CHAPTER

23 TRANSLATIONS AND REFLECTIONS

SKILLS TO KNOW

- Basic reflectional and rotational symmetry
- How to identify the translation/reflection of a given graph or function
- How to translate/reflect a given function and/or select correct graph, equation, or name new points

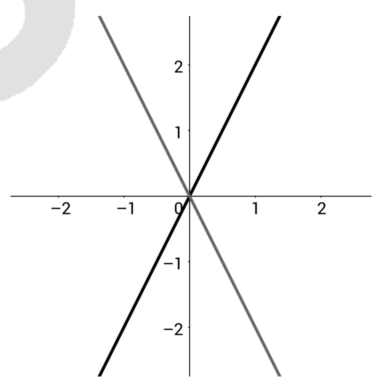
Translating an equation can be split into 3 categories: **shifting**, **compressing**, and r**eflecting/ rotating**.

Two of these sections we covered already in **Graph Behavior**, **shifting** (vertical and horizontal shifts via constants such as "h" and "k") and **compressing** (altering a graph's width or apparent slope with a constant such as "a"). For this chapter, we will focus on problems that involve **reflections** and **rotations**.

REFLECTIONS AND ROTATIONS

Reflectional symmetry: the property a figure has if half of the figure is congruent to the other half over an axis.

Rotational symmetry: also known as radial symmetry—the property a figure has if it is congruent to itself after some rotation less than 360°.



This figure has both reflectional and rotational symmetry over the x or y axis.

How to "reflect" a function algebraically

For equations, negative signs that impact the entire equation can denote reflections.

To reflect over the x-axis: Solve for y or $f(x)$ and distribute a negative sign to the entire other side of the equation. Alternatively, simply place a negative sign in front of every instance of y (no rearrangement necessary).

To reflect over the y-axis: Solve for x and distribute a negative sign to the entire other side of the equation. Alternatively, simply place a negative sign in front of every instance of x (no rearrangement necessary).

As you can see in the picture above, simple reflection could be $y = 2x$ reflected to $y = -2x$. This reflection would be over the x-axis because we solved for y and applied a negative sign to the other half of the equation. But you might notice this reflection is also over the y-axis. That's because a direct variation equation (one with no "y" intercept that goes through the origin) reflects over both axis when you add a negative to either side of the equation. We can see this algebraically by solving for x per our rules above (that denote a y-axis reflection requires first solving for x) to get $\frac{1}{2}y = x$ and then applying the negative to get $-\frac{1}{2}y = x$. We could then multiply both sides by negative two to see that this is the same as $y = -2x$.

Symmetry Questions

Oftentimes on the ACT®, you are asked more conceptual questions that require you to understand the notion of reflectional or rotational symmetry.

Which of the following letters of the alphabet does NOT have at least 1 reflectional symmetry and at least 1 rotational symmetry?
(Note: the angles of rotation for the rotational symmetry must be less than 360°.)

A. H **B.** I **C.** O **D.** U **E.** X

Let's take this problem one half at a time. First up is reflectional symmetry. H, I, O, and X are symmetrical about a centered horizontal axis, and U is symmetrical about a centered vertical axis. So no eliminations here.

Now, for rotational symmetry, H, I, O, and X are symmetrical at 180°. However, U is not symmetrical at any rotation less than 360° meaning that it does not have any rotational symmetry.

Answer: **D.**

TRANSLATIONS

Translation problems we first covered in Graph Behavior, but we'll look at more complex examples here that blend the idea of reflections and or rotations with traditional horizontal/vertical shifts and compression.

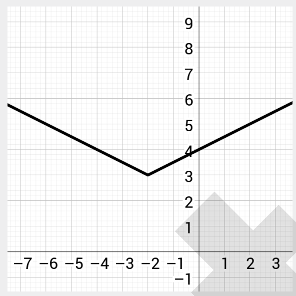
Now, let's take everything we learned and do full translations.

1. Determine vertical shift
2. Determine horizontal shift
3. Determine compression
4. Determine any reflections across the axis or rotations about the origin.

