one can also use the Fierz identities to simplify the supercoordinates. After a long tedious calculation, the final result yields

$$
\begin{aligned}
\int d^4\theta V\Sigma =& \int d^4\theta\theta^2\bar{\theta}^2(-8i)[D(y)-i\partial_\mu A^\mu]\sigma(y) \\
&+\frac{1}{16}[-4\times 2\times 8\bar{\chi}\chi]+\frac{1}{16}[4i\times 2\times 8\epsilon^{\mu\nu\omega}\partial_\mu A_\nu A_\omega(y)] \\
&+\frac{1}{16}[4\times 2\times 8\partial_\mu\sigma(y)A^\mu] \\
=&-8iD\sigma-4\chi\bar{\chi}+4i\epsilon^{\mu\nu\omega}\partial_\mu A_\nu A_\omega+4\partial_\mu\sigma(y)A^\mu-8\partial_\mu A^\mu\sigma(y)
\end{aligned}
\tag{130}
$$

If we neglect the blue term, then one can obtain

$$
S_{CS}\propto\frac{k}{4\pi}\int d^3x\big(\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+i\chi\bar{\chi}-2D\sigma\big). \tag{131}
$$

Where if we follow the normalization as in [51], one would get the correct pre-factor. If we instead use the Eq. (125), then the integrand becomes

$$\begin{aligned}V\Sigma =&\theta^2\bar{\theta}^2(D(y)-i\partial_\mu A^\mu)(-4i\sigma(y))+(-2\sqrt{2}i)\theta\bar{\chi}(y)(-\sqrt{2}i\bar{\theta}^2)(\theta\chi(y))\\&-2\sqrt{2}i\chi(y)\bar{\theta}(\sqrt{2}i\theta^2\bar{\theta}\bar{\chi})+4\theta\bar{\theta}D(y)\left(2i\theta\bar{\theta}\sigma(y)+2\theta\gamma^\nu\bar{\theta}A_\nu\right)\\&-4\theta\gamma^\mu\bar{\theta}\partial_\mu\sigma(y)[2i\theta\bar{\theta}\sigma(y)+2\theta\gamma^\omega\bar{\theta}A_\omega(y)]\\&-4i\theta\gamma^{\mu\nu}\bar{\theta}\partial_\mu A_\nu[2i\theta\bar{\theta}\sigma(y)+2\theta\gamma^\lambda\bar{\theta}A_\lambda(y)]\end{aligned}\tag{132}$$

use $(\theta\gamma^\mu\bar{\theta})(\theta\gamma^\nu\bar{\theta})=\frac{1}{2}\eta^{\mu\nu}\theta^2\bar{\theta}^2$, one can write the red term in a total derivative. Let's check the green term indeed vanishes:

$$\begin{aligned}\bar{D}^a\bar{D}_a(green)=&\#\bar{D}^a\bar{D}_a(\theta\gamma^{\mu\nu}\bar{\theta})\theta\bar{\theta}\\=&\frac{1}{2}\bar{D}^a\bar{D}_a\theta^\alpha(\gamma^\mu)_\alpha{}^\beta(\gamma^\nu)_\beta{}^\lambda\bar{\theta}_\lambda\theta^\sigma\bar{\theta}_\sigma-(\mu\leftrightarrow\nu)\\=&\frac{1}{2}\theta^\alpha(\gamma^\mu)_\alpha{}^\beta(\gamma^\nu)_\beta{}^\lambda\epsilon_{\lambda b}(-\delta^b_a)\theta^\sigma(-\delta^a_b)\\&+\frac{1}{2}\theta^\alpha(\gamma^\mu)_\alpha{}^\beta(\gamma^\nu)_\beta{}^\lambda(-\delta^a_\lambda)\theta^\sigma\epsilon_{\sigma c}(\delta^c_a)-(\mu\leftrightarrow\nu)\\=&\theta\gamma^\mu\gamma^\nu\theta-\theta\gamma^\mu\gamma^\nu\theta-(\mu\leftrightarrow\nu)\\=&0.\end{aligned}\tag{133}$$

So the Chern-Simons action is indeed correct in Eq. (131).

2.3.2.3 *The matter theory*

The gauged matter Lagrangian take the same form as in $\mathcal{N}=1$ SYM

$$\mathcal{L}_{matter}=\int d^4\theta\,\bar{\Phi}e^{\mathcal{V}}\Phi+\left(\int d^2\theta W(\Phi)+\int d^2\bar{\theta}\bar{W}(\bar{\Phi})\right)\tag{134}$$

In SYM, the first term takes the form $\bar{\Phi}e^{\mathcal{V}}\Phi=\bar{\Phi}\Phi+\bar{\Phi}\mathcal{V}\Phi+\frac{1}{2}\bar{\Phi}\mathcal{V}^2\Phi$, yields

$$\bar{\Phi}e^V\Phi\big|_{\theta^2\bar{\theta}^2}\propto(D_M\bar{\phi})D^M\phi-i\bar{\psi}\bar{\sigma}^M D_M\psi+\bar{F}F+\frac{i}{\sqrt{2}}\bar{\phi}\lambda\psi-\frac{i}{\sqrt{2}}\bar{\phi}\bar{\lambda}\psi+\frac{1}{2}\bar{\phi}D\phi\tag{135}$$

Where $D_M\Phi=\partial_M\Phi+iv_M\Phi$. Using our dictionary from the dimensional reduction of vector multiplet, one would guess in 3d Chern-Simons case, the first term would look like

$$\bar{\Phi}e^V\Phi\big|_{\theta^2\bar{\theta}^2}\propto(D_\mu\bar{\phi})D^\mu\phi-\bar{\phi}\sigma^2\phi+i\bar{\psi}\gamma^\mu D_\mu\psi-\bar{\psi}\sigma\psi+\bar{F}F+\frac{i}{\sqrt{2}}\bar{\phi}\lambda\psi-\frac{i}{\sqrt{2}}\bar{\phi}\bar{\lambda}\psi+\frac{1}{2}\bar{\phi}D\phi\tag{136}$$

The superpotential term $W(\Phi)$ must have the R charge 2 in order to Eq. (134) invariant under R symmetry. In the meantime, it needs to be of mass dim 3 to be marginal. After the superspace integration, it will give us the F term W_F and $\bar{W}_F$, it needs to be quartic for scale invariance, then they give terms like $\phi^2\psi^2$ and $\phi^3 F$.

2.3.3 *Non-Abelian superconformal Chern-Simons theories*

The nonabelian $\mathcal{N}=2$ Chern-Simons action looks quiet different from SYM, it takes to form [54]

$$S=\int d^3x\int d^4\theta\{\frac{k}{2\pi}\int_0^1 dt\,\mathrm{Tr}[\mathcal{V}\bar{\mathcal{D}}^\alpha(e^{t\mathcal{V}}\mathcal{D}_\alpha e^{-t\mathcal{V}})]+\bar{\Phi}e^{\mathcal{V}}\Phi+\int d^3x(\int d^2\theta W(\Phi)+\int d^2\bar{\theta}\bar{W}(\bar{\Phi}))\} \tag{137}$$

One would later see that the t parameter is just used to get $2/3$ in our non-Abelian case. Recall that the field strength transform as $F_{\mu\nu}\mapsto UF_{\mu\nu}U^{-1}$ under gauge transformation, let's check for supersymmetric field strength

$$\begin{aligned}\Sigma_{nAb}&=\bar{\mathcal{D}}^\alpha\int_0^1 dt\big(e^{t\mathcal{V}}\mathcal{D}_\alpha e^{-t\mathcal{V}}\big)\\ &\mapsto\bar{\mathcal{D}}^\alpha\int_0^1 dt\Big(e^{i\Lambda}e^{t\mathcal{V}}e^{-i\bar{\Lambda}}\mathcal{D}_\alpha[e^{i\bar{\Lambda}}e^{-t\mathcal{V}}e^{-i\Lambda}]\Big)\\ &\mapsto e^{i\Lambda}\Big(\bar{\mathcal{D}}^\alpha\int_0^1 dt\big(e^{t\mathcal{V}}\mathcal{D}_\alpha e^{-t\mathcal{V}}\big)\Big)e^{-i\Lambda}\end{aligned} \tag{138}$$

Here Σ_{nAb} is just a notation, doesn't stands for linear multiplet. Actually it doesn't give us the component that contain non-Abelian field strength since one cannot use $A\wedge F$ for in non-Abelian case. Anyway, let's calculate the components of Σ_{nAb}

$$\begin{aligned}\Sigma_{nAb}&=\bar{\mathcal{D}}^\alpha\Big(\int_0^1 dt(1+t\mathcal{V}+\frac{1}{2}t^2\mathcal{V}^2)\mathcal{D}_\alpha(1-t\mathcal{V}+\frac{1}{2}t^2\mathcal{V}^2)\Big)\\ &=\bar{\mathcal{D}}^\alpha\Big(\int_0^1 dt(-t\mathcal{D}_\alpha\mathcal{V}+\frac{1}{2}t^2\mathcal{D}_\alpha\mathcal{V}^2-t^2\mathcal{V}\mathcal{D}_\alpha\mathcal{V})\Big)\\ &=\bar{\mathcal{D}}^\alpha\Big(-\frac{1}{2}\mathcal{D}_\alpha\mathcal{V}+\frac{1}{6}\mathcal{D}_\alpha\mathcal{V}^2-\frac{1}{3}\mathcal{V}\mathcal{D}_\alpha\mathcal{V})\Big)\\ &=-\frac{1}{2}\bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha\mathcal{V}+\frac{1}{6}\bar{\mathcal{D}}^\alpha[\mathcal{D}_\alpha\mathcal{V},\mathcal{V}]\end{aligned} \tag{139}$$

The first term takes the same form as in Abelian case. Let's calculate the second term

$$
\begin{aligned}
\mathcal{D}_\alpha \mathcal{V} \cdot \mathcal{V} = & 2\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta A_\mu(y)\big(2\theta\gamma^\nu\bar{\theta}A_\nu(y) + 2i\theta\bar{\theta}\sigma(y) + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi}(y)\big) \\
& + 2i\bar{\theta}_\alpha\sigma(y)\big(2\theta\gamma^\nu\bar{\theta}A_\nu(y) + 2i\theta\bar{\theta}\sigma(y) + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi}(y)\big) \\
& + 2\sqrt{2}i\theta_\alpha\bar{\theta}\bar{\chi}(y)\big(2\theta\gamma^\nu\bar{\theta}A_\nu(y) + 2i\theta\bar{\theta}\sigma(y)\big)
\end{aligned} \tag{140}
$$

so the anti-commutator will give us

$$
\begin{aligned}
[\mathcal{D}_\alpha \mathcal{V}, \mathcal{V}] = & 4(\gamma^\mu\bar{\theta})_\alpha\theta\gamma^\nu\bar{\theta}[A_\mu, A_\nu] + 4\sqrt{2}i\theta_\alpha\theta\gamma^\mu\bar{\theta}\bar{\theta}^\beta[\bar{\chi}_\beta, A_\mu] \\
& + 2\sqrt{2}\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\theta^2\bar{\theta}^\omega[A_\mu, \bar{\chi}_\omega] + 4i\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\theta\bar{\theta}[A_\mu, \sigma(y)] \\
& + 4i\bar{\theta}_\alpha\theta\gamma^\mu\bar{\theta}[\sigma(y), A_\mu(y)] - 2\sqrt{2}\theta^2\bar{\theta}_\alpha\bar{\theta}^\beta[\sigma(y), \bar{\chi}_\beta(y)] \\
& - 4\sqrt{2}\theta_\alpha\theta\bar{\theta}\bar{\theta}^\beta[\bar{\chi}_\beta, \sigma(y)]
\end{aligned} \tag{141}
$$

where the blue term won't contribute to the action after the superspace integration. Now take the second derivative, one would get

$$
\begin{aligned}
I = & \bar{\mathcal{D}}^\alpha[\mathcal{D}_\alpha \mathcal{V}, \mathcal{V}] \\
= & 4\gamma^\mu_{\alpha\beta}\epsilon^{\alpha\beta}\theta\gamma^\nu\bar{\theta}[A_\mu, A_\nu] + 4\epsilon^{\alpha\lambda}\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\theta^\omega(\gamma^\nu)_{\omega\lambda}[A_\mu, A_\nu] \\
& + 4\sqrt{2}i\theta\gamma^\mu\theta\bar{\theta}[\bar{\chi}, A_\mu] + 4\sqrt{2}i\theta\gamma^\mu\bar{\theta}\theta[\bar{\chi}, A_\mu] \\
& + 2\sqrt{2}i\theta^2[A_\mu, \bar{\chi}\gamma^\mu\bar{\theta}] + 2\sqrt{2}i\epsilon^{\alpha\rho}\gamma_{\alpha\rho}(\dots) \\
& + 16i\theta\gamma^\mu\bar{\theta}[A_\mu, \sigma(y)] \\
= & 0 - 4\theta\gamma^{\mu\nu}\bar{\theta}[A_\mu, A_\nu] + 4\sqrt{2}i\theta\gamma^\mu\theta\bar{\theta}[\bar{\chi}, A_\mu] + 4\sqrt{2}i\theta\gamma^\mu\bar{\theta}\theta[\bar{\chi}, A_\mu] \\
& + 2\sqrt{2}i\theta^2[A_\mu, \bar{\chi}\gamma^\mu\bar{\theta}] + 0 + 16i\theta\gamma^\mu\bar{\theta}[A_\mu, \sigma(y)]
\end{aligned} \tag{142}
$$

