Chair of AUTOMATIC CONTROL ENGINEERING	DYNAMICAL SYSTEMS
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Exercise 1: Sliding Mode Controller

A DC/DC converter is supposed to be controlled using sliding mode. A desired output voltage should be reached by appropriately driving a transistor. The system is given by the following equations:

$$\dot{x}_1 = -\frac{1}{L}x_2 + \frac{V_0}{L}u$$
$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{RC}x_2$$

This is a test!

where R, L, C, $V_0 > 0$.
The control signal u can only take on binary values, i.e., $u \in \{0, 1\}$.
The state x_1 denotes the current, x_2 denotes the output voltage. The constant input voltage is V_0.

1.1 Calculate the current x_1^* which is required in steady-state to achieve the desired putput voltage $x_2^* = V_\mathsf{d}$.

Now consider $s = x_1 - x_1^*$ and $u = \frac{1}{2}(1 - \mathrm{sgn(s)})$.

1.2 Calculate the system behavior $\dot{\underline{x}}_\mathsf{av} = \underline{f}_\mathsf{av}(x)$, $\underline{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$ in $s = 0$ using Filippov's method.

1.3 Sketch the behavior of $x_2(t)$ after reaching $s = 0$.

1.4 Check for existence of sliding mode.

Exercise 2:

Consider the nonlinear second order system

$$\boxed{\begin{aligned} &\dot{\underline{x}} = A\underline{x} + \underline{b}(\underline{x})u, \quad \underline{x} \in \mathbb{R}^2,\ u \in \mathbb{R} \\ &A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} \sin^2 y \\ \cos^2 y \end{bmatrix}, \quad y = [1\ 1]\,\underline{x}\,. \end{aligned}}$$

Look at this nice formula

2.1 Design a sliding mode controller (target value $y = 0$).

2.2 Investigate the behavior on $s = 0$ using equivalent control.

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Exercise:

$\dot{\underline{x}} = 0$

$\dot{\underline{x}}^* = 0$

Determine EP for unforced System: $u = 0$

Filippous method:

$$\dot{x}_{av} = \alpha \underline{f}^+ + (1-\alpha)\underline{f}^-$$

$$\alpha = \frac{\frac{ds}{dx} f^-}{\frac{ds}{dx}(f^- - f^+)}$$

$$s = x_1 - x_1^*$$

$$\frac{ds}{dx} = [1 \quad 0]^T$$

new controller $u^+ = \frac{1}{2}(1-1) = 0$

$$u^- = \frac{1}{2}(1+1) = 1$$

$$f^+ = \begin{bmatrix} -\frac{1}{L} x_2 \\ \frac{1}{C} x_1 - \frac{1}{RC} x_2 \end{bmatrix}$$

$$f^- = \begin{bmatrix} -\frac{1}{L} x_2 + \frac{V_0}{L} \\ \frac{1}{C} x_1 - \frac{1}{RC} x_2 \end{bmatrix}$$

