

Exercise 1: Sliding Mode Controller

A DC/DC converter is supposed to be controlled using sliding mode. A desired output voltage should be reached by appropriately driving a transistor. The system is given by the following equations:

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{L}x_2 + \frac{V_0}{L}u \\ \dot{x}_2 &= \frac{1}{C}x_1 - \frac{1}{RC}x_2\end{aligned}$$

This is a test!

where  $R, L, C, V_0 > 0$ .

The control signal  $u$  can only take on binary values, i.e.,  $u \in \{0, 1\}$ .

The state  $x_1$  denotes the current,  $x_2$  denotes the output voltage. The constant input voltage is  $V_0$ .

- 1.1 Calculate the current  $x_1^*$  which is required in steady-state to achieve the desired putput voltage  $x_2^* = V_d$ .

Now consider  $s = x_1 - x_1^*$  and  $u = \frac{1}{2}(1 - \text{sgn}(s))$ .

- 1.2 Calculate the system behavior  $\dot{\underline{x}}_{av} = \underline{f}_{av}(\underline{x})$ ,  $\underline{x} = [x_1 \ x_2]^\top$  in  $s = 0$  using Filippov's method.

- 1.3 Sketch the behavior of  $x_2(t)$  after reaching  $s = 0$ .

- 1.4 Check for existence of sliding mode.

Exercise 2:

Consider the nonlinear second order system

$$\dot{\underline{x}} = A\underline{x} + \underline{b}(\underline{x})u, \quad \underline{x} \in \mathbb{R}^2, \quad u \in \mathbb{R}$$

$$A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} \sin^2 y \\ \cos^2 y \end{bmatrix}, \quad y = [1 \ 1] \underline{x}.$$

- 2.1 Design a sliding mode controller (target value  $y = 0$ ).

- 2.2 Investigate the behavior on  $s = 0$  using equivalent control.

Exercise:

$$\dot{\underline{x}} = 0$$

$$\dot{\underline{x}}^* = 0$$

Determine EP for unforced  
System:  $u = 0$

Filippov's method:

$$\dot{x}_{av} = \alpha f^+ + (1 - \alpha) f^-$$

$$\alpha = \frac{\frac{ds}{dx} f^-}{\frac{ds}{dx} (f^- - f^+)}$$

$$s = x_1 - x_1^*$$

$$\frac{ds}{dx} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$

$$\text{new controller } u^+ = \frac{1}{2} (1 - 1) = 0$$

$$u^- = \frac{1}{2} (1 + 1) = 1$$

$$f^+ = \begin{bmatrix} -\frac{1}{L} x_2 \\ \frac{1}{C} x_1 - \frac{1}{RC} x_2 \end{bmatrix}$$

$$f^- = \begin{bmatrix} -\frac{1}{L} x_2 + \frac{u_b}{L} \\ \frac{1}{C} x_1 - \frac{1}{RC} x_2 \end{bmatrix}$$