So

$$
\begin{aligned}
\Sigma_{nAb} =& -\frac{1}{2}\Sigma + \frac{1}{6}I \\
=& \Big(2i\sigma(y) + \sqrt{2}i\theta\bar{\chi}(y) + \sqrt{2}i\chi\bar{\theta} - 2\theta\bar{\theta}D(y) \\
&+ 2\theta\gamma^{\mu}\bar{\theta}\partial_{\mu}\sigma(y) + \sqrt{2}\theta^2(\partial_{\mu}\bar{\chi}\gamma^{\mu}\bar{\theta}) + 2i\theta\gamma^{\mu\nu}\bar{\theta}\partial_{\mu}A_{\nu}\Big) \\
&+ \Big(-\frac{2}{3}\theta\gamma^{\mu\nu}\bar{\theta}[A_{\mu}, A_{\nu}] + \frac{4}{6}\sqrt{2}i\theta\gamma^{\mu}\theta\bar{\theta}[\bar{\chi}, A_{\mu}] \\
&+ \frac{4}{6}\sqrt{2}i\theta\gamma^{\mu}\bar{\theta}\theta[\bar{\chi}, A_{\mu}] + \frac{2}{6}\sqrt{2}i\theta^2[A_{\mu}, \bar{\chi}\gamma^{\mu}\bar{\theta}] + \frac{16}{6}i\theta\gamma^{\mu}\bar{\theta}[A_{\mu}, \sigma(y)]\Big) \\
=& 2i\sigma(y) + \sqrt{2}i\theta\bar{\chi}(y) + \sqrt{2}i\chi\bar{\theta} - 2\theta\bar{\theta}D(y) \\
&+ 2\theta\gamma^{\mu}\bar{\theta}\partial_{\mu}\sigma(y) + {\color{blue}\frac{8}{3}i\theta\gamma^{\mu}\bar{\theta}[A_{\mu}, \sigma(y)]} + {\color{red}\sqrt{2}\theta^2\big(\partial_{\mu}\bar{\chi}\gamma^{\mu}\bar{\theta} + \frac{i}{3}[A_{\mu}, \bar{\chi}\gamma^{\mu}\bar{\theta}]\big)} \\
&+ i\theta\gamma^{\mu\nu}\bar{\theta}\big(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + \frac{2i}{3}[A_{\mu}, A_{\nu}]\big)
\end{aligned}
\tag{143}
$$

where the blue term vanishes after multiplying with $\mathcal{V}$ due to identities

$$
\begin{aligned}
\theta\gamma^{\mu}\bar{\theta}\theta^2 &= 0, \\
\theta\gamma^{\mu}\bar{\theta}\theta\gamma^{\nu}\bar{\theta}\ \mathrm{Tr}(A_{\mu}[A_{\nu}, \sigma(y)]) &= 0.
\end{aligned}
\tag{144}
$$

The red vanishes identically after multiplying with $\mathcal{V}$. Compare Eq. (143) with Eq. (125), one would expect that

$$
S \propto \int \frac{k}{4\pi}\mathrm{Tr}\big(A \wedge dA + \frac{2i}{3}A^3 + i\bar{\chi}\chi - 2D\sigma(y)\big) \tag{145}
$$

3

ABJM THEORY AND ITS DEFECT VERSION

In this Chapter we will discuss an example of AdS/CFT duality involving a three-dimensional superconformal gauge theory with $\mathcal{N} = 6$ supersymmetry, known as the ABJM theory. We will first discuss the $\mathcal{N} = 2$ superspace formulation of the $\mathcal{N} = 8$ superconformal Bagger-Lambert-Gustavsson theory, and of the $\mathcal{N} = 6$ superconformal Aharony-Bergman-Jafferis-Maldacena $U(N) \times U(N)$ Chern-Simons theory. Then we will touch upon the planar limit of ABJM theory, and layout the integrability of it. In the end, we will talk about the $1/2$ BPS domain wall version of the ABJM theory, and how to relate the integrable boundary to the integrable states.

3.1 ADS/CFT

To begin with, we give a short review of the AdS/CFT correspondence. AdS/CFT correspondence is a realization of the holographic principle. This principle goes back to an observation by 't Hooft that the large N limit of a gauge theory looks like a string theory by matching the corresponding partition function [56]. Specifically, suppose one calculates vacuum diagrams in an ordinary $U(N)$ gauge theory, where we take N to be large. Then one can naturally arrange the expansion in powers of $1/N$ in terms of diagrams of different Euler characteristic, $\chi = V - E + F$ with E is the number of propagators, V is the number of vertices, F is the number of faces. Planar diagrams have a contribution scaling as N^2, genus one diagrams as N^0, etc.. In string theory, there also exists a sum over Riemann surfaces, and the higher genus diagrams are suppressed by the powers of the string coupling g_s^2. This suggests there is a correspondence between string theories and gauge theories, with $g_s \sim 1/N$, such that large N limit corresponds to a weakly coupled string theory. Insertion of source in the gauge theory would map to the insertion of vertex operators in the string theory. This idea was made precise after the discovery of D-branes in supersymmetric string theory. Consider N coincident D_3-branes (in type IIB string theory). At weak coupling $g_s N \ll 1$, the branes live on the flat 10-dimensional spacetime and we have open string ending on the D-branes as well as closed strings propagating in the bulk. The effective field theory for the massless excitations of this system is a supersymmetric Yang-Mills theory. In the case of N D_3 branes in flat space, one obtains maximally symmetric $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$. On the other hand, at strong coupling $g_s N \gg 1$, the branes curve the spacetime substantially, sourcing the extremal black 3-brane geometry:

$$ds^2 = f(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + f(r)^{1/2}(dr^2 + r^2 d\Omega_5^2), \quad f(r) = 1 + \frac{4\pi g_s N \ell_s^4}{r^4} \tag{146}$$

where x^μ denote the 4 coordinates along the D_3-brane worldvolume and $d\Omega_5^2$ is the metric of a unit S^5. The solution is supported by a self-dual 5-form field strength, which has flux on the S^5. Since the gauge theory is well-defined at any coupling, it is natural to conjecture that this description in fact applies even when $g_s N$ is large, i.e. in the same regime as where the closed string description holds. This leads to Maldacena to conjecture that type IIB string theory on a space that is asymptotically $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory [1].

3.2 ONE M_2-BRANE AND ABELIAN CHERN-SIMONS THEORY

Recall previously we have already laid out the Chern-Simons action for theories with a single $U(1)$ gauge group. Here we will give the explicit action for it again in the superspace formalism [54]

$$S = \int d^3x \int d^4\theta \Big(\frac{k}{4\pi} \mathcal{V}\Sigma + \sum_{i=1}^{N_f} \Phi_i e^{q_i \mathcal{V}} \Phi_i \Big), \quad k \in \mathbb{Z} \tag{147}$$

where Φ_i are chiral matter superfields transforming under the gauge group with charge q_i, and where

$$\begin{aligned}\mathcal{V} &= 2i\theta\bar{\theta}\sigma + 2\theta\gamma^{\mu}\bar{\theta}A_{\mu} + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi} - \sqrt{2}i\bar{\theta}^2\theta\chi + \theta^2\bar{\theta}^2 D,\\ \Sigma &= \bar{D}^{\alpha}D_{\alpha}\mathcal{V}.\end{aligned} \tag{148}$$

The vector superfield $\mathcal{V}$ is composed of a gauge field A_{μ}, a two-component Dirac spinor χ, a scalar field σ, which comes from the A_3 component of the gauge field when we do dimensional reduction from the $3+1$ dimensional theory, and scalar field D. The parameter k is called the Chern-Simons level, which should be integer for gauge invariant theory. After we integrate out the superspace, the action can be written as:

$$\begin{aligned}S_{CS} &= \frac{k}{4\pi}\int d^3x\Big(\epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda} + i\bar{\chi}\chi - 2D\sigma\Big),\\ S_{matter} &= \int d^3x\sum_{i=1}^{N_f}\Big(-\mathcal{D}_{\mu}\phi_i^{\dagger}\mathcal{D}^{\mu}\phi_i - i\zeta_i^{\dagger}\mathcal{D}\zeta_i + q_i\phi_i^{\dagger}D\phi_i - q_i^2\phi_i^{\dagger}\sigma^2\phi_i\\ &\qquad - q_i\zeta_i^{\dagger}\sigma\zeta_i + iq_i\phi_i^{\dagger}\bar{\chi}\zeta_i - iq_i\zeta_i^{\dagger}\chi\phi_i\Big),\end{aligned} \tag{149}$$

where $\mathcal{D}_{\mu}$ represents the covariant derivative, and where ϕ_i and ζ_i represent the scalar and the fermionic part of the chiral matter field Φ_i respectively. Since the fields in the vector multiplet are non-dynamical, hence auxiliary field, we can choose which one to integrate out by imposing equation of motion (here we integrate out scalar field D). The theory we wish to pursue has similar structure as above gauge theory, which has gauge group $U(1)\times U(1)$, Chern-Simons levels $(k,-k)$ ensuring parity invariance and four chiral superfields transforming under these groups in bifundamental representation. The matter action for this theory in superspace formalism are

$$S_{matter} = \int d^3x\int d^4\theta\Big(\bar{\mathcal{Z}}_A e^{-\mathcal{V}}\mathcal{Z}^A e^{\hat{\mathcal{V}}} + \bar{\mathcal{W}}^B e^{-\hat{\mathcal{V}}}\mathcal{W}_B e^{\mathcal{V}}\Big), \quad A,B = 1,2, \tag{150}$$

where $\mathcal{Z}^A$ and $\mathcal{W}_B$ are chiral multiplets, whose lowest components are scalar fields that we denote as Z^A and W_B. Follow the same procedure, we integrate out the superspace, then we integrate out the equations of motion for the auxiliary fields. We find the D-term potential for this theory vanishes. Since the F-term potential vanishes for Abelian theory, we could say that the moduli space is just $\mathbb{C}^4$, but it is not the case since we are considering Chern-Simons theory. Combine the following fields [51]

$$Y^A = \{Z^A, W^{\dagger A}\}, \quad Y_A^{\dagger} = \{Z_A^{\dagger}, W_A\}, \tag{151}$$

the fields Y^A transform in the same way as Z^A under the gauge group. Do a redefinition $A^{\pm}_{\mu} = A_\mu \pm \hat{A}_\mu$, the gauge transformation becomes

$$A^{-}_{\mu} \to A^{-}_{\mu} + \partial_\mu \Lambda^-, \quad Y^A \to e^{i\Lambda^-} Y^A, \tag{152}$$

such transformation would introduce an extra boundary term of bosonic action that can be written as [10]

$$\delta S = \frac{k}{4\pi} \Lambda^- \int_{S^2} F^+, \tag{153}$$

with F^+ is a two form dA^+. Since the fluxes are quantized, to make the action gauge invariant, the following condition must be satisfied

$$\Lambda^- = \frac{2\pi l}{k}, \quad l \in \mathbb{Z}. \tag{154}$$

This gives an identification of the matter fields, indicating the moduli space for the theory is $\mathbb{C}^4/\mathbb{Z}_k$.

