CHAPTER

11 PROBABILITY

SKILLS TO KNOW

- Basic Probability
- Finding the probability something will not happen
- Independent Events & Dependent Events
- And Situations/Or Situations
- Probability & Permutations
- Finding expected values
- Probability Notation, Union & Intersection

BASIC PROBABILITY

The probability of an event occurring is the likelihood that something will happen.

Probability is expressed as a decimal or fraction between 0 and 1 inclusive. If the probability of an event is 1, it will happen with 100% certainty. The closer the probability of an event is to 1, the more likely it will occur. The closer the probability of an event is to zero, the less likely it is to occur.

When the selection of an outcome is **at random**, we can calculate probability by creating a fraction:

$$\frac{\textit{Number of possible desired outcomes}}{\textit{Number of total possible outcomes}}$$

For example, if we want to know the chances of choosing a blue marble from a box when there are 3 blue marbles and 10 marbles total, that would be $\frac{3}{10}$.

This can also be expressed as:

$$\frac{\textit{Number of "successful" outcomes possible}}{\textit{Number of "successful" outcomes possible} + \textit{Number of "failed" outcomes possible}}$$

For example, if there are 3 blue marbles and 7 other colored marbles, there are 3 ways to succeed and 7 ways to fail if I pick one marble out of the box and want a blue one. The probability is thus:

$$\frac{3}{3+7}$$

There are 52 cards in a deck of cards. There are 4 suits (spades, clubs, diamonds, and hearts), each with 13 cards, 3 of which are face cards. 2 suits are red, and 2 are black. If a card is drawn at random, what is the probability that it is a red face card?

To solve, we use the fraction formula for finding probability:

$$\frac{\textit{Number of possible desired outcomes}}{\textit{Number of total possible outcomes}}$$

We need to find two things:

1. **The numerator:** the number of red face cards (what we want).

   We have 3 face cards per suit. Two of the suits are red. That means we have $2(3)$ or 6 total red face cards.

2. **The denominator:** the total number of cards in the deck (total possible outcomes).

   Per the question, we have 52 cards in the deck. We now divide the value for #1 above by that of #2:

$$\frac{6}{52}=\frac{3}{26}$$

Answer: $\frac{3}{26}$.

PROBABILITY THAT SOMETHING *WON'T* HAPPEN

Finding the probability something *will* happen can also be solved by calculating the probability that something *won't* happen. In the world of probability, success and failure are mutually exclusive concepts: either something happens or it doesn't; two options exist. As a result, the probability that something will happen and that something won't happen always sum to 1. For example, if there is a $\frac{3}{5}$ chance you'll pick a red marble from a bucket, there is a $\frac{2}{5}$ chance you won't.

If you need to know the probability that something happens, but it's easier to find the probability of that something *not happening*, solve for the latter probability and subtract from one. Likewise, if you're asked to find the probability of something *not happening* and it's easier to solve for probability of something *happening*, solve for that and subtract from one.

> There are 100 slips of paper numbered 1 through 100 inclusive in a hat. If one slip is drawn at random, what is the probability the number drawn is not a perfect square?

I know $1,4,9,16,25,36,\ldots,81,100$ are the list of perfect squares. I omitted the middle, because I know how each is formed: squaring a number 1 to 10 inclusive, as 1 is 1 squared, and 100 is 10 squared. In between, I'll have $2^2,3^2,4^2$, etc. Thus I know there will be 10 of these numbers in the list. As a result, the probability of getting a perfect square as my number on the slip is:

$$\frac{10}{100}=\frac{1}{10}$$

Now I subtract this value from 1 to find the answer: $1-\frac{1}{10}=\frac{9}{10}$.

Answer: $\frac{9}{10}$.

INDEPENDENT VS. DEPENDENT EVENTS

Independent events in probability are similar to those we covered in the previous chapter on counting & arrangements. Remember independent events don't affect the outcome of other events. Coin flips, dice rolls, and situations with wording such as "with replacement" or "items/digits can repeat" are typically **independent events.**

Dependent events in probability are also the same as covered in the previous chapter on counting. In general, we need to be careful to adjust our calculations as we step through each possibility when we have **dependent events**—typically as we choose a possibility, that possibility cannot be chosen again so we have to reduce the number of available choices with each blank. Drawing three letters from a bag of lettered tiles, choosing people for a team, or selecting songs to sing at a recital are all dependent events. You wouldn't sing the same song twice at your voice recital, so what you pick for the first song affects what you choose for the 2nd. If you had 4 songs to choose from, after you choose one song you'll only have 3 songs to choose from. Often, you'll see words like "distinct," "unique," or "without replacement" when encountering problems that involve **dependent events**.

