CHAPTER

# 10 COUNTING AND ARRANGEMENTS

## SKILLS TO KNOW

- The Fundamental Counting Principle
- Independent & Dependent Events
- Factorial
- Unique Elements/Order Matters
- Permutations (order matters)
- Order Doesn't Matter
- Combinations (order doesn't matter)
- Finding the number of possible arrangements (hybrid problems)
- Spatial arrangements

Oftentimes on the ACT®, you'll need to calculate the number of possibilities for a certain outcome. For example, maybe you need to count up all the possible sandwiches you can build at a sandwich shop. Maybe you need to know how many ways you can arrange the letters in CHILLAXIN. Or maybe you're choosing four peers for an Ultimate Frisbee team, and you want to count the number of possible teams.

In all of these situations, you are counting. That is what this chapter is all about. Regardless of what kind of counting problem you are working on, it's important to understand a general principle of counting:

### THE FUNDAMENTAL COUNTING PRINCIPLE

> If there are $m$ ways to do one thing, and $n$ ways to do another thing, and assuming that each "thing" is unique (or that the order of these things matters), there are $m$ times $n$ ways to do both.

In other words, for most counting problems:*

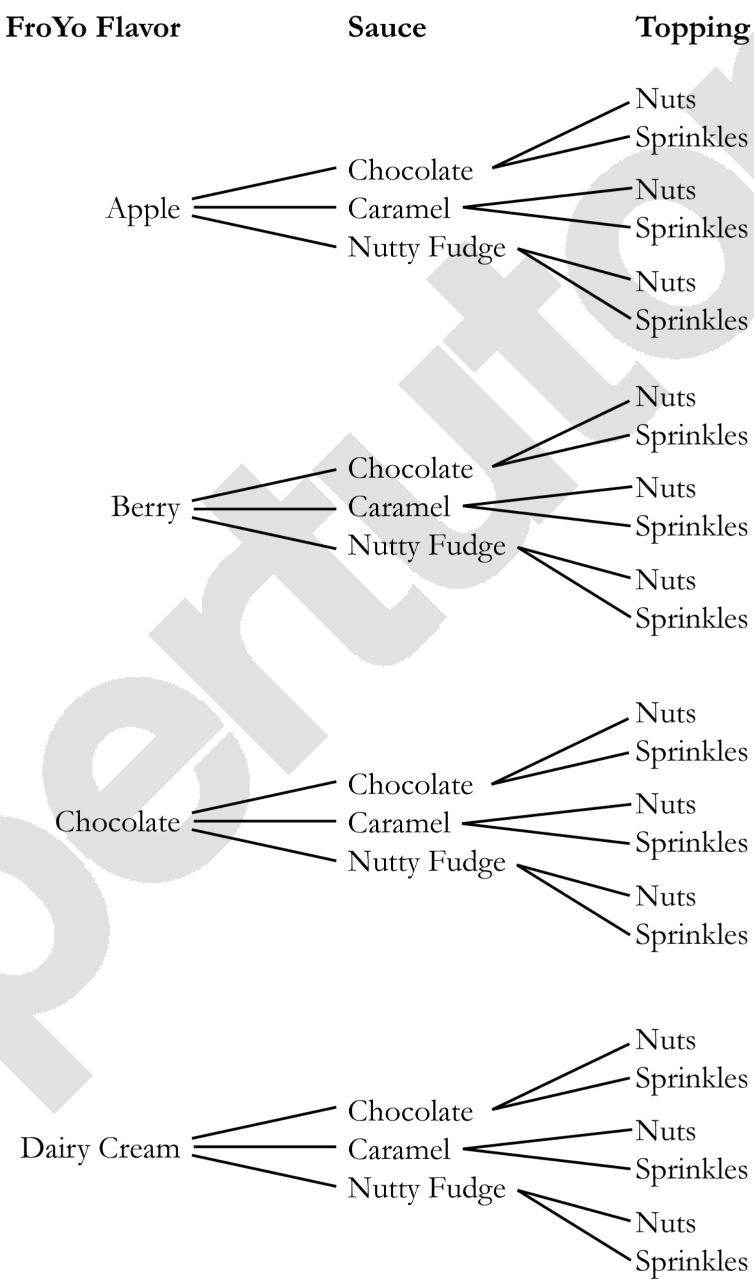
**STEP 1**: figure out the number of possibilities of each unique condition, choice, or event
**STEP 2**: multiply those possibilities together

When counting, you constantly ask yourself, HOW MANY OPTIONS DO I HAVE? Then multiply the results of your thinking to get the answer.