3.3 MULTI-BRANES AND NON-ABELIAN CHERN-SIMON THEORY

The non-Abelian Chern-Simons theory for multi-M_2 branes is found by Bagger, Lambert and Gustavsson with a novel algebra called "3-algebra" [5, 9]. Even though it possesses the maximal supersymmetry in $2+1$ dimension, it is a specific example of a Chern-Simons gauge theory with gauge group $SU(2) \times SU(2)$ with opposite Chern-Simons levels (here $\mathcal{L}_{matter}$ already contain F-term from superpotential):

$$\begin{aligned} \mathcal{L}_{CS} &= \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \mathrm{Tr}\big(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda\big), \\ \mathcal{L}_{matter} &= -(\mathcal{D}^\mu X^I)^\dagger \mathcal{D}_\mu X^I + i\bar{\Psi}^\dagger \Gamma^\mu \mathcal{D}_\mu \Psi - \frac{4i\pi}{k} \bar{\Psi}^\dagger \Gamma^{IJ} \big(X^I X^{J\dagger} \Psi + X^J \Psi^\dagger X^I + \\ &\quad \Psi X^{I\dagger} X^J\big) - \frac{32\pi^2}{3k^2} \mathrm{Tr}\big(X^{[I} X^{\dagger J} X^{K]} X^{\dagger [K} X^J X^{\dagger I]}\big), \end{aligned} \tag{155}$$

where the covariant derivative is

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + i A_\mu X^I - i X^I \hat{A}_\mu, \quad I = 1, \ldots, 8. \tag{156}$$

The bifundamental scalars X^I satisfy the reality condition

$$X^* = -\varepsilon X \varepsilon, \tag{157}$$

where $\varepsilon = i\sigma_2$. The above condition only works for the gauge group $SU(2) \times SU(2)$, which is exactly why such formulation cannot be applied to general N. The Lagrangian in Eq. (155) can again be formulated in the $\mathcal{N} = 2$ superspace. The gauge fields A and $\hat{A}$ is the components of two gauge vector superfields $\mathcal{V}$ and $\hat{V}$, their component expansions in Wess-Zumino gauge are

$$\mathcal{V} = 2i\theta\bar{\theta}\sigma + 2\theta\gamma^\mu\bar{\theta}A_\mu + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi} - \sqrt{2}i\bar{\theta}^2\theta\chi + \theta^2\bar{\theta}^2 D, \tag{158}$$

and corresponding $\hat{\mathcal{V}}$. The matter fields will appear in chiral superfields $\mathcal{Z}$ and anti-chiral superfields $\bar{\mathcal{Z}}$, which transform in fundamental and anti-fundamental representation of flavor group $SU(4)$. The components read

$$\begin{aligned} \mathcal{Z} &= Z + \sqrt{2}\theta\zeta + \theta^2 F, \\ \bar{\mathcal{Z}} &= Z^\dagger - \sqrt{2}\bar{\theta}\zeta^\dagger - \bar{\theta}^2 F^\dagger. \end{aligned} \tag{159}$$

The identifications between Z and the scalars in Eq. (156) are

$$Z^A = X^A + iX^{A+4}, \quad \text{for} \quad A = 1, \ldots, 4. \tag{160}$$

The superspace action $S = S_{CS} + S_{matter} + S_{sp}$ is made of a Chern-Simons action, a matter action and a superpotential action (not vanishing for non-Abelian case)

$$\begin{aligned} S_{CS} &= -i\frac{k}{8\pi}\int d^3x d^4\theta \int_0^1 dt \, \mathrm{Tr}[\mathcal{V}\bar{D}^\alpha(e^{t\mathcal{V}}D_\alpha e^{-t\mathcal{V}}) - \hat{\mathcal{V}}\bar{D}^\alpha(e^{t\hat{\mathcal{V}}}D_\alpha e^{-t\hat{\mathcal{V}}})], \\ S_{matter} &= -\int d^3x d^4\theta \, \mathrm{Tr}(\bar{\mathcal{Z}}_A e^{-\mathcal{V}}\mathcal{Z}^A e^{\hat{\mathcal{V}}}), \\ S_{sp} &= \frac{8\pi}{k}\int d^3x d^2\theta W(\mathcal{Z}) + \frac{8\pi}{k}\int d^3x d^2\bar{\theta}\bar{W}(\bar{\mathcal{Z}}), \end{aligned} \tag{161}$$

where the superpotential can be written in term of chiral multiplet $\mathcal{Z}_a^A$ transforming under the $SO(4)$ gauge group

$$W = -\frac{1}{8\cdot 4!}\epsilon_{ABCD}\epsilon^{abcd}\mathcal{Z}_a^A\mathcal{Z}_b^B\mathcal{Z}_c^C\mathcal{Z}_d^D, \tag{162}$$

and the anti-chiral one. After the integration of superspace and some manipulation of the auxiliary space, one would end up with the Lagrangian as stated in Eq. (155) by the identification Eq. (160).

To promote the above formalism to the general gauge group is not possible due the reality condition Eq. (157). Instead we do the following identification

$$Z^1 = X^1 + iX^5, \quad W_1 = X^{3\dagger} + iX^{7\dagger},$$
$$Z^2 = X^2 + iX^6, \quad W_2 = X^{4\dagger} + iX^{8\dagger}. \tag{163}$$

And assuming these field are components of chiral superfields, then the action for super-potential can be written as

$$S_{sp} = \frac{8\pi}{k} \int d^3x d^2\theta W(\mathcal{Z}, \mathcal{W}) + \frac{8\pi}{k} \int d^3x d^2\bar{\theta} \bar{W}(\bar{\mathcal{Z}}, \bar{\mathcal{W}}), \tag{164}$$

with

$$W = \frac{1}{4}\epsilon_{AC}\epsilon^{BD}\mathrm{Tr}\, \mathcal{Z}^A \mathcal{W}_B \mathcal{Z}^C \mathcal{W}_D, \quad \bar{W} = \frac{1}{4}\epsilon^{AC}\epsilon_{BD}\mathrm{Tr}\, \bar{\mathcal{Z}}^A \bar{\mathcal{W}}_B \bar{\mathcal{Z}}^C \bar{\mathcal{W}}_D. \tag{165}$$

The chiral superfield and anti-chiral superfield is exactly the same as in Eq. (159), for $\mathcal{W}$ and $\bar{\mathcal{W}}$ we have

$$\mathcal{W} = W + \sqrt{2}\theta\omega + \theta^2 G,$$
$$\bar{\mathcal{W}} = \bar{W} - \sqrt{2}\bar{\theta}\omega^\dagger - \bar{\theta}^2 G^\dagger. \tag{166}$$

The matter action will change to

$$S_{matter} = \int d^3x d^4\theta\, \mathrm{Tr}\Big(- \bar{\mathcal{Z}}_A e^{-\mathcal{V}} \mathcal{Z}^A e^{\hat{\mathcal{V}}} - \bar{\mathcal{W}}_A e^{-\hat{\mathcal{V}}} \mathcal{W}^A e^{\mathcal{V}} \Big). \tag{167}$$

After the integration over superspace, and imposing the E.O.M over auxiliary fields, one would end up with an action without manifest $SU(4)$ R symmetry. To restore the $SU(4)$ symmetry, or rather to say to make the $SU(4)$ symmetry manifest, we consider the following combinations

$$Y^A = \{Z^A, W^{\dagger A}\}, \quad Y_A^\dagger = \{Z_A^\dagger, W_A\}, \quad A = 1, \ldots, 4, \tag{168}$$

as well as the fermions

$$\psi_A = \{\epsilon_{AB}\zeta^B e^{-i\pi/4}, -\epsilon_{AB}\omega^{\dagger B} e^{i\pi/4}\}, \quad \psi^{A\dagger} = \{-\epsilon^{AB}\zeta_B^\dagger e^{i\pi/4}, \epsilon^{AB}\omega_B e^{-i\pi/4}\}, \tag{169}$$

we can write the potential as

$$
\begin{aligned}
V^{bos} &= -\frac{4\pi^2}{3k^2}\mathrm{Tr}\,[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\
&\quad + 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger] \\
V^{ferm} &= \frac{2i\pi}{k}\mathrm{Tr}\,[Y_A^\dagger Y^A \psi^{B\dagger}\psi_B - Y^A Y_A^\dagger \psi_B \psi^{B\dagger} + 2Y^A Y_B^\dagger \psi_A \psi^{B\dagger} - 2Y_A^\dagger Y^B \psi^{A\dagger}\psi_B \\
&\quad - \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon_{ABCD} Y^A \psi^{B\dagger} Y^C \psi^{D\dagger}].
\end{aligned}
\tag{170}
$$

And the full action reads as following

$$
\begin{aligned}
S = \int d^3x\ \mathrm{Tr}\Big[\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\big(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\big) \\
- \mathcal{D}_\mu Y_A^\dagger \mathcal{D}^\mu Y^A - i\psi^{A\dagger}\mathcal{D}\psi_A \\
- V^{bos} - V^{ferm}\Big].
\end{aligned}
\tag{171}
$$

As we have already mention in the Abelian theory that the moduli space is actually the $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$ with Chern-Simons levels $(k, -k)$. The corresponding supersymmetry transformation can be found in [57], we will layout here as

$$
\begin{aligned}
\delta Y^A &= i\omega^{AB}\psi_B, \\
\delta Y_A^\dagger &= i\psi^{\dagger B}\omega_{AB}, \\
\delta\psi_A &= -\gamma_\mu\omega_{AB}\mathcal{D}_\mu Y^B + \frac{2\pi}{k}\big(-\omega_{AB}(Y^C Y_C^\dagger Y^B - Y^B Y_C^\dagger Y^C) + 2\omega_{CD} Y^C Y_A^\dagger Y^D\big), \\
\delta\psi^{A\dagger} &= \mathcal{D}_\mu Y_B^\dagger \omega^{AB}\gamma_\mu + \frac{2\pi}{k}\big(-(Y_B^\dagger Y^C Y_C^\dagger - Y_C^\dagger Y^C Y_C^\dagger)\omega^{AB} + 2Y_D^\dagger Y^A Y_C^\dagger \omega^{CD}\big), \\
\delta A_\mu &= \frac{\pi}{k}(-Y^A\psi^{B\dagger}\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A Y_B^\dagger), \\
\delta\hat{A}_\mu &= \frac{\pi}{k}(-\psi^{A\dagger}Y^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu Y_A^\dagger\psi_B),
\end{aligned}
\tag{172}
$$

where ψ and ω_{AB} have lower spinor indices, while $\psi^\dagger$ and ω^{AB} have upper spinor indices. The 6 Majorana $(2+1)$-dimensional spinors, ϵ_i $(i = 1, \dots, 6)$, which are the $\mathcal{N} = 6$ SUSY generators, the ω_{AB} is given by

$$
\begin{aligned}
\omega_{AB} &= \epsilon_i(\Gamma^i)_{AB}, \\
\omega^{AB} &= \epsilon_i((\Gamma^i)^*)^{AB},
\end{aligned}
\tag{173}
$$

in which the A, B indices are anti-symmetric and the 4 by 4 matrices Γ^i is the following:

$$\begin{aligned} &\Gamma^1 = \sigma_2 \otimes \mathbb{I}_2, \quad \Gamma^4 = \sigma_1 \otimes \sigma_2, \\ &\Gamma^2 = -i\sigma_2 \otimes \sigma_3, \quad \Gamma^5 = \sigma_3 \otimes \sigma_2, \\ &\Gamma^3 = i\sigma_2 \otimes \sigma_1, \quad \Gamma^6 = -i\mathbb{I}_2 \otimes \sigma_2, \end{aligned} \tag{174}$$

with the following properties

$$\begin{aligned} &\{\Gamma^i, \Gamma^{j\dagger}\} = 2\delta_{ij}, \quad (\Gamma^i)_{AB} = -(\Gamma^i)_{BA}, \\ &\frac{1}{2}\epsilon^{ABCD}\Gamma^i_{CD} = -(\Gamma^{i\dagger})^{AB} = ((\Gamma^i)^*)^{AB}. \end{aligned} \tag{175}$$

ABJM theory has a gravitational dual by the gauge gravity duality. The corresponding gravitational dual can be described by a stack of coincident M_2 branes, the spacetime symmetry preserved by the existence of a electric-charged M_2-brane (or N of them coincidentally) would by

$$\Big(T^2 \otimes SO^+(1,2)\Big) \otimes SO(8), \tag{176}$$

where T^2 is the translational symmetry in two directions. One can assume the form of the spacetime metric taking a similar form as Reissner-Nordstrom metric:

$$ds^2 = f(r)\Big(-dt^2 + \sum_{i=1}^{2} dx_i^2\Big) + h(r)\Big(dr^2 + r^2 d\Omega_7^2\Big), \tag{177}$$

where $f(r)$ and $h(r)$ can be determined by the E.O.M of 11D supergravity, which takes the form

$$\begin{aligned} h^3(r) = f^{-3/2}(r) = 1 + \frac{L^6}{r^6}, \quad L^6 = 32\pi^2 N l_p^6 \\ F_4 = dC_3 = dt \wedge dx_1 \wedge dx_2 \wedge dh^{-3}(r), \end{aligned} \tag{178}$$

with a non-trivial 4-form field strength charged under C_3, where l_p is the 11D Planck length.

3.4 ABJM THEORY AND INTEGRABILITY

Soon after the formulation of ABJM theory, evidence emerged that ABJM in the planar (large N) limit is an integrable model. The discovery of planar integrability in ABJM opened the door to applying powerful tools like the Bethe ansatz to compute anomalous dimensions of operators. In this section, we will review the ingredients of the integrability of planar ABJM following the footsteps of Minahan and Zarembo [15, 16].

