Simply Typed Lambda Calculus

From Untyped to Simply Typed Lambda Calculus

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Untyped Lambda Calculus

Untyped Lambda Calculus - Recapitulation

We can boil down computation to a tiny calculus

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All we need is:

- Function Definition / Abstraction ($\lambda x.e$)
- Function Application ($e\ e$)
- Parameters / Variables (x)

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Then we get:

- Booleans
- Numerals
- Data Structures
- Control Flow
- ...

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Turing Completeness

- If it can be computed, it can be computed in Lambda Calculus!

Example - $(\lambda p.\lambda q.p)\ a\ b$

$$(\lambda p. \qquad \lambda q.p \qquad) \qquad a \qquad b$$

Meaning

$\lambda p.\lambda q.p$	Is a function that returns a function ($\lambda q.p$)
a, b	Some variables (defined somewhere else)
p	Is a variable that is bound to the parameter with the same name

Example - $(\lambda p.\lambda q.p)\ a\ b$

$(\lambda p. \quad \lambda q.p \quad) \quad a \quad b$ Substitute $p \mapsto a$

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Example - $(\lambda p.\lambda q.p)$ a b

$$\begin{array}{llllll} (\lambda p. & \lambda q.p &) & a & b & \text{Substitute } p \mapsto a \\ (& \lambda q.a &) & & b & \end{array}$$

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Example - $(\lambda p.\lambda q.p)\ a\ b$

$(\lambda p.$	$\lambda q.p$	$)$	a	b	Substitute $p \mapsto a$
$($	$\lambda q.a$	$)$		b	Substitute $q \mapsto b$
$($	a	$)$			

Meaning

$\lambda p.\lambda q.p$	Is a function that returns a function ($\lambda q.p$)
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Build an Interpreter

Let's build an interpreter

- Deepen our intiution
- Later move on to the *Simply Typed Lambda Calculus*
    - Why do we need types?
    - How does a type checker work?
    - How does it restrict the programs we might write?
- We'll do *Math Driven Development*
    - Look at the concepts in math first, then translate them to Haskell

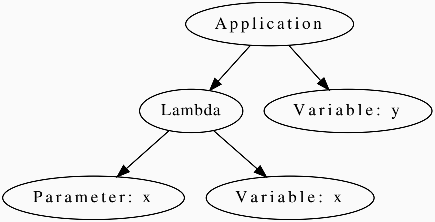
Structure

$e ::=$	Expressions:
x	Variable
$\lambda x.e$	Abstraction
$e\ e$	Application

$\lambda x.e$	Function Definition
$e\ e$	Function Application

Abstract Syntax Tree

$(\lambda x.x)\ y$



Meaning:

- Identity function ($\lambda x.x$) is applied to a variable (y)

Interpreter - Syntax

```
module UntypedSyntax where

type Name = String

data Expr                     -- e ::=       Expressions:
  = Var Name                  --       x     Variable
  | Lambda Name               --       λx.e  Abstraction
           Expr
  | App Expr                  --       e e   Application
        Expr
  deriving (Eq, Show)
```

Interpreter - Syntax - Examples

```haskell
module UntypedSyntaxExamples where

import UntypedSyntax

-- true ≡ λp.λq.p
true :: Expr
true = Lambda "p" (Lambda "q" (Var "p"))

-- false ≡ λp.λq.q
false :: Expr
false = Lambda "p" (Lambda "q" (Var "q"))

-- and ≡ λp.λq.p q p
and :: Expr
and = Lambda "p" $ Lambda "q" $ App (App (Var "p") (Var "q")) (Var "p")
```

Natural Deduction

Notation

$$\frac{}{\mathit{Axiom}} \tag{A1}$$

$$\frac{\mathit{Antecedent}}{\mathit{Conclusion}} \tag{A2}$$

Meaning:

Axiom	Rule without Precondition
Antecedent	Precondition - if it's fulfilled this rule applies.
Conclusion	What follows from this rule.
A1, A2	Names for the rules

Proof: 2 is a Natural Number

$$\frac{}{0 : \texttt{Nat}} \qquad \text{(A1)}$$

$$\frac{n : \texttt{Nat}}{\texttt{succ}(n) : \texttt{Nat}} \qquad \text{(A2)}$$

Meaning:

A1 0 is a natural number (by definition)