**Be careful, as some kinds of counting problems (i.e. combinations in which order <u>does not matter</u> or problems involving <u>non-unique</u> elements) are more complex, even though they build on this same idea. Read on for more.*

Students are selling gluten-free, probiotic-enriched, artisanal frozen yogurt ("fro-yo") sundaes at a football game. Each sundae consists of one flavor of frozen yogurt, one sauce, and one topping. If there are 4 flavors of fro-yo, 3 sauces, and 2 different toppings, how many different sundaes could be created?

In middle school, you likely created a tree diagram to solve questions like these.




That works, but there's a faster way. Because FroYo, Sauce and Topping choices are all unique choices we can use the Fundemental Counting Principle.

I create a blank for each unique "event" or choice, and ask myself, "how many options do I have?"

$\underline{\qquad\quad}$ $\underline{\qquad\quad}$ $\underline{\qquad\quad}$

FroYo Sauce Topping

Based on our counting principle, we just put the number of options for each in the blanks, and multiply together.

$$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 2 \quad} = 24 \text{ sundaes possible}$$

FroYo Sauce Topping

As you can see, this is something of a condensed "tree" diagram. The tree lists out the number of items, and at each branching we essentially multiply by the number of options. Using the counting principle and a few blanks, however, is more efficient.

Answer: **24 sundaes.**

## INDEPENDENT AND DEPENDENT EVENTS

When you're using the counting principle, you must take caution when you ask yourself "how many options do I have?" How many options you have depends on whether events are **independent** or **dependent**.

### Independent Events

**Independent events** don't affect the outcome of each other. For example, if I select a bingo letter at random, and then put it back in the bucket before I select another, each selection of a letter is an independent event. Whenever you see the words **"with replacement"** or **"can repeat"** in a word problem, you are dealing with independent events.

Common independent events include coin flips (the coin "resets" each time you flip) and dice rolls (again each number is available on each roll).

How many numbers exist between $300-600$ that only include odd digits?

$$\underline{\qquad} \times \underline{\qquad} \times \underline{\qquad}$$

My choice of the digits is independent. What I choose for the first digit doesn't affect what digits are available for the 2nd or 3rd digits. Let's use our Fundamental Counting Principle and some blanks to solve this problem.

For the first blank, I can choose $3$ or $5$ ($4$ is even so it won't create an odd digit number, $6$ only creates $600$, an even number with all even digits), so I have two choices:

$$\underline{\quad 2 \quad} \times \underline{\qquad} \times \underline{\qquad}$$

For the 2nd blank, I can choose any digit of $1, 3, 5, 7,$ or $9$. That's $5$ choices.

$$\underline{\quad 2 \quad} \times \underline{\quad 5 \quad} \times \underline{\qquad}$$

For the 3rd blank, I can again choose any digit of $1, 3, 5, 7,$ or $9$. That's $5$ choices. Because these events are independent, I am free to put the exact same digit I chose in the 2nd blank if I want to. There are no additional limits.

$$\underline{\quad 2 \quad} \times \underline{\quad 5 \quad} \times \underline{\quad 5 \quad}$$

I multiply $2(5)(5)$ to get $50$ possible numbers in all.

Answer: $50$.

### Dependent Events

**Dependent events** affect the outcome of subsequent events. If I offer you a dessert from a tray with two vegan oatmeal cookies, one chocolate chip cookie, and a crispy rice treat, and you take the chocolate chip, the next person to choose a cookie can't have that cookie. Your choosing of a cookie and the girl-sitting-next-to-you's choice are dependent events: what you choose impacts what she can choose. If you see words in the problem describing each outcome as **"different"** or **"distinct,"** or are told that letters or numbers **"cannot repeat"** you likely are dealing with dependent events.

> Three students each select a different book to read from a list of $10$ summer reading books. How many different ways can the students select their books?

Because the students are selecting "different" books, we know these events are dependent. If June selects Othello, then Max cannot select Othello. We can set up blanks for each student, and think about the number of choices we have:

$$\underset{\text{June}}{\underline{\quad 10 \quad}} \times \underset{\text{Max}}{\underline{\quad 9 \quad}} \times \underset{\text{Silas}}{\underline{\quad 8 \quad}} = 720 \text{ choices}$$

June has $10$ choices, then whatever she picks, Max can't pick, so Max has $9$ choices. Silas can't pick either of the two books June and Max picked, so he has $8$ choices. Each choice affects the number of books available for the next choice.

