Untyped Lambda Calculus

Structure and Evaluation, Currying, Church Encodings

Sven Tennie

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Dream IT
`https://dreamit.de`

Introduction

Lambda Calculus

- Invented by Alonzo Church (1920s)
- Equally expressive to the Turing Machine(s)
- Formal Language
- Computational Model
  - Lisp (1950s)
  - ML
  - Haskell
- "Lambda Expressions" in almost every modern programming language

Why should I care?

- Simple Computational Model
  - to describe structure and behaviour (E.g. Operational Semantics, Type Systems)
  - to reason and prove

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- Explains why things in FP are like they are
  - Pure Functions
  - Higher-Order Functions
  - Currying
  - Lazy Evaluation

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  - to reason and prove
- Explains why things in FP are like they are
  - Pure Functions
  - Higher-Order Functions
  - Currying
  - Lazy Evaluation
- Understand FP Compilers
  - Introduce FP stuff into other languages
  - Write your own compiler
  - GHC uses an enriched Lambda Calculus internally

Basics

Untyped Lambda Calculus

$t ::=$	Terms:
x	Variable
$\lambda x.t$	Abstraction
$t\ t$	Application

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Example - Identity

Lambda Calculus

$$\underbrace{\underbrace{\lambda x.x}_{\text{Abstraction}} \quad \underbrace{y}_{\text{Variable}}}_{\text{Application}} \rightarrow y$$

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Lambda Calculus

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Javascript

$$\underbrace{(\underbrace{\mathit{function}\ (x)\{\mathit{return}\ x;\}}_{\mathit{Abstraction}})\ (\underbrace{y}_{\mathit{Variable}})}_{\mathit{Application}}$$

Example - $(\lambda x.\lambda y.x\ y)\ a\ b$

Abstractions
Think: Function Definitions

$(\underline{\lambda x.\underline{\lambda y.x\ y}})\ a\ b$

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Abstractions
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$(\underline{\lambda x.\underline{\lambda y.x\ y}})\ a\ b$

Variables
Think: Parameters

$(\lambda x.\lambda y.\underline{x}\ \underline{y})\ \underline{a}\ \underline{b}$

Example - $(\lambda x.\lambda y.x\ y)\ a\ b$

Abstractions
Think: Function Definitions

$(\underline{\lambda x.\underline{\lambda y.x\ y}})\ a\ b$

Variables
Think: Parameters

$(\lambda x.\lambda y.\underline{x}\ \underline{y})\ \underline{a}\ \underline{b}$

Applications
Think: Function Calls

$\underline{(\lambda x.\lambda y.\underline{x\ y})\ a}\ \underline{b}$

Example - $(\lambda x.\lambda y.x\ y)\ a\ b$

$$(\lambda x. \quad \lambda y.x \quad y) \quad a \quad b$$

Example - $(\lambda x.\lambda y.x\ y)\ a\ b$

$(\lambda x. \quad \lambda y.x \quad y) \quad a \quad b$ Substitute $x \mapsto a$

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Example - $(\lambda x.\lambda y.x\ y)\ a\ b$

$$
\begin{array}{lllllll}
 & (\lambda x. & \lambda y.x & y) & a & b & \text{Substitute } x \mapsto a \\
\to & & (\lambda y.a & y) & & b &
\end{array}
$$

Example - $(\lambda x.\lambda y.x\ y)\ a\ b$

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\to & & a & b & & &
\end{array}
$$

Notational Conventions

- We use parentheses to clearify what's meant
- Applications associate to the left

$$s\ t\ u \equiv (s\ t)\ u$$

- Abstractions expand as much to the right as possible

$$\lambda x.\lambda y.x\ y\ x \equiv \lambda x.(\lambda y.(x\ y\ x))$$

Scope

$$\lambda x.\lambda y.x\ y\ z$$

Bound* and *Free

λy y is *bound*, x and z are *free*

λx x and y are *bound*, z is *free*

λx**,** λy *binders*

Scope

$$\lambda x.\lambda y.x\ y\ z$$

Bound* and *Free

λy	y is *bound*, x and z are *free*
λx	x and y are *bound*, z is *free*
λx**,** λy	*binders*

A term with no free variables is *closed*

- A *combinator*
- $id \equiv \lambda x.x$
- Y, S, K, I ...

