

Untyped Lambda Calculus

Structure and Evaluation, Currying, Church Encodings

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November 25, 2018

Dream IT

<https://dreamit.de>

Introduction

Lambda Calculus

- Invented by Alonzo Church (1920s)
- Equally expressive to the Turing Machine(s)
- Formal Language
- Computational Model
 - Lisp (1950s)
 - ML
 - Haskell
- "Lambda Expressions" in almost every modern programming language

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- Simple Computational Model
 - to describe structure and behaviour (E.g. Operational Semantics, Type Systems)
 - to reason and prove

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- Explains why things in FP are like they are
 - Pure Functions
 - Higher-Order Functions
 - Currying
 - Lazy Evaluation

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 - to describe structure and behaviour (E.g. Operational Semantics, Type Systems)
 - to reason and prove
- Explains why things in FP are like they are
 - Pure Functions
 - Higher-Order Functions
 - Currying
 - Lazy Evaluation
- Understand FP Compilers
 - Introduce FP stuff into other languages
 - Write your own compiler
 - GHC uses an enriched Lambda Calculus internally

Basics

Untyped Lambda Calculus

$t ::=$

x

$\lambda x. t$

$t \ t$

Terms:

Variable

Abstraction

Application

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Example - Identity

Lambda Calculus

$$\underbrace{\underbrace{\lambda x. x}_{\text{Abstraction}} \quad \underbrace{y}_{\text{Variable}}}_{\text{Application}} \rightarrow y$$

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Lambda Calculus

$\underbrace{\underbrace{\lambda x. x}_{\text{Abstraction}} \underbrace{y}_{\text{Variable}}}_{\text{Application}} \rightarrow y$

Javascript

$\underbrace{\underbrace{(\text{function } (x) \{ \text{return } x; \})}_{\text{Abstraction}} \underbrace{(y)}_{\text{Variable}}}_{\text{Application}}$

Example - $(\lambda x. \lambda y. x y) a b$

Abstractions

Think: Function Definitions

$(\lambda x. \lambda y. x y)$ $a b$

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Variables

Think: Parameters

$(\lambda x. \lambda y. \underline{x} \underline{y}) \underline{a} \underline{b}$

Example - $(\lambda x. \lambda y. x \ y) \ a \ b$

Abstractions

Think: Function Definitions

$(\lambda x. \lambda y. x \ y)$ $a \ b$

Variables

Think: Parameters

$(\lambda x. \lambda y. \underline{x} \ \underline{y}) \ \underline{a} \ \underline{b}$

Applications

Think: Function Calls

$(\lambda x. \lambda y. x \ y)$ $\underline{a} \ \underline{b}$

Example - $(\lambda x. \lambda y. x \ y) \ a \ b$

$(\lambda x. \quad \lambda y. x \quad y) \quad a \quad b$

Example - $(\lambda x. \lambda y. x \ y) \ a \ b$

$(\lambda x. \quad \lambda y. x \quad y) \quad a \quad b$ Substitute $x \mapsto a$

Example - $(\lambda x. \lambda y. x y) a b$

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Example - $(\lambda x. \lambda y. x y) a b$

$$\begin{array}{llll} (\lambda x. & \lambda y. x & y) & a \quad b \quad \text{Substitute } x \mapsto a \\ \rightarrow & (\lambda y. a & y) & b \end{array}$$

Example - $(\lambda x. \lambda y. x \ y) \ a \ b$

$$\begin{array}{llll} (\lambda x. \lambda y. x \ y) \ a \ b & \text{Substitute } x \mapsto a \\ \rightarrow (\lambda y. a \ y) \ b & \text{Substitute } y \mapsto b \end{array}$$

Example - $(\lambda x. \lambda y. x \ y) \ a \ b$

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 $\rightarrow (\lambda y. a \ y) \ b$ Substitute $y \mapsto b$
 $\rightarrow a \ b$