$$\alpha = \frac{-\frac{1}{L}x_2 + \frac{V_b}{L}}{\frac{V_o}{L}}$$

$$\mathring{x}_{av} = \begin{bmatrix} eq_1 \\ eq_2 \end{bmatrix} = \ldots$$

$$u^-, u^+ = \quad s > 0 : U = \frac{1}{2}\left(1 - \mathrm{sign}(s)\right)$$

$$s < 0 : U = \frac{1}{2}\left(1 - \mathrm{sing}(s)\right)$$

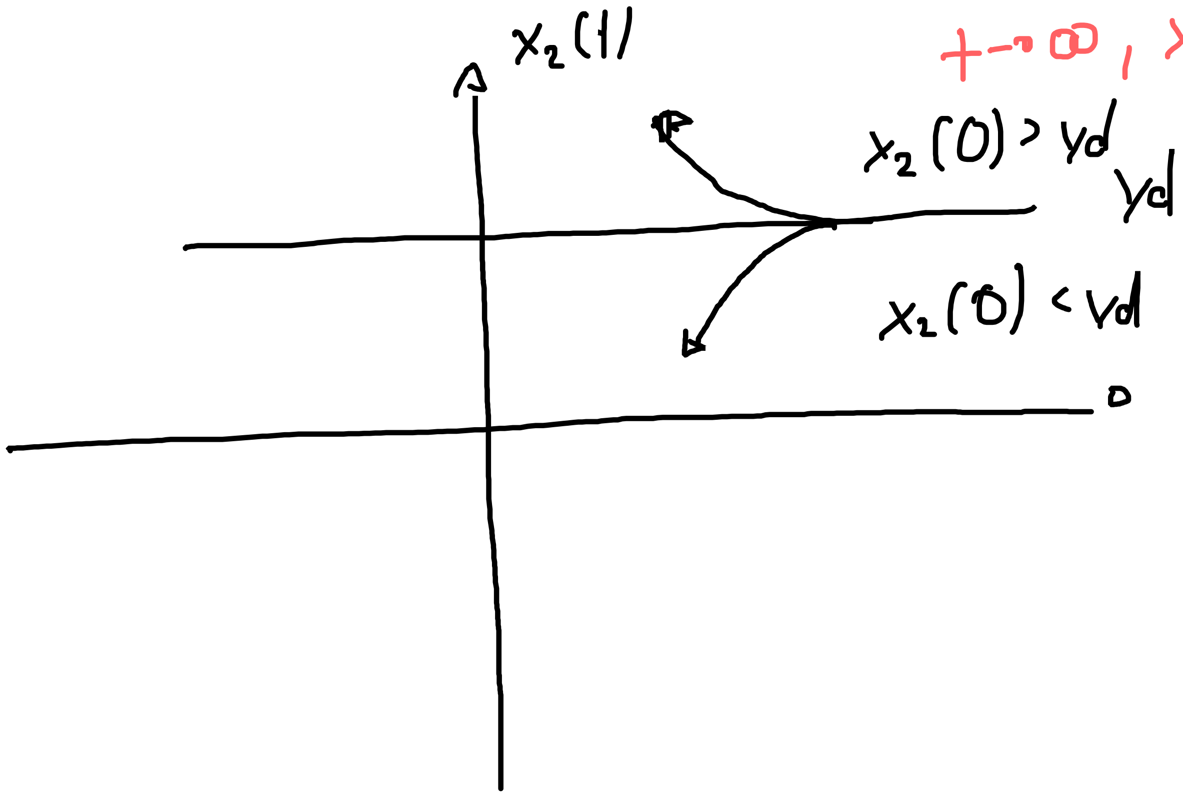
1.3. Sketch behaviour of $x_2(t)$

$$s = 0 : \quad x_1 = \lambda_1^* = \frac{V_p}{R}$$

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{RC}x_2 = \frac{V_p}{RC} - \frac{1}{RC}x_2$$

$$x_2(t) = V_d + \left(x_2(0) - V_d\right)e^{-\frac{t}{RC}}$$

$t \to \infty,\ x_2 = V_d$



1.4. Case study for ekis mice

$$s > 0 : \dot{s} = \dot{x}_1 = -\frac{1}{L}x_2 + \frac{V_0}{L}u^+ \qquad u^+ = 0$$

$$= -\frac{1}{L}x_2 < 0$$

$$\Omega_1 = \{x_2 : x_2 > 0\}$$

$$s < 0 : \dot{s} = \dot{x}_1 = -\frac{1}{L}x_2 + \frac{V_0}{L}u^-, \quad u^- = 0$$

$$= -\frac{1}{L}x_2 + \frac{V_0}{L} > 0$$

$$\Omega_2 := \{x_2 : x_2 < V_0\}$$

$$\lim_{s \to 0^+} -\frac{1}{L}x_2 < 0 \qquad \Omega_3 = \Omega_1$$

limit value equals to itself

$$\lim_{s \to 0^-} -\frac{1}{L}x_2 + \frac{V_0}{L} > 0 . \quad \Omega_4 = \Omega_2$$

$$\Omega = \Omega_1 \wedge \Omega_2 \wedge \Omega_3 \wedge \Omega_4 = [x_2 : 0 < x_2 < V_0]$$

In Ω sliding mode exists

Additional excerase:

1) $-2x_1^* - 6u x_1^* = 0$ if $u \neq -\frac{1}{3}$ $x_1^* = 0$

$x_2^* = 0$

2) $u = \frac{1}{3}$ second equation holds

forever $\to$ no information

$2x_1^* + 3x_2^* = 0$

$x_1^* = t,\ x_2^* = -\frac{2}{3}t$ $x^* = \begin{bmatrix} t \\ -\frac{2}{3}t \end{bmatrix}$

2.2 - Lyapunov indirect method

if $Re(\lambda(A)) < 0$ then the system would be stable

$\Rightarrow$ Approximation with Taylor

1. Determine matrix A
2. Determine $\lambda(A)$ solving $|\lambda I - A| = 0$
3. Check weather all $Re(\lambda(A)) < 0$

Z.3. $S = x_1 + x_2$, $u = \text{sign}(x_1 s)$

$s > 0$: $\dot{s} = \dot{x}_1 + \dot{x}_2 = \dots$

$x_1 < 0$, $x_1 s < 0$, $u = \text{sign}(x_1 s) = -1$

Then $\dot{s} =$

Ω_1

$s < 0$: $\dot{s} = \dots$

Ω_2

Limits:

Z4. α:

Exercise 3: Simple Inverted Pendulum

Consider an inverted pendulum, which is described by the following differential equations:

$$J\ddot{\theta} - mgl\sin\theta = \tau \qquad J,\ m,\ g,\ l > 0$$

3.1 Use the control signal $\tau = -\tau_0\,\mathrm{sign}(s_1)$ and the sliding manifold $s_1 = c_1\dot{\theta} + c_2\theta,\ c_1, c_2 > 0$. Investigate the existence of sliding mode.