3.4.1 *ABJM theory and alternating spin chain*

The global symmetry group of ABJM theory, for $k > 2$, is given by the orthosymplectic supergroup $\mathrm{OSp}(6|4)$ and the "baryonic" $U(1)_b$ (unlike the case for $\mathcal{N} = 4$ super Yang-Mills theory, the $U(1)$ subgroup can decouple from $SU(N)$ group, for which all the fields are neutral, and it describes the center of mass of coincident N D-branes). The bosonic part of the global symmetry are the R-symmetry group $SO(6) \cong SU(4)_R$ and the 3d conformal group $Sp(4) \cong SO(2,3)$. The conformal group has subgroups the spacetime rotations $SO(1,2)$ and dilatations $U(1)_\Delta$. The fermionic part generates the $\mathcal{N} = 6$ supersymmetry transformations. Last but not least, the baryonic charge $U(1)_b$ is $+1$ for bi-fundamental fields, -1 for anti-fundamental fields, and 0 for adjoint fields. For later calculation convenience, instead of using fundamental and contravariant representation of $SL(2,\mathbb{R})$ to denote the fermions, we will choose to use the Dirac fermions. We choose a field normalization such that the Lagrangian for level k is just $\frac{k}{4\pi}$ multiplying level 1 Lagrangian [18, 58]

$$\begin{aligned}\mathcal{L} =& \frac{k}{4\pi}\mathrm{Tr}\big[\varepsilon^{\mu\nu\lambda}\Big(- A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda + \hat{A}_\mu\partial_\nu\hat{A}_\lambda + \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\Big) \\ & \mathcal{D}_\mu Y_A^\dagger \mathcal{D}^\mu Y^A + i\bar{\psi}^{A\dagger}\mathcal{D}\psi_A - V^{bos} - V^{ferm}\big],\end{aligned} \tag{179}$$

where the potential terms

$$\begin{aligned} V^{bos} =& -\frac{1}{12}\big(- Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger - Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\ & + 6Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger - 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger\big), \\ V^{ferm} =& \frac{1}{2}Y_A^\dagger Y^A \bar{\psi}^{B\dagger}\psi_B - \frac{1}{2}\bar{\psi}^{A\dagger}Y^B Y_B^\dagger \psi_A - Y_A^\dagger Y^B \bar{\psi}^{A\dagger}\psi_B \\ & + \bar{\psi}^{A\dagger}Y^B Y_A^\dagger\psi_B - \frac{i}{2}\epsilon^{ABCD}Y_A^\dagger\bar{\psi}_B Y_C^\dagger\psi_D + \frac{i}{2}\epsilon_{ABCD}Y^A\bar{\psi}^{B\dagger}Y^C\psi^{D\dagger},\end{aligned} \tag{180}$$

with $A = 1, 2, 3, 4$ are $SU(4)$ R-symmetry indices. A conformal field theory is completely determined by its two- and three-point correlators. One can choose a basis of local gauge invariant operators such that the two-point correlators takes the form

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(x)\rangle = \frac{\delta_{IJ}}{|x-y|^{2\Delta_I}}, \tag{181}$$

where Δ_I is the scaling dimension of operator $\mathcal{O}_I$. It can split into two parts

$$\Delta_I = \Delta_I^{(0)} + \gamma_I, \tag{182}$$

where $\Delta_I^{(0)}$ is the classical scaling dimension by purely dimension analysis, and γ_I is the anomalous dimension from quantum corrections. Additional conformal date like structure constant can be obtained from three-point functions, an example in the $SU(2) \times SU(2)$ sector can be found in [59].The operators $\mathcal{O}_I$ and their scaling dimensions can be thought of as the eigenstates and eigenvalues of an dilatation operator D, for which we only concern about the quantum correction part, denoted as Γ. If a CFT has a large-N expansion, then Γ has the following topological expansion [60]:

$$\Gamma(\lambda, 1/N) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{l=1} \lambda^l \Gamma_{l,g}, \tag{183}$$

where λ is the 't Hooft coupling, g is the genus number. If we focus on the weak coupling limit $\lambda \ll 1$, then Γ can be computed perturbatively by computing the renormalization matrix Z which cancels the the divergences of the two-point correlators, then taking the logarithmic derivative of Z with respect to the renormalization scale. In the large N limit, one the $g = 0$ term in Eq. (183) survives, which is known as planar limit. For the calculation of anomalous dimension in non-planar limit, a study has been performed by [61]. Also in this limit, multi-trace operators decouple so one only considers single-trace operator. Fo As we will show below, such mixing matrix Γ can be identified with the Hamiltonian of a quantum spin chain. Since the matter fields in the ABJM theory are in the bifundamental representation of the gauge group $U(N) \times \hat{U}(N)$, gauge invariant operators are constructed by taking the trace of an even number of fields which alternate between the $(N, \bar{N})$ and

$(\bar{N}, N)$ representation. The simplest gauge invariant operators constructed out of such matter fields are *single-trace* operators, which in general take the following form:

$$\mathcal{O}^{I_1,\ldots,I_L}_{J_1,\ldots,J_L} = \mathrm{Tr}(Y^{I_1}\bar{Y}_{J_1}\ldots Y^{I_L}\bar{Y}_{J_L}). \tag{184}$$

To find the anomalous dimension, we first calculate the two-point function made of the above single trace operators at two-loop order, and then extract the logarithmically divergent pieces from it. The divergences due to quantum corrections require the renormalization [62]

$$\mathcal{O}^{ren}_a = \mathcal{Z}^b_a(\lambda, \Lambda)\mathcal{O}^{bare}_b, \tag{185}$$

Λ is a renormalization scale with dimension of mass. The matrix $\mathcal{Z}^b_a$ cancels the divergences and implies there is a mixing of bare operators. The operator mixing matrix Γ can be viewed as the matrix representation of the dilation operator, which is given by

$$\Gamma = \mathcal{Z}^{-1}\frac{d\mathcal{Z}}{d\ln\Lambda}. \tag{186}$$

The dilatation operator in basis Eq. (184) up to two-loop calculation in the scalar sector is given by [15]

$$\Gamma = \frac{\lambda^2}{2}\sum_{l=1}^{2L}\left(2 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}\right), \tag{187}$$

where P_{ab} and K_{ab} are permutation and trace operators acting on the a-th and b-th sites. The above dilatation operator Γ is nothing but the Hamiltonian of $SU(4)$ alternating spin chain. The odd and even sites sit in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representation of $SU(4)$ group, correspond to fields Y^A and $Y^\dagger_A$. The corresponding vacuum of the dilatation operator is the operator

$$\mathcal{O}_{vac} = \mathrm{Tr}(Y^1 Y^\dagger_4)^L. \tag{188}$$

The spectrum of Γ can be solved by the Bethe ansatz, and the Bethe states are described by two sets of momentum carrying Bethe roots $\mathbf{u}$ and $\mathbf{v}$, as well as auxiliary Bethe roots $\mathbf{w}$. The number of rapidities of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are denoted by $K_{\mathbf{u}}$, $K_{\mathbf{v}}$, $K_{\mathbf{w}}$ respectively. The rapidities satisfy the Bethe equations which can be obtained by nested Algebraic Bethe ansatz [63]

$$\begin{aligned}
1 &= \left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L \prod_{k=1,\neq j}^{K_{\mathbf{u}}} S(u_j,u_k) \prod_{k=1}^{K_{\mathbf{w}}} \tilde{S}(u_j,w_k),\\
1 &= \prod_{k=1,\neq j}^{K_{\mathbf{w}}} S(w_j,w_k) \prod_{k=1}^{K_{\mathbf{u}}} \tilde{S}(w_j,u_k) \prod_{k=1}^{K_{\mathbf{v}}} \tilde{S}(w_j,v_k),\\
1 &= \left(\frac{v_j+\frac{i}{2}}{v_j-\frac{i}{2}}\right)^L \prod_{k=1,\neq j}^{K_{\mathbf{v}}} S(v_j,v_k) \prod_{k=1}^{K_{\mathbf{w}}} \tilde{S}(v_j,w_k),
\end{aligned} \tag{189}$$

where the S-matrix $S(u,v)$ and $\tilde{S}(u,v)$ are given by

$$S(u,v) = \frac{u-v-i}{u-v+i}, \quad \tilde{S}(u,v) = \frac{u-v+\frac{i}{2}}{u-v-\frac{i}{2}}. \tag{190}$$

And the two-loop anomalous dimension of the operator like Eq. (184) is given by

$$\gamma = \lambda^2\Big(\sum_{k=1}^{K_{\mathbf{u}}} \frac{1}{u_k^2+\frac{1}{4}} + \sum_{k=1}^{K_{\mathbf{v}}} \frac{1}{v_k^2+\frac{1}{4}}\Big). \tag{191}$$

3.4.2 *Domain wall in ABJM theory and one-point functions*

In the presence of defect, the one-point functions are not vanishing due to the breaking of translational symmetry. As in $\mathcal{N}=4$ super Yang-Mills theory, we consider a co-dimensional one defect (which turns out to be integrable), in this setup the one-point functions are fixed up to a constant C [64]

$$\langle \mathcal{O}_I(x)\rangle = \frac{C_I}{z^{\Delta_I}}, \tag{192}$$

where z denotes the distance from x to the interface. We will discuss a defect version of ABJM theory with a co-dimension 1 defect, which is built by introducing a domain wall in the theory. This defect ABJM theory is holographic dual to a string theory with D_2-D_4 probe brane system. In analogy to super Yang-Mills theory, the one-point functions one wish to calculate can be mapped to the overlap between certain integrable boundary states and Bethe eigenstates.

3.4.2.1 1/2-*BPS solutions as integrable domain wall*

The corresponding BPS condition can be got by demanding half of the supersymmetry transformation of fermions vanishing, i.e. $\delta\psi_A = 0$. By further assuming that only Y^1 and Y^2 picking up vacuum expectation value (as expected from supersymmetry) will give us the correct BPS equation [57]. Here we choose a different but equivalent approach, we use the trick by Bogomol'nyi [34]. The energy density can be reformulated by a total square plus a total derivative term, then by minimizing the energy we end up with a set of solution

$$\frac{dY^\alpha}{dx} = \frac{1}{2}Y^\alpha Y^\dagger_\beta Y^\beta - \frac{1}{2}Y^\beta Y^\dagger_\beta Y^\alpha, \quad \alpha = 1,2, \tag{193}$$

where x denotes x^2, which is the coordinate transverse to the domain wall. The solution for the half-space $x > 0$ takes the form

$$Y^\alpha(x) = \frac{y^\alpha}{\sqrt{x}}, \quad x > 0, \tag{194}$$

where y^α are given by

$$(y^1)_{ij} = \delta_{i,j-1}\sqrt{i}, \quad (y^2)_{ij} = \delta_{i,j}\sqrt{N-i}, \quad i = 1,\dots,q-1 \quad j = 1,\dots,q, \tag{195}$$

satisfying the following algebra (see also [65])

$$\begin{aligned} y^1 &= y^2 y_2^\dagger y^1 - y^1 y_2^\dagger y^2, \\ y^2 &= y^1 y_1^\dagger y^2 - y^2 y_1^\dagger y^1. \end{aligned} \tag{196}$$

Here we assuming $q \leq N$, the rest elements in Y^α are zeros, so we actually have the solution of the form

$$Y^\alpha_{cl}(z) = \frac{1}{\sqrt{x}} \begin{pmatrix} y^\alpha_{(q-1)\times q} & 0_{(q-1)\times(N-q)} \\ 0_{(N-q+1)\times q} & 0_{(N-q+1)\times(N-q)} \end{pmatrix}, \quad z > 0. \tag{197}$$

From the vacuum solutions, the gauge symmetry explicitly breaks down to $U(N-q+1)\times\hat{U}(N-q)$ and will restore to the full gauge symmetry as $x \to \infty$. And the global R-symmetry breaks from $SU(4)$ down to $SU(2)\times SU(2)\times U(1)$. Defining

$$\Phi^\alpha{}_\beta = y^\alpha y^\dagger_\beta, \tag{198}$$

one can derive the following commutation relations

$$[\Phi^{\alpha}{}_{\beta}, \Phi^{\gamma}{}_{\theta}] = \delta^{\alpha}_{\theta}\Phi^{\beta}{}_{\gamma} - \delta^{\beta}{}_{\gamma}\Phi^{\alpha}{}_{\theta}, \tag{199}$$

which is nothing but the commutation relation for $U(2)$. Forming the linear combination

$$t_i = \frac{1}{2}(\sigma_i)^{\alpha}{}_{\beta}\Phi^{\beta}{}_{\alpha}, \tag{200}$$

one can show that

$$[t_i, t_j] = i\varepsilon_{ijk}t_k, \tag{201}$$

which means that the t_i are $N \times N$ matrices with $\mathfrak{su}(2)$ generators of size $(q-1) \times (q-1)$ in their upper left hand corner. One can also define

$$\hat{\Phi}_{\alpha}^{\beta} = y^{\dagger}_{\alpha}y^{\beta}, \tag{202}$$

and find the similar commutation relations

$$[\hat{\Phi}_{\alpha}^{\beta}, \hat{\Phi}_{\gamma}^{\theta}] = -\delta_{\alpha}^{\theta}\hat{\Phi}_{\gamma}^{\beta} + \delta^{\beta}{}_{\gamma}\hat{\Phi}_{\alpha}^{\theta}. \tag{203}$$

Then the matrices $\hat{t}_i$ defined by

$$\hat{t}_i = -\frac{1}{2}(\sigma_i)_{\alpha}^{\beta}\hat{\Phi}_{\beta}^{\alpha}, \tag{204}$$

contain a q-dimensional representation of $\mathfrak{su}(2)$ in their upper left hand corner, with

$$[\hat{t}_i, \hat{t}_j] = i\varepsilon_{ijk}\hat{t}_k. \tag{205}$$

In the appendix we will see that the classical solutions Eq. (196) will serve as the basic ingredients for a deformed version fuzzy sphere basis.