Notational Conventions

- We use parentheses to clarify what's meant
- Applications associate to the left

$$s\ t\ u \equiv (s\ t)\ u$$

- Abstractions expand as much to the right as possible

$$\lambda x. \lambda y. x\ y\ x \equiv \lambda x. (\lambda y. (x\ y\ x))$$

$\lambda x. \lambda y. x \ y \ z$

Bound and Free

λy y is *bound*, x and z are *free*

λx x and y are *bound*, z is *free*

$\lambda x, \lambda y$ *binders*

$\lambda x. \lambda y. x \ y \ z$

Bound and Free

λy y is *bound*, x and z are *free*

λx x and y are *bound*, z is *free*

$\lambda x, \lambda y$ *binders*

A term with no free variables is *closed*

- A *combinator*
- $id \equiv \lambda x. x$
- $Y, S, K, I \dots$

Higher Order Functions

- Functions that take or return functions
 - Are there "by definition"

$$\underbrace{\underbrace{\lambda x.x}_{\text{Abstraction}} \underbrace{\lambda y.y}_{\text{Abstraction}}}_{\text{Application}} \rightarrow \underbrace{\lambda y.y}_{\text{Abstraction}}$$

Idea

- Take a function with n arguments
- Create a function that takes one argument and returns a function with $n - 1$ arguments

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Example

- $(+1)$ Section in Haskell
- $(\lambda x. \lambda y. + x y) 1 \rightarrow \lambda y. + 1 y$

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- Take a function with n arguments
- Create a function that takes one argument and returns a function with $n - 1$ arguments

Example

- (+1) Section in Haskell
- $(\lambda x. \lambda y. + x y) 1 \rightarrow \lambda y. + 1 y$
- Partial Function Application is there "by definition"
 - You can use this stunt to "curry" in every language that supports "Lambda Expressions"

Alpha Conversion

$$\lambda x.x \rightarrow_{\alpha} \lambda y.y$$

Reductions and Conversions

Alpha Conversion

$$\lambda x.x \rightarrow_{\alpha} \lambda y.y$$

Beta Reduction

$$(\lambda x.x) y \rightarrow_{\beta} y$$

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Eta Conversion

Iff (if and only if) x is not free in f :

$$\begin{array}{c} \lambda x.f x \rightarrow_{\eta} f \\ (\lambda x. \underbrace{(\lambda y.y)}_f x) a \rightarrow_{\eta} \underbrace{(\lambda y.y)}_f a \end{array}$$

Reductions and Conversions

Alpha Conversion

$$\lambda x.x \rightarrow_{\alpha} \lambda y.y$$

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$$(\lambda x.x) y \rightarrow_{\beta} y$$

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$$\begin{aligned} \lambda x.f x &\rightarrow_{\eta} f \\ (\lambda x. \underbrace{(\lambda y.y)}_f x) a &\rightarrow_{\eta} \underbrace{(\lambda y.y)}_f a \end{aligned}$$

If x is free in f , η conversion not possible:

$$\lambda x. \underbrace{(\lambda y.y \overset{\text{Bound}}{\downarrow} x)}_f x \not\rightarrow_{\eta} (\lambda y.y \overset{\text{Free?!}}{\downarrow} x)$$

- Everything (Term) is an Expression
 - No statements
- No "destructive" Assignments
 - The reason why FP Languages promote pure functions
 - But you could invent a built-in function to manipulate "state"...

Evaluation

- We learned how to write down and talk about Lambda Calculus Terms
- How to evaluate them?
- Different Strategies
 - Interesting outcomes

Full Beta-Reduction

- RedEx
 - **R**educible **E**xpression
 - Always an Application

$$\begin{array}{c} (\lambda x.x) ((\lambda y.y) (\lambda z. (\lambda a.a) z)) \\ \underbrace{\hspace{10em}}_{\text{RedEx}} \\ \underbrace{\hspace{10em}}_{\text{RedEx}} \\ \underbrace{\hspace{10em}}_{\text{RedEx}} \end{array}$$

Full Beta-Reduction

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 - Always an Application

$$\underbrace{(\lambda x.x) \underbrace{((\lambda y.y) \underbrace{(\lambda z. (\lambda a.a) z))}_{RedEx})}_{RedEx}}_{RedEx}$$

Full Beta-Reduction

- Any RedEx, Any Time
- Like in Arithmetics
- Too vague for programming...

