CHAPTER

17 EXPONENTS AND RADICALS

SKILLS TO KNOW

- How to use exponents and how radicals relate to exponents
- Applying exponent rules
- Applying radical rules
- How to add, subtract, multiply, and divide with exponents

RADICAL AND EXPONENT RULES

EXPONENT RULES:

Zero power rule: $a^0 = 1$

Negative exponent rule: $a^{-b} = \frac{1}{a^b}$

Power of a power: $\left(a^b\right)^c = a^{bc}$

Product of powers: $a^b a^c = a^{b+c}$

Power of a product: $(ab)^c = a^c b^c$

Power of a quotient: $\frac{a^b}{c^b} = \left(\frac{a}{c}\right)^b$

Quotient of powers: $\frac{a^b}{a^c} = a^{b-c}$

RADICAL RULES:

Fractional exponent conversion: $\sqrt[c]{a^b} = a^{\frac{b}{c}}$

Product of radicals: $\sqrt[c]{a}\sqrt[c]{b} = \sqrt[c]{ab}$

On the ACT®, you'll need to be able to work with all the basic rules of exponents and radicals listed above. If these rules confuse you, or you can't remember them straight, **work on understanding the principle behind each formula** so that even if you're stuck you can derive the rule: make up some actual numbers and step through the function to remember the rule you've forgotten.

Ideally you won't need to do this on the ACT® itself (it's better to use AND memorize formulas—time on the ACT® is too short to always use the long way), but you may need to as you work through the problems in this chapter to refresh your memory, or if you have time to double check your work. <u>On the following pages are mini proofs of each of these rules—create your own when you get stuck!</u> You can also make up little mnemonics (memory tricks) to remember rules, or worst case do the problem by hand, writing each number out.

POWER OF A POWER

$$\left(a^b\right)^c = a^{bc}$$

To remember this rule, you can think of how when the exponents are separated only by a <u>parenthesis,</u> they are very close together so it's multiplication—in multiplication you omit the multiplication sign and separate items with <u>parentheses</u>. Here's a mini-proof that shows this rule to be true:

$$\left(5^2\right)^3 = \left(5*5\right)^3 = \underbrace{\overbrace{(5*5)}^{2\ 5's}\overbrace{(5*5)}^{2\ 5's}\overbrace{(5*5)}^{2\ 5's}}_{3\text{ times}} = 5^6 = 5^{(2*3)}$$

Simplify $\left(x^4\right)^3$

We simply apply the rule: $\left(x^4\right)^3 = x^{12}$

If necessary, you could write it out as above:

$$\left(x^4\right)^3 = \left(x^4\right)\left(x^4\right)\left(x^4\right) = \underbrace{\overbrace{(x*x*x*x)}^{4x's}\overbrace{(x*x*x*x)}^{4x's}\overbrace{(x*x*x*x)}^{4x's}}_{3\text{ sets}} = x^{3*4} = x^{12}$$

PRODUCT OF POWERS

$$a^b a^c = a^{b+c}$$

For this one, remember that the b and c seem farther apart. Imagine that there's room for an addition sign between them. Here's a mini proof/example:

$$7^3 7^5 = \overbrace{(7*7*7)}^{3\ 7's}\overbrace{(7*7*7*7*7)}^{5\ 7's} = 7^{3+5} = 7^8$$

Which of the following expressions is equivalent to $x^9 x^{(-2)^2}$?