Answer: $720$.

## FACTORIAL

One type of mathematical function you need to know is factorial. Denoted by an **exclamation point**, a factorial is the product of a positive integer and each positive integer that is less than that integer.

#### DEFINITION OF A FACTORIAL

If $n$ is a positive integer, then $n! = n(n-1)(n-2)(n-3)...$

Example: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is written $5!$ and is read $5$ factorial.
By definition, $0! = 1$.

**CALCULATOR TIP**: Most graphing calculators have the factorial function built in. On a TI-84, hit MATH then select the PROB menu and key down to the exclamation point!

If $x$ and $n$ are positive integers greater than $3$, and $x$ is two more than $n$, then

$$\frac{x!}{n!}=?$$

**A.** $(x-n)!$ **B.** $n+2$ **C.** $(n)(n-1)$ **D.** $(x)(x+1)$ **E.** $(n+2)(n+1)$

To solve this problem, we must understand what factorial means. Knowing that, we can make up some numbers that adhere to the situation to help make the problem easier to solve.

**TIP**: Whenever you see variables in answer choices, you can make up numbers to help solve.

Let's say $x$ is $7$ and $n$ is $5$.

$$\frac{7!}{5!}=\frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}=\frac{7\cdot6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

We can see that everything cancels out except the $7$ times $6$.

$$7\cdot6=42$$

We get $42$. But I look up and now have to figure out which answer yields $42$ when I plug in $x=7$ and $n=5$.

| | | |
|---|---|---|
| A. $(x-n)!$ | $(7-5)!=2!=2$ | No. |
| B. $(n+2)$ | $(5+2)=7$ | No. |
| C. $(n)(n-1)$ | $5(5-1)=20$ | No. |
| D. $(x)(x+1)$ | $7(7+1)=56$ | No. |
| E. $(n+2)(n+1)$ | $(5+2)(5+1)=42$ | Yes. |

Answer: **E.**

We can also think of this algebraically: I know $x=n+2$, so I can substitute $n+2$ in for $x$ in my original expression. (This adheres to the equation→expression pattern you may remember from our Algebra chapters.)

$$\frac{x!}{n!}=\frac{(n+2)!}{n!}=\frac{(n+2)(n+1)(n)(n-1)(n-2)\ldots(2)(1)}{(n)(n-1)(n-2)\ldots(2)(1)}$$

Here we can see what will happen: from the term $n$ downward, all the terms on the top match all the terms on the bottom and will cancel out:

$$\frac{(n+2)(n+1)\cancel{(n)}\cancel{(n-1)}\cancel{(n-2)}\ldots\cancel{(2)}\cancel{(1)}}{\cancel{(n)}\cancel{(n-1)}\cancel{(n-2)}\ldots\cancel{(2)}\cancel{(1)}}=(n+2)(n+1)$$

Using the algebraic method can be a bit faster, but may be more challenging; do what works for you.

## UNIQUE ELEMENTS/ORDER MATTERS

Before we get too deep into counting and arrangement problems, you must be able to identify what kind of problem you are trying to solve. One of the first things you need to analyze when confronting one of these problems is whether the items you are counting or arranging ARE UNIQUE and thus ORDER MATTERS or whether they are NON-UNIQUE or ORDER DOESN'T MATTER.

So far, all of the problems in this chapter have dealt with situations in which **either the elements counted are unique OR in which the order of the elements matters.** When one or both of these conditions are true, we can use the **Fundamental Counting Principle.**

### What do I mean by "unique" and "order matters"?

Anytime we have events that occur and they are different in nature, we consider them **unique.** Examples: choosing a hat and choosing a scarf. Choosing a meat, a bread, and a cheese for a sandwich.

**We call problems in which order matters "permutations."** Any time we create a particular order of items or select a few items from a larger set to arrange in order or in unique positions, we have a permutation. Examples: Choosing an order for the Battle of the Bands, Choosing three dogs to win 1st, 2nd, and 3rd place in a dog show, Choosing four officers (President, Treasurer, Secretary, Vice President) from a group of eligible students.

We can treat **unique events** and those in which **order matters** the same: **create blanks, fill them with the applicable number of choices, and multiply.**

## PERMUTATIONS

Again, **permutations** involve finding the number of ways you can arrange **in order** a certain set or number of items. For easy problems (until say question 30), the ideas above are all you need to solve permutations. For those aiming for a 27+ score, you may need to know a bit more, though, about permutations and how you can solve them.