Higher Order Functions

- Functions that take or return functions
  - Are there "by definition"

$$\underbrace{\underbrace{\lambda x.x}_{Abstraction} \quad \underbrace{\lambda y.y}_{Abstraction}}_{Application} \rightarrow \underbrace{\lambda y.y}_{Abstraction}$$

Currying

Idea

- Take a function with n arguments
- Create a function that takes one argument and returns a function with $n - 1$ arguments

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Example

- `(+1)` Section in Haskell
- $(\lambda x.\lambda y. + x\ y)\ 1 \rightarrow \lambda y. +\ 1\ y$

Currying

Idea

- Take a function with n arguments
- Create a function that takes one argument and returns a function with $n-1$ arguments

Example

- `(+1)` Section in Haskell
- $(\lambda x.\lambda y. + x\ y)\ 1 \rightarrow \lambda y. +\ 1\ y$

- Partial Function Application is there "by definition"
  - You can use this stunt to "curry" in every language that supports "Lambda Expressions"

Reductions and Conversions

Alpha Conversion

$$\lambda x.x \rightarrow_{\alpha} \lambda y.y$$

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Beta Reduction

$$(\lambda x.x)\ y \rightarrow_{\beta} y$$

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$$\lambda x.x \rightarrow_\alpha \lambda y.y$$

Beta Reduction

$$(\lambda x.x)\ y \rightarrow_\beta y$$

Eta Conversion
Iff (if and only if) x is not free in f:

$$\lambda x.f\,x \rightarrow_\eta f$$

$$(\lambda x.\underbrace{(\lambda y.y)}_{f}\ x)\ a \rightarrow_\eta \underbrace{(\lambda y.y)}_{f}\ a$$

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If x is free in f, η conversion not possible:

$$\lambda x.\underbrace{(\lambda y.y \quad \overset{\text{Bound}}{\overset{\downarrow}{x}} \quad)}_{f}\ x \not\rightarrow_\eta (\lambda y.y \quad \overset{\text{Free?!}}{\overset{\downarrow}{x}} \quad)$$

Remarks

- Everything (Term) is an Expression
  - No statements
- No "destructive" Assignments
  - The reason why FP Languages promote pure functions
  - But you could invent a built-in function to manipulate "state"...

Evaluation

Operational Semantics

- We learned how to write down and talk about Lambda Calculus Terms
- How to evaluate them?
- Different Strategies
    - Interesting outcomes

Full Beta-Reduction

- RedEx
  - **Red**ucible **Ex**pression
  - Always an Application

$$\underbrace{(\lambda x.x)\ \underbrace{((\lambda y.y)\ (\lambda z.\underbrace{(\lambda a.a)\ z}_{\textit{RedEx}}))}_{\textit{RedEx}}}_{\textit{RedEx}}$$

Full Beta-Reduction

- RedEx
  - **Red**ucible **Ex**pression
  - Always an Application

$$\underbrace{(\lambda x.x)\ \underbrace{((\lambda y.y)\ (\lambda z.\underbrace{(\lambda a.a)\ z}_{RedEx}))}_{RedEx}}_{RedEx}$$

Full Beta-Reduction

- Any RedEx, Any Time
- Like in Arithmetics
- Too vague for programming...

Normal Order Reduction

$$(\lambda x.x)\ ((\lambda y.y)\ (\lambda z.(\lambda a.a)\ z))$$

Normal Order Reduction

- Left-most, Outer-most RedEx

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\rightarrow&\lambda z.z
\end{aligned}
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Normal Order Reduction

- Left-most, Outer-most RedEx

Call-by-Name

$$(\lambda x.x)\ ((\lambda y.y)\ (\lambda z.(\lambda a.a)\ z))$$

Call-by-Name

- like Normal Order Reduction, but no reductions inside Abstractions
  - Abstractions are values
- lazy, non-strict
  - Parameters are not evaluated before they are used
- Optimization: Save results $\rightarrow$ *Call-by-Need*

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Call-by-Value

$$(\lambda x.x)\ ((\lambda y.y)\ (\lambda z.(\lambda a.a)\ z))$$

Call-by-Value

- Outer-most, only if right-hand side was reduced to a value
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Church Encodings

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- Encode Data into the Lambda Calculus
- To simplify our formulas, let's say that we have declarations

$$id \equiv \lambda x.x$$

$$id\ y \rightarrow y$$

Booleans

$$
\begin{aligned}
\mathit{true} &\equiv \qquad \lambda t.\lambda f.t \\
\mathit{false} &\equiv \qquad \lambda t.\lambda f.f
\end{aligned}
$$

