DISTANCE AND MIDPOINT

ACT Math: Lesson and Problem Set

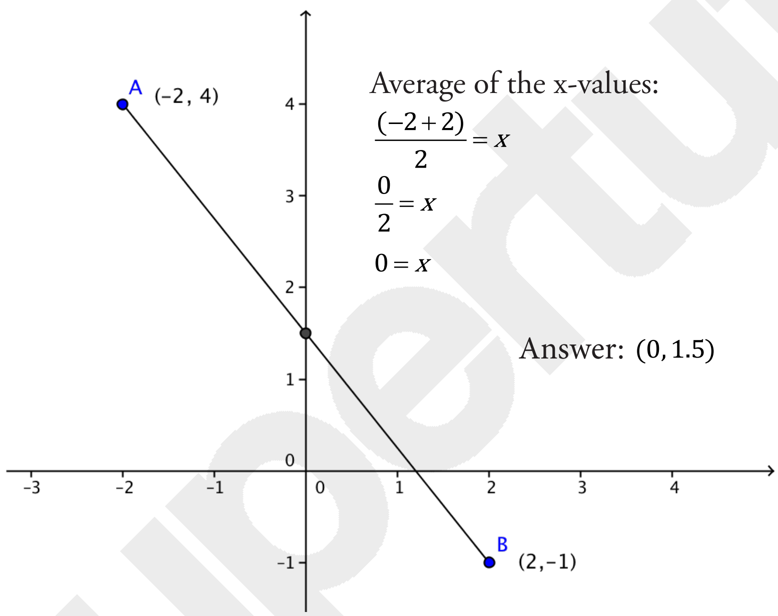
SKILLS TO KNOW

- How to find the midpoint of a line (Midpoint Formula)
- How to find the distance between two coordinate points (Distance Formula)

MIDPOINT

You may not remember midpoint from coordinate geometry, but don't worry, you can do these problems without a complicated formula by using a little common sense to memorize the idea.

The midpoint is essentially the **average** of the x-values and the **average** of the y-values.



Average of the x-values:

$$\frac{(-2+2)}{2}=x$$

$$\frac{0}{2}=x$$

$$0=x$$

Average of y-values:

$$\frac{(4+-1)}{2}=y$$

$$\frac{3}{2}=y$$

$$1.5=y$$

Answer: $(0, 1.5)$

That's right. It's that easy. If you have two points you add the x's together and divide by two, and then add the y's together and divide by two. Above you take the 2 and -2 and find halfway between—that's the average of 2 and -2, which is 0. Same with the y—find the halfway point or average between 4 and -1, which is 1.5.

For the record, the midpoint formula is: $\left(\frac{(x_1+x_2)}{2},\frac{(y_1+y_2)}{2}\right)$

TIP: Many questions on the ACT that deal with midpoint will NOT give you the end points, but rather may give you one end point, the midpoint, and ask for the other endpoint. Always read the questions carefully! Use algebra to solve after setting up an equation.

In the standard (x, y) coordinate plane, the midpoint of a line segment is $(5,7)$ and an endpoint of that segment is located at $(9,-3)$. If (x, y) are the coordinates of B, what is the value of $x + y$?

Here we know our midpoint and endpoint, so we set up our formula to solve for the missing endpoint, (x, y).

$$\left(\frac{(x_1+x_2)}{2},\frac{(y_1+y_2)}{2}\right)=(\text{midpoint } x \text{ value, midpoint } y \text{ value})$$