$$\alpha = \frac{-\frac{1}{L}x_2 + \frac{V_b}{L}}{\frac{V_o}{L}}$$

$$\dot{x}_{av} = \begin{bmatrix} eq_1 \\ eq_2 \end{bmatrix} = \dots$$

$$u^-, u^+ : s > 0 : u = \frac{1}{2} (1 - \text{sign}(s))$$

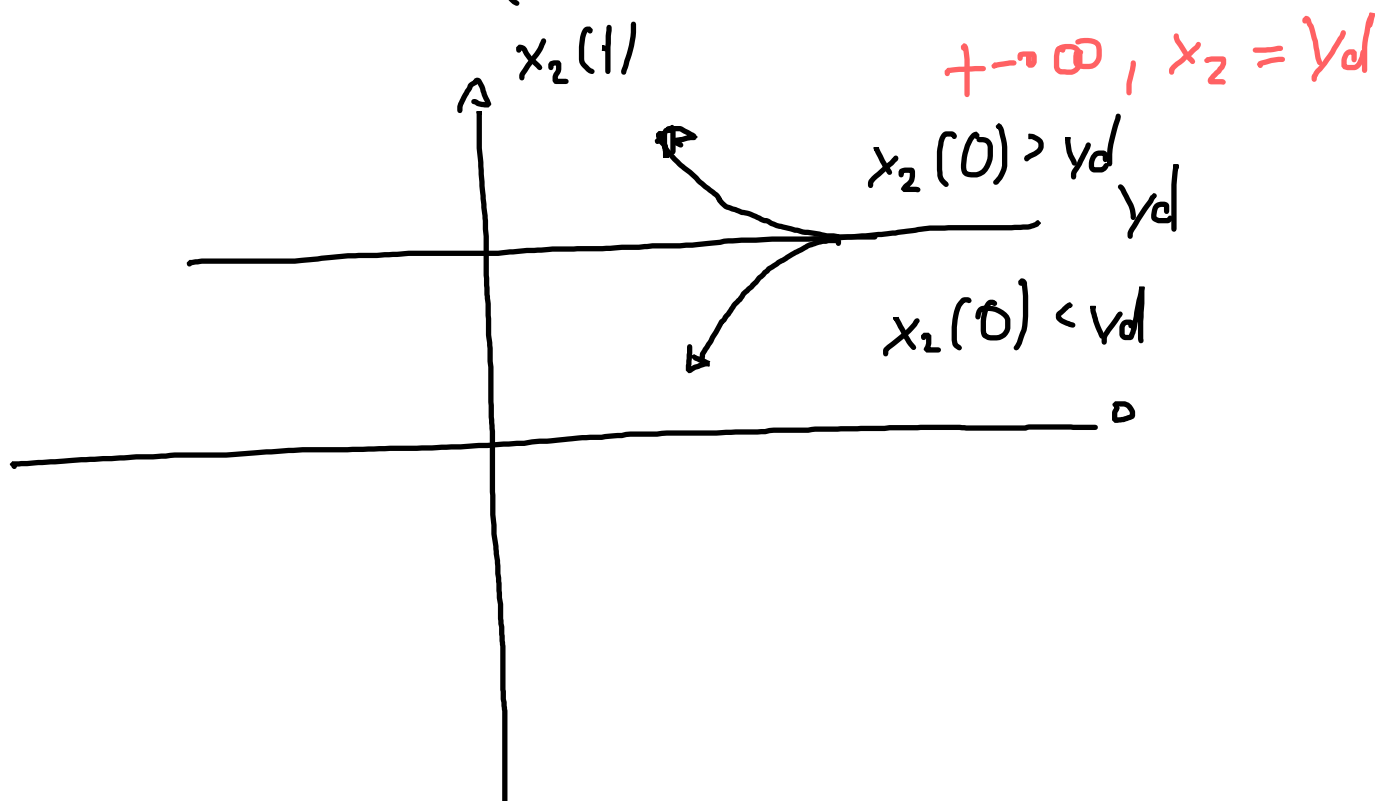
$$s < 0 : u = \frac{1}{2} (1 + \text{sign}(s))$$

1.3. Sketch behaviour of  $x_2(t)$

$$s = 0 : x_1 = \lambda_1^* = \frac{V_p}{R}$$

$$\dot{x}_2 = \frac{1}{C} x_1 - \frac{1}{RC} x_2 = \frac{V_p}{RC} - \frac{1}{RC} x_2$$

$$x_2(t) = V_d + (x_2(0) - V_d) e^{-\frac{t}{RC}}$$



# 1.4. Case study for existence

$$S > 0 : \dot{S} = \dot{x}_1 = -\frac{1}{L}x_2 + \frac{V_0}{L}u^+ \quad u^+ = 0 \\ = -\frac{1}{L}x_2 < 0$$

$$\Omega_1 = \{x_2 : x_2 > 0\}$$

$$S < 0 : \dot{S} = \dot{x}_1 = -\frac{1}{L}x_2 + \frac{V_0}{L}u^-, \quad u^- = 0 \\ = -\frac{1}{L}x_2 + \frac{V_0}{L} > 0$$

$$\Omega_2 = \{x_2 : x_2 < V_0\}$$

$$\lim_{S \rightarrow 0^+} -\frac{1}{L}x_2 < 0 \quad \Omega_3 = \Omega_1$$

limit value equals to itself

$$\lim_{S \rightarrow 0^-} -\frac{1}{L}x_2 + \frac{V_0}{L} > 0 \quad \Omega_4 = \Omega_2$$

$$\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 = [x_2 : 0 < x_2 < V_0]$$

In  $\Omega$  sliding mode exists

Additional exercise:

$$1) -2x_1^* - 6u x_1^* = 0 \quad \text{if } u \neq -\frac{1}{3} \quad x_1^* = 0$$

$$x_2^* = 0$$

2)  $u = \frac{1}{3}$  second equation holds  
forever  $\rightarrow$  no information

$$2x_1^* + 3x_2^* = 0$$

$$x_1^* = t, \quad x_2^* = -\frac{2}{3}t \quad x^* = \begin{bmatrix} t \\ -\frac{2}{3}t \end{bmatrix}$$

2.2 - Lyapunov indirect method

if  $\operatorname{Re}(\lambda(A)) < 0$  then the system  
would be stable

2. Approximation with Taylor

1. Determine matrix  $A$
2. Determine  $\lambda(A)$  solving  $|\lambda I - A| = 0$
3. Check whether all  $\operatorname{Re}(\lambda(A)) < 0$

$$Z.3. \quad S = x_1 + x_2, \quad u = \text{sign}(x_1 s)$$

$$s > 0: \quad \dot{s} = \dot{x}_1 + \dot{x}_2 = \dots$$

$$x_1 < 0, \quad x_1 s < 0, \quad u = \text{sign}(x_1 s) = -1$$

$$\text{Then } \dot{s} =$$

$$\Omega_1$$

$$s < 0: \quad \dot{s} = \dots$$

$$\Omega_2$$

Limits:

$$Z.4. \quad \alpha:$$

### Exercise 3: Simple Inverted Pendulum

Consider an inverted pendulum, which is described by the following differential equations:

$$J\ddot{\theta} - mgl \sin \theta = \tau \quad J, m, g, l > 0$$

- 3.1 Use the control signal  $\tau = -\tau_0 \operatorname{sign}(s_1)$  and the sliding manifold  $s_1 = c_1\dot{\theta} + c_2\theta$ ,  $c_1, c_2 > 0$ . Investigate the existence of sliding mode.