Booleans

$$
\begin{aligned}
\mathit{true} &\equiv \lambda t.\lambda f.t \\
\mathit{false} &\equiv \lambda t.\lambda f.f
\end{aligned}
$$

$$
\mathit{test} \equiv \lambda c.\lambda t.\lambda f.c\ t\ f
$$

Booleans

$$test\ true\ a\ b$$

$$\begin{aligned} true &\equiv \lambda t.\lambda f.t \\ false &\equiv \lambda t.\lambda f.f \end{aligned}$$

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\to\ & (\lambda f.a)\ b \\
\to\ & a
\end{aligned}
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And

$$\begin{aligned} \mathit{true} &\equiv \lambda t.\lambda f.t \\ \mathit{false} &\equiv \lambda t.\lambda f.f \end{aligned}$$

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$$
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\mathit{true} &\equiv \lambda t.\lambda f.t \\
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$$
\mathit{and} \equiv \lambda p.\lambda q.p\ q\ p
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And

$$\mathit{and}\ \mathit{true}\ \mathit{false}$$

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And

$$\textit{and true false}$$

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$$\begin{aligned} \mathit{true} &\equiv \lambda t.\lambda f.t \\ \mathit{false} &\equiv \lambda t.\lambda f.f \end{aligned}$$

$$\mathit{and} \equiv \lambda p.\lambda q.p\ q\ p$$

$$\begin{aligned} &\mathit{and}\ \mathit{true}\ \mathit{false} \\ \equiv &(\lambda p.\lambda q.p\ q\ p)\ \mathit{true}\ \mathit{false} \\ \rightarrow &(\lambda q.\mathit{true}\ q\ \mathit{true})\ \mathit{false} \end{aligned}$$

And

$$\begin{aligned} \textit{true} &\equiv \lambda t.\lambda f.t \\ \textit{false} &\equiv \lambda t.\lambda f.f \end{aligned}$$

$$\textit{and} \equiv \lambda p.\lambda q.p\ q\ p$$

$$\begin{aligned} &\textit{and true false} \\ \equiv &(\lambda p.\lambda q.p\ q\ p)\ \textit{true false} \\ \rightarrow &(\lambda q.\textit{true}\ q\ \textit{true})\ \textit{false} \\ \rightarrow &\textit{true false true} \end{aligned}$$

And

$$\begin{aligned} true &\equiv \lambda t.\lambda f.t \\ false &\equiv \lambda t.\lambda f.f \\ \\ and &\equiv \lambda p.\lambda q.p\ q\ p \end{aligned}$$

$$\begin{aligned} & and\ true\ false \\ \equiv & (\lambda p.\lambda q.p\ q\ p)\ true\ false \\ \rightarrow & (\lambda q.true\ q\ true)\ false \\ \rightarrow & true\ false\ true \end{aligned}$$

And

$$\begin{aligned} \mathit{true} &\equiv \lambda t.\lambda f.t \\ \mathit{false} &\equiv \lambda t.\lambda f.f \\ \\ \mathit{and} &\equiv \lambda p.\lambda q.p\ q\ p \end{aligned}$$

$$\begin{aligned} & \mathit{and}\ \mathit{true}\ \mathit{false} \\ \equiv & (\lambda p.\lambda q.p\ q\ p)\ \mathit{true}\ \mathit{false} \\ \to & (\lambda q.\mathit{true}\ q\ \mathit{true})\ \mathit{false} \\ \to & \mathit{true}\ \mathit{false}\ \mathit{true} \\ \equiv & (\lambda t.\lambda f.t)\ \mathit{false}\ \mathit{true} \end{aligned}$$

And

$$
\begin{aligned}
\mathit{true} &\equiv \lambda t.\lambda f.t \\
\mathit{false} &\equiv \lambda t.\lambda f.f \\
\\
\mathit{and} &\equiv \lambda p.\lambda q.p\ q\ p
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{and}\ \mathit{true}\ \mathit{false} \\
\equiv&(\lambda p.\lambda q.p\ q\ p)\ \mathit{true}\ \mathit{false} \\
\rightarrow&(\lambda q.\mathit{true}\ q\ \mathit{true})\ \mathit{false} \\
\rightarrow&\mathit{true}\ \mathit{false}\ \mathit{true} \\
\equiv&(\lambda t.\lambda f.t)\ \mathit{false}\ \mathit{true}
\end{aligned}
$$