### General Permutations

What I'll call "general permutations" require you to count the number of ways to arrange a particular number of items *in order* taken from a set of unique items.

GENERAL PERMUTATIONS FORMULA

The formula for general permutations (with distinct items that are dependent events) is:

$${}_nP_r = \frac{n!}{(n-r)!}$$ where $n$ is the number of items taken $r$ at a time.

Sue has 4 picture books and must choose three, in order, for a toddler story time event at the library. How many ways can she do so?

If using this formula, our variable $n$ would equal the number of items we are choosing from, $4$, and $r$ would equal the number we are selecting and placing in order, $3$. You could solve by plugging these numbers into the formula.

$$_4P_3=\frac{4!}{(4-3)!}$$

And then solving:

$$_4P_3=\frac{4\cdot 3\cdot 2\cdot 1}{(1)}=24$$

But I'm not a huge fan of formulas. This particular formula will only work if you have dependent events, order matters, and every item you select is unique. True, more formulas exist for other conditions, but that's just more to memorize.

An easier way to approach this question is to use blanks and solve the problem manually, as explained above.

| | | | | |
|---|---|---|---|---|
| $\underline{\quad 4 \quad}$ | $\times$ | $\underline{\quad 3 \quad}$ | $\times$ | $\underline{\quad 2 \quad}$ |
| First Book | | Second Book | | Third Book |

We then multiply these amounts together $(4\times 3\times 2)$ to get $24$.

As I ask myself "how many choices do I have," I automatically consider whether the events are dependent (i.e. you lose one possibility with each successive blank) or independent (i.e. you have just as many choices each time). As long as I correctly answer that question, I can intuitively fill in blanks rather than strain my brain with formulas and not worry about which formula works when.

You can also solve "general permutations" like this one with your calculator.

We are taking $4$ books $3$ at a time, and order matters, so we use the function $_nP_r={}_4P_3$.

On a TI-83 or TI-84, First enter the value of $n$ (in this case "$4$") on your display. Then, press MATH, go to the PROB menu, and find $_nP_r$. Select it and hit enter. Then enter the number for $r$, here that is $3$. Click enter and your calculator will compute the answer: $24$.

For other calculators, do a quick internet search to determine if this function is available or how to access it.

A car can seat $6$ people. $6$ friends want to take a road trip. How many unique ways can they be seated?

Let's solve it with blanks first. Line up the choices, placing each choice in order on a blank, and multiply:

$$6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=720 \text{ people}$$

We have six "slots" or seats and for the first seat, six people to choose from. After one person is assigned a seat, we have five to choose from for the next seat, and so on.

We can also solve using the formula method:

In this problem, $n=6$ because we have $6$ choices, and $r=6$, since we're choosing $6$ seats.

$$
\begin{aligned}
{}_nP_r &= \frac{6!}{(6-6)!} \\
&= \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(0)!}
\end{aligned}
$$

But what is $0!$? Well, by definition it is $1$ as we mentioned earlier. But that is yet one more thing to memorize and to confuse you (can you tell I don't love formulas!). In any case we multiply out the top to get: $72$.

Your calculator can likely do this problem, as well. On a TI-84, enter $6$ as your value for $n$, then go to MATH then hit PRB and select ${}_nP_r$ from the menu. Click enter and finally enter your value $6$ for $r$.

Answer: $720$.

What expression gives the number of permutations of $9$ objects taken $3$ at a time?

**A.** $\frac{9!}{(9-3)!}$ **B.** $\frac{3!}{(9-3)!}$ **C.** $\frac{9!}{(9-3)!3!}$ **D.** $\frac{(9-7)!}{(3)!}$ **E.** $9^3$

Ok, ACT®, you got me. This is the one kind of question for which the general permutation formula **does** come in handy.

Using the same formula as above, the answer is $\frac{9!}{(9-3)!}$, because out of the $9$ choices we have, we are choosing $3$ of them. Still, in a pinch, we could use our blanks and back solve (calculating each individual answer).

We need $(9)(8)(7)=504$. Choice A simplifies to $504$.

Answer: **A**.

#### Permutations: Independent Events

Some permutations include items that repeat. Often these repeating items are called independent events. We approach these again using blanks and multiplying. **You can't use the previous formula when items you are arranging can repeat or when some of the items you are arranging are identical (while others may not be).** Though formulas do exist for some of these cases, I don't recommend using them if you don't already know them.

How many three-digit combinations can you make on a combination lock that includes three digits zero through nine?