"AND" SITUATIONS

If the probability of two independent events are A and B, then the probability of both A "AND" B occurring (assuming each event is unique, or order matters) **is A times B.** In short, when you have two **"AND"** situations (Event A is true AND Event B is true) **you multiply.**

> Ned is throwing a coin to see if it's heads or tails. What is the probability that he will throw heads three times in a row?

The probability of getting heads or tails is 1 in 2 chances, so the chance of getting heads once is $\frac{1}{2}$. Since Ned will be doing this three times and order matters (to get three heads in a row we need each subsequent toss to be heads, the first, the second, and the third) we multiply $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ to get $\frac{1}{8}$.

Answer: $\frac{1}{8}$.

A similar trend emerges with **dependent events**. We still **multiply probabilities together** when both **event A AND event B** are true to find the probability both are true (assuming **order matters**). However, We must take event A into account when we calculate the probability of B. In other words, we multiply (Probability of Event A) (Probability of event B given Event A happening first). It sounds a bit confusing, but in practice it's simple.

> Marlin is choosing three toys at random to take to the beach from her basket of 10 beach toys. If $\frac{1}{2}$ of the beach toys are plastic, what is the probability she chooses all plastic toys?

We start by remembering the fraction that defines probability:

$$\frac{\textit{Number of possible desired outcomes}}{\textit{Number of total possible outcomes}}$$

These are dependent events, so as I work I must take into account the previous choices. I want to find the probability that three particular events occur in a row: she chooses a plastic toy first, a plastic toy second, and a plastic toy third. For all of these events to be true, we have **an "AND" situation**. Each of these events **must occur** in order to get my desired outcome. Thus we can find the probability of each case and then must **multiply these individual probabilities together**.

Her probability of choosing a plastic toy the first is $\frac{5}{10}$, but the second is $\frac{4}{9}$. When she chooses the 2nd toy, the 1st toy is no longer an option; there are only 4 plastic toys to choose from and 9 toys left in total to choose from. When she chooses the third toy, there is a $\frac{3}{8}$ chance she chooses a plastic one. At this point, she will only have 3 plastic toys left to choose from and 8 toys in total. As you can see, we reduce the numerator and denominator accordingly as we go: $\frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} = \frac{1}{12}$.

Why can we solve this problem using a "permutation" when it sounds like a combination?

Though order doesn't matter here in one sense (she is selecting a few items from a group), it DOES matter in terms of her selecting a toy that is plastic on each turn, i.e. each moment when she chooses a toy a certain event must occur: she picks a plastic toy each time. Thus we can still treat this as a permutation and not a combination. We could actually solve this problem using combinations as well. For that method, we'd rely again on the fraction that defines probability, and calculate our numerator as the number of ways to choose 3 plastic toys from 5 (${}_5C_3$ or "5 choose 3") and then divide that by the number of ways to choose 3 toys from 10 (${}_{10}C_3$ or "10 choose 3"). Using our calculator's built in combinations function we get: $\frac{{}_3C_5}{{}_5C_{10}}$ and get $\frac{10}{120}$ or $\frac{1}{12}$.

"OR" SITUATIONS

Sometimes we can find a probability by adding together the probability of all the unique ways we could get what we want. These cases are essentially **"OR" scenarios**. Situation A is true or B is true or C is true, for example. When we have **"OR"** situations in probability, and our elements are unique (i.e. "mutually exclusive") we **ADD the probabilities together.** Add the probability of all the unique cases that produce your desired outcome to find the overall probability of that outcome.

Let's say we want to know the probability of flipping one head and one tail when two coins are flipped. For example, if the chance I get heads then tails on two coin flips is $\frac{1}{4}$ and the chance I get tails then heads on two coin flips is $\frac{1}{4}$, then the chance in two flips of a coin that result in one heads and one tails is $\frac{1}{4}+\frac{1}{4}$ or $\frac{1}{2}$.

Important: If you add probabilities together, you must be certain the outcomes included within each probability you've calculated **DO NOT OVERLAP**. Events must be distinct or **"mutually exclusive"** if you're going to add their probabilities. I can't be in 4th grade and in 5th grade, those are mutually exclusive events that do not overlap. But I could be in 4th grade and female. Those are NOT mutually exclusive events.