$$\left(\frac{(x+9)}{2},\frac{(y-3)}{2}\right)=(5,7)$$

$$\frac{(x+9)}{2}=5 \qquad \frac{(y-3)}{2}=7$$

$$x+9=10 \qquad y-3=14$$

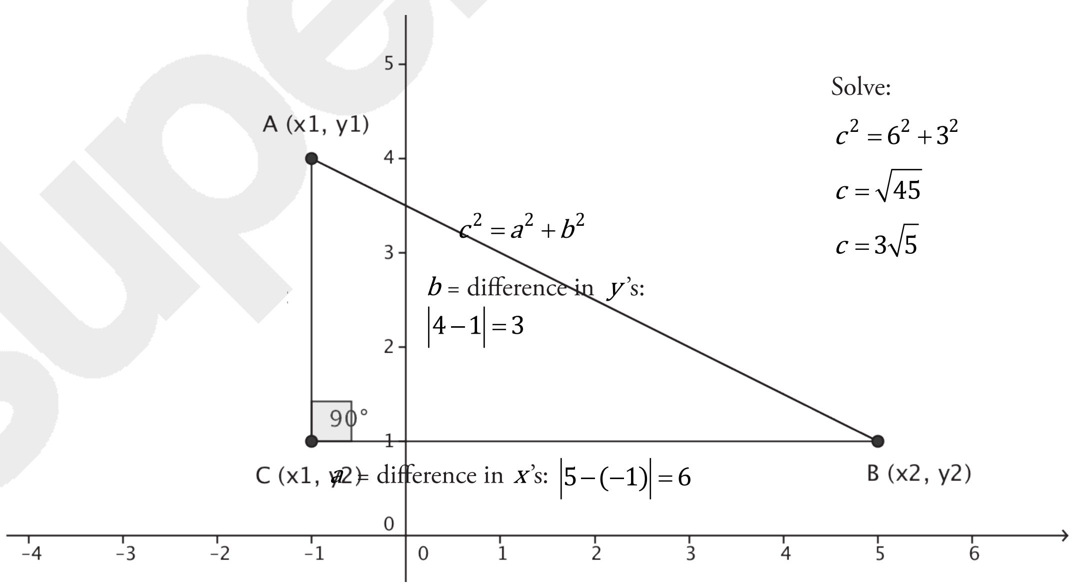
$$x=1 \qquad y=17$$

$$x+y=1+17\rightarrow 18$$

Answer: 18

THE DISTANCE FORMULA

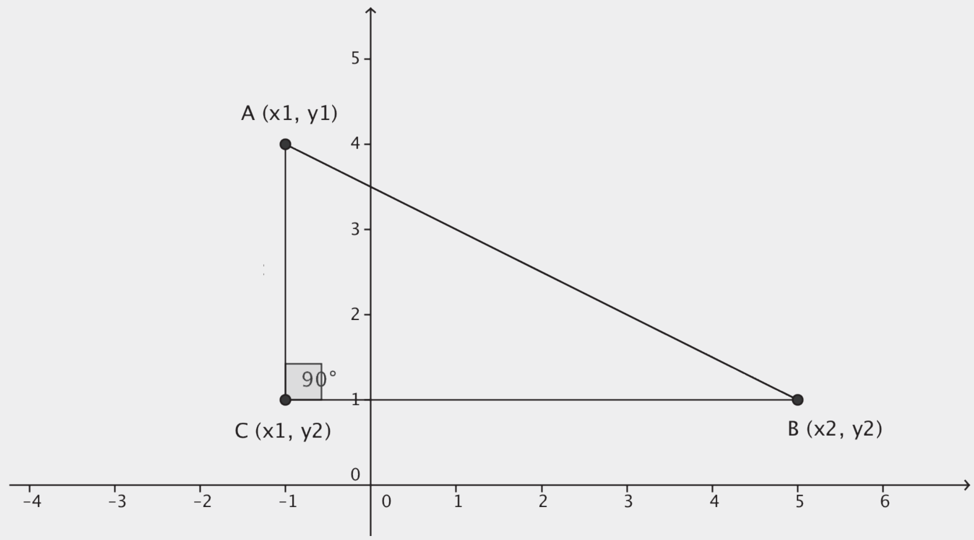
To find the distance between two coordinate points, all you have to do is think about the Pythagorean Theorem. Essentially, it's the same thing. If you wanted to find distance between A & B in the picture below, you could draw a right triangle and then use the Pythagorean Theorem:



In fact, as long as you remember that the distance formula is essentially the Pythagorean Theorem, you can rely on that idea to solve. If you're a formula person, however, there's a formula you can memorize. You can even program that formula into your calculator if you like (see chapter on Calculator Programs).

If you've got two points, (x_1, y_1) and (x_2, y_2), then: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This is the official distance formula.

What is the distance in coordinate units between the points $(5,4)$ and $(10,7)$ in the standard (x, y) coordinate plane?



The easiest way to approach this, if you get confused by formulas or tend to make careless errors, is to sketch it out. Then count out the difference from 4 to 7 and from 5 to 10—voila! You get 3 and 5.

Enter the Pythagorean Theorem: $a^2 + b^2 = c^2$

Square each, and you get $9 + 25 = 34 = c^2$ Solve for c and you get $\sqrt{34}$.

Now sometimes on the test things won't be that easy.

(x,y) and $(-4,2)$ are 10 units away from each other in the standard coordinate (x,y) plane. If y is eight less than x, than which of the following could be the value of x?

A. 4 **B.** −10 **C.** −2 **D.** 6 **E.** 8

You could do this problem by sketching it out. But you can also do this algebraically.

Regardless, first translate the "y is eight less than x" into $y = x - 8$. Then substitute $x - 8$ for y into (x,y). Now you have $(x,x-8)$ as the point you are looking for.