Again we make three blanks and ask ourselves for each element of the combination, how many choices do we have? Here, numbers can repeat so for each digit we can choose from ten options (zero through nine is ten different digits.

$$\underline{10}\times\underline{10}\times\underline{10}=1000 \text{ ways}$$

For these problems, we have $n$ different choices, and every time we choose, we still have $n$ choices. Thus, we can simply multiply the number of choices, $n$, by the amount of times we choose to get the answer.

How many three digit numbers exist?

For this problem I can't use the formulas, because we don't have the same number of options for each blank. If I falsely assumed we could use that formula, it would only cause trouble. Here, the first digit could only be one of nine digits, because it can't be zero. I only have $9$ choices, $1-9$, as zero is not an option.

$$\underline{\ 9\ }\times\underline{\quad}\times\underline{\quad}$$

But the 2nd and 3rd blanks have $10$ possibilities, because $0$ is now an option, in addition to $1-9$.

$$\underline{\ 9\ }\times\underline{\ 10\ }\times\underline{\ 10\ }=900$$

Answer: $900$.

#### Permutations & Repetition

You may have noticed that earlier I discussed the idea of "unique" choices. Sometimes our choices are not totally unique, even if order matters, such as when some elements repeat. For example, finding the number of ways to arrange the letters in the word MOOD would be a task in which order matters; however, I see that the two O's are not unique. Because these elements are essentially identical, the problem is more complex and we must account for that repetition.

How many ways can you arrange the letters in the word **CHILLAXIN**?

We have 9 letters to arrange in order, but two of them are L's and two are I's. That makes things a bit complicated. When we have non-unique elements in our arrangements, I'll call these "repeats."

Basically, if I start off approaching this as a typical permutation, and I pretend each letter is unique (all 9 of them, one C, one H, two I's, two L's, an A, an X and an N), I would have:

$$\underline{9}\times\underline{8}\times\underline{7}\times\underline{6}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}$$

But there's a problem. We're treating the problem as if the two I's are unique. Let's imagine one I is upper case and one is lower case so we can visualize how this impacts our calculation. We're treating CHiLLAXIN and CHILLAXiN as two separate arrangements when in reality they only represent a single way of arranging the letters: our I's are NOT unique. We've counted two different options as possibilities that should actually only be counted once. To account for this difference, we must "divide out the repeats," i.e. divide our original calculation by $2$.

The same is true of the letter L. Below I've used bold to distinguish L number 1 and L number 2:

CHI**L**LAXIN and CHIL**L**AXIN would be the same arrangement, but with our permutation would be counted twice. Again the two L's double our count unnecessarily. So I need to divide by $2$ a second time to account for the two L's.

Thus my answer is:

$$\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot2}=90720$$

Answer: **90720.**

But what if letters repeat more than two times?

How many ways can one arrange the letters in **SCISSORS**?

I have four S's one C, one I, one O and one R. The four S's are not unique.

Again, let's start with the traditional permutation for the 8 letters as if the S's WERE unique:

$$8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$$

In any given arrangement, say SSSSCIOR, for example, my S's could swap positions and they would be repeated. The question I want to ask myself, is if I listed out ALL the arrangements I counted that look exactly like this one (SSSSCIOR) how many arrangements would that be? This time, I'll visualize it as $S_1S_2S_3S_4\text{CIOR}$. That's the same as $S_1S_4S_2S_3\text{CIOR}$ or any other version that rearranges those first four letters in any combination. What I actually want to know is how many ways can I arrange 4 unique items in those first four slots, i.e. to arrange 4 items ($S_1S_3S_2S_4$) taken 4 at a time. I can calculate ${}_nP_r$ ($4!$) or think of four slots and how many options I have for each slot:

$$4\cdot3\cdot2\cdot1=24$$

Regardless of where the 4 S's land, every word in my original count will always be one of 24 "identical" arrangements, because the four slots that contain S's can be rearranged in 24 different ways.

This reveals an important principle:

**Whenever you want to divide out repeats, and $p$ is the number of times an element in an ordered arrangement repeats, divide your original permutation by $p$!**

Thus to find our answer, we divide our original calculation by $4!$ or $24$:

$$\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1}=\frac{8\cdot7\cdot6\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

We can see that everything cancels except:

$$8\cdot7\cdot6\cdot5=1680$$

Answer: **1680.**

There is also a formula to deal with permutations that involve repetition:

The number of permutations of $n$ items of which $p$ are alike and $q$ are alike is:

$$\frac{n!}{p!q!}$$

#### Permutations with limitations

If you're not aiming for a 34+ on the math, feel free to skip this one (it's rare on the ACT).