A. $x^{\frac{9}{4}}$ **B.** x^5 **C.** x^{13} **D.** x^{36} **E.** $x^9 + x^4$

A couple of rules are at play—first start with the exponent $x^{(-2)^2}$ and simplify i.e. $(-2)^2 = 4$, so it simplifies to x^4. (Remember that when negative numbers are in parenthesis and squared (or taken to any even power), the answer is always positive—so the answer is NOT negative four!)

$$x^9 x^{(-2)^2} = x^9 x^4$$

Now we apply our Products of Powers Property:

$$x^9 x^4 = x^{9+4} = x^{13}$$

Answer: **C.**

$\left[\left(x^3\right)\left(x^2\right)^2\right]^3$ is equivalent to:

A. x^{10} **B.** x^{21} **C.** x^{27} **D.** x^{30} **E.** x^{36}

To solve this, we would use the rules $\left(a^b\right)^c = a^{bc}$, $\left(ab\right)^c = a^c b^c$, and $a^b a^c = a^{b+c}$. We can first simplify $\left[\left(x^3\right)\left(x^2\right)^2\right]^3$ to $\left(x^3\right)^3\left(x^2\right)^{2\times 3}$ which is $\left(x^3\right)^3\left(x^2\right)^6 \to \left(x^9\right)\left(x^{12}\right) \to x^{21}$.

If you have trouble remembering the property, you can write the process out by hand:

$$x^3x^5 = \left(x * x * x\right)\left(x * x * x * x * x\right) = x^8$$

Answer: **B.**

TIP: **CREATE A MINI PROOF IF MEMORY FAILS!**
If you get a problem too challenging to write out by hand, say $x^{99}x^4$, and aren't 100% sure of the formula, then write a "mini proof" as with 7, 3 and 5 above—they're more manageable number than 99! You'll soon remember the formula and can apply it (the answer is x^{103}).

POWER OF A PRODCUT

$$\boxed{\left(ab\right)^c = a^c b^c}$$

Proof: $\left(ab\right)^c = \underbrace{\left(ab\right)\left(ab\right)...\left(ab\right)}_{c \text{ times}} = \underbrace{\left(a\right)\left(a\right)...\left(a\right)}_{c \text{ times}} * \underbrace{\left(b\right)\left(b\right)...\left(b\right)}_{c \text{ times}} = a^c * b^c = a^c b^c$

<u>Think the distributive property—but for exponents.</u>

$\left(2xy\right)^4$ is equivalent to:

A. $2x^4y^4$ **B.** $8x^5y$ **C.** $16x^4y^4$ **D.** $2xy^4$ **E.** $\frac{x^4}{2}$

Here we distribute the 4 to the 2 and the x and then simplify: $\left(2xy\right)^4 = 2^4 * x^4 * y^4 = 16x^4y^4$

Answer: **C.**

MISTAKE ALERT: Don't forget to distribute to EACH piece, and to write parenthesis down when you are recopying problems. Choice (A) doesn't distribute to the 2! (D) doesn't distribute!

POWER OF A QUOTIENT

$$\frac{a^b}{c^b}=\left(\frac{a}{c}\right)^b$$

Proof: $\frac{a^c}{b^c}=a^c b^{-c}=a^c\left(-b\right)^c=\left(\frac{a}{b}\right)^c$

This one is pretty much the same as Power of a Product—think of the distributive property. Distribute to the top and bottom.

Simplify $\frac{8^2}{4^2}$

$$\frac{8^2}{4^2}=\left(\frac{8}{4}\right)^2=\left(2\right)^2=4$$

$$\text{Alternatively: } \frac{8^2}{4^2}=\frac{64}{16}=4$$

Answer: 4.

QUOTIENT OF POWERS

$$\frac{a^b}{a^c}=a^{b-c}$$

Mini proof/example: Simplify $\frac{3^5}{3^3}$:

$$\left(\frac{3*3*3*3*3}{3*3*3}\right)=\left(\frac{3*3*\cancel{3}*\cancel{3}*\cancel{3}}{\cancel{3}*\cancel{3}*\cancel{3}}\right)=\frac{3*3}{1}=9$$

$$\frac{3^5}{3^3}=3^{5-3}=3^2=9$$

For all nonzero a and b, $\frac{\left(6a^5b^4\right)\left(8a^3b^7\right)}{4a^7b^9}=?$