ABSOLUTE VALUE

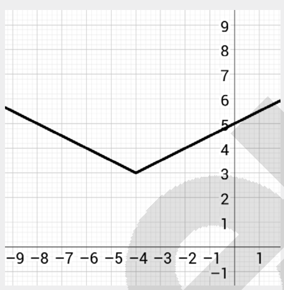
For absolute values, most of the rules still apply, but reflections over the y-axis are more complex, as it's tough to solve for x. As a result, your best move is to apply the negative sign to the **variable x itself.** I.e. to reflect across the y-axis, put a negative sign on every instance of x in the equation. To reflect across the x-axis, put a negative on the y-value.

The function $y=\left|\frac{x}{2}+1\right|+3$ is graphed in the standard (x,y) coordinate plane below.

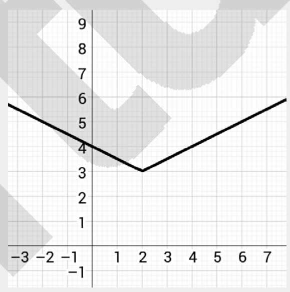


One of the following graphs in the standard (x,y) coordinate plane shows the result of shifting the function 6 units to the right and reflecting it about the y-axis. Which graph is it?

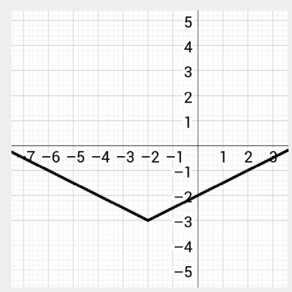
A.



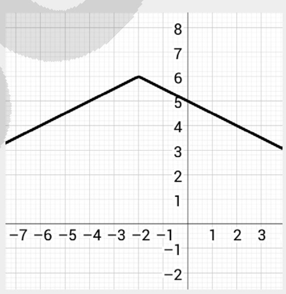
B.



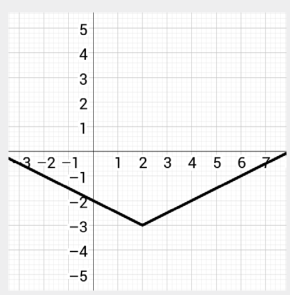
C.



D.



E.



First, we need to factor out the coefficient of the x term from both values inside the absolute value. Remember, the "a" in any equation should be isolated from the "$x-h$" term in order to see our horizontal shift properly. We don't want $ax-h$, we want $x-h$ to be its own little piece of the equation. Here the coefficient is $\frac{1}{2}$, because x is divided by 2, so I factor out $\frac{1}{2}$ from both $\frac{x}{2}$ and 1 to form this equivalent equation:

$$y = \left| \frac{1}{2}(x+2) \right| + 3$$

Now I can move my 3 if I'd like to the other side to see the effect on y (when I look for vertical shifts, I want $y-k$ to replace any instance of y in the root version of the equation).

$$(y-3) = \left| \frac{1}{2}(x+2) \right|$$

Now I'm ready to make another transformation, applying that transformation the $x-h$ and $y-k$ terms:

To shift the equation to the right 6 units, we need to subtract 6 from the $x+2$ term. Remember, from the last chapter, we replace every instance of x with $(x-h)$ where h is the number of units to the right—but here we already have an $(x+2)$ term indicating. Still, we can replace x with our new "$x-h$" in position and get what we need.

$$(y-3) = \left| \frac{1}{2}((x-6)+2) \right|$$

Which simplifies to:

$$(y-3) = \left| \frac{1}{2}(x-4) \right|$$

Finally, we reflect it over the y-axis by placing a negative sign in front of the x:

$$(y-3) = \left| \frac{1}{2}(-x-4) \right|$$

Now, I need to factor out that negative sign in front of the x so I get a clean "$x-h$" term. In doing so, I also factor the -1 out of -4:

$$(y-3) = \left| -\frac{1}{2}(x+4) \right|$$

Furthermore, I know the "negative" will disappear given the absolute value, so this is equivalent to:

$$(y-3) = \left| \frac{1}{2}(x+4) \right|$$

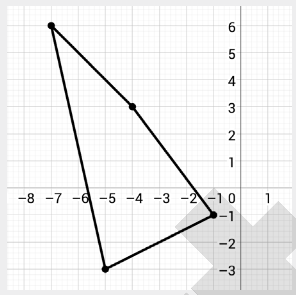
Now we can use our knowledge of horizontal and vertical shift to find our graph. First, our vertex of an absolute value equation would be $(0,0)$. We can find the horizontal and vertical shift of this vertex by looking at our new h and k: h is now indicating we move 4 to the left ($+4$) so our x value of the vertex is -4. k indicates that we move 3 up, so our y value of the vertex is positive 3. We also know the slope of the line should be positive $\frac{1}{2}$ and negative $\frac{1}{2}$ before and after the vertex. Additionally, we can assume this is upward facing given that there is no negative in front of the absolute value expression.