A2 The successor of a natural number is a natural number

Proof: 2 is a Natural Number

$$\frac{}{0 : \texttt{Nat}} \quad \text{(A1)}$$

$$\frac{n : \texttt{Nat}}{\texttt{succ}(n) : \texttt{Nat}} \quad \text{(A2)}$$

$$\cfrac{\cfrac{\cfrac{}{0 : \mathit{Nat}}\ \text{(A1)}}{\mathit{succ}(0) : \mathit{Nat}}\ \text{(A2)}}{\mathit{succ}(\mathit{succ}(0)) : \mathit{Nat}}\ \text{(A2)}$$

Meaning:

- **A1** 0 is a natural number (by definition)
- **A2** The successor of a natural number is a natural number
- → Thus the sucessor of the successor of 0 (2) must be a natural number

Evaluation Rules

Evaluation Rules - Call by Value - E-App1

$$\frac{e_1 \rightarrow e_1'}{e_1\, e_2 \rightarrow e_1'\, e_2} \qquad \text{E-App1}$$

Meaning:

- Under the condition that e_1 can be reduced further, do it.

Evaluation Rules - Call by Value - E-App2

$$\frac{e_2 \rightarrow e_2'}{v_1\, e_2 \rightarrow v_1\, e_2'} \qquad \text{E-App2}$$

Meaning:

- Under the condition that e_2 can be reduced further and v_1 is a value, do it.
- "Bare" Untyped Lambda Calculus:
  - Only Lambdas (functions) are values.
  - But you can add Ints, Booleans, etc. ("Enriched Untyped Lambda Calculus")

Evaluation Rules - Call by Value - E-AppLam

$$(\lambda x.e)v \rightarrow [x/v]e \qquad \text{E-AppLam}$$

Meaning:

- If a lambda (function) is applied to a value, substitute that value for it's parameter.
- ”substitute” : replace it for every occurence in the lambda's body

Interpreter - Evaluation

```haskell
module UntypedEval where

import UntypedSyntax

eval :: Expr -> Expr
-- No rule for variables
eval variable@(Var _) = variable
-- No rule for lambdas
eval lambda@(Lambda _ _) = lambda
```

Interpreter - Evaluation

```
eval (App e1 e2)
--
-- $\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2}$   $(E - App1)$
--
 =
  let e1' = eval e1
--
-- $\frac{e_2 \to e_2'}{v_1\, e_2 \to v_1\, e_2'}$   $(E - App2)$
--
   in let e2' = eval e2
       in case e1'
                of
--
-- $(\lambda x.e)v \to [x/v]e$   $(E - AppLam)$
--
            (Lambda name e1'_body) -> eval $ substitute name e2' e1'_body
            e1' -> App e1' e2'
```

Interpreter - Substitution

```haskell
substitute :: Name -> Expr -> Expr -> Expr
substitute name substitution var@(Var varName)
  | name  == varName = substitution
  | otherwise = var
substitute name substitution (App term1 term2) =
  App (substitute name substitution term1) (substitute name substitution term2)
substitute name substitution (Lambda varName term) =
  if name == varName
    then Lambda varName term
    else Lambda varName (substitute name substitution term)
```

Tests

```
:load UntypedEval

eval $ Var "a"
```

```
Ghci> Var "a"
```

```
-- true ≡ λp.λq.p
true :: Expr
true = Lambda "p" (Lambda "q" (Var "p"))

eval $ App (App true (Var "a")) (Var "b")
```

```
Ghci> Var "a"
```

Tests

```haskell
-- false ≡ λp.λq.q
false :: Expr
false = Lambda "p" (Lambda "q" (Var "q"))

-- and ≡ λp.λq.p q p
and :: Expr
and = Lambda "p" $ Lambda "q" $ App (App (Var "p") (Var "q")) (Var "p")

eval $ App (App and true) false
```

```
Ghci> Lambda "p" (Lambda "q" (Var "q"))
```

Simply Typed Lambda Calculus

Structure

$e ::=$	Expressions:
x	Variable
$\lambda x : \tau . e$	Abstraction
$e\ e$	Application

τ Type of the parameter x

- ‘Bool‘, ‘Int‘, …

What’s a Type?

A Type is a set of values that an expression may return:

Bool	True, False
Int	$[-2^{29}..2^{29}-1]$ (in Haskell, ‘Data.Int‘)

Simple types don’t have parameters, no polymorphism:

`Bool`, `Int`	no parameters → simple types
`Maybe a`	*a* is a type parameter → not a simple type
`id :: a -> a`	*a* is a type parameter → not a simple type

Type Safety = Progress + Preservation

Progress : If an expression is well typed then either it is a value, or it can be further evaluated by an available evaluation rule.