$$(\lambda x.x) ((\lambda y.y) (\lambda z.(\lambda a.a) z))$$

Normal Order Reduction

- Left-most, Outer-most RedEx

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Normal Order Reduction

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$$\begin{aligned} & (\lambda x.x) ((\lambda y.y) (\lambda z.(\lambda a.a) z)) \\ \rightarrow & (\lambda y.y) (\lambda z.(\lambda a.a) z) \end{aligned}$$

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Normal Order Reduction

- Left-most, Outer-most RedEx

$$(\lambda x.x) ((\lambda y.y) (\lambda z.(\lambda a.a) z))$$

Call-by-Name

- like Normal Order Reduction, but **no reductions inside Abstractions**
 - Abstractions are values
- lazy, non-strict
 - **Parameters are not evaluated before they are used**
- Optimization: Save results \rightarrow *Call-by-Need*

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Call-by-Value

- Outer-most, only if right-hand side was reduced to a value
- No reductions inside Abstractions
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- eager, strict
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Church Encodings

- Encode Data into the Lambda Calculus
- To simplify our formulas, let's say that we have declarations

$$id \equiv \lambda x.x$$

$$id\ y \rightarrow y$$

Booleans

true \equiv $\lambda t.\lambda f.t$

false \equiv $\lambda t.\lambda f.f$

Booleans

true \equiv $\lambda t.\lambda f.t$

false \equiv $\lambda t.\lambda f.f$

test \equiv $\lambda c.\lambda t.\lambda f.c\ t\ f$

Booleans

test true a b

true \equiv $\lambda t.\lambda f.t$

false \equiv $\lambda t.\lambda f.f$

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Booleans

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$test \equiv \lambda c. \lambda t. \lambda f. c \ t \ f$

$test \ true \ a \ b$
 $\equiv (\lambda c. \lambda t. \lambda f. c \ t \ f) \ true \ a \ b$

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$\rightarrow true \ a \ b$

$\equiv (\lambda t. \lambda f. t) \ a \ b$

$\rightarrow (\lambda f. a) \ b$

$\rightarrow a$

And

true \equiv $\lambda t.\lambda f.t$

false \equiv $\lambda t.\lambda f.f$

And

$true \equiv \lambda t. \lambda f. t$

$false \equiv \lambda t. \lambda f. f$

$and \equiv \lambda p. \lambda q. p \ q \ p$

And

and true false

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$\rightarrow (\lambda q.\text{true}\ q\ \text{true})\ \text{false}$

$\rightarrow \text{true}\ \text{false}\ \text{true}$

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$true \equiv \lambda t. \lambda f. t$

$false \equiv \lambda t. \lambda f. f$

$and \equiv \lambda p. \lambda q. p \ q \ p$

$and \ true \ false$

$\equiv (\lambda p. \lambda q. p \ q \ p) \ true \ false$

$\rightarrow (\lambda q. true \ q \ true) \ false$

$\rightarrow true \ false \ true$

$\equiv (\lambda t. \lambda f. t) \ false \ true$

$\rightarrow (\lambda f. false) \ true$

$\rightarrow false$

Or

$\lambda p. \lambda q. p \ p \ q$

$$\textit{pair} \equiv \lambda x. \lambda y. \lambda z. z \times y$$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

Pairs

$pair_{AB} \equiv$

$pair\ a\ b$

$pair \equiv \lambda x. \lambda y. \lambda z. z\ x\ y$

$first \equiv (\lambda p. p)\ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p)\ \lambda x. \lambda y. y$

Pairs

$pair_{AB} \equiv$

pair *a b*

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

$pair_{AB} \equiv$ *pair* $a \ b$
 $\equiv (\lambda x. \lambda y. \lambda z. z \ x \ y) \ a \ b$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

$pair_{AB} \equiv$ $pair \ a \ b$
 \equiv $(\lambda x. \lambda y. \lambda z. z \ x \ y) \ a \ b$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

$pair_{AB} \equiv$ $pair \ a \ b$
 \equiv $(\lambda x. \lambda y. \lambda z. z \ x \ y) \ a \ b$
 \rightarrow $(\lambda y. \lambda z. z \ a \ y) \ b$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