And

$$true \equiv \lambda t.\lambda f.t$$
$$false \equiv \lambda t.\lambda f.f$$

$$and \equiv \lambda p.\lambda q.p\ q\ p$$

$$
\begin{aligned}
& and\ true\ false \\
\equiv & (\lambda p.\lambda q.p\ q\ p)\ true\ false \\
\rightarrow & (\lambda q.true\ q\ true)\ false \\
\rightarrow & true\ false\ true \\
\equiv & (\lambda t.\lambda f.t)\ false\ true \\
\rightarrow & (\lambda f.false)\ true
\end{aligned}
$$

And

$$true \equiv \lambda t.\lambda f.t$$
$$false \equiv \lambda t.\lambda f.f$$

$$and \equiv \lambda p.\lambda q.p\ q\ p$$

$$\begin{aligned}
& and\ true\ false \\
\equiv & (\lambda p.\lambda q.p\ q\ p)\ true\ false \\
\rightarrow & (\lambda q.true\ q\ true)\ false \\
\rightarrow & true\ false\ true \\
\equiv & (\lambda t.\lambda f.t)\ false\ true \\
\rightarrow & (\lambda f.false)\ true
\end{aligned}$$

And

$$\begin{aligned} \mathit{true} &\equiv \lambda t.\lambda f.t \\ \mathit{false} &\equiv \lambda t.\lambda f.f \end{aligned}$$

$$\mathit{and} \equiv \lambda p.\lambda q.p\ q\ p$$

$$\begin{aligned} & \mathit{and}\ \mathit{true}\ \mathit{false} \\ \equiv & (\lambda p.\lambda q.p\ q\ p)\ \mathit{true}\ \mathit{false} \\ \to & (\lambda q.\mathit{true}\ q\ \mathit{true})\ \mathit{false} \\ \to & \mathit{true}\ \mathit{false}\ \mathit{true} \\ \equiv & (\lambda t.\lambda f.t)\ \mathit{false}\ \mathit{true} \\ \to & (\lambda f.\mathit{false})\ \mathit{true} \\ \to & \mathit{false} \end{aligned}$$

Or

$$\lambda p.\lambda q.p\ p\ q$$

Pairs

$$pair \equiv \quad \lambda x.\lambda y.\lambda z.z\ x\ y$$

Pairs

$$\begin{aligned}
\mathit{pair} &\equiv \quad \lambda x.\lambda y.\lambda z.z\ x\ y \\
\mathit{first} &\equiv \quad (\lambda p.p)\ \lambda x.\lambda y.x \\
\mathit{second} &\equiv \quad (\lambda p.p)\ \lambda x.\lambda y.y
\end{aligned}$$

Pairs

$$pair_{AB} \equiv \qquad pair\ a\ b$$

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

Pairs

$$pair_{AB} \equiv \qquad\qquad \textcolor{orange}{pair}\ a\ b$$

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

Pairs

$$\begin{aligned} pair_{AB} &\equiv \color{orange}{pair}\color{black}{}\ a\ b \\ &\equiv (\lambda x.\lambda y.\lambda z.z\ x\ y)\ a\ b \end{aligned}$$

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

Pairs

$$pair_{AB} \equiv pair\ a\ b$$
$$\equiv (\lambda x.\lambda y.\lambda z.z\ x\ y)\ a\ b$$

$$pair \equiv \lambda x.\lambda y.\lambda z.z\ x\ y$$
$$first \equiv (\lambda p.p)\ \lambda x.\lambda y.x$$
$$second \equiv (\lambda p.p)\ \lambda x.\lambda y.y$$

Pairs

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

$$\begin{aligned} pair_{AB} &\equiv \textcolor{orange}{pair}\ a\ b \\ &\equiv \textcolor{orange}{(\lambda x.\lambda y.\lambda z.z\ x\ y)\ a}\ b \\ &\to (\lambda y.\lambda z.z\ a\ y)\ b \end{aligned}$$

Pairs

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

$$\begin{aligned} pair_{AB} &\equiv pair\ a\ b \\ &\equiv (\lambda x.\lambda y.\lambda z.z\ x\ y)\ a\ b \\ &\rightarrow (\lambda y.\lambda z.z\ a\ y)\ b \end{aligned}$$