> A multiple-choice quiz has four answer options for each of five questions. What is the probability of choosing answer choices at random and missing exactly one question?

Because getting a question right or wrong is independent of how I did on the last question (assuming I'm guessing at random), the events are independent. To solve this problem, we pretend order matters and break it into multiple cases in which order matters that produce the combination we want.

To get four right and one wrong could look like this:

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}=\frac{3}{1024}$$

This is the probability that I miss ONE question, and that the question I miss is the LAST question. But there are 5 different orders this could happen in, i.e. the "wrong question" (our $\frac{3}{4}$ in the string above) could also be 1st, 2nd, 3rd, or 4th:

Case 1: I get question 1 wrong:

$$\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{1024}$$

Case 2: I miss question 2:

$$\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{1024}$$

Case 3: I miss question 3:

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{1024}$$

Case 4: I miss question 4:

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}=\frac{3}{1024}$$

For each of these cases the fractions are the same as the case I initially wrote out, but just in a different order. Each of the five cases has a probability of $\frac{3}{1024}$.

I need the sum of all these possibilities, as each one is a distinct way I could make my desired condition true. I could add $\frac{3}{1024}$ to itself five times, or simply multiply $\frac{3}{1024}$ to get the answer:

$$5\left(\frac{3}{1024}\right)=\frac{15}{1024}$$

Answer: $\frac{15}{1024}$.

USING PERMUTATIONS IN PROBABILITY PROBLEMS

More complex probability problems synthesize your knowledge of counting problems and of probability. To solve these problems, remember probability is always found by finding:

$$\frac{\textit{Total number of outcomes that fulfill the desired parameters}}{\textit{Total number of possible outcomes}}$$

Oftentimes, we can use permutations (or the fundamental counting principle or even combinations) to solve for each of these two values, and then in turn solve for the probability. The ACT® rarely requires you to know how to do these with combinations, so I'll focus on permutations. Still, the same principle would work if the problem you confront involves combinations.

> In the bleachers of a football stadium, 2 boys and 3 girls are seated together in a random order. What is the probability that the 2 boys are seated next to each other?

First, we know we need to solve for two values:

1. Total number of ways to arrange 2 boys and 3 girls such that the 2 boys are always next to each other (numerator of our probability)
2. Total number of ways to arrange 5 kids (I could say 2 boys and 3 girls, but each person is actually unique, so this is really just how to arrange 5 kids; thinking this way makes the math easier).

Both of these elements can be solved using permutations and a bit of creativity.

Let's start with #1:

When I need to keep 2 items next to each other in a permutation, one way I can think of this is Case 1/Case 2. Let's name our boys Brian and Max, and our girls Leah, Wei Wei, and Ann.

Case 1: Brian is seated directly to the left of Max
Case 2: Brian is seated directly to the right of Max

Now I can solve for the number of permutations I have, pretending that Brian and Max are "glued" together and essentially are one person. I'll just calculate the number of ways to arrange for each case and add all the possibilities together.

Case 1: I can choose from the following four taken four at a time:

Brian/Max, Leah, Wei Wei, Ann

$$\underline{\;4\;} \cdot \underline{\;3\;} \cdot \underline{\;2\;} \cdot \underline{\;1\;} = 24$$

Case 2: I can choose from the following four taken four at a time:

Max/Brian, Leah, Wei Wei, Ann

$$\underline{\;4\;} \cdot \underline{\;3\;} \cdot \underline{\;2\;} \cdot \underline{\;1\;} = 24$$

I add these together and get 48 (I also could have seen that Case 1 & 2 will have the same number of options, and thus could have simply multiplied 24 by 2). In any case, I know my numerator: 48.

Now for Step #2:
How many ways can I arrange 5 kids, in order? 5 kids taken 5 at a time when order matters is simply:

$$\underline{\;5\;} \cdot \underline{\;4\;} \cdot \underline{\;3\;} \cdot \underline{\;2\;} \cdot \underline{\;1\;} = 120$$

120 is my denominator.

Now I simplify the fraction:

$$\frac{\textit{Total number of outcomes that fulfill the desired parameters}}{\textit{Total number of possible outcomes}}$$

$$= \frac{48}{120} = \frac{2}{5} \textit{ or } .4$$

EXPECTED VALUES

Finding the expected value is like finding a weighted average. Here's an example.