$pair_{AB} \equiv$ $pair \ a \ b$
 \equiv $(\lambda x. \lambda y. \lambda z. z \ x \ y) \ a \ b$
 \rightarrow $(\lambda y. \lambda z. z \ a \ y) \ b$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

$pair_{AB} \equiv$ $pair \ a \ b$

\equiv $(\lambda x. \lambda y. \lambda z. z \ x \ y) \ a \ b$

\rightarrow $(\lambda y. \lambda z. z \ a \ y) \ b$

\rightarrow $\lambda z. z \ a \ b$

Pairs

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$second \equiv (\lambda p. p) \ \lambda x. \lambda y. y$

$pair_{AB} \equiv$ $pair \ a \ b$
 \equiv $(\lambda x. \lambda y. \lambda z. z \ x \ y) \ a \ b$
 \rightarrow $(\lambda y. \lambda z. z \ a \ y) \ b$
 \rightarrow $\lambda z. z \ a \ b$
 \equiv $pair'_{ab}$

Pairs (continued)

$$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$$

$$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$$

$$pair'_{ab} \equiv \lambda z. z \ a \ b$$

Pairs (continued)

first *pair'*_{ab}

pair $\equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

first $\equiv (\lambda p. p) \ \lambda x. \lambda y. x$

*pair'*_{ab} $\equiv \lambda z. z \ a \ b$

Pairs (continued)

first $pair'_{ab}$

$pair \equiv \lambda x. \lambda y. \lambda z. z \ x \ y$

$first \equiv (\lambda p. p) \ \lambda x. \lambda y. x$

$pair'_{ab} \equiv \lambda z. z \ a \ b$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ &\equiv (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv &(\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \end{aligned}$$

Pairs (continued)

$$\begin{aligned} \text{pair} &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ \text{first} &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ \text{pair}'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &\text{first } \text{pair}'_{ab} \\ &\equiv (\lambda p. p) \ (\lambda x. \lambda y. x) \ \text{pair}'_{ab} \\ &\rightarrow \text{pair}'_{ab} \ (\lambda x. \lambda y. x) \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv &(\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ \rightarrow &pair'_{ab} \ (\lambda x. \lambda y. x) \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv & (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ \rightarrow & pair'_{ab} \ (\lambda x. \lambda y. x) \\ \equiv & (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv & (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ \rightarrow & pair'_{ab} \ (\lambda x. \lambda y. x) \\ \equiv & (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \end{aligned}$$

Pairs (continued)

$$\begin{aligned} \text{pair} &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ \text{first} &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ \text{pair}'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &\text{first } \text{pair}'_{ab} \\ &\equiv (\lambda p. p) \ (\lambda x. \lambda y. x) \ \text{pair}'_{ab} \\ &\rightarrow \text{pair}'_{ab} \ (\lambda x. \lambda y. x) \\ &\equiv (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \\ &\rightarrow (\lambda x. \lambda y. x) \ a \ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv & (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ \rightarrow & pair'_{ab} \ (\lambda x. \lambda y. x) \\ \equiv & (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \\ \rightarrow & (\lambda x. \lambda y. x) \ a \ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv & (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ \rightarrow & pair'_{ab} \ (\lambda x. \lambda y. x) \\ \equiv & (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \\ \rightarrow & (\lambda x. \lambda y. x) \ a \ b \\ \rightarrow & (\lambda y. a) \ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ \equiv & (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ \rightarrow & pair'_{ab} \ (\lambda x. \lambda y. x) \\ \equiv & (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \\ \rightarrow & (\lambda x. \lambda y. x) \ a \ b \\ \rightarrow & (\lambda y. a) \ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x. \lambda y. \lambda z. z \ x \ y \\ first &\equiv (\lambda p. p) \ \lambda x. \lambda y. x \\ pair'_{ab} &\equiv \lambda z. z \ a \ b \end{aligned}$$
$$\begin{aligned} &first \ pair'_{ab} \\ &\equiv (\lambda p. p) \ (\lambda x. \lambda y. x) \ pair'_{ab} \\ &\rightarrow pair'_{ab} \ (\lambda x. \lambda y. x) \\ &\equiv (\lambda z. z \ a \ b) \ (\lambda x. \lambda y. x) \\ &\rightarrow (\lambda x. \lambda y. x) \ a \ b \\ &\rightarrow (\lambda y. a) \ b \\ &\rightarrow a \end{aligned}$$