Pairs

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

$$\begin{aligned} pair_{AB} &\equiv pair\ a\ b \\ &\equiv (\lambda x.\lambda y.\lambda z.z\ x\ y)\ a\ b \\ &\to (\lambda y.\lambda z.z\ a\ y)\ b \\ &\to \lambda z.z\ a\ b \end{aligned}$$

Pairs

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ second &\equiv (\lambda p.p)\ \lambda x.\lambda y.y \end{aligned}$$

$$\begin{aligned} pair_{AB} &\equiv & pair\ a\ b \\ &\equiv & (\lambda x.\lambda y.\lambda z.z\ x\ y)\ a\ b \\ &\rightarrow & (\lambda y.\lambda z.z\ a\ y)\ b \\ &\rightarrow & \lambda z.z\ a\ b \\ &\equiv & pair'_{ab} \end{aligned}$$

Pairs (continued)

$$\begin{aligned} \mathit{pair} \equiv \quad & \lambda x.\lambda y.\lambda z.z\ x\ y \\ \mathit{first} \equiv \quad & (\lambda p.p)\ \lambda x.\lambda y.x \\ \mathit{pair}'_{ab} \equiv \quad & \lambda z.z\ a\ b \end{aligned}$$

Pairs (continued)

$$\textit{first pair}'_{ab}$$

$$\begin{aligned} \textit{pair} &\equiv \quad \lambda x.\lambda y.\lambda z.z\ x\ y \\ \textit{first} &\equiv \quad (\lambda p.p)\ \lambda x.\lambda y.x \\ \textit{pair}'_{ab} &\equiv \quad \lambda z.z\ a\ b \end{aligned}$$

Pairs (continued)

$$first \ pair'_{ab}$$

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\quad & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\quad & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \rightarrow\quad & pair'_{ab}\ (\lambda x.\lambda y.x) \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \rightarrow\ & pair'_{ab}\ (\lambda x.\lambda y.x) \end{aligned}$$

Pairs (continued)

$$
\begin{aligned}
pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\
first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\
pair'_{ab} &\equiv \lambda z.z\ a\ b
\end{aligned}
$$

$$
\begin{aligned}
& first\ pair'_{ab} \\
\equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\
\rightarrow\ & pair'_{ab}\ (\lambda x.\lambda y.x) \\
\equiv\ & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x)
\end{aligned}
$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \rightarrow\ & pair'_{ab}\ (\lambda x.\lambda y.x) \\ \equiv\ & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x) \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \rightarrow\ & pair'_{ab}\ (\lambda x.\lambda y.x) \\ \equiv\ & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x) \\ \rightarrow\ & (\lambda x.\lambda y.x)\ a\ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\quad & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \rightarrow\quad & pair'_{ab}\ (\lambda x.\lambda y.x) \\ \equiv\quad & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x) \\ \rightarrow\quad & (\lambda x.\lambda y.x)\ a\ b \end{aligned}$$

Pairs (continued)

$$
\begin{aligned}
pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\
first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\
pair'_{ab} &\equiv \lambda z.z\ a\ b
\end{aligned}
$$

$$
\begin{aligned}
& first\ pair'_{ab} \\
\equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\
\rightarrow\ & pair'_{ab}\ (\lambda x.\lambda y.x) \\
\equiv\ & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x) \\
\rightarrow\ & (\lambda x.\lambda y.x)\ a\ b \\
\rightarrow\ & (\lambda y.a)\ b
\end{aligned}
$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \rightarrow\ & pair'_{ab}\ (\lambda x.\lambda y.x) \\ \equiv\ & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x) \\ \rightarrow\ & (\lambda x.\lambda y.x)\ a\ b \\ \rightarrow\ & (\lambda y.a)\ b \end{aligned}$$

Pairs (continued)

$$\begin{aligned} pair &\equiv \lambda x.\lambda y.\lambda z.z\ x\ y \\ first &\equiv (\lambda p.p)\ \lambda x.\lambda y.x \\ pair'_{ab} &\equiv \lambda z.z\ a\ b \end{aligned}$$

$$\begin{aligned} & first\ pair'_{ab} \\ \equiv\ & (\lambda p.p)\ (\lambda x.\lambda y.x)\ pair'_{ab} \\ \to\ & pair'_{ab}\ (\lambda x.\lambda y.x) \\ \equiv\ & (\lambda z.z\ a\ b)\ (\lambda x.\lambda y.x) \\ \to\ & (\lambda x.\lambda y.x)\ a\ b \\ \to\ & (\lambda y.a)\ b \\ \to\ & a \end{aligned}$$