Algebraic method:

Plug in to the ordered pair: $(x,x-8)$ and your original pair, $(-4,2)$ into the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

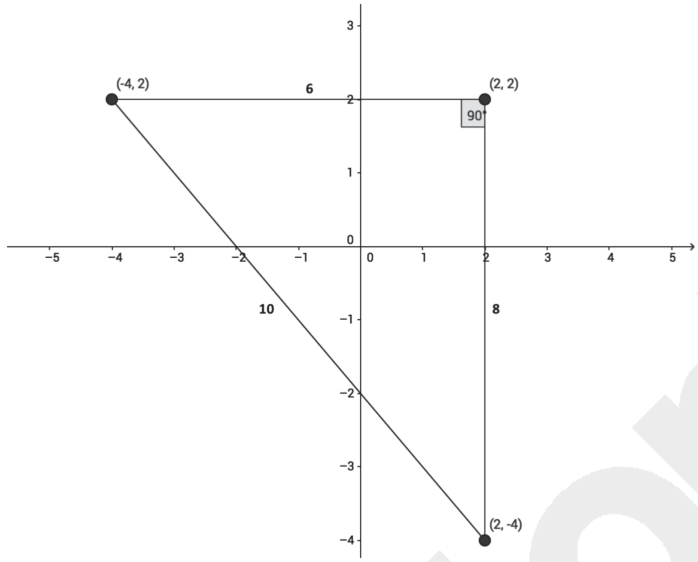
$$10 = \sqrt{(-4-x)^2 + (2-(x-8))^2}$$

$100 = (-1(4+x))^2 + (2-x+8)^2$	Square both sides, factor out -1, distribute negative sign to $(x-8)$
$100 = (-1)^2(16+8x+x^2) + (10-x)^2$	Expand: $(a+b)^2 = a^2 + 2ab + b^2$ (special product), distribute exponent, add like terms
$100 = x^2 + 8x + 16 + 100 - 20x + x^2$	Expand: $(a-b)^2 = a^2 - 2ab + b^2$ (special product); Commutative property
$0 = 2x^2 - 12x + 16$	Simplify
$0 = x^2 - 6x + 8$	Divide both sides by 2
$0 = (x-4)(x-2)$	Factor
$0 = (x-4)$ *or* $0 = (x-2)$	Apply Zero Product Property
$x = 4$ *or* 2	Solve for x

Sketch method:

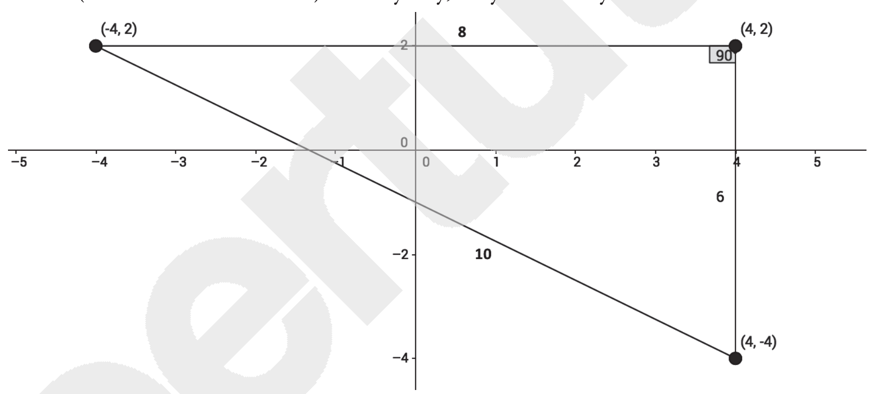
Remember distance (units away) is like the hypotenuse of a right triangle. Since our distance is 10—I have a hunch this may involve a Pythagorean triple like $6, 8, 10$. The fastest way to do this is actually to draw a picture and hope to get lucky—you can do this and eyeball it—plot points that are 6 away in one dimension (x or y) and 8 away in the other (y *or* x) from $(-4,2)$.

For instance, here's one triangle I could quickly sketch:



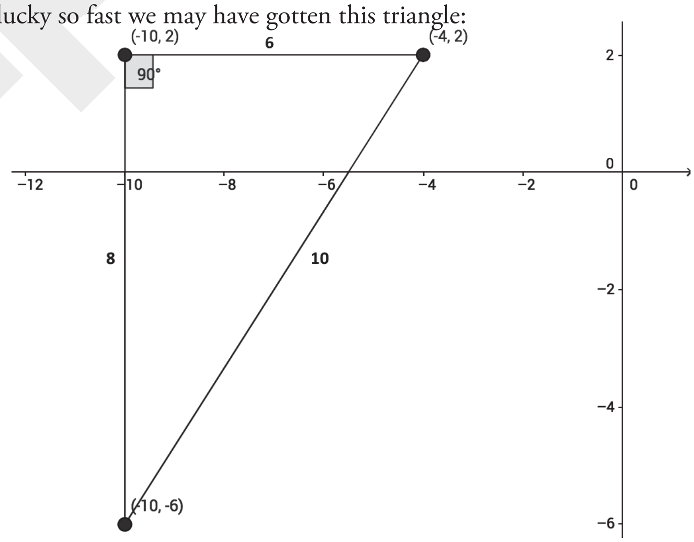
Well that didn't work as $x = 2$ is not available. Remember we are looking for a value that COULD be right, not the only possible value, so don't get discouraged if your first triangle isn't there.