Numerals

Peano Axioms

Every natural number can be defined with 0 and a successor function

$$0 \equiv \lambda f. \lambda x. x$$

$$1 \equiv \lambda f. \lambda x. f \ x$$

$$2 \equiv \lambda f. \lambda x. f \ (f \ x)$$

$$3 \equiv \lambda f. \lambda x. f \ (f \ (f \ x))$$

Meaning

0 f is evaluated 0 times

1 f is evaluated once

x can be every lambda term

Numerals Example - Successor

$0 \equiv \lambda f. \lambda x. x$

$1 \equiv \lambda f. \lambda x. f\ x$

Numerals Example - Successor

$$0 \equiv \lambda f. \lambda x. x$$

$$1 \equiv \lambda f. \lambda x. f\ x$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f\ (n\ f\ x)$$

Numerals Example - Successor

successor 1

$0 \equiv \lambda f. \lambda x. x$

$1 \equiv \lambda f. \lambda x. f\ x$

$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f\ (n\ f\ x)$

Numerals Example - Successor

successor 1

$0 \equiv \lambda f. \lambda x. x$

$1 \equiv \lambda f. \lambda x. f\ x$

$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f\ (n\ f\ x)$

Numerals Example - Successor

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x && \equiv \text{successor } 1 \\ 1 &\equiv \lambda f. \lambda x. f \, x && \equiv (\lambda n. \lambda f. \lambda x. f \, (n \, f \, x)) \, 1 \end{aligned}$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f \, (n \, f \, x)$$

Numerals Example - Successor

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x && \equiv \text{successor } 1 \\ 1 &\equiv \lambda f. \lambda x. f \, x && \equiv (\lambda n. \lambda f. \lambda x. f \, (n \, f \, x)) \, 1 \end{aligned}$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f \, (n \, f \, x)$$

Numerals Example - Successor

$$\begin{array}{llll} & & & \text{successor } 1 \\ 0 \equiv & \lambda f. \lambda x. x & \equiv & (\lambda n. \lambda f. \lambda x. f (n f x)) 1 \\ 1 \equiv & \lambda f. \lambda x. f x & \rightarrow & \lambda f. \lambda x. f (1 f x) \end{array}$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f (n f x)$$

Numerals Example - Successor

$$\begin{array}{llll} & & & \text{successor } 1 \\ 0 \equiv & \lambda f. \lambda x. x & \equiv & (\lambda n. \lambda f. \lambda x. f (n f x)) \ 1 \\ 1 \equiv & \lambda f. \lambda x. f x & \rightarrow & \lambda f. \lambda x. f (1 f x) \end{array}$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f (n f x)$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$\begin{array}{llll} & & & \text{successor } 1 \\ 0 \equiv & \lambda f. \lambda x. x & \equiv & (\lambda n. \lambda f. \lambda x. f (n \ f \ x)) \ 1 \\ 1 \equiv & \lambda f. \lambda x. f \ x & \rightarrow & \lambda f. \lambda x. f \ (\textcolor{brown}{1} \ f \ x) \\ & & \equiv & \lambda f. \lambda x. f \ ((\lambda f. \lambda x. f \ x) \ f \ x) \end{array}$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f \ (n \ f \ x)$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$\begin{array}{llll} & & & \text{successor } 1 \\ 0 \equiv & \lambda f. \lambda x. x & \equiv & (\lambda n. \lambda f. \lambda x. f (n f x)) 1 \\ 1 \equiv & \lambda f. \lambda x. f x & \rightarrow & \lambda f. \lambda x. f (1 f x) \\ & & \equiv & \lambda f. \lambda x. f ((\lambda f. \lambda x. f x) f x) \end{array}$$