3.4.2.2 *One-point functions from integrable boundary states*

In the presence of $1/2$-BPS domain wall, one-point functions Eq. (194) in ABJM theory at tree level can be calculated in a rather simple way. Directly plugging in the classical solutions, one would have the tree level expectation value

$$\langle \mathcal{O}(z) \rangle_{tree} = \frac{1}{x^L}\Psi^{J_1 \ldots J_L}_{I_1 \ldots I_L}\mathrm{Tr}\,(y^{I_1}y^{\dagger}_{J_1}y^{I_2}y^{\dagger}_{J_2}\ldots y^{I_L}y^{\dagger}_{J_L}), \tag{206}$$

where the coefficients $\Psi^{J_1\dots J_L}_{I_1\dots I_L}$ can be determined by the Bethe ansatz equations given $\mathcal{O}$ is an eigenstate of the dilation operator. It turns out the evaluation of Eq. (206) can be converted into an overlap between integrable boundary states (stands for the domain wall) and the eigenstates of the corresponding spin chain. The definition of integrable boundary states [25] share the same feature as in the case of defect $\mathcal{N} = 4$ super symmetric Yang-Mills theory, that is

$$Q_{2n+1}|\psi\rangle = 0, \quad n = 1, 2, \dots, \tag{207}$$

where Q_{2n+1} denotes the local conserved charge with interaction range $2n+1$. It turns out a special type of integrable boundary states, called matrix product states (MPS), fall into above definition, which is relevant to us. A generic periodic MPS can be defined as

$$|\mathrm{MPS}\rangle = \sum_{i_1,\dots,i_L}^{d} \mathrm{Tr}\,[A_1^{(i_1)} A_2^{(i_2)} \dots A_L^{(i_L)}]|i_1, i_2, \dots, i_L\rangle, \tag{208}$$

where d is the dimension of the physical space and $A_n^{(i_n)}$ are $d_{n-1} \times d_n$ dimensional matrices with d_n dubbed as bond dimensions. The corresponding MPS we are interested can be obtained by replacing matrices $A_n^{(i_n)}$ with the classical solution of the scalar fields, takes the form

$$|\mathrm{MPS}\rangle = \sum_{I_k, J_k=1}^{2} \mathrm{Tr}\,[y^{I_1} y^\dagger_{J_1} y^{I_2} y^\dagger_{J_2} \dots y^{I_L} y^\dagger_{J_L}]|I_1, J_1, \dots, I_L, J_L\rangle. \tag{209}$$

One can similar find that the Bethe states (eigenstate of the spin chain Hamiltonian) can be written as

$$|\mathbf{u}\rangle = \Psi^{J_1\dots J_L}_{I_1\dots I_L}|I_1, J_1, \dots, I_L, J_L\rangle, \tag{210}$$

where the coefficients $\Psi^{J_1\dots J_L}_{I_1\dots I_L}$ are same as before, can be determined by the Bethe ansatz equations. Then one would naturally find that

$$\langle \mathrm{MPS}|\mathbf{u}\rangle = \Psi^{J_1\dots J_L}_{I_1\dots I_L}\mathrm{Tr}\,(y^{I_1} y^\dagger_{J_1} y^{I_2} y^\dagger_{J_2} \dots y^{I_L} y^\dagger_{J_L}), \tag{211}$$

which in the other hand gives us the exact form of the one-point functions

$$\langle \mathcal{O}(x)\rangle_{tree} = \frac{1}{x^L}\frac{1}{\lambda^L\sqrt{L}}\frac{\langle \mathrm{MPS}|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{1/2}}, \quad x > 0. \tag{212}$$

The overlap between integrable boundary states and Bethe states are analytically calculable by the technique of the boundary Yang-Baxter equation, for which there is a recursion relation of the overlap between lower bond dimension and higher bond dimension, and such recursion relation will give us a determinant form of the overlap.

3.4.2.3 *The gravitational dual*

In the weak coupling limit, ABJM theory describes the world-volume theory of a stack of N D2 branes in type IIA superstring theory. It can be deformed by introducing probe branes in the string theory. The $1/2$-BPS domain wall version ABJM theory has a dual string theory description where a $D4$ probe brane is embedded in the type IIA background $\mathrm{AdS}_4 \times \mathbb{CP}^3$. The resulting D2-D4 probe brane system consists of N coincident D2-branes and one single probe D4-brane inserted. The orientation is summarized into the table [66]

	x_0	x_1	x_2	z	χ	θ_1	ϕ_1	θ_2	ϕ_2	ψ
D2	•	•	•							
D4	•		•	•		•	•			

Table 1: Coordinate table for D2 and D4 branes.

The probe brane system has the geometry $AdS_3 \times \mathbb{CP}^1$ with q units of world volume gauge field flux on the $\mathbb{CP}^1 \subset \mathbb{CP}^3$, which matches with the $SU(2) \times SU(2) \times U(1)$ symmetry of the vevs of the scalar fields in ABJM. It preserves half of the supersymmetry in the string theory. The integrability of the string boundary conditions on the probe D4-brane within the Green-Schwarz sigma model was checked in [67]

To have a better understanding of the above embedding, we choose a new parametrization of S^7 [68]

$$\begin{aligned} ds^2_{S^7} =& d\xi^2 + \frac{\cos^2\xi}{4}[(d\chi_1 + \cos\theta_1 d\phi_1)^2 + d\theta_1^2 + \sin^2\theta_1 d\phi_1^2] \\ &+ \frac{\sin^2\xi}{4}[(d\chi_2 + \cos\theta_2 d\phi_2)^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2], \end{aligned} \tag{213}$$

where $\xi \in [0, \frac{\pi}{2}), \theta_1, \theta_2 \in [0, \pi], \phi_1, \phi_2 \in [0, 2\pi]$ and $\chi_1, \chi_2 \in [0, 4\pi)$. Introducing new coordinates by

$$\chi_1 = 2y + \psi, \quad \chi_2 = 2y - \psi, \tag{214}$$

with range $y \in [0, 2\pi]$, $\psi \in [-2\pi, 2\pi]$. By making the following identification,

$$y \sim y + \frac{2\pi}{k}, \tag{215}$$

one can recover the orbifold $S^7/\mathbb{Z}_k$. The metric of S^7 is

$$ds^2_{S^7} = ds^2_{\mathbb{CP}^3} + (dy + A)^2 \tag{216}$$

with

$$A = \frac{1}{2}(\cos^2\xi - \sin^2\xi)d\psi + \frac{1}{2}\cos^2\xi \cos\theta_1 d\phi_1 + \frac{1}{2}\sin^2\xi \cos\theta_2 d\phi_2, \tag{217}$$

and

$$\begin{aligned} ds^2_{\mathbb{CP}^3} =& d\xi^2 + \cos^2\xi \sin^2\xi \Big(d\psi + \frac{\cos\theta_1}{2}d\phi_1 - \frac{\cos\theta_2}{2}d\phi_2\Big)^2 \\ &+ \frac{1}{4}\cos^2\xi\big(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\big) + \frac{1}{4}\sin^2\xi\big(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\big). \end{aligned} \tag{218}$$

For AdS_4 one use the Poincare metric

$$ds^2_{AdS_4} = \frac{1}{z^2}dz^2 + z^2(-dx_0^2 + dx_1^2 + dx_2^2), \tag{219}$$

with the boundary sitting at $z \to \infty$. Then the type IIA string theory is described by the metric

$$ds^2 = \tilde{R}^2\big(ds^2_{AdS_4} + 4ds^2_{\mathbb{CP}^3}\big), \tag{220}$$

where

$$\frac{\tilde{R}^2}{\alpha'} = \pi\sqrt{\frac{2N}{k}}. \tag{221}$$

The background comes with a RR 2-form field as well as a RR 4-form field which are given by

$$\begin{aligned} F^{(2)} =& k\Big(- \cos\xi \sin\xi d\xi \wedge (2d\psi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2) \\ & - \frac{1}{2}\cos^2\xi \sin\theta_1 d\theta_1 \wedge d\phi_1 - \frac{1}{2}\sin^2\xi \sin\theta_2 d\theta_2 \wedge d\phi_2\Big), \\ F^{(4)} =& \frac{3R^3}{8}\epsilon_{AdS_4}, \end{aligned} \tag{222}$$

where ϵ_{AdS_4} is the unit volume form of the AdS_4 space, and $\left(\frac{R}{l_p}\right)^3 = 4k\left(\frac{\tilde{R}}{\sqrt{\alpha'}}\right)^2$. The D4 probe brane is placed at $\xi = 0$ (and θ_2, ϕ_2, ψ are constants) and we take its world volume coordinates to be z, x_0, x_1, θ_1, ϕ_1, while the embedding coordinate x_2 is supposed to be non-constant but to depend only on z, the dependence is related to the magnetic flux on $\mathbb{CP}^1$. The $\xi = 0$ condition denotes the wrapping $\mathbb{CP}^1 \subset \mathbb{CP}^3$ with parametrization θ_1 and ϕ_1. The brane warps an $\mathrm{AdS}_3 \subset \mathrm{AdS}_4$ parametrized by the coordinates z, x_0 and x_1.

3.4.3 *Semi-classical analysis*

An obvious problem would be the extension of the above tree level overlap formula for one-point functions to higher perturbative orders and eventually to the non-perturbatively situation. One could follow the same strategy of the 1/2-BPS domain wall version of $\mathcal{N} = 4$ super Yang-Mills theory [69], where one uses fuzzy spherical harmonics from the vacuum algebraic structure, to diagonalize the mixing between color and flavor indices. Here we follow the details in [70]. We expand the action around the classical configuration, i.e. we make the following replacement

$$Y^\alpha \to \frac{y^\alpha}{\sqrt{z}} + \tilde{Y}^\alpha, \tag{223}$$

where y^α is the classical solution given in Eq. (195) (instead of using x, we use z here), and $\tilde{Y}^\alpha$ denotes the corresponding quantum fluctuation. For simplicity we will leave out the tilde on the quantum field in the following. The expanded action then reads

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{\text{int}}, \tag{224}$$

where $\mathcal{L}_b$ and $\mathcal{L}_f$ are Gaussian terms for bosonic and fermionic fields respectively and where $\mathcal{L}_{\text{int}}$ are interaction terms. The linear terms vanish by the BPS conditions. The non-vanishing vevs introduce a highly non-trivial mixing between both flavour and colour components of various fields. Our aim will be to disentangle this mixing and find the spectrum of the quantum fluctuations which is a necessary prerequisite for enabling perturbative computations. Instead of adding gauge fixing condition as in

$\mathcal{N} = 4$, we do not add gauge fixing terms at the present stage (due to the background field cannot unwind the coupling between kinetic and mass term), but fix the gauge by imposing constraints on the solutions later. The bosonic Gaussian action can be organized as

$$\begin{aligned}\mathcal{L}_b =& \frac{k}{4\pi}\mathrm{Tr}\{-\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu + \epsilon^{\mu\rho\nu}\hat{A}_\mu\partial_\rho\hat{A}_\nu + \left(A - \hat{A}\text{ mixing}\right) \\ &+ \partial_\mu Y^\alpha\partial^\mu Y^\dagger_\alpha + \left(Y^\dagger - Y\text{ mixing}\right) + \left(Y - A\text{ mixing}\right)\},\end{aligned} \tag{225}$$

where the mixing terms involving gauge fields are given by

$$\left(A - \hat{A}\text{ mixing}\right) = -\frac{2}{z}\hat{A}_\mu y^\dagger_\alpha A^\mu y^\alpha + \frac{1}{z}y^\dagger_\alpha A_\mu A^\mu y^\alpha + \frac{1}{z}\hat{A}_\mu y^\dagger_\alpha y^\alpha \hat{A}^\mu, \tag{226}$$

as well as

$$\begin{aligned}\left(Y - A\text{ mixing}\right) = 2i\left(\partial_\mu\frac{y^\alpha}{\sqrt{z}}\right)(\hat{A}^\mu Y^\dagger_\alpha - Y^\dagger_\alpha A^\mu) + 2i\left(\partial_\mu\frac{y^\dagger_\alpha}{\sqrt{z}}\right)\left(A^\mu Y^\alpha - Y^\alpha\hat{A}^\mu\right) \\ + \frac{i}{\sqrt{z}}\partial_\mu A^\mu\left(Y^\alpha y^\dagger_\alpha - y^\alpha Y^\dagger_\alpha\right) + \frac{i}{\sqrt{z}}\partial_\mu\hat{A}^\mu\left(Y^\dagger_\alpha y^\alpha - y^\dagger_\alpha Y^\alpha\right).\end{aligned} \tag{227}$$