Maybe I can eyeball $x = 4$ (an answer that is there) and say hey, why don't I try 8 across instead of 6 across:



Voila! That one works! $x = 4$! But we're not done yet—we still have to double check that this answer works with all the question's parameters. Looking at the sketch, our *y*-value is (-4). We need to ensure that this *y* value (-4) is 8 less than the x-value (remember $\left(x, x-8\right)$?). $4 - 8 = -4$, so this point is our answer.

However, if we were not so lucky so fast we may have gotten this triangle:



1. In the standard (x, y) coordinate plane, approximately how many coordinate units is $(4,5)$ from $(9,-2)$?

A. 9
B. 10
C. 7
D. 4
E. 6

2. What is the distance, in coordinate units, between points P $(-4,-6)$ and Q $(1,3)$ in the standard (x, y) coordinate plane?

A. $3\sqrt{2}$
B. $3\sqrt{10}$
C. $\sqrt{106}$
D. 10
E. 12

3. What is the distance, in coordinate units, between points **P** $(6,2)$ and **Q** $(3,-7)$ in the standard (x, y) coordinate plane?

A. $9\sqrt{10}$
B. $3\sqrt{10}$
C. 9
D. 10
E. 3

4. On a map of Truetown in a standard (x, y) coordinate plane, where 1 coordinate unit represents 1 block, the firehouse is at $(-7,2)$ and the police station is at $(3,-4)$ What is the straight-line distance, in blocks between the firehouse and the police station?

A. 8
B. 4
C. $2\sqrt{34}$
D. 136
E. $2\sqrt{5}$

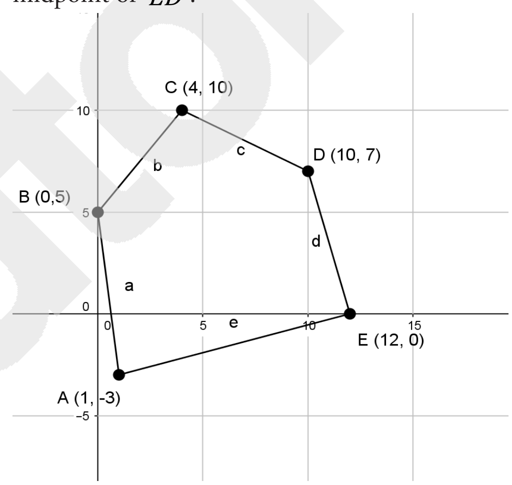
5. A middle school is located 2 miles south and 3 miles east of the city library. A high school is located 4 miles north and 2 miles west of the library. Which of the following is the shortest distance, in miles, between the middle school and the high school?

A. $\sqrt{65}$
B. 12
C. $\sqrt{11}$
D. $\sqrt{61}$
E. $\sqrt{51}$

6. Tomas is standing 50 feet due west of a picnic table at his local park, and 120 feet due south of a water fountain outside a city recreation center. How far apart, in feet, are the picnic table and the water fountain?

A. 130
B. 170
C. 70
D. $10\sqrt{17}$
E. $\sqrt{119}$

7. In the pentagon $ABCDE$ shown in the standard (x, y) coordinate plane below, what is the distance, in coordinate units, from the midpoint of $\overline{AC}$ to the midpoint of $\overline{ED}$?

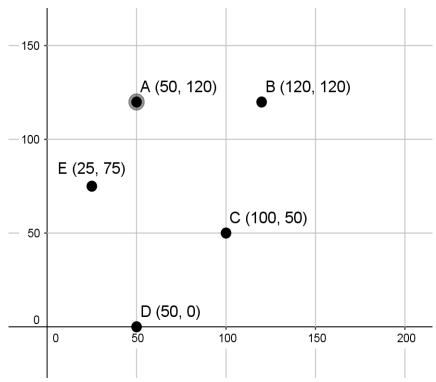


A. 7
B. 8.5
C. 10.8
D. 11.4
E. 12

8. Margaret, Alice, and Trevor start walking from the same point in the center of the schoolyard. If Margaret walks 5 feet due south, and 6 feet due east, Alice walks 7 feet due north and 8 feet due west, and Trevor walks to the point half way between the Margaret and Alice, how far from the original starting point is Trevor, in feet?