Whenever you have limitations on what can go where, **fill the most restricted slots first**.

> Mary is arranging five books on her shelf: Geometry, Algebra, US History, English and Biology. If she doesn't want her two math books on either end of the shelf, how many ways can she arrange them?

When permutations have restrictions, you cannot use a formula. Set up your blanks, and then fill the most restricted slots first. That means the first and last slot.

$$\underline{\ 3\ }\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\ 2\ }$$

For the first restricted slot, we can choose from three books (US History, English, Biology). For the last restricted slot, we'll choose from the two remaining that we didn't put in the first slot. Now we have one of these three left over. Now add back in the two math books to this one remaining book. Now I have three books to choose from as I start to fill the three center slots, then two books, then one book:

$$\underline{\ 3\ }\cdot\underline{\ 3\ }\cdot\underline{\ 2\ }\cdot\underline{\ 1\ }\cdot\underline{\ 2\ }=36$$

Answer: 36.

#### What do I mean by "order doesn't matter"?

Anytime we are looking to count events in which **order doesn't matter**, counting possibilities generally becomes more complicated.

Finding the number of ways I can have two dice rolled and sum to $5$ is a situation in which **order doesn't matter.** Each die can be in either order (i.e. I can roll a $2$ then a $3$ or a $3$ then a $2$).

When we have elements that are **"non-unique"** or **"order doesn't matter,"** we generally start by calculating the situation as if order *does* matter, and then dividing out the repeats.

## COMBINATIONS

**We call arrangements in which order doesn't matter "combinations."** A combination typically involves choosing a certain number of items from a set. Typical combinations involve pulling from a single group to create a separate (typically smaller) group. Examples: Choosing three pizza toppings from 10, selecting four winners who all win the same prize from a raffle, picking four people to attend a quiz bowl meet from a team of 8.

**Combinations without repetition are most common.** These occur when every item you choose from is unique. For example, if four girls are vying for two spots on the prom committee, Jenny can't be both members of the committee. Thus there is no repetition, as all items in our final selection are unique.

### COMBINATIONS FORMULA USING UNIQUE ELEMENTS

The formula for combinations when choosing from unique elements is:

$$_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Where $n$ is the number of choices we have and $r$ is how many elements we select. We can also say we take $n$ items $r$ at a time, or $n$ choose $r$.

But alas, formulas are hard to remember! So let's learn how to not use the formula! (Or at least how to derive it ☺).

**TWO STEPS to solving combinations (with unique elements that don't repeat):**

1. Solve as if it were a permutation: make blanks, fill in number of choices, multiply.
2. Divide out the repeats.

Panda Salad Emporium is offering a salad trio dish, which invites customers to choose one portion of three different salads from their 6 summer salad options. Customers can choose between Chinese Chicken, Sesame Ginger Spinach, Fruit Salad, Tuna Salad, Edamame Breeze Salad, and Pasta? Pasta! Salad.

How many different salad trios are possible?

**Step 1: Run the permutation (pretend order matters…)**

$$\underline{\ 6\ } \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } = 120$$

**Step 2: Divide out the repeats**

Now let's think. I now have a list of all the salad trios as if order matters. But there's a problem. C S F seems like the same trio as S F C and C F S, etc. How many repeats would this make?

We figure out how many ways can you arrange three salads in a row, if you know which salads they are.

$$\underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 6 \text{ or } 3!$$

Now, we "divide out" these repeats:

$$\frac{120}{6} = 20$$

Answer: 20.

TIP: **the repeats in a combination are always the number of blanks factorial.**

In other words, to find a combination, first find the permutation, then divide by the number of blanks factorial.

**CALCULATOR TIP**: You can also solve this or other simple combination problems using your calculator. On a TI-83 or TI-84, First enter the value of $n$, the items you are choosing from, (in this case "6") on your display. Then, press MATH, go to the PROB menu, and find ${}_{n}C_{r}$. Select it and hit enter. Then enter the number for $r$, how many items you are selecting at a time; here that is $3$. Press enter and your calculator will compute the answer: $20$.

For other calculators, do a quick internet search to determine if this function is available.

### Complex Combination/Permutation problems

The ACT® generally avoids really tricky arrangement problems, but with the influx of so many arrang… …mplex situat… …I will be us…

> A deck of Tujeon cards, traditional playing cards from Korea, contains eighty cards in eight suits. Each suit contains nine numeral cards and one General (jang) card. In how many ways can someone select six cards of a single suit?