For the scalar fields there is no mixing in the kinetic terms but the mass-like terms can be naturally split in the following way according to the mixing pattern

$$\left(Y^\dagger - Y\text{ mixing}\right) = m^2_{Y^\alpha Y^\dagger_\alpha} + m^2_{Y^\alpha Y^\dagger_\beta} + m^2_{YY}, \tag{228}$$

where

$$\begin{aligned}m^2_{Y^A Y^\dagger_A} =& \frac{1}{4z^2}Y^A Y^\dagger_A y^\beta y^\dagger_\beta y^\gamma y^\dagger_\gamma - \frac{1}{2z^2}Y^A Y^\dagger_A y^\beta y^\dagger_\gamma y^\gamma y^\dagger_\beta - \frac{1}{2z^2}Y^A y^\dagger_\beta y^\beta Y^\dagger_A y^\gamma y^\dagger_\gamma \\ &+ \frac{1}{z^2}Y^A y^\dagger_\beta y^\gamma Y^\dagger_A y^\beta y^\dagger_\gamma + \frac{1}{4z^2}Y^\dagger_A Y^A y^\dagger_\beta y^\beta y^\dagger_\gamma y^\gamma - \frac{1}{2z^2}Y^\dagger_A Y^A y^\dagger_\beta y^\gamma y^\dagger_\gamma y^\beta,\end{aligned} \tag{229}$$

$$\begin{aligned}m^2_{Y^\alpha Y^\dagger_\beta} =& \frac{1}{4z^2}\left(Y^\alpha y^\dagger_\alpha y^\beta Y^\dagger_\beta y^\gamma y^\dagger_\gamma + y^\dagger_\alpha Y^\alpha Y^\dagger_\beta y^\beta y^\dagger_\gamma y^\gamma + Y^\alpha y^\dagger_\alpha y^\beta y^\dagger_\beta y^\gamma Y^\dagger_\gamma + Y^\alpha y^\dagger_\beta y^\beta Y^\dagger_\gamma y^\gamma y^\dagger_\alpha\right) \\ &- \frac{1}{2z^2}\left(Y^\alpha Y^\dagger_\beta y^\beta y^\dagger_\alpha y^\gamma y^\dagger_\gamma + Y^\alpha y^\dagger_\beta y^\beta y^\dagger_\alpha y^\gamma Y^\dagger_\gamma + y^\alpha y^\dagger_\beta Y^\beta Y^\dagger_\alpha y^\gamma y^\dagger_\gamma + y^\alpha y^\dagger_\beta y^\beta Y^\dagger_\alpha Y^\gamma y^\dagger_\gamma\right. \\ &\left. + y^\alpha y^\dagger_\beta Y^\beta y^\dagger_\alpha y^\gamma Y^\dagger_\gamma + y^\alpha Y^\dagger_\beta y^\beta y^\dagger_\alpha Y^\gamma y^\dagger_\gamma\right) + \frac{1}{z^2}\left(Y^\alpha Y^\dagger_\beta y^\gamma y^\dagger_\alpha y^\beta y^\dagger_\gamma + Y^\alpha y^\dagger_\beta y^\gamma y^\dagger_\alpha y^\beta Y^\dagger_\gamma\right),\end{aligned} \tag{230}$$

$$\begin{aligned}m^2_{YY} =& \frac{1}{4z^2}\left(Y^\alpha y^\dagger_\alpha Y^\beta y^\dagger_\beta y^\gamma y^\dagger_\gamma + Y^\alpha y^\dagger_\beta Y^\beta y^\dagger_\gamma y^\gamma y^\dagger_\alpha + y^\alpha Y^\dagger_\alpha y^\beta Y^\dagger_\beta y^\gamma y^\dagger_\gamma + y^\alpha Y^\dagger_\beta y^\beta Y^\dagger_\gamma y^\gamma y^\dagger_\alpha\right) \\ &- \frac{1}{2z^2}\left(Y^\alpha y^\dagger_\beta Y^\beta y^\dagger_\alpha y^\gamma y^\dagger_\gamma + Y^\alpha y^\dagger_\beta y^\beta y^\dagger_\alpha Y^\gamma y^\dagger_\gamma + y^\alpha Y^\dagger_\beta y^\beta Y^\dagger_\alpha y^\gamma y^\dagger_\gamma + y^\alpha y^\dagger_\beta y^\beta Y^\dagger_\alpha y^\gamma Y^\dagger_\gamma\right. \\ &\left. + y^\alpha y^\dagger_\beta Y^\beta y^\dagger_\alpha Y^\gamma y^\dagger_\gamma + y^\alpha Y^\dagger_\beta y^\beta y^\dagger_\alpha y^\gamma Y^\dagger_\gamma\right) + \frac{1}{z^2}\left(Y^\alpha y^\dagger_\beta Y^\gamma y^\dagger_\alpha y^\beta y^\dagger_\gamma + y^\alpha Y^\dagger_\beta y^\gamma Y^\dagger_\alpha y^\beta y^\dagger_\gamma\right).\end{aligned} \tag{231}$$

The subscripts indicate the type of coupling between the scalar fields in the various terms. We notice that the terms in Eq. (229) do not involve any flavour mixing whereas the terms in Eq. (230) and Eq. (231) do. The ABJM action is already quadratic in the fermionic fields ψ, hence the perturbative Gaussian action $\mathcal{L}_f$ is obtained by simply replacing the scalar fields Y in the fermionic part of Eq. (179) by their classical solutions, and we do not repeat the expression here. Before the diagonalization, we perform a classification of our fields, first in flaour space and subsequently in color space. As for flavour space we notice that the scalar fields $Y^{\alpha=3,4}$ do not appear due to the corresponding classical fields are vanishing. Hence, these scalars do not mix with the gauge field components and also not with the scalars $Y^{\alpha=1,2}$. We shall therefore denote $Y^{\alpha=3,4}$ as easy bosons. As opposed to these the scalar fields $Y^{\alpha=1,2}$ mix with the z-component of the gauge field via the terms in Eq. (227) and with all components of the gauge field via the kinetic Chern-Simons terms. Hence, we separate the bosonic fields in the following two categories

$$\begin{aligned} &\text{easy bosons:} && Y^{\tilde{\alpha}}, \quad \tilde{\alpha} = 3, 4, \\ &\text{complicated bosons:} && Y^{\alpha},\ A_\mu,\ \hat{A}_\mu, \quad \alpha = 1, 2,\ \mu = 0, 1, z. \end{aligned}$$

In a standard Yang-Mills gauge theory, one would seek to eliminate mixing terms involving derivatives of fields such as some of the terms in Eq. (227) by working in background field gauge. This gauge choice, however, does not considerably simplify the mixing problem as we are still left with the kinetic Chern-Simons terms. We shall therefore not add gauge fixing terms at the present stage but solve the equations of motions for the various fields and impose the appropriate gauge constraints on the solutions later. Solving the equations of motion suffices for determining the scaling behavior of a field near the AdS boundary and thus the conformal dimension of its dual operator. See appendix C for more about the propagators in AdS space.

In the same way we split the fermionic fields into easy and complicated according to the complexity of their mixing as follows (from now on unless explicitly stated, α and $\tilde{\alpha}$ will denote flavor indices),

easy fermions: $\psi_\alpha, \quad \alpha = 1, 2,$

complicated fermions: $\psi_{\tilde{\alpha}}, \quad \tilde{\alpha} = 3, 4.$

In color space it is convenient to perform a block decomposition of the quantum fields

$$X = \begin{pmatrix} X(\nwarrow) & X(\nearrow) \\ X(\swarrow) & X(\searrow) \end{pmatrix}, \tag{232}$$

where the sizes of the various blocks of a given X is defined by the size of its upper left hand block which can be found in the list below

- $Y^A(\nwarrow)$ is of size $(q-1) \times q$,
- $A^\mu(\nwarrow)$ is of size $(q-1) \times (q-1)$,
- $\hat{A}^\mu(\nwarrow)$ is of size $q \times q$,
- $\psi_A(\nwarrow)$ is of size $(q-1) \times q$.

This splitting makes the breaking of the gauge symmetry manifest. For example,

$$\mathrm{Tr}\left(Y^\alpha Y^\dagger_\beta y^\beta y^\dagger_\alpha\right) = \mathrm{Tr}\left(Y^\alpha(\nwarrow) Y^\dagger_\beta(\nwarrow) y^\beta y^\dagger_\alpha\right) + \mathrm{Tr}\left(Y^\alpha(\nearrow) Y^\dagger_\beta(\swarrow) y^\beta y^\dagger_\alpha\right), \tag{233}$$

which highlights the fact that the unbroken gauge group is $U(N-q+1) \times U(N-q)$. Due to the matrix structure of the classical fields, y^α, the block decomposition above is also useful for exposing which field components couple to each other. E.g. in Eq. (233), the diagonal $(\nwarrow, \searrow)$ and off-diagonal terms $(\nearrow, \swarrow)$ decouple. We likewise have the following useful observation

$$\mathrm{Tr}\left(Y^\alpha y^\dagger_\beta Y^\beta y^\dagger_\alpha y^\gamma y^\dagger_\gamma\right) = \mathrm{Tr}\left(Y^\alpha(\nwarrow) y^\dagger_\beta Y^\beta(\nwarrow) y^\dagger_\alpha y^\gamma y^\dagger_\gamma\right), \tag{234}$$

with similar relations being valid for other quantum fields. One can easily observe that only the mixing terms where the quantum fields are adjacent contribute to the mixing between off-diagonal blocks. The diagonalization process is a battle with weapons of representation theory and structure of fields in AdS spacetime. For which we leave it in our publication.

4

CONCLUSION AND OUTLOOK

In this thesis, we have presented the Chern-Simons theory in three spacetime dimension with many novel features, especially the topological invariant quantity. When combined with supersymmetry, one makes the Chern-Simons theory become superconformal, which is essential in the construction of bulk theory of M branes. In addition, D-brane and heterotic string constructions often induce Chern-Simons couplings, understanding supersymmetric Chern-Simons dynamics was important for consistency with string dualities and for constructing brane intersection models.

The culmination of these efforts was the ABJM theory, for which describes the bulk theory of a stack of coincident of M_2 branes. In a certain limit, this theory is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ and therefore represents a new example of the AdS/CFT correspondence. Even though $\mathcal{N} = 4$ super Yang-Mills theory and ABJM theory have common features, notably they are both superconformal field theories which have gravity duals, the ABJM theory exhibits many new properties. As a result, the techniques for computing anomalous dimensions or spectrum that were developed for $\mathcal{N} = 4$ cannot be trivially applied to the ABJM theory. ABJM theory lives in three dimensions, and the Chern-Simons term is essential for construction of conformal theory in three dimension by purely dimension analysis. Another exotic feature of three dimensions is the lack of chirality, which makes the Chern-Simons theory change signs under parity, to calculate parity invariant quantities, one needs to be careful. For example, to calculate anomalous dimensions in planar limit, one needs to go to at two-loop order for non-vanishing results. The ABJM is not maximal supersymmetric, only has $3/4$ maximal supersymmetry. This has a profound effect on many quantities due to the lack constraints, such as the magnon dispersion relation and the all-loop Bethe Ansatz, are dependent on an interpolation function $h(\lambda)$ which has different asymptotics at weak and strong coupling. The matter fields in ABJM theory are in the bifundamental representation of the gauge group $U(N) \times U(N)$ instead of adjoint representations. Then the double line notation for Feynman diagrams is still workable but with two different types of lines. Moreover, gauge invariant operators in the ABJM theory must contain an even number of fields which alternate between the representation $(N, \bar{N})$ and $(\bar{N}, N)$, which agree with the double line notations. If such operators are mapped to spin chain language, then there are two types of elementary excitations (momentum carrying modes), those sit at even sites and odd sites respectively.

Due to recent development in the integrable boundary states, lots of attention is paid to a holographical realization of such system. In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the configuration D3-D5 probe brane system would introduce such integrable boundary states when we calculate the one-point functions. The tree-level one-point functions can be mapped to the overlap between Bethe states and matrix product states (a type of integrable boundary states), and the matrix product states can be constructed by the solutions of certain Boundary Yang-Baxter equation. The one-point functions can be found in a closed form by the techniques of twisted Yangian algebra. One would naturally desire to extend the same formalism into the lower dimensional AdS/CFT correspondence, namely the ABJM theory. The holographic dual turns out to be a D2-D4 probe brane system, with corresponding $1/2$ BPS domain wall version of ABJM theory in the gauge theory side. Again, the tree-level one-point functions has already been found in a closed form [71], and its extension to higher rank with the use of KT relation [72].

This thesis studies the quantum fluctuation around the BPS domain wall solution in ABJM theory, it serves as the first step toward the weak coupling perturbative computation of around the above vacuum. Even though we do not preform the one-point functions calculation, it laid some groundwork for the perturbative computation, and eventually obtain non-perturbative results based on the intuition of how the integrable structure found at tree level gets deformed at the loop level, i.e. the calculation of the g-functions [73]. In mass-deformed ABJM theory, there exists similar structures of the vacuum expectation value. It would be interesting to generalize the above procedure to the case of mass-deformed ABJM, which could link with the centrally-extended spin chain. One would also be interested in the study of supersymmetric Chern-Simons theories in the presence of other defects, such as the Wilson line defect [74, 75].

Besides our calculations, there are lots of open problems in the area of ABJM theory that are still uncharted areas. Unlike in $\mathcal{N} = 4$ super Yang-Mills theory, ABJM theory doesn't possess maximal supersymmetry, which is related to many novel phenomenon. The interpolation function $h(\lambda)$ is much less understood via the BES equation, since it is not protected, asymptotic behavior of it at weak and strong coupling is quite different functions of λ. There is no known closed-form expression that smoothly interpolates between these regimes by localization or bootstrap technique. On the string theory side, integrability of the classical sigma model is established, but quantum integrability is not fully understood. Quantization of the worldsheet theory is difficult also due to the lack of maximal supersymmetry, which complicates the Green-Schwartz formulation, gauge fixing and cancellation of divergences. The string action in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ can be fitted into a coset only in the bosonic part, not the full superspace. The background includes Ramond-Ramond fluxes, which is notoriously hard in the Green-Schwartz formalism.