A. 2
B. 1
C. $\sqrt{2}$
D. $\sqrt{85}$
E. 13

9. The Rosa Parks Community Center is planning a community jobs fair. Several booths lettered **A, B, C, D,** and **E** are to be placed at the center according to the representation on the standard (x, y) coordinate plane below, in which each coordinate unit represents 1 yard. If the general information booth is to be placed directly halfway between booths **C** and **E**, what is the approximate distance from the general information booth to booth **B**?



A. 63.6
B. 81.3
C. 88.3
D. 115
E. 130

10. The points P, Q, R, S, and T lie in that order on a straight line. The midpoint of $\overline{PR}$ is Q, and the midpoint of $\overline{PT}$ is S. The length of $\overline{PQ}$ is n inches, and the length of $\overline{ST}$ is $3n+4$, what is the length of $\overline{RS}$?

A. n
B. $n+4$
C. $2n+4$
D. $3n+4$
E. 4

11. A line segment in the standard (x, y) coordinate plane has endpoints $(-5,8)$ and $(1,2)$. What is the x-coordinate of the midpoint of this line segment

A. -3
B. -2
C. $\frac{3}{2}$
D. 0
E. 5

12. What is the midpoint of the line segment with endpoints $(-3,10)$ and $(10,-28)$ in the standard (x, y) coordinate plane?

A. $\left(-\frac{7}{2}, 9\right)$
B. $\left(\frac{7}{2}, -9\right)$
C. $(4,18)$
D. $\left(\frac{13}{2}, -9\right)$
E. $(13,-18)$

13. The point $(6,-5)$ is the midpoint of the line segment in the standard (x, y) coordinate plane joining the point $(9,1)$ and the point (a,b). Which of the following is (a,b)?

A. $(-3,11)$
B. $(3,7)$
C. $(3,8)$
D. $(3,-11)$
E. $(7.5,-2)$

14. In the standard (x, y) coordinate plane, the midpoint of a line segment is $(19,2)$ and an endpoint of that segment is located at $(1,-3)$. If (x, y) are the coordinates of B, what is the value of $x+y$?

A. 44
B. 37
C. 30
D. 7
E. -30

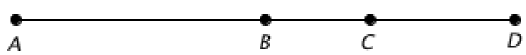
15. In the standard (x, y) coordinate plane, $(8,15)$ is halfway between $(5z, a-3)$ and $(3z, z+1)$. What is the value of z?

A. 0
B. 2
C. 3
D. 4
E. 6

16. Points $P, Q, R,$ and S lie on a line in the order given. The midpoint of $\overline{PR}$ is Q. The length of $\overline{PQ}$ is b and the length of $\overline{PR}$ is 16 inches, and the length of $\overline{RS}$ is $3b-12$ inches. What is the approximate length, in inches, of $\overline{QS}$?

A. 8
B. 12
C. 20
D. 26.67
E. 30

17. Points $A, B, C,$ and D lie on the real number line as shown below. The length of $\overline{AD}$ is 60 units; $\overline{AC}$ is 39 units long; and $\overline{BD}$ is 32 units long. How many units long, if it can be determined, is $\overline{BC}$?

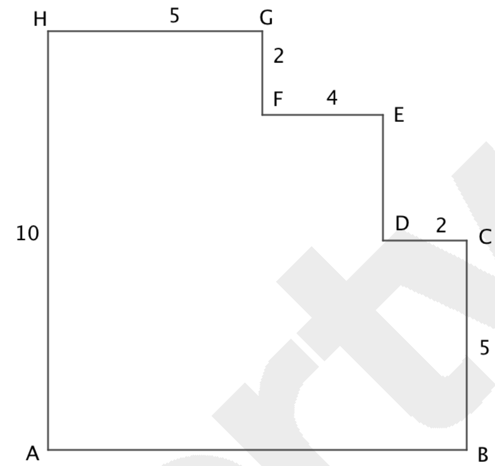


A. 10
B. 11
C. 12
D. 13
E. Cannot be determined from the given information.

18. Points P, Q, R, S and T lie on a line in the order given. The coordinate of R is 0, Q is the midpoint of $\overline{PR}$ and S is the mid-point of $\overline{PT}$. $\overline{RS}$ is 4 units longer than $\overline{PQ}$ and $\overline{ST}$ is 8 units longer than double the length of $\overline{QR}$. What is the coordinate of S?

A. 4
B. 8
C. 12
D. 20
E. Cannot be determined from the given information.

For Questions 9-11, refer to the figure below.