A. $12ab$ **B.** $12ab^2$ **C.** $12a^{\frac{8}{7}}b^{\frac{11}{9}}$ **D.** $12a^{\frac{15}{7}}b^{\frac{28}{9}}$ **E.** $12a^8b^{19}$

We'll have to apply a few rules—let's start by simplifying the top of the fraction:

$$\left(6a^5b^4\right)\left(8a^3b^7\right)$$

We can group like terms together and simplify:

$$\left(6\times 8\right)\left(a^5a^3\right)\left(b^4b^7\right)$$
$$=48\left(a^5a^3\right)\left(b^4b^7\right)$$

Now we can use the Product of Powers Property $\left(a^ba^c=a^{b+c}\right)$, to add the appropriate exponents.

$$48\left(a^5a^3\right)\left(b^4b^7\right)$$
$$=48a^{5+3}b^{4+7}$$
$$=48a^8a^{11}$$

Now insert this simplified expression back into the numerator: $\dfrac{48a^8b^{11}}{48a^7b^9}$

Because the answers have no fractions, we use the quotient of powers on all like terms, $\dfrac{a^b}{a^c}=a^{b-c}$:

$$\frac{48}{4}\times\frac{a^8}{a^7}\times\frac{b^{11}}{b^9}$$
$$=12\times a^{8-7}\times b^{11-9}$$
$$=12a\times b^2$$
$$=12ab^2$$

Answer: **B.**

RADICAL RULES

Remember that a fractional exponent is the same as the reciprocal root, i.e. $a^{\frac{1}{b}}=\sqrt[b]{a}$, or $4^{\frac{1}{2}}=\sqrt{4}$, $4^{\frac{1}{3}}=\sqrt[3]{4}$ etc. For more complicated combinations of exponents and roots, the following pattern applies:

FRACTIONAL EXPONENT CONVERSION

$$\sqrt[c]{a^b}=a^{\frac{b}{c}}$$

Proof: $\sqrt[c]{a^b}=\left(a^b\right)^{\frac{1}{c}}=a^{\frac{b}{c}}$

PRODUCT OF RADICALS

$$\sqrt[c]{a}\sqrt[c]{b}=\sqrt[c]{ab}$$

Proof: $\sqrt[c]{a}\sqrt[c]{b}=a^{\frac{1}{c}}*b^{\frac{1}{c}}=(ab)^{\frac{1}{c}}=\sqrt[c]{ab}$

Once you know the rules, there are a few principles that will help you solve the following problems:

Which is a solution of $\sqrt[5]{32x^{10}}=800$?

A. 4 **B.** 20 **C.** 40 **D.** 400 **E.** $\sqrt[10]{400}$

Using the rule $\sqrt[c]{a^b}=a^{\frac{b}{c}}$, we know that $\sqrt[5]{32x^{10}}$ can be rewritten as $32^{\frac{1}{5}}x^{\frac{10}{5}}$. This simplifies to $32^{\frac{1}{5}}x^2$. We can use our calculator to find out that $32^{\frac{1}{5}}=2$. Now we can write the equation that we're trying to solve:

$$2x^2=800$$
$$x^2=400$$
$$x=20$$

Answer: **B.**

Which expression is equivalent to $\sqrt[4]{81x^7}$?

A. $3x^3$ **B.** $3x^{\frac{7}{4}}$ **C.** $3x^7$ **D.** $81x^3$ **E.** $81x^{\frac{7}{4}}$

We can solve this by using the rule $\sqrt[c]{a}\sqrt[c]{b}=\sqrt[c]{ab}$ (in the opposite direction). Thus, we can rewrite $\sqrt[4]{81x^7}$ to $\sqrt[4]{81}\sqrt[4]{x^7}$. We can use the calculator to figure out that $\sqrt[4]{81}=3$, and using the rule $\sqrt[c]{a^b}=a^{\frac{b}{c}}$, we can rewrite $\sqrt[4]{x^7}$ as $x^{\frac{7}{4}}$. Thus, our answer is $3x^{\frac{7}{4}}$.

Answer: **B.**