Integrability in ABJ theory, a $U(N)_k \times U(M)_{-k}$ gauge theory, is a subtle topic because of the parity violation, which would introduce a new parameter of the difference of two ranks. The difference of rank $\frac{M-N}{k}$ (also noted as "parity violation parameter") plays a role similar to a twist or deformation in the integrable structure [76]. At planar level, integrability appears to survive under certain conditions, especially when the rank difference is small compared to N, M, k. Two-loop dilatation operator in scalar sector is identical to ABJM at planar level, and the all-loop asymptotic Bethe ansatz can be extended to ABJ theory by including the appropriate twist. It is still unclear if integrability persists non-perturbatively or in the full spectrum.

The study of superconformal Chern-Simons theories is still in its early stages, including various deformations, this subject should continue to provide novel perspectives on integrability and the gauge gravity duality.

Part III

APPENDICES

A

APPENDIX A A DIRECT PROOF OF CHERN-SIMONS LEVEL

This appendix serves as a direct proof of the Chern-Simons level for the case of $SU(2)$ fundamental representation, the generalization to group of higher-rank should be similar. The quantity we wish to calculate is

$$-\frac{1}{24\pi^2}\int_{\mathcal{M}} \mathrm{tr}(g^{-1}dg)^3. \tag{235}$$

A general group element in defining representation of $SU(2)$ would take the form

$$g = \begin{pmatrix} \cos(\beta)e^{i\phi} & \sin(\beta)e^{i\psi} \\ -\sin(\beta)e^{-i\psi} & \cos(\beta)e^{-i\phi} \end{pmatrix}, \tag{236}$$

where $0 \le \beta \le \frac{\pi}{2}, 0 \le \phi, \psi \le 2\pi$. We start now from the "Euler" form in the defining representation [77]

$$\begin{aligned} g &= e^{i\frac{\alpha}{2}\sigma_3} e^{i\frac{\beta'}{2}\sigma_2} e^{i\frac{\gamma}{2}\sigma_3} \in SU(2) \\ &= \begin{pmatrix} \cos(\frac{\beta'}{2})e^{i(\frac{\alpha+\gamma}{2})} & \sin(\beta')e^{i(\frac{\alpha-\gamma}{2})} \\ -\sin(\frac{\beta'}{2})e^{-i(\frac{\alpha-\gamma}{2})} & \cos(\frac{\beta'}{2})e^{-i(\frac{\alpha+\gamma}{2})} \end{pmatrix}, \end{aligned} \tag{237}$$

we need to find the range of α, β', γ, compare Eq. (237) with Eq. (236), one would find

$$\beta = \frac{1}{2}\beta',\ \phi = \frac{1}{2}(\alpha+\gamma),\ \psi = \frac{1}{2}(\alpha-\gamma). \tag{238}$$

One can see that the range of $\{\alpha, \gamma\}$ is half of the range $\{\phi, \psi\}$, since the Jacobian is

$$J(\frac{\phi, \psi}{\alpha, \gamma}) = \frac{1}{2}. \tag{239}$$

Therefore we choose

$$0 \leq \beta' \leq \pi,\ 0 \leq \alpha \leq 2\pi,\ 0 \leq \gamma \leq 4\pi. \tag{240}$$

Now let define so-called canonical right one-forms on the group $SU(2)$, which are also called Maurer-Cartan one-forms (also useful for classical construction of integrable system)

$$R = g^{-1}dg = \frac{i}{2}\sigma_k R_k, \tag{241}$$

with

$$\begin{aligned} R_1 &= -\sin\gamma d\beta' + \cos\gamma \sin\beta' d\alpha, \\ R_2 &= -\cos\gamma d\beta' + \sin\gamma \sin\beta' d\alpha, \\ R_3 &= d\gamma + \cos\beta' d\alpha. \end{aligned} \tag{242}$$

Then

$$\text{Tr } (g^{-1}dg)^3 = -\frac{3}{2} d\gamma \wedge \sin\beta' d\beta' \wedge d\alpha, \tag{243}$$

so

$$\int_{\mathcal{M}} \frac{1}{24\pi^2} \text{tr } (g^{-1}dg)^3 = -1. \tag{244}$$

B

APPENDIX B FUZZY SPHERICAL HARMONIC

Fuzzy spheres, as a class of non-commutative spaces, natrually arise from the dynamics of certain D-brane systems to describe matrix degrees of freedom. Dirichlet branes (D-branes) are solitonic, membrane-like objects of various dimensions in string theory. A Dp-brane is a $p+1$ dimensional hyper-plane in 10 dimensional space-time, where open strings can end. Dp-branes are half-BPS objects and carry an elementary unit of charge with respect to the $p+1$ form gauge potential, coming from the Ramond-Ramond (RR) sector of the type II superstring.

Dp-branes are dynamical objects, which realize gauge theories on their world-volume [78]. The massless spectrum of the open string living on the brane can be described by a maximally supersymmetric $U(1)$ gauge theory in $p+1$ dimensions. There is a vector field and the $9-p$ massless real scalar fields that are present in the super-multiplet, can be associated with the Goldstone modes, descrbing the fluctuation of the brane in the transverse directions. Therefore, when branes sit on top of each other, then the vacuum expectation values (vevs) for the scalar fields vanish. A set of N coincident D-branes provides N^2 possibilities for the endpoints of open strings, since the latter can begin or end on any one of them. The spcetrum is described by $U(N)$ maximally supersymmetric gauge theory. When N is large, such object would be massive that it deforms the geometry.

Since we have identified the world-volume scalars with the transverse spatial coordinates, for matrix valued scalars it hints that the space-time geometry becomes non-commutative. A key concept is from the study of configuration of Matrix theory, that is the geometry of the membrane world-volume is not conventional: It is described by non-commutative geometry, which is a generalization of classical/commutative geometry [79]. The very first description of such a system was by de Wit, Hoppe and Nicolai [80]. The authors found that when quantizing in light gauge, the spherical supermembrane has a residual invariance under area preserving diffeomorphisms on the world-volume. The symmetry group can be identified with $U(N)$ group in the large-N limit. The most essential part in the derivation was the construction of an exact correspondence between the functions of the spherical membrane and $U(N)$ matrices: Functions on the sphere, which are functions of the euclidean coordinates, can be described in Matrix theroy by the equivalent symmetrised polynomials in the generators of the N-dimensional representation of $SU(2)$.

In this section, we will first review the matrix description of 2-sphere, the fuzzy spheres in S^2, then describe a non-square deformation of it, for which we used in the diagonalization of coupling of flavour and colour space in the main text.

B.1 THE FUZZY 2-SPHERE

The fuzzy 2-sphere is defined as the algebra generated by the three elements t_3, t_+, t_- obeying the relations of the $SU(2)$ Lie algebra [81]:

$$[t_+, t_-] = 2t_3, \quad [t_3, t_+] = t_+, \quad [t_3, t_-] = -t_-, \tag{245}$$

together with a constraint on the Casimir:

$$t_3^2 + \frac{1}{2}(t_+t_- + t_-t_+) = J(J+1). \tag{246}$$

This algebra is infinite dimensional, i.e. t_-^n, for any n, are independent elements. However, there is a isomorphism between the algebra of $q \times q$ matrices and a finite dimensional quotient of such infinite algebra with identification $q = n+1,\ n = 2J$. We call this finite dimensional truncation $\hat{\mathcal{A}}_n(S^2)$, as an algebra over the complex numbers, is isomorphic to the algebra of $q \times q$ matrices. The $\hat{\mathcal{A}}_n(S^2)$ can be decomposed as a direct sum of representations of integer spin s, where s ranging over from 1 to n.

$$\hat{\mathcal{A}}_n = \oplus_{s=0}^{n} V_s. \tag{247}$$

Rewriting $t_1 = \frac{1}{2}(t_+ + t_-)$ and $t_2 = \frac{1}{2i}(t_+ - t_-)$, representations of spin s correspond to matrices of the form

$$f_{a_1,a_2,\ldots,a_s} t^{a_1} t^{a_2} \ldots t^{a_s}, \tag{248}$$

where the indices $a_1, \ldots, a_n$ run from 1 to 3, and f is a traceless symmetric tensor. The number of independent component of completely symmetric, rank-s traceless tensor $f_{a_1,a_2,\ldots,a_s}$ is $\begin{pmatrix} s+2 \\ s \end{pmatrix} - \begin{pmatrix} s \\ s-2 \end{pmatrix} = 2s+1$. By summing over all ranks we get that the set of function $\hat{\mathcal{A}}_n(S^2)$ is an $\sum_{s=0}^{q-1}(2s+1) = q^2$ dimensional vector space. The explicit construction of the above irreducible representations can be realized by the spherical harmonics $\hat{Y}_\ell^m$ with magnetic quantum number $m = -\ell, \ldots, \ell$, and angular momentum $\ell = 1, \ldots, q-1$. They satisfy the relations

$$L_3 \hat{Y}_\ell^m = m \hat{Y}_\ell^m, \quad L^2 \hat{Y}_l^m = \ell(\ell+1) \hat{Y}_l^m, \tag{249}$$

where

$$L_i X = t_i X - X t_i, \quad L^2 = \sum_{i=1}^{3} L_i L_i. \tag{250}$$

B.2 MODIFIED FUZZY SPHERICAL HARMONICS

As mentioned above, the standard fuzzy spherical harmonics can be served as the basis traceless symmetric matrix. The standard fuzzy sphere harmonics $Y_\ell^m(\hat{t})$ for the q-dimensional irreducible representation fulfill

$$[\hat{t}_3, Y_\ell^m(\hat{t})] = m Y_\ell^m(\hat{t}), \quad \ell = 0, 1, \ldots, q-1, \tag{251}$$

which motivates us to define $\hat{Y}^m_\ell = (Y^m_\ell(\hat{t}))^T$ such that

$$[-\hat{t}^T_3, \hat{Y}^m_\ell] = m\hat{Y}^m_\ell, \tag{252}$$

while the action of the ladder operators reads

$$\begin{aligned}
[-\hat{t}^T_+, \hat{Y}^m_\ell] &= \sqrt{(\ell-m)(\ell+m+1)}\hat{Y}^{m+1}_\ell, \\
[-\hat{t}^T_-, \hat{Y}^m_\ell] &= \sqrt{(\ell+m)(\ell-m+1)}\hat{Y}^{m-1}_\ell.
\end{aligned} \tag{253}$$

Let us define L_i in $\pi_{q-1} \otimes \pi_q$ representation of $\mathfrak{su}(2)$

$$L_i X = t_i X + X\hat{t}^T_i. \tag{254}$$

Then the vacuum equation, or the BPS condition can be reformulated as

$$\begin{aligned}
L_3 y^1 &= -\frac{1}{2}y^1, \quad L_+ y^1 = -y^2, \quad L_- y^1 = 0, \\
L_3 y^2 &= \frac{1}{2}y^1, \quad L_+ y^2 = 0, \quad L_- y^2 = -y^1,
\end{aligned} \tag{255}$$

where $L_\pm = L_1 \pm iL_2$. Then noticing that ($i = 1, 2, 3, \pm$ and $\alpha = 1, 2$)

$$L_i y^\alpha \hat{Y}^m_\ell = (L_i y^\alpha)\hat{Y}^m_\ell + y^\alpha[-\hat{t}^T_i, \hat{Y}^m_\ell], \tag{256}$$

one easily finds

$$\begin{aligned}
L_3\, y^1 \hat{Y}^m_\ell &= (m - \frac{1}{2})y^1\hat{Y}^m_\ell, \quad L_+ y^1 \hat{Y}^\ell_\ell = -y^2\hat{Y}^\ell_\ell, \quad L_-\, y^1\hat{Y}^{-\ell}_\ell = 0, \\
L_3\, y^2 \hat{Y}^m_\ell &= (m + \frac{1}{2})y^2\hat{Y}^m_\ell, \quad L_+\, y^2 \hat{Y}^\ell_\ell = 0, \quad L_-\, y^2\hat{Y}^{-\ell}_\ell = -y^1\hat{Y}^{-\ell}_\ell,
\end{aligned} \tag{257}$$

Now, by writing the Casimir in terms of L_3 and the ladder operators

$$L_i L_i = L_3^2 - L_3 + L_+ L_- = L_3^2 + L_3 + L_- L_+, \tag{258}$$

one further obtains

$$L_i L_i\, y^1\hat{Y}^{-\ell}_\ell = (\ell + \frac{1}{2})(\ell + \frac{3}{2})y^1\hat{Y}^{-\ell}_\ell, \quad L_i L_i\, y^2\hat{Y}^\ell_\ell = (\ell + \frac{1}{2})(\ell + \frac{3}{2})y^2\hat{Y}^\ell_\ell, \tag{259}$$

which implies that $y^1\hat{Y}^{-\ell}_\ell$ is the lowest state in the $2(\ell+1)$-dimensional irrep, while $y^2\hat{Y}^\ell_\ell$ is the highest state in the $2(\ell+1)$-dimensional irrep. Thus, $y^2\hat{Y}^\ell_\ell$ ($y^1\hat{Y}^{-\ell}_\ell$) with $\ell = 0, 1, \ldots, q-2$ include the highest (lowest) states of all the irreps deduced from $\pi_q \otimes \pi_{q-1}$, and we can obtain the entire set of states in each irrep by acting on $y^2\hat{Y}_\ell^\ell$ $(y^1\hat{Y}_\ell^{-\ell})$ with L_- (L_+). The states constructed in the two ways are respectively given by $(n = 0, 1, \ldots, 2\ell + 1)$

$$(L_-)^n y^2\hat{Y}_\ell^\ell = \sqrt{\frac{(2\ell)!(n-1)!}{(2\ell-(n-1))!}} \left(-ny^1\hat{Y}_\ell^{\ell-(n-1)} + \sqrt{(2\ell-n+1)n}\, y^2\hat{Y}_\ell^{\ell-n} \right), \tag{260}$$

$$(L_+)^n y^1\hat{Y}_\ell^{-\ell} = \sqrt{\frac{(2\ell)!(n-1)!}{(2\ell-(n-1))!}} \left(-ny^2\hat{Y}_\ell^{-\ell+(n-1)} + \sqrt{(2\ell-n+1)n}\, y^1\hat{Y}_\ell^{-\ell+n} \right). \tag{261}$$

One easily finds $(L_-)^n y^2\hat{Y}_\ell^\ell \propto (L_+)^{2(\ell+1)-n} y^1\hat{Y}_\ell^{-\ell}$ which verifies the consistency of the two ways of generating states. Hence, up to a normalization factor, the states in the spin $\ell + 1/2$ irrep can be expressed as

$$T_{\ell+1/2}^{m+1/2} = -\sqrt{l-m}\, y^1\hat{Y}_\ell^{m+1} + \sqrt{\ell+m+1}\, y^2\hat{Y}_\ell^m, \tag{262}$$

where $\ell = 0, 1, \ldots, q-2$ and $m = -\ell-1, -\ell, \ldots, \ell$. We also assume $\hat{Y}_\ell^{-\ell-1} = \hat{Y}_\ell^{\ell+1} = 0$. These states carry different eigenvalues of the Casimir and L_3 so they are automatically orthogonal to each other.