The shape above has only right angles. In a coordinate plane (not shown), $A=(0,0)$ and all other points are the given distances in coordinate units. Point J (not shown) is half-way between A and E.

19. What is the perimeter of the figure?

A. 28
B. 31
C. 39
D. 42
E. 45

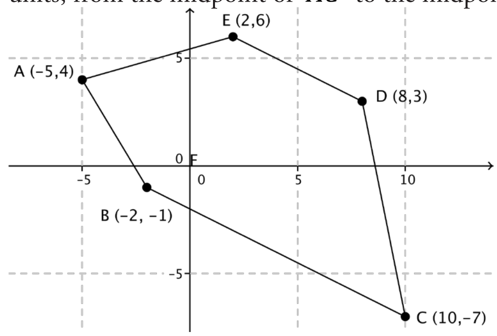
20. What are the coordinates of J?

A. $(4,4.5)$
B. $(4.5,4)$
C. $(5,5)$
D. $(8,9)$
E. $(9,8)$

21. Approximately how far, in coordinate units, is J from B?

A. 6
B. 6.7
C. 7
D. 7.6
E. 8.2

22. In pentagon $ABCDE$ shown in the standard (x, y) coordinate plane below, what is the distance, in coordinate units, from the midpoint of $\overline{AC}$ to the midpoint of $\overline{BD}$?



A. $\frac{\sqrt{2}}{2}$

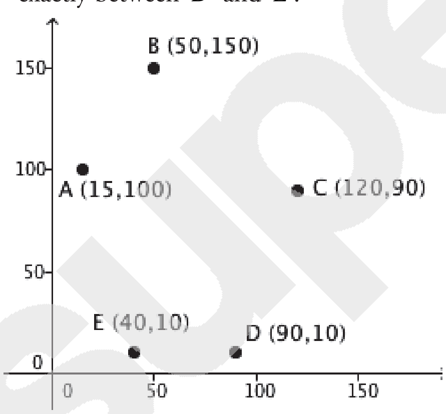
B. $\frac{\sqrt{26}}{2}$

C. $\frac{\sqrt{122}}{4}$

D. $\frac{\sqrt{146}}{2}$

E. 6.5

23. Sarah is planning on throwing a birthday party at her local park. In the figure below, the locations for several bounce houses are represented as points $A, B, C, D,$ and E on the standard (x, y) coordinate plane, where each coordinate unit represents 1 meter. The water fountain is exactly between B and E.



How many meters will the water fountian be from A?

A. 30

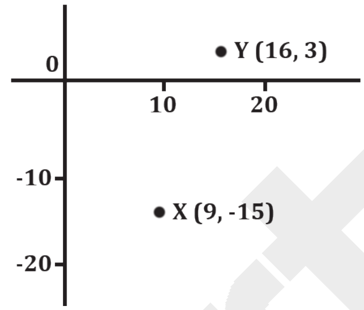
B. $\sqrt{931.25}$

C. $\sqrt{1000}$

D. $\sqrt{1125}$

E. $\sqrt{1300}$

24. In the standard (x, y) coordinate plane shown below, what is the distance in the y direction, in units, from point X to point Y?



A. 7

B. 18

C. $\sqrt{373}$

D. -12

E. -18

25. What is the distance, in coordinate units, from $(3,4)$ to $(9,-3)$ in the standard (x, y) coordinate plane?

A. $\sqrt{17}$

B. $\sqrt{37}$

C. $\sqrt{39}$

D. $\sqrt{41}$

E. $\sqrt{85}$

ANSWER KEY

1. A	**2. C**	**3. B**	**4. C**	**5. D**	**6. A**	**7. B**	**8. C**	**9. B**	**10. B**	**11. B**	**12. B**	**13. D**	**14. A**
15. B	**16. C**	**17. B**	**18. B**	**19. D**	**20. B**	**21. D**	**22. B**	**23. E**	**24. B**	**25. E**			

ANSWER EXPLANATIONS

1. **A.** We can solve this problem using the Pythagorean theorem. The horizontal and vertical distances between the two points make up the two legs of our triangle. We find the horizontal distance as the difference between the x coordinates: $9-4=5$, and the vertical distance as the difference between the y coordinates: $-2-5=-7$. We plug 5 and -7 into our formula to get $5^2+(-7)^2=c^2$ where c is the distance we are looking for. Simplify into $25+49=c^2$ which becomes $c^2=74$. Thus, $c\approx 8.6$ which rounds to 9.

2. **C.** We solve this problem the same way we solved question 1, only this time the answers are exact. Plug in the difference between x values: $1-(-4)=5$ and the difference between y values: $3-(-6)=9$ into the Pythagorean theorem: $5^2+9^2=c^2$. $c^2=25+81=106$. Thus, $c=\sqrt{106}$.