As the torque  $\tau$  of the real system cannot be discontinuous, the above system model is extended by the dynamics of a DC motor:

$$L \frac{di}{dt} + Ri + K_n \dot{\theta} = u \quad \tau = K_m i$$

$L, R, K_n, K_m > 0$  are motor parameters,  $i$  is the motor current and  $u$  is the input voltage.

- 3.2 Determine the current  $i^*$  which forces the following pendulum dynamics:

$$\ddot{\theta} = -\alpha_1 \theta - \alpha_2 \dot{\theta}.$$

- 3.3 Investigate the existence and stability of sliding mode for the controller  $u = -u_0 \operatorname{sign}(s_2)$  with  $s_2 = i - i^*$ .

Look at this.

### Additional Exercise: System with Disturbance

Consider the following second order system

$$\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 \\ \dot{x}_2 &= -2x_1 - 6x_2(u + d) \end{aligned} \tag{1}$$

with states  $x_1 \in \mathbb{R}$ ,  $x_2 \in \mathbb{R}$ , input signal  $u \in \mathbb{R}$  and disturbance  $d \in \mathbb{R}$ . For the disturbance,  $\underline{d=0}$  initially holds.

- Z.1 Determine the equilibrium states  $\underline{x}^*$  of the system depending on the input  $u$ , which is considered to be constant.

Hint: Perform a case analysis for  $u$ .

In the following, the system (1) should be stabilized using the switching controller

$$u = \operatorname{sign}(x_1 s) \tag{2}$$

on the sliding surface

$$s = x_1 + x_2 \tag{3}$$

- Z.2 Analyze the stability of the equilibrium point  $\underline{x}^* = \underline{0}$  of system (1) using Lyapunov's indirect method for  $u = 1$  and  $u = -1$ , respectively.

Z.3 Determine the domain  $\mathcal{B} \subset \mathbb{R}^2$  in the state space in which sliding mode can exist for  $x_1 < 0$  using the condition  $s\dot{s} < 0$  and the sliding surface (3). Mark the area in the diagram below.

Hint: The non-disturbed case  $d = 0$  is considered.

Z.4 Determine the dynamics in sliding mode ( $s = 0$ ) using Filippov's method and simplify as much as possible.

Z.5 Is the dynamics in sliding mode Lyapunov stable, asymptotically Lyapunov stable, or instable? Justify your choice.

A limited disturbance  $d_- < d < d_+$  is now acting on the system.

Z.6 Determine the (constant) values of  $d_-$  and  $d_+$  for which the system can still reach sliding mode.

Hint: Use the boundary value requirements and only consider the case  $x_1 < 0$ .

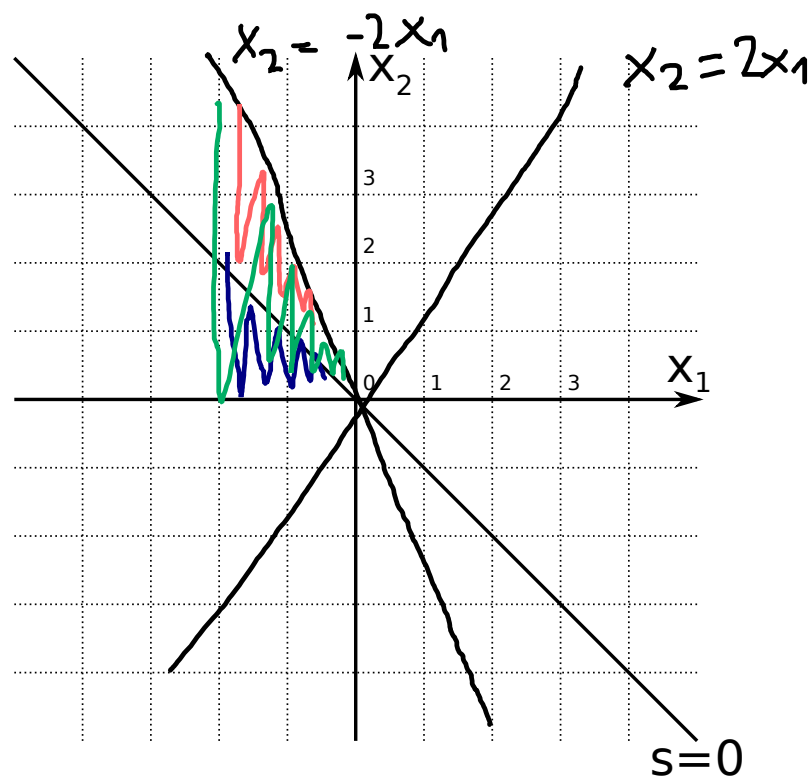


Figure 1: Coordinate frame for exercise Z.3