This question is trickier. We need to solve it either piecemeal as a combination OR as a permutation and then manually divide out the repeats. I will solve both ways.

**Method 1: Add together all the possible cases.**

I'm going to first approach this problem one suit at a time. First, I'm going to ask: how many ways can I select six cards from a single group of 10 cards (one full suit)? Order doesn't matter, so that would be $10$ choose $6$, ${}_{10}C_{6}$. I could also solve by doing $10\cdot9\cdot8\cdot7\cdot6\cdot5$, and dividing by $6!$ (the repeats, or number of blanks factorial).

On my calculator, I input ${}_{10}C_{6}$ (key "10," then MATH > PROB menu > ${}_{n}C_{r}$, then "6") to get $210$. Now, I'm going to have that many options for each of the eight suits, so I multiply $210$ by $8$ to get $1680$.

**Method 2: Approach as a permutation, then take out the repeats.**

Here, I'll imagine I'm choosing $6$ of one suit from all $80$ cards to start.

In my first blank (pretending order matters), I can chose ANY of the $80$ cards. If my goal is to pick $6$ cards of a single suit, I can pick any card of any suit to start. I'm not restricted in my choice until the 2nd blank, when that card must match the suit of the first card I chose.

$$\underline{80}\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_$$

Now for the 2nd blank, I have $9$ choices. Regardless of what my first choice was, I now have to "follow suit" (haha) and select a card from the same suit. As all suits have $10$ cards, and I've already burned one of the cards in that first blank, I have $9$ left to choose from. Then $8$ to choose from in the third blank, then $7$ and so on…

$$80\cdot9\cdot8\cdot7\cdot6\cdot5=1209600$$

Now I must divide out the repeats. The repeats are equal to the number of blanks factorial, because I can rearrange this permutation in $6!$ ways and still have all the same elements. I can use the built in factorial function in my calculator to make this fast and easy: $6! = 720$

I divide my first permutation ($1209600$) by my repeats ($720$):

$$\frac{1209600}{720} = 1680$$

Answer: $1680$.

#### Combinations & Repetition

Combinations with repetition rely on multiple duplicate items in our pool we choose from. For example, if we shop for five spiral notebooks at Sav-R-Store, the story may have many colors of the same notebook on sale. We can buy black notebooks, red notebooks, green notebooks or blue ones, but we could also choose all black, $2$ black and $3$ red, etc. Which exact black notebook I pick doesn't matter, they are mass-produced and all essentially the same. Most of the time, these kind of situations are NOT dependent events (they might be partially dependent if stock is low on something, say you only have $2$ red notebooks left). If Sav-R-Store has $200$ of each color in stock, I can choose as many of any as I want, or choose three different colors, so that would be independent. **When the items you choose from are NOT unique** (like pens, coins that are gathered not flipped, notebooks at the store, etc.) **be careful and avoid formulas unless you are 100% sure you're using the right one.** Often these problems must be computed manually or approached creatively because they are potentially complicated.

> Jessica has $3$ coins in her pocket. She knows they are some combination of quarters, dimes, nickels, and pennies, but doesn't know which type(s) she has. How many different combinations of coins could she have in her pocket?

For this problem, we're best off finding the solution manually. By manually, I mean we're at least using blanks, or possibly counting out individual cases.

Because I can have repeats, dividing out the repeats and using blanks may not work. For instance, if order mattered, I know I would have $4 \cdot 4 \cdot 4$ options, or $64$ permutations. But I can't simply divide by $3!$ Not only would that give me a non-integer answer, but it doesn't encompass the number of repeats because QQQ would only be counted once in my "order matters" calculation, but PQQ, QPQ, and QQP would be a single combination counted three times.

Thus my best solution is to write out all the possibilities, but do so in a systematic way.

**Three of a kind:** QQQ, PPP, NNN, DDD—**4 options**

**Two of a kind:** QQ(P/N/D), PP(Q/N/D), NN(Q/P/D), DD(Q/P/N)—**12 options**

If I have two quarters, I can then have one penny/one nickel/one dime as my third coin. That makes 3 options. I could also think of this as a permutation.

The same is true when I have two pennies, two nickels or two dimes. I have three options in each of these cases, too. I'm choosing "two" coins but order matters as the first coin is doubled and the second isn't. ${}_4P_2 = 12$ if I use my calculator (or I can do $\underline{4} \cdot \underline{3}$).