3. **B.** We solve this problem exactly how we solved 2. Plug in the difference between the x values: $6-3=3$ and the difference between the y values: $2-(-7)=9$ into the Pythagorean theorem: $3^2+9^2=c^2$. $c^2=9+81=90$. $c=\sqrt{90}=\sqrt{9}\sqrt{10}=3\sqrt{10}$.

4. **C.** This problem is just like the ones in the easy section, we are simply looking for the distance between two points. The difference between the x coordinates is $3-(-7)=10$ and the difference between the y coordinates is $-4-2=-6$. Plug into the Pythagorean theorem: $10^2+(-6)^2=c^2\rightarrow c^2=100+36=136\rightarrow$ $c=\sqrt{136}=\sqrt{4}\sqrt{34}=2\sqrt{34}$.

5. **D.** If needed, sketch the word problem to gain a clearer understanding. The total North-South distance between the middle school and high school is $4+2=6$ miles. The total East-West distance is $2+3=5$ miles. Plug these values into the Pythagorean theorem to get $5^2+6^2=c^2\rightarrow c^2=25+36=61\rightarrow$ $c=\sqrt{61}$.

6. **A.** We can see from a sketch of the problem that the water fountain, the picnic table, and Tomas form a right triangle with legs of lengths 120 and 50. This triangle is similar to the right triangle whose sides are the Pythagorean triple 5-12-13. Thus, we can deduce that the hypotenuse of the larger triangle is 130.

7. **B.** The midpoint between A and C is $\left(\frac{1+4}{2},\frac{-3+10}{2}\right)=(2.5,3.5)$. The midpoint between D and E is $\left(\frac{10+12}{2},\frac{7+0}{2}\right)=(11,3.5)$. The distance between $(2.5,3.5)$ and $(11,3.5)$ is only the difference between the x values, since the y values are equal. The distance is $11-2.5=8.5$.

8. **C.** If we define the starting point as the origin $(0,0)$, we can express the position of Margaret as $(6,-5)$ and the position of Alice as $(-8,7)$. We find the position of Trevor as the midpoint: $\left(\frac{6-8}{2},\frac{-5+7}{2}\right)=\left(\frac{-2}{2},\frac{2}{2}\right)=(-1,1)$. We find the distance between Trevor and the origin using the Pythagorean theorem: $(-1)^2+1^2=c^2.c^2=1+1=2$. $c=\sqrt{2}$.

9. **B.** The coordinates of the general information booth are $\left(\frac{25+100}{2},\frac{50+75}{2}\right)=(62.5,62.5)$. The distance between this and B, $(120,120)$ is given by $\sqrt{(120-62.5)^2+(120-62.5)^2}=\sqrt{57.5^2+57.5^2}=\sqrt{3306.25+3306.25}=\sqrt{6612.5}\approx 81.3$.

10. B. Since S is the midpoint of $\overline{PT}$, $\overline{PQ}+\overline{QR}+\overline{RS}=\overline{ST}$. We know that $\overline{PQ}=\overline{QR}$ since Q is the midpoint of $\overline{PR}$. $\overline{PQ}=n$ and $\overline{ST}=3n+4$, as given. By substitution: $n+n+\overline{RS}=3n+4$, which becomes $\overline{RS}+2n=3n+4$. Simplifying, we find that $\overline{RS}=n+4$.

11. B. Plug the coordinates into the midpoint formula: $\left(\frac{-5+1}{2},\frac{8+2}{2}\right)=\left(\frac{-4}{2},\frac{10}{2}\right)=(-2,5)$. The x-coordinate of this point is -2.

12. B. Plug the coordinates into the midpoint formula: $\left(\frac{-3+10}{2},\frac{10+(-28)}{2}\right)=\left(\frac{7}{2},\frac{-18}{2}\right)=\left(\frac{7}{2},-9\right)$.

13. D. Plug the coordinates into the midpoint formula: $\left(\frac{a+9}{2},\frac{b+1}{2}\right)=(6,-5)$. By multiplying the x and y coordinates on either side by 2, $(a+9,b+1)=(12,-10)$. We simplify this to $(a,b)=(3,-11)$. Note that when solving, we do not add or subtract the same value from the x and y coordinates, only one or the other.

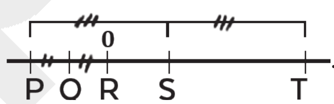
14. A. We solve this the same way we solved the question before. Plug in the values given into the midpoint formula: $\left(\frac{x+1}{2},\frac{y+(-3)}{2}\right)=(19,2)$. This becomes $(x+1,y-3)=(38,4)$. Thus, $(x,y)=(37,7)$. $x+y=37+7=44$.