Numerals

Peano Axioms
Every natural number can be defined with 0 and a successor function

$$\begin{aligned} 0 &\equiv \lambda f.\lambda x.x \\ 1 &\equiv \lambda f.\lambda x.f\,x \\ 2 &\equiv \lambda f.\lambda x.f\,(f\,x) \\ 3 &\equiv \lambda f.\lambda x.f\,(f\,(f\,x)) \end{aligned}$$

Meaning

0 f is evaluated 0 times

1 f is evaluated once

x can be every lambda term

Numerals Example - Successor

$$0 \equiv \lambda f.\lambda x.x$$

$$1 \equiv \lambda f.\lambda x.f\,x$$

Numerals Example - Successor

$$
\begin{aligned}
0 &\equiv \lambda f.\lambda x.x \\
1 &\equiv \lambda f.\lambda x.f\ x \\
\\
successor &\equiv \lambda n.\lambda f.\lambda x.f\ (n\ f\ x)
\end{aligned}
$$

Numerals Example - Successor

$$successor\ 1$$

$$\begin{aligned} 0 &\equiv \lambda f.\lambda x.x \\ 1 &\equiv \lambda f.\lambda x.f\ x \end{aligned}$$

$$successor \equiv \lambda n.\lambda f.\lambda x.f\ (n\ f\ x)$$

Numerals Example - Successor

$\mathit{successor}\ 1$

$$
\begin{aligned}
0 &\equiv \lambda f.\lambda x.x \\
1 &\equiv \lambda f.\lambda x.f\ x
\end{aligned}
$$

$$\mathit{successor} \equiv \lambda n.\lambda f.\lambda x.f\ (n\ f\ x)$$

Numerals Example - Successor

$$\textit{successor}\ 1$$

$$\equiv \quad (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1$$

$$0 \equiv \lambda f.\lambda x.x$$

$$1 \equiv \lambda f.\lambda x.f\,x$$

$$\textit{successor} \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)$$

Numerals Example - Successor

$$\begin{aligned} 0 &\equiv \lambda f.\lambda x.x \\ 1 &\equiv \lambda f.\lambda x.f\,x \end{aligned}$$

$$\begin{aligned} & \mathit{successor}\ 1 \\ \equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \end{aligned}$$

$$\mathit{successor} \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)$$

Numerals Example - Successor

$$0 \equiv \lambda f.\lambda x.x$$
$$1 \equiv \lambda f.\lambda x.f\,x$$

$$successor \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)$$

$$\begin{aligned} & successor\ 1 \\ \equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\ \rightarrow\ & \lambda f.\lambda x.f\,(1\ f\ x) \end{aligned}$$

Numerals Example - Successor

$$
\begin{aligned}
0 &\equiv \lambda f.\lambda x.x \\
1 &\equiv \lambda f.\lambda x.f\,x
\end{aligned}
$$

$$
\begin{aligned}
& \mathit{successor}\ 1 \\
\equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\
\rightarrow\ & \lambda f.\lambda x.f\,(1\ f\ x)
\end{aligned}
$$

$$
\mathit{successor} \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)
$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$0 \equiv \lambda f.\lambda x.x$$

$$1 \equiv \lambda f.\lambda x.f\, x$$

$$successor \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)$$

$$\begin{aligned}
& successor\ 1 \\
\equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\
\to\ & \lambda f.\lambda x.f\,(1\ f\, x) \\
\equiv\ & \lambda f.\lambda x.f\,((\lambda f.\lambda x.f\, x)\ f\, x)
\end{aligned}$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$0 \equiv \lambda f.\lambda x.x$$
$$1 \equiv \lambda f.\lambda x.f\,x$$

$$\begin{aligned} & \mathit{successor}\ 1 \\ \equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\ \rightarrow\ & \lambda f.\lambda x.f\,(1\ f\,x) \\ \equiv\ & \lambda f.\lambda x.f\,((\lambda f.\lambda x.f\,x)\ f\,x) \end{aligned}$$

$$\mathit{successor} \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$
\begin{aligned}
0 &\equiv \lambda f.\lambda x.x \\
1 &\equiv \lambda f.\lambda x.f\,x \\
\\
successor &\equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)
\end{aligned}
$$