Preservation : If an expression e has type τ, and is evaluated to e', then e' has type τ.

- $e \equiv (\lambda x : Int.x)1$ and $e' \equiv 1$ have both the same type: ‘Int‘

Evaluation

Evalution rules stay the same!

- Type checking is done upfront

Typing Rules - Variables

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \text{T-Var}$$

Meaning

Γ	The Typing Environment, a list of $(\mathit{Variable} : \mathit{Type})$ pairs (associations)
$x : \sigma \in \Gamma$	If (x, σ) is in the Typing Environment
$\Gamma \vdash x : \sigma$	x has type σ

Typing Rules - Constants

$$\Gamma \vdash n : \mathsf{Int} \qquad \text{T-Int}$$

$$\Gamma \vdash \mathsf{True} : \mathsf{Bool} \qquad \text{T-True}$$

$$\Gamma \vdash \mathsf{False} : \mathsf{Bool} \qquad \text{T-False}$$

Meaning

`True`, `False` literals / constants are of type `Bool`

n number literals / constants are of `Int`

Why do we need Γ here?

- We handle Type Constructors like variables
- Think: $\Gamma \equiv \emptyset, \mathit{True} : \mathit{Bool}, \mathit{False} : \mathit{Bool}, 0 : \mathit{Int}, 1 : \mathit{Int}, \ldots$

Typing Rules - Lambdas

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . e : \tau_1 \to \tau_2} \quad \text{T-Lam}$$

Meaning

Condition With $x : \tau_1$ in the Typing Environment, e has type τ_2

Conclusion $\lambda x : \tau_1 . e$ has type $\tau_1 \to \tau_2$

Because e has type τ_2 if x has type τ_1

Typing Rules - Applications

$$\frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1 e_2 : \tau_2} \qquad \text{T-App}$$

Meaning

Condition If e_1 is a function of type $\tau_1 \rightarrow \tau_2$ and e_2 has type τ_2

Conclusion Then the type of $e_1 e_2$ (function application) is τ_2

Typing Rules - Applications

$$\frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1 e_2 : \tau_2} \qquad \text{T-App}$$

Meaning

Condition If e_1 is a function of type $\tau_1 \rightarrow \tau_2$ and e_2 has type τ_2

Conclusion Then the type of $e_1 e_2$ (function application) is τ_2

```
id' :: Int -> Int
id' i = i

1 :: Int
(id' 1) :: Int
```

Type Checker - Expressions

```haskell
module TypedSyntax where

import qualified Data.Map.Strict as Map

type Name = String

data Expr
  = IntValue Int
  | BoolValue Bool
  | Var Name
  | App Expr
        Expr
  | Lambda Name
           Type
           Expr
  deriving (Eq, Show)
```

Type Checker - Types

```
type Environment = Map.Map Name Type

data Type
  = TInt
  | TBool
  | TArr Type
         Type
  deriving (Eq, Show)
```

Type Checker - Literals

```
module TypedCheck where

import Data.Either.Extra
import qualified Data.Map.Strict as Map

import TypedSyntax

check :: Environment -> Expr -> Either Name Type
--
-- Γ ⊢ n : Int   (T-Int)
--
check _ (IntValue _) = Right TInt
--
-- Γ ⊢ True : Bool   (T-True)
--
check _ (BoolValue True) = Right TBool
--
-- Γ ⊢ False : Bool   (T-False)
--
check _ (BoolValue False) = Right TBool
```

Type Checker - Lambda & Application

```
--
-- $\frac{\Gamma, x{:}\tau_1 \vdash e{:}\tau_2}{\Gamma \vdash \lambda x{:}\tau_1 . e{:}\tau_1 \to \tau_2}$   (T-Lam)
--
check env (Lambda name atype e) = do
  t <- check (Map.insert name atype env) e
  return $ TArr atype t
--
-- $\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\, e_2 : \tau_2}$   (T-App)
--
check env (App e1 e2) = do
  (TArr ta1 ta2) <- check env e1
  t2 <- check env e2
  if ta1 == t2
    then Right ta2
    else Left $ "Expected " ++ (show ta1) ++ " but got : " ++ (show t2)
```

Type Checker - Variables

```haskell
--
-- $\frac{x:\sigma \in \Gamma}{\Gamma \vdash x:\sigma}$   (T-Var)
--
check env (Var name) = find env name

find :: Environment -> Name -> Either Name Type
find env name = maybeToEither "Var not found!" (Map.lookup name env)
```