C

APPENDIX C PROPAGATORS IN AdS_{d+1}

In AdS/CFT duality, there is a dictionary between boundary theory and bulk theory by matching of the Hilbert space. The conformal dimension of an operator is mapped to the mass in the bulk, for with one can see details below.

C.1 SCALARS IN AdS_{d+1} SPACETIME

Consider a scalar field $\phi(X)$ on $\mathcal{M} = AdS_{d+1}$ background, the action for it would be

$$S = -\frac{1}{2}\int d^{d+1}X\sqrt{-g}(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi), \tag{263}$$

where in Poincare coordinates $X^M = (z, x^\mu)$ (with x^μ parameterizing the d-dimensional boundary directions and $z > 0$ the AdS radial coordinate), the metric is

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \tag{264}$$

the equation of motion of ϕ is

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\phi\right) - m^2\phi = 0. \tag{265}$$

Given translation symmetries in x^μ directions, we make the ansatz for the solution:

$$\phi(z, x^\mu) = \int \frac{d^dk}{(2\pi)^d}e^{ik\cdot x}\tilde{\phi}(z,k), \quad (k\cdot x = \eta_{\mu\nu}k^\mu x^\nu). \tag{266}$$

Substitute this ansatz into Eq. (265), we get

$$z^{d+1}\partial_z(z^{1-d}\partial_z\tilde{\phi}) - k^2z^2\tilde{\phi} - m^2R^2\tilde{\phi} = 0, \tag{267}$$

where $k^\mu = (\omega, \vec{k})$ and $k^2 = -\omega^2 + \vec{k}^2$. Let us rewrite Eq. (267) in a more explicit form

$$z^2\partial_z^2\tilde{\phi} + (1-d)z\partial_z\tilde{\phi} - (k^2z^2 + m^2R^2)\tilde{\phi} = 0, \tag{268}$$

which is almost the modified Bessel equation. Make a rescaling $\tilde{\phi}(k,z) = z^{d/2}f(kz)$ (where one introduces the dimensionless parameter kz), one can get

$$(kz)^2\partial_{kz}^2 f + kz\partial_{kz} f - \left(\frac{d^2}{4} + m^2R^2 + k^2z^2\right) f = 0. \tag{269}$$

Assuming $k^2 > 0$, one can reads the solution for $f(kz)$:

$$f(kz) = a_k K_\nu(kz) + b_k I_\nu(kz), \tag{270}$$

where we have defined the parameter ν as

$$\nu = \sqrt{\frac{d^2}{4} + m^2R^2}. \tag{271}$$

Finally the solution for $\tilde{\phi}(k,z)$ reads

$$\tilde{\phi}(k,z) = a_k(kz)^{d/2}K_\nu(kz) + b_k(kz)^{d/2}I_\nu(kz). \tag{272}$$

The conformal invariance will be more explicit if we switch to Euclidean signature, consider the dimension $D = d+1$ with Euclidean signature and the half-space metric [82, 83]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{x_0^2}(dx_0^2 + dx_i^2). \tag{273}$$

We will use the notation $x = (x_0, \vec{x})$, $\vec{x} = (x_i)$, $\quad i = 1, \ldots, d$. The action for a massive scalar would be

$$I = \frac{1}{2}\int d^{d+1}x\sqrt{g_0}[(\partial_\mu\phi)^2 + m^2\phi^2], \tag{274}$$

then the Green's function for construction of correlation functions would be

$$(\partial^2 - m^2)G(x,y) = -\frac{1}{\sqrt{g_0}}\delta(x-y). \tag{275}$$

More explicitly, in the metric system Eq. (273), the above equation becomes:

$$x_0^{d+1}\partial_\mu\left(x_0^{-d+1}\partial_\mu G(x,y)\right) - m^2 G(x,y) = -\delta(\vec{x}-\vec{y})\delta(x_0-y_0)y_0^{d+1}. \tag{276}$$

Make a rescaling $G(x,y) = x_0^{d/2}H(x,y)$, then

$$(\Delta_\nu + \partial_i^2)H(x,y) = -y_0^{\frac{d-2}{2}}\delta(\vec{x}-\vec{y})\delta(x_0-y_0), \tag{277}$$

where $\nu^2 = m^2 + \frac{1}{4}d^2$ and the operator Δ_ν defined by

$$\Delta_\nu = \partial_0^2 + \frac{1}{x_0}\partial_0 - \frac{\nu^2}{x_0^2}, \tag{278}$$

has Bessel functions as its eigenfunctions

$$\Delta_\nu J_\nu(wx_0) = -w^2 J_\nu(wx_0). \tag{279}$$

Let us rewrite the delta-function in terms of the above Bessel functions (for which are orthonormal)

$$\delta(x_0-y_0) = y_0\int_0^\infty dww J_\nu(wx_0)J_\nu(wy_0), \tag{280}$$

then the Green function can be rewrite as

$$H(x,y) = \int \frac{d^dk}{(2\pi)^d}e^{i\vec{k}\cdot\vec{x}}\int_0^\infty dww J_\nu(wx_0)\tilde{H}(w,k;y). \tag{281}$$

Substitute the above expression into Eq. (277), one can determine

$$\tilde{H}(w,k;y) = y_0^{d/2}\frac{J_\nu(wy_0)}{w^2+k^2}e^{-i\vec{k}\cdot\vec{y}}. \tag{282}$$

The scalar Green function in Euclidean spacetime is given by

$$\begin{aligned} G(x,y) &= (x_0y_0)^{d/2}\int \frac{d^dk}{(2\pi)^d}\int_0^\infty dww\frac{1}{w^2+k^2}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}J_\nu(wx_0)J_\nu(wy_0) \\ &= (x_0y_0)^{d/2}\int \frac{d^dk}{(2\pi)^d}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}I_\nu(kx_0^<)K_\nu(kx_0^>) \\ &\sim rz^{-\lambda}F(\lambda,\nu-\frac{1}{2};2\nu+1,z^{-1}), \end{aligned} \tag{283}$$

where $x_0^<$ ($x_0^>$) is the smaller (larger) number among x_0 and y_0, r is some normalization constant, F is the hypergeometric function and $z = \frac{(x_0+y_0)^2+(\vec{x}-\vec{y})^2}{4x_0y_0}$. The conformal invariance is manifest from the hypergeometric function.

C.2 SOLUTIONS OF A CLASS OF MASSIVE CHERN-SIMONS EQUATION OF MOTION

The $1/2$ BPS domain wall introduced will deform the space-time symmetry into $AdS_3 \times \mathbb{CP}^1$, such fact also shows in the equation of motion of our defect ABJM theory. We here consider a more generic EOM of a 3d massive Chern-Simons theory, which is useful for the diagonalization of several of our blocks. Suppose we have the EOM

$$\left(c\epsilon_{\mu\rho\nu}\partial^\rho - \frac{d}{z}\eta_{\mu\nu}\right)A^\nu - i\left(\frac{1}{z^{1/2}}\partial_\mu + \frac{1}{2z^{3/2}}\delta^z_\mu\right)Y = 0, \tag{284}$$

where $\mu = 0, 1, z$. Going to momentum space in the longitudinal directions $a = 0, 1$ by introducing momenta k^0, k^1 a possible solutions reads (with $ik = \sqrt{-k_a k^a}$)

$$\begin{aligned}
A^a &= -\frac{1}{2k}\epsilon^{ab}k_b f K_{\frac{d}{c}}(ikz), \\
A^z &= \frac{1}{4}f\left(K_{\frac{d}{c}-1}(ikz) - K_{\frac{d}{c}+1}(ikz)\right), \\
Y &= \frac{c\sqrt{z}}{4i}f\left(K_{\frac{d}{c}-1}(ikz) + K_{\frac{d}{c}+1}(ikz)\right),
\end{aligned} \tag{285}$$

where $f \propto e^{-ik_a x^a}$ should be determined by normalization conditions and $K_\nu(x)$ is the modified Bessel function. One can easily verify this solution by making use of the recurrence relation of the modified Bessel function

$$x\partial_x K_\nu(x) = \pm\nu K_\nu(x) - xK_{\nu\pm1}(x), \tag{286}$$

$$x\partial_x K_{\nu\pm1}(x) = \mp(\nu\pm1)K_{\nu\pm1}(x) - xK_\nu(x). \tag{287}$$

There exists (as usual) also a solution given in terms of modified Bessel functions of type $I_\nu(x)$ which we give below. For the purpose of determining the spectrum we only need to worry about about one of these solutions as we can read off the mass parameter from the index of either of the Bessel functions. In order to construct the propagator we of course need both solutions, cf. Eq. (283). In order to identify the quantum fluctuations with a specific value of ν we organize A^z, Y into linear combinations given in terms of a single Bessel function

$$X^+ = fK_{\frac{d}{c}-1}(ikz), \quad X^- = -fK_{\frac{d}{c}+1}(ikz), \tag{288}$$

where

$$X^{\pm} = 2A^z \pm \frac{2i}{c\sqrt{z}}Y. \tag{289}$$

One has another set of independent solutions also expressed entirely in terms of K_ν's (which can be eliminated by gauge fixing).

$$\begin{aligned} A^a &= -\frac{1}{k}\eta^{ab}k_b f K_{\frac{d}{c}}(ikz), \\ A^z &= \frac{1}{4}f(K_{\frac{d}{c}-1}(ikz) + K_{\frac{d}{c}+1}(ikz)), \\ Y &= \frac{c\sqrt{z}}{4i}f(K_{\frac{d}{c}-1}(ikz) - K_{\frac{d}{c}+1}(ikz)). \end{aligned} \tag{290}$$

For completeness, let us also give the solution of Eq. (284) in terms of $I_\nu(x)$ which differs slightly in the signs of some of the terms involved,

$$\begin{aligned} A^a &= \frac{1}{2k}\epsilon^{ab}k_b g(I_{\frac{d}{c}}(ikz)), \\ A^z &= \frac{1}{4}g\left(I_{\frac{d}{c}-1}(ikz) - I_{\frac{d}{c}+1}(ikz)\right), \\ Y &= \frac{c\sqrt{z}}{4i}g\left(I_{\frac{d}{c}-1}(ikz) + I_{\frac{d}{c}+1}(ikz)\right), \end{aligned} \tag{291}$$

where $g \propto e^{-ik_a x^a}$ also should be determined by normalization conditions. Above equations can be verified by making use of the recurrence relations of the modified Bessel functions of type $I_\nu(x)$

$$x\partial_x I_\nu(x) = \pm\nu I_\nu(x) + xI_{\nu\pm1}(x), \tag{292}$$

$$x\partial_x I_{\nu\pm1}(x) = \mp(\nu\pm1)I_{\nu\pm1}(x) + xI_\nu(x). \tag{293}$$

Also in this case there exists a related solution (which can be eliminated by the gauge fixing), namely

$$\begin{aligned} A^a &= \frac{1}{k}\eta^{ab}k_b g I_{\frac{d}{c}}(ikz), \\ A^z &= \frac{1}{4}g(I_{\frac{d}{c}-1}(ikz) + I_{\frac{d}{c}+1}(ikz)), \\ Y &= \frac{c\sqrt{z}}{4i}g(I_{\frac{d}{c}-1}(ikz) - I_{\frac{d}{c}+1}(ikz)). \end{aligned} \tag{294}$$

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