**One of a kind:** QPN QPD QND PND—**4 options**

When I say "do manually," I sometimes will speed up using permutation or combination principles once I get into the problem. For this last case, one of a kind, I realize that this task is actually a good old regular combination! I am taking $4$ unique items $3$ at a time, and nothing repeats! I could do this part quickly by entering ${}_4C_3$ in my calculator, which equals $4$. Using this combination function will keep me from careless errors that occur more often with manual listing.

**Fun Fact:** As you might notice, from Three of a Kind and One of a Kind above, choosing $3$ items from a group of $4$ has the same number of options as choosing $1$ item from $4$. Think about it: choosing $3$ to select is the same as choosing $1$ to leave out. In other words, ${}_nC_r = {}_nC_{(n-r)}$. You don't have to know memorize this, but it's a fun fact that may be exploited in a tough question on the test.

Now I just add together: $4+12+4=20$.

Answer: $20.$

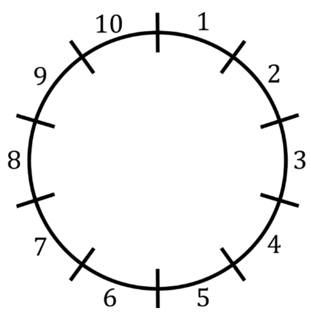
## MISCELLANEOUS PATTERNS & SPATIAL ARRANGEMENT PROBLEMS

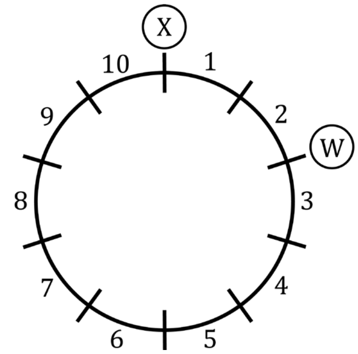
Sometimes problems involve finding patterns but the traditional mold of combinations and permutations doesn't work so well. For these problems, draw it out, use logic, make a chart, or try to figure it out manually.

Four points $W, X, Y,$ and $Z$, lie on a circle with a circumference of $10$ units. $W$ is $2$ units clockwise from $X$. $Z$ is $8$ units counter-clockwise from $W$. $Y$ is $6$ units clockwise from $W$. What is the order of the points starting with $X$ and going clockwise around the circle?

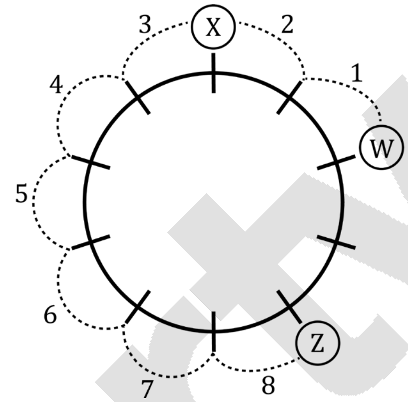
Here I draw out a circle, and mark off ten segments separated by hash marks, with each segment representing a unit. Be sure to make $10$ segments NOT ten hash marks!

Now I start drawing out each option one at a time, systematically. I know clockwise means around to the right, and counterclockwise around to the left. I count each "hop" manually for every unit I move left or right. See my work below.

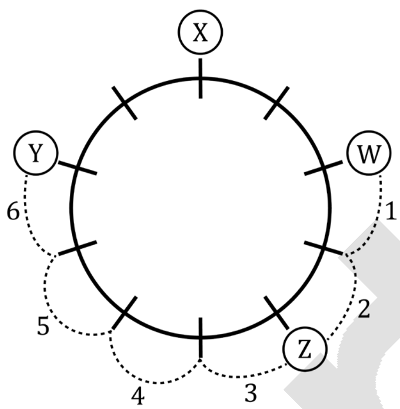


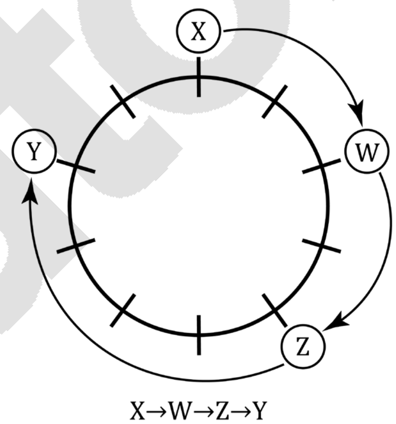

$W$ is $2$ units clockwise from $X$




$Z$ is $8$ units counterclockwise from $W$




$Y$ is $6$ units clockwise from $W$




What is the order starting with $X$, going clockwise?

Answer: $X, W, Z, Y$.