The probability distribution of the discrete random variable Y is shown in the table below. What is the expected value of Y?

y	Probability $P(Y=y)$
0	0.15
1	0.26
2	0.29
3	0.11
4	0.19

First of all, don't be thrown by the language **"discrete random variable."** That just means that Y is not continuous as a possible value. For example, the number of people in an elevator is a discrete number because you can't have half a person; every number is a whole number. If I listed out probabilities of the number of people in the office elevator on any given trip, my values would all be discrete. Discrete variables don't have to be integers, but the point is that you don't have to worry about a bunch of values in between what is on your chart.

Whenever I have problems with a probability chart like this, I always double check that the given probabilities add to one. If not, the chart is not a complete depiction of what is going on and I must account for that. Occasionally these problems will only give you the "first few values" of this variable and not all of them.

Here, I see that all my probabilities sum to 1. Thus I know to find my expected value, I simply multiply the value of y times its probability of occurring. Then I add all these little values up:

$$\begin{aligned} &0(0.15)+1(0.26)+2(0.29)+3(0.11)+4(0.19) \\ &=0+0.26+0.58+0.33+0.76 \\ &=1.93 \end{aligned}$$

The answer should essentially be the weighted average of the values in your chart. If your answer doesn't seem about "average," go back and check your work. Here this makes sense. 1 and 2 occur most often.

Answer: 1.93.

PROBABILITY NOTATION, UNION & INTERSECTION

(FYI, this is <u>NOT frequently tested</u>. Learn the rest first!).

Occasionally, the ACT® may use certain notation to denote probability. <u>Typically, however, the ACT® will define this notation for you if you are expected to use or understand it.</u> That means you shouldn't worry too much about remembering everything below; just be familiar with it.

We say that the probability of Event A occurring is $P(A)$. What that means is that if I write "$P(A)$" that represents the fraction or decimal probability that something happens. Similarly the probability of Event B occurring would be $P(B)$, of Event C, $P(C)$ and so on.

> Let $P(A)$ represent the probability of event A occurring. If event R occurs when three coins are flipped and all are heads, calculate $P(\textit{not } R)$.

For this problem, we simply calculate the odds of getting all heads by multiplying the independent probability of each event. Since the probability of flipping heads is $\frac{1}{2}$, we multiply:

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$$

Because we need "not R" we now subtract the probability of getting R from 1:

$$1-\frac{1}{8}=\frac{8}{8}-\frac{1}{8}=\frac{7}{8}$$

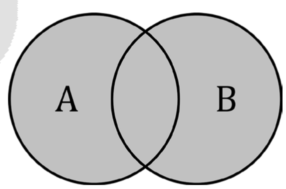
Answer: $\frac{7}{8}$.

<u>"UNION"</u>

You may occasionally see a U-like symbol in probability problems. When it is facing upwards, we call this "union." For example, the probability of A union B is denoted by:

$$P(A\cup B)$$

All this is asking for is the combined probability of A and B, i.e. the probability that A happens, B happens, or both happen. **Union occurs when A OR B occur.** It is always greater than or equal to the probability of A alone or B alone. The picture below is one way to visualize events A and B if there is some overlap.



As you can see, we include all of A and all of B, being careful to only count the elements that overlap once (if applicable).

Unlike the "mutually exclusive" situations discussed earlier (i.e. when there is no overlap, such as having blue eyes or brown eyes), when we simply add together probabilities to find the probability of one OR the other occurring, Union situations often involve overlap (though they need not involve overlap). For example, if event A is having brown hair and event B is having blue eyes, some people have both traits, and that would be "overlap" if we counted those who had A **OR** B. **When events have some overlap we call these "inclusive events."**

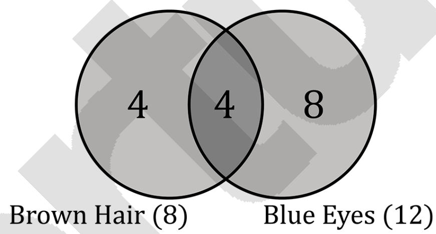
If Events A and B are inclusive, then the probability that A or B occurs is the sum of their probabilities minus the probability that both occur. I.e. to find the combined probability of inclusive events, we add the individual probabilities together and subtract the overlap.

$$P\left(A\,or\,B\right)=P\left(A\right)+P\left(B\right)-P\left(A\,and\,B\right)$$

To help understand or remember the formula, use a **Venn diagram** to visualize these problems.