$$
\begin{aligned}
& successor\ 1 \\
\equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\
\rightarrow\ & \lambda f.\lambda x.f\,(1\ f\ x) \\
\equiv\ & \lambda f.\lambda x.f\,((\lambda f.\lambda x.f\,x)\ f\ x) \\
\rightarrow\ & \lambda f.\lambda x.f\,((\lambda x.f\,x)\ x)
\end{aligned}
$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$0 \equiv \lambda f.\lambda x.x$$
$$1 \equiv \lambda f.\lambda x.f\,x$$

$$successor \equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)$$

$$\begin{aligned} & successor\ 1 \\ \equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\ \rightarrow\ & \lambda f.\lambda x.f\,(1\ f\,x) \\ \equiv\ & \lambda f.\lambda x.f\,((\lambda f.\lambda x.f\,x)\ f\,x) \\ \rightarrow\ & \lambda f.\lambda x.f\,((\lambda x.f\,x)\ x) \end{aligned}$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$
\begin{aligned}
0 &\equiv \lambda f.\lambda x.x \\
1 &\equiv \lambda f.\lambda x.f\,x \\
\\
successor &\equiv \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)
\end{aligned}
$$

$$
\begin{aligned}
& successor\ 1 \\
\equiv\ & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\
\rightarrow\ & \lambda f.\lambda x.f\,(1\ f\,x) \\
\equiv\ & \lambda f.\lambda x.f\,((\lambda f.\lambda x.f\,x)\ f\,x) \\
\rightarrow\ & \lambda f.\lambda x.f\,((\lambda x.f\,x)\ x) \\
\rightarrow\ & \lambda f.\lambda x.f\,(f\,x)
\end{aligned}
$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - Successor

$$
\begin{aligned}
0 &\equiv & \lambda f.\lambda x.x \\
1 &\equiv & \lambda f.\lambda x.f\,x \\
\\
successor &\equiv & \lambda n.\lambda f.\lambda x.f\,(n\ f\ x)
\end{aligned}
$$

$$
\begin{aligned}
& & successor\ 1 \\
&\equiv & (\lambda n.\lambda f.\lambda x.f\,(n\ f\ x))\ 1 \\
&\rightarrow & \lambda f.\lambda x.f\,(1\ f\,x) \\
&\equiv & \lambda f.\lambda x.f\,((\lambda f.\lambda x.f\,x)\ f\,x) \\
&\rightarrow & \lambda f.\lambda x.f\,((\lambda x.f\,x)\ x) \\
&\rightarrow & \lambda f.\lambda x.f\,(f\,x) \\
&\equiv & 2
\end{aligned}
$$

Note
We use *Normal Order Reduction* to reduce inside abstractions!

Numerals Example - 0 + 0

$$0 \equiv \qquad \lambda f.\lambda x.x$$

Numerals Example - 0 + 0

$$\begin{aligned} 0 &\equiv \lambda f.\lambda x.x \\ plus &\equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x) \end{aligned}$$

Numerals Example - 0 + 0

$$plus\ 0\ 0$$

$$\begin{aligned} 0 &\equiv \lambda f.\lambda x.x \\ plus &\equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x) \end{aligned}$$

Numerals Example - 0 + 0

$plus \; 0 \; 0$

$$\begin{aligned} 0 &\equiv \lambda f.\lambda x.x \\ plus &\equiv \lambda m.\lambda n.\lambda f.\lambda x.m \; f \; (n \; f \; x) \end{aligned}$$

Numerals Example - 0 + 0

$$\begin{aligned} & \textit{plus}\ 0\ 0 \\ \equiv\ & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \end{aligned}$$

$$0 \equiv \lambda f.\lambda x.x$$

$$\textit{plus} \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

Numerals Example - 0 + 0

$$\begin{aligned} & \mathit{plus}\ 0\ 0 \\ \equiv\quad & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \end{aligned}$$

$$0 \equiv \lambda f.\lambda x.x$$

$$\mathit{plus} \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

Numerals Example - 0 + 0

$$\begin{aligned} & \mathit{plus}\ 0\ 0 \\ \equiv \quad & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \\ \rightarrow \quad & (\lambda n.\lambda f.\lambda x.0\ f\ (n\ f\ x))\ 0 \end{aligned}$$