15. B. Plug in the values given into the midpoint formula: $\left(\frac{5z+3z}{2},\frac{a-3+z+1}{2}\right)=(8,15)$. This becomes $(5z+3z,a-3+z+1)=(16,30)$. $(8z,a+z-2)=(16,30)$. We cannot do anything about the expression with a in it, but we can pull out $8z=16$, which yields $z=2$.

16. C. We know that $\overline{PR}=2\overline{PQ}$, since Q is the midpoint of $\overline{PR}$. $\overline{PR}=16=2b$, so $b=8$. $\overline{RS}=3b-12=3(8)-12=12$. $\overline{QR}+\overline{RS}=\overline{QS}$. $\overline{QS}=8+12=20$.

17. B. $\overline{AC}=39$. $\overline{BD}=32$. $\overline{AC}+\overline{BD}-\overline{BC}=60$, which means $30+32-\overline{BC}=60$. Solve to find that $\overline{BC}=11$.

18. B. Drawing this out, we get:



Instead of getting jumbled up with our points, let's label the $\overline{PQ}$ (which is the same as $\overline{QR}$ because Q is the midpoint) n. We are told that $\overline{RS}=4+n$, so we can say that $\overline{PS}=n+n+(4+n)$. We also know that $\overline{ST}=2\overline{QR}+8$, which can also be written as $2n+8$. Because we know that $\overline{PS}=\overline{ST}$, we can set these equations equal to each other: $n+n+(4+n)=2n+8$. Simplifying this, we have $3n+4=2n+8\rightarrow n=4$. Because we're looking for the coordinate of S, we are really looking for the distance of $\overline{RS}$, since we know R is at 0. $\overline{RS}=4+n\rightarrow 4+4=8$. The coordinate of S is 8.

19. D. The sum of horizontal and vertical lengths on opposite sides of this shape are equal. As an example, notice how in this figure, $\overline{GE}, \overline{ED}, and\, \overline{CB}$ correspond exactly to sections of $\overline{AH}$ to the left. Thus, the perimeter is equal to 2 times the vertical length of the shape, 10, plus 2 times the horizontal length of the shape, $5+4+2$. The perimeter is equal to $2(10)+2(5+4+2)=2(10)=2(11)=20+22=42$.

20. B. The vertical lengths $\overline{GE}$, $\overline{ED}$, and CB are equal to 10. $GF=2$ and $CB=5$, so by substitution, $2+ED+5=10$. Simplifying, $ED=3$. The horizontal distance between A and E is equal to $HG+FE=5+4=9$. The vertical distance between A and E is equal to $BC+ED=5+3=8$. Since A is at $(0,0)$, the coordinates of E are $(9,8)$. Point J is the midpoint of the two, so it is at $\left(\frac{0+9}{2},\frac{0+8}{2}\right)=(4.5,4)$.

21. D. B is 11 units to the right of $(0,0)$, giving it the coordinates $(11,0)$. The distance between J and B is given by the Pythagorean theorem. J is at $(4.5,4)$.

We can plug that in as $\sqrt{(11-4.5)^2+(0-4)^2}=\sqrt{6.5^2+(-4)^2}=\sqrt{42.25+16}=\sqrt{58.25}\approx 7.6$.

22. B. The midpoint of $\overline{AC}$, given by the midpoint formula, is $\left(\frac{-5+10}{2},\frac{4+(-7)}{2}\right)=\left(\frac{5}{2},\frac{-3}{2}\right)$. The midpoint of $\overline{BD}$ is $\left(\frac{-2+8}{2},\frac{-1+3}{2}\right)=\left(\frac{6}{2},\frac{2}{2}\right)=(3,1)$.

The distance between the midpoints is $\sqrt{\left(3-\frac{5}{2}\right)^2+\left(1-\left(-\frac{3}{2}\right)\right)^2}=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{25}{4}}=\sqrt{\frac{26}{4}}=\frac{\sqrt{26}}{2}$.

23. E. The water fountain is at the midpoint between B and E: $\left(\frac{40+50}{2},\frac{10+150}{2}\right)=\left(\frac{90}{2},\frac{160}{2}\right)=(45,80)$. The distance between this point and A is $\sqrt{(45-15)^2+(80-100)^2}=\sqrt{900+400}=\sqrt{1300}$.

24. B. Keep in mind that distance is always a positive quantity. The distance in the y direction is $3-(-15)=18$.

25. E. Plug the coordinates into the distance formula: $\sqrt{(9-3)^2+(-3-4)^2}=\sqrt{6^2+(-7)^2}=\sqrt{36+49}=\sqrt{85}$.