As the torque τ of the real system cannot be discontinuous, the above system model is extended by the dynamics of a DC motor:

$$L\frac{di}{dt} + Ri + K_n\dot{\theta} = u \qquad \tau = K_{\mathrm{m}}i$$

$L,\ R,\ K_n,\ K_{\mathrm{m}} > 0$ are motor parameters, i is the motor current and u is the input voltage.

3.2 Determine the current i^* which forces the following pendulum dynamics:

$$\ddot{\theta} = -\alpha_1\theta - \alpha_2\dot{\theta}\ .$$

$$\boxed{\text{3.3 Investigate the existence and stability of sliding mode for the controller } u = -u_0\,\mathrm{sign}(s_2) \text{ with } s_2 = i - i^*.}$$

This formula is super important!

Additional Exercise: System with Disturbance

Consider the following second order system

$$\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 \\ \dot{x}_2 &= -2x_1 - 6x_1(u+d) \end{aligned} \tag{1}$$

with states $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, input signal $u \in \mathbb{R}$ and disturbance $d \in \mathbb{R}$. For the disturbance, $\underline{d = 0}$ initially holds.

Z.1 Determine the equilibrium states $\underline{x}^*$ of the system depending on the input u, which is considered to be constant.
Hint: Perform a case analysis for u.

In the following, the system (1) should be stabilized using the switching controller

$$u = \mathrm{sign}(x_1 s) \tag{2}$$

on the sliding surface

$$s = x_1 + x_2 \tag{3}$$

Z.2 Analyze the stability of the equilibrium point $\underline{x}^* = \underline{0}$ of system (1) using Lyapunov's indirect method for $u = 1$ and $u = -1$, respectively.

Z.3 Determine the domain $\mathcal{B} \subset \mathbb{R}^2$ in the state space in which sliding mode can exist for $x_1 < 0$ using the condition $s\dot{s} < 0$ and the sliding surface (3). Mark the area in the diagram below.
Hint: The non-disturbed case $d = 0$ is considered.

Z.4 Determine the dynamics in sliding mode $(s = 0)$ using Filippov's method and simplify as much as possible.

Z.5 Is the dynamics in sliding mode Lyapunov stable, asymptitically Lyapunov stable, or instable? Justify your choice.

A limited disturbance $d_- < d < d_+$ is now acting on the system.

Z.6 Determine the (constant) values of d_- und d_+ for which the system can still reach sliding mode.
Hint: Use the boudary value requirements and only consider the case $x_1 < 0$.

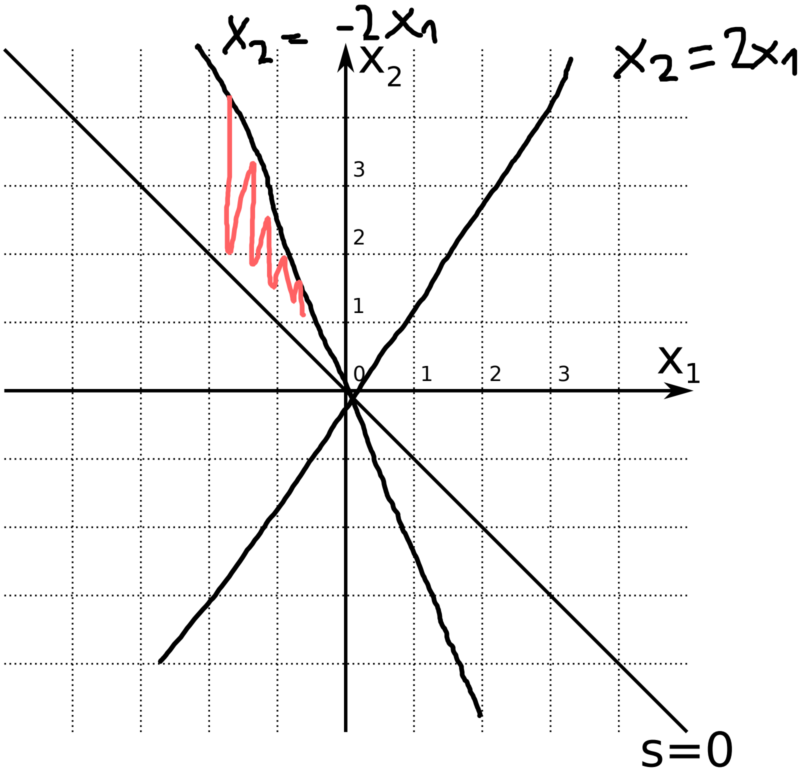


Figure 1: Coordinate frame for exercise Z.3