$$0 \equiv \lambda f.\lambda x.x$$

$$\mathit{plus} \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

Numerals Example - 0 + 0

$$
\begin{aligned}
& \mathit{plus}\ 0\ 0 \\
\equiv\quad & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \\
\rightarrow\quad & (\lambda n.\lambda f.\lambda x.0\ f\ (n\ f\ x))\ 0
\end{aligned}
$$

$$0 \equiv \lambda f.\lambda x.x$$

$$\mathit{plus} \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

Numerals Example - 0 + 0

$$0 \equiv \lambda f.\lambda x.x$$

$$plus \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

$$\begin{aligned} & plus\ 0\ 0 \\ \equiv\ & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \\ \rightarrow\ & (\lambda n.\lambda f.\lambda x.0\ f\ (n\ f\ x))\ 0 \\ \rightarrow\ & \lambda f.\lambda x.0\ f\ (0\ f\ x) \end{aligned}$$

Numerals Example - 0 + 0

$$0 \equiv \lambda f.\lambda x.x$$

$$plus \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

$$\begin{aligned} & plus\ 0\ 0 \\ \equiv\ & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \\ \rightarrow\ & (\lambda n.\lambda f.\lambda x.0\ f\ (n\ f\ x))\ 0 \\ \rightarrow\ & \lambda f.\lambda x.0\ f\ (0\ f\ x) \end{aligned}$$

Numerals Example - 0 + 0

$$0 \equiv \lambda f.\lambda x.x$$

$$plus \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\,(n\ f\ x)$$

$$\begin{aligned}
& plus\ 0\ 0 \\
\equiv\ & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\,(n\ f\ x))\ 0\ 0 \\
\rightarrow\ & (\lambda n.\lambda f.\lambda x.0\ f\,(n\ f\ x))\ 0 \\
\rightarrow\ & \lambda f.\lambda x.0\ f\,(0\ f\ x) \\
\equiv\ & \lambda f.\lambda x.(\lambda f.\lambda x.x)\ f\,(0\ f\ x)
\end{aligned}$$

Numerals Example - 0 + 0

$$0 \equiv \lambda f.\lambda x.x$$

$$plus \equiv \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$$

$$\begin{aligned}
& plus\ 0\ 0 \\
\equiv\ & (\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))\ 0\ 0 \\
\rightarrow\ & (\lambda n.\lambda f.\lambda x.0\ f\ (n\ f\ x))\ 0 \\
\rightarrow\ & \lambda f.\lambda x.0\ f\ (0\ f\ x) \\
\equiv\ & \lambda f.\lambda x.(\lambda f.\lambda x.x)\ f\ (0\ f\ x)
\end{aligned}$$

Numerals Example - 0 + 0

$$0 \equiv \lambda f.\lambda x.x$$

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$$\begin{aligned}
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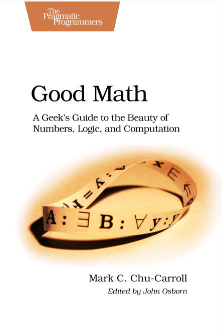
\rightarrow\ & \lambda f.\lambda x.(\lambda x.x)\ x \\
\rightarrow\ & \lambda f.\lambda x.x \\
\equiv\ & 0
\end{aligned}$$

End

Thanks

- Hope you enjoyed this talk and learned something new.
- Hope it wasn't too much math and dusty formulas ... :)

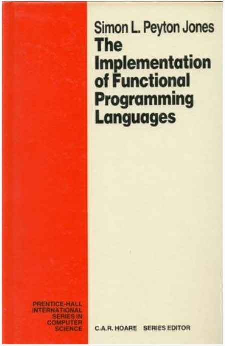
Good Math



/"A Geek's Guide to the Beauty of Numbers, Logic, and Computation"/

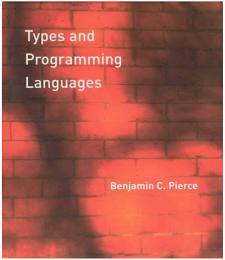
- Easy to understand

The Implementation of Functional Programming Languages



- How to compile to the Lambda Calculus?
- Out-of-print, but freely available
  - https://www.microsoft.com/en-us/research/publication/the-implementation-of-functional-programming-languages/

Types and Programming Languages



- Types systems explained by building interpreters and proving properties
- Very "mathematical", but very complete and self-contained