In in a class of 24 students, if 12 students have blue eyes, the probability of which is denoted by $P\left(A\right)$, 8 students have brown hair, the probability of which is denoted by $P\left(B\right)$, and 4 students have both, the probability of which is denoted by $P\left(A\,and\,B\right)$, what is the probability that students have brown hair or blue eyes, denoted by $P\left(A\,or\,B\right)$?

First, don't be thrown by all the notation. It's only there to confuse you. Just solve the problem. I know probability is the number of desired outcomes divided by the possible outcomes. I know I have 24 kids in the class, so that is my denominator. My numerator is the number of blue eyed and brown haired students *inclusive*, i.e. anyone who has either trait or both: those with brown hair and not blue eyes, those with both blue eyes and brown hair, and those with blue eyes and not brown hair. I can start by figuring out each of these cases using the Venn Diagram below. I subtract 4 (overlap) from 8 (number of students who have brown hair) to find the number who have brown hair but not blue eyes and subtract 4 from 12 to find blue-eyed kids without brown hair (8):



I can now add each segment in my Venn Diagram to find the number of people who have either brown hair, blue eyes, or both ($4+4+8=16$).

I can also find this by taking $8+12-4=16$, per the formula we discussed earlier.

Now I place 16 over 24: $\frac{16}{24}=\frac{4}{6}=\frac{2}{3}$.

Answer: $\frac{2}{3}$.

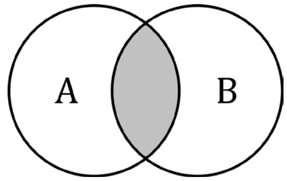
“INTERSECTION”

Another notation you might see is an upside down “U.” We call this an intersection. The probability of A intersection B is denoted by:

$$P\left(A\cap B\right)$$

An intersection occurs when **A AND B** are both simultaneously true. It is always less than or equal to the probability of A or B alone.

The picture below depicts what A intersection B looks like; it is the "overlap" of values in both sets:



Let A and B be independent events. Denote $P(A)$ as the probability that event A will occur, $P(A \cup B)$ as the probability that event A or B or both will occur. Which of the following equations *must* be true? (Note: $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.)

A. $P(A)P(B) = P(A \cup B)$

B. $P(A) > P(A \cup B)$

C. $P(B) - P(A) = P(A \cap B) - 2P(A) + P(A)P(B)$

D. $P(A \cup B) > P(A) + P(B)$

E. $P(A)P(A \cup B) = P(A)^2 - P(A)^2 P(B) + P(A)P(B)$

Let's go through each choice.

A. $P(A)P(B) = P(A \cup B)$

When we multiply two independent probabilities together, we find the chances that *both* occur. Thus this is a calculation of the intersection, NOT the union. A is incorrect.

B. $P(A) > P(A \cup B)$

We know that union essentially adds any items in set B to set A. Thus the probability of A alone cannot be greater than a probability that has at least as many if not more options that make it true. B is also incorrect.

C. $P(B) - P(A) = P(A \cap B) - 2P(A) + P(A)P(B)$

At first, this might look like an algebraic manipulation of the original, given equation. Except it includes the symbol for INTERSECTION not UNION. Be careful! All U's are not the same!

D. $P(A \cup B) > P(A) + P(B)$

Again the union is the set of all the items in A plus all the items in B, minus any overlap (if applicable). If there were *no* overlap, the two sides of this expression would be equal. Since that is possible, I know this is not something that MUST be true. In fact, it can't be true. If there *is* overlap, then $P(A) + P(B)$ will overestimate the value of the union (as it is the total before subtraction of the overlapping elements in the sets). Reversing the inequality sign and making it "or equal to" would make this expression true (i.e. $P(A \cup B) \leq P(A) + P(B)$).

E. Looking at the left of the equation, we see that we've simply multiplied the union by $P(A)$. Let's use substitution and expand this expression, using the information in the "Note":

$$P(A)P(A \cup B)$$

We now plug in $P(A)+P(B)-P(A)P(B)$ for $P(A\cup B)$ (this is given in the problem).

$$P(A)\big(P(A)+P(B)-P(A)P(B)\big)$$

Using the distributive property we get:

$$P(A)^2+P(A)P(B)-P(A)^2P(B)$$

Now we rearrange using the commutative property:

$$P(A)^2-P(A)^2P(B)+P(A)P(B)$$

Answer: **E.**