$$\text{successor} \equiv \lambda n. \lambda f. \lambda x. f (n f x)$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x && \equiv \text{successor } 1 \\ 1 &\equiv \lambda f. \lambda x. f \ x && \equiv (\lambda n. \lambda f. \lambda x. f \ (n \ f \ x)) \ 1 \\ &&& \rightarrow \lambda f. \lambda x. f \ (1 \ f \ x) \\ &&& \equiv \lambda f. \lambda x. f \ ((\lambda f. \lambda x. f \ x) \ f \ x) \\ &&& \rightarrow \lambda f. \lambda x. f \ ((\lambda x. f \ x) \ x) \\ \text{successor} &\equiv \lambda n. \lambda f. \lambda x. f \ (n \ f \ x) \end{aligned}$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x && \equiv \text{successor } 1 \\ 1 &\equiv \lambda f. \lambda x. f \ x && \equiv (\lambda n. \lambda f. \lambda x. f \ (n \ f \ x)) \ 1 \\ &&& \rightarrow \lambda f. \lambda x. f \ (1 \ f \ x) \\ &&& \equiv \lambda f. \lambda x. f \ ((\lambda f. \lambda x. f \ x) \ f \ x) \\ &&& \rightarrow \lambda f. \lambda x. f \ ((\lambda x. f \ x) \ x) \\ \text{successor} &\equiv \lambda n. \lambda f. \lambda x. f \ (n \ f \ x) \end{aligned}$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$\begin{array}{llll} & & & \text{successor } 1 \\ 0 \equiv & \lambda f. \lambda x. x & \equiv & (\lambda n. \lambda f. \lambda x. f (n \ f \ x)) \ 1 \\ 1 \equiv & \lambda f. \lambda x. f \ x & \rightarrow & \lambda f. \lambda x. f (1 \ f \ x) \\ & & \equiv & \lambda f. \lambda x. f ((\lambda f. \lambda x. f \ x) \ f \ x) \\ \text{successor} \equiv & \lambda n. \lambda f. \lambda x. f (n \ f \ x) & \rightarrow & \lambda f. \lambda x. f ((\lambda x. f \ x) \ x) \\ & & \rightarrow & \lambda f. \lambda x. f (f \ x) \end{array}$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$\begin{array}{llll} & & & \text{successor } 1 \\ 0 \equiv & \lambda f. \lambda x. x & \equiv & (\lambda n. \lambda f. \lambda x. f (n \ f \ x)) \ 1 \\ & & \rightarrow & \lambda f. \lambda x. f (1 \ f \ x) \\ 1 \equiv & \lambda f. \lambda x. f \ x & \equiv & \lambda f. \lambda x. f ((\lambda f. \lambda x. f \ x) \ f \ x) \\ & & \rightarrow & \lambda f. \lambda x. f ((\lambda x. f \ x) \ x) \\ \text{successor} \equiv & \lambda n. \lambda f. \lambda x. f (n \ f \ x) & \rightarrow & \lambda f. \lambda x. f (f \ x) \\ & & \equiv & 2 \end{array}$$

Note

We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - $0 + 0$

$0 \equiv$

$\lambda f. \lambda x. x$

Numerals Example - 0 + 0

$$0 \equiv \lambda f. \lambda x. x$$

$$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$$

Numerals Example - 0 + 0

plus 0 0

$0 \equiv \lambda f. \lambda x. x$

$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$

Numerals Example - 0 + 0

plus 0 0

$0 \equiv \lambda f. \lambda x. x$

$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$

Numerals Example - $0 + 0$

$$\begin{aligned} & \text{plus } 0 \ 0 \\ \equiv & \ (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \end{aligned}$$

$$0 \equiv \lambda f. \lambda x. x$$

$$\text{plus} \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$$

Numerals Example - $0 + 0$

$$\begin{aligned} & \text{plus } 0 \ 0 \\ \equiv & \ (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \end{aligned}$$