Given these parameters, the answer is A.

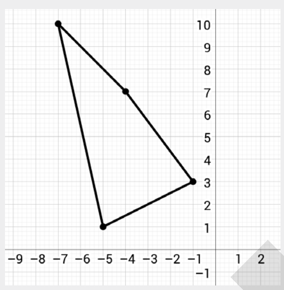
This answer could also be reached by counting out a few individual points (like the vertex) on the graph itself and converting each to a new position, one step at a time.

Answer: **A**.

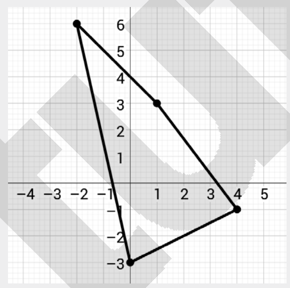
Quadrilateral $ABCD$ has vertices $(-1,-1)$, $(-5,-3)$, $(-4,3)$, and $(-7,6)$ in the standard (x,y) coordinate plane. Suppose $ABCD$ is translated 5 units to the right and 4 units up, forming quadrilateral $A'B'C'D'$. Which of the following shows the coordinates of the vertices of $A'B'C'D'$?



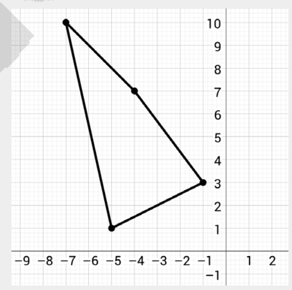
A.



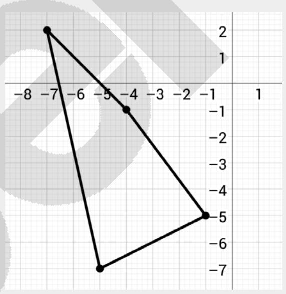
B.



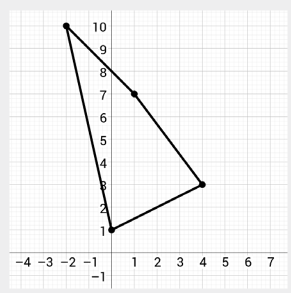
C.



D.



E.



For this question, we can increase every x value by 5 and every y value by 4. We can also just count on the graph for each point and find the transformed quadrilateral. Thus, the new vertices are : $(-1+5,-1+4)$, $(-5+5,-3+4)$, $(-4+5,3+4)$, $(-7+5,6+4)$ = $(4,3)$, $(0,1)$, $(1,7)$, $(-2,10)$.

Answer: **E**.

In the standard (x, y) coordinate plane the graph of $y = \sqrt{x}$ is shifted 3 units up and 4 units to the left, and is then reflected over the y-axis. Which of the following is an equation of the translated graph?

A. $y = \sqrt{(-x+3)} + 4$

B. $y = \sqrt{(-x+4)} + 3$

C. $y = \sqrt{-(x+4)} + 3$

D. $y = \sqrt{(-x+4)} - 3$

E. $y = \sqrt{-(x+3)} - 4$

For this question, we know that in $y = m(x-h)+k$, the k determines vertical shift, and the h determines horizontal shift. Because the equation shifts up by 3, we know k is 3; similarly, because the equation is moving 4 units to the left, $h = -4$. We can plug this into $y = \sqrt{(x-h)} + k$. This gives us $y = \sqrt{(x+4)} + 3$. Now we need to reflect over the y- axis. This can be done by adding a negative sign in front of (and only in front of) x. Thus, we know that the transformed equation is $y = \sqrt{(-x+4)} + 3$.

Answer: **B**.