$$0 \equiv \lambda f. \lambda x. x$$

$$\text{plus} \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$$

Numerals Example - 0 + 0

plus 0 0

$\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0$

$\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0$

$0 \equiv \lambda f. \lambda x. x$

$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$

Numerals Example - $0 + 0$

plus 0 0

$\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0$

$\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0$

$0 \equiv \lambda f. \lambda x. x$

$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$

Numerals Example - $0 + 0$

plus 0 0

$\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0$

$\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0$

$\rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x)$

$0 \equiv \lambda f. \lambda x. x$

$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$

Numerals Example - $0 + 0$

plus 0 0

$\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0$

$\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0$

$\rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x)$

$0 \equiv \lambda f. \lambda x. x$

$plus \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)$

Numerals Example - $0 + 0$

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x \\ \text{plus } 0 \ 0 &\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\ &\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\ &\rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\ &\equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\ \text{plus} &\equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \end{aligned}$$

Numerals Example - $0 + 0$

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x \\ \text{plus } 0 \ 0 &\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\ &\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\ &\rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\ &\equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\ \text{plus} &\equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \end{aligned}$$

Numerals Example - $0 + 0$

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x \\ plus &\equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \end{aligned}$$
$$\begin{aligned} &plus \ 0 \ 0 \\ &\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\ &\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\ &\rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\ &\equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\ &\rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \end{aligned}$$

Numerals Example - $0 + 0$

$$\begin{aligned} & \text{plus } 0 \ 0 \\ & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\ & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\ & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\ 0 & \equiv \lambda f. \lambda x. x \\ & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\ & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\ \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \end{aligned}$$

Numerals Example - 0 + 0

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) & \rightarrow \lambda f. \lambda x. 0 \ f \ x
 \end{aligned}$$

Numerals Example - 0 + 0

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \rightarrow \lambda f. \lambda x. 0 \ f \ x
 \end{aligned}$$

Numerals Example - 0 + 0

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ x
 \end{aligned}$$

Numerals Example - $0 + 0$

$$\begin{aligned}
 0 &\equiv \lambda f. \lambda x. x \\
 plus &\equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \\
 plus \ 0 \ 0 &\equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 &\rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 &\rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 &\equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 &\rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 &\equiv \lambda f. \lambda x. 0 \ f \ x \\
 &\equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ x
 \end{aligned}$$

Numerals Example - $0 + 0$

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ x \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ x
 \end{aligned}$$

Numerals Example - $0 + 0$

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ x \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ x
 \end{aligned}$$

Numerals Example - 0 + 0

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ x \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ x \\
 & \rightarrow \lambda f. \lambda x. x
 \end{aligned}$$

Numerals Example - 0 + 0

$$\begin{aligned}
 & \text{plus } 0 \ 0 \\
 & \equiv (\lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x)) \ 0 \ 0 \\
 & \rightarrow (\lambda n. \lambda f. \lambda x. 0 \ f \ (n \ f \ x)) \ 0 \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ (0 \ f \ x) \\
 0 & \equiv \lambda f. \lambda x. x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ (0 \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ (0 \ f \ x) \\
 \text{plus} & \equiv \lambda m. \lambda n. \lambda f. \lambda x. m \ f \ (n \ f \ x) \\
 & \rightarrow \lambda f. \lambda x. 0 \ f \ x \\
 & \equiv \lambda f. \lambda x. (\lambda f. \lambda x. x) \ f \ x \\
 & \rightarrow \lambda f. \lambda x. (\lambda x. x) \ x \\
 & \rightarrow \lambda f. \lambda x. x \\
 & \equiv 0
 \end{aligned}$$

End

Thanks

- Hope you enjoyed this talk and learned something new.
- Hope it wasn't too much math and dusty formulas ... :)

img/good_math.jpg

/"A Geek's Guide to the Beauty of
Numbers, Logic, and Computation"/

- Easy to understand

The Implementation of Functional Programming Languages

- How to compile to the Lambda Calculus?

img/the-implementation-of-functional-programming-languages

- Out-of-print, but freely available
 - <https://www.microsoft.com/en-us/research/publication/the-implementation-of-functional-programming-languages/>

Types and Programming Languages

- Types systems explained by building interpreters and proving properties
- Very "mathematical", but very complete and self-contained

`img/types-and-programming-languages.jpg`