CHAPTER

6 MATRIX ALGEBRA

SKILLS TO KNOW

- Adding and subtracting matrices
- Multiplying a matrix by a scalar
- Solving for variable values in equivalent matrices
- Multiplying matrices
- Finding basic determinants (when given the formula)*

MATRIX BASICS

Many students may not have had matrix algebra in school, while others forget the rules of matrices. So matrix questions can pose difficulty for many students. These questions do not occur frequently on the test—on any given exam you have perhaps less than a 50% chance of encountering one of them. Nonetheless, you are responsible for knowing the several matrix related tasks above.

First let's familiarize ourselves with the matrix:

m x n Matrix
3 x 3

$$\begin{array}{cc} & \text{Columns} \\ \text{Cells} & \\ \text{Rows} & \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \end{array}$$

A matrix is a rectangular array of quantities or values that is usually enclosed by brackets. We usually name matrices with capital letter variables—for example: A, B, C.

Matrices are measured RISE by RUN. RISE counts the number of rows, and RUN the number of columns. The dimensions of a few matrices are listed below:

$$1\times3 \text{ Matrix}: \begin{bmatrix} 4 & 3 & 5 \end{bmatrix}$$ – ONE row by THREE columns

$$4\times2 \text{ Matrix}: \begin{bmatrix} 3 & 4 \\ 6 & 5 \\ 8 & 2 \\ 3 & 4 \end{bmatrix}$$ – FOUR rows by TWO columns

Sometimes we refer to the individual positions in a matrix based on the row and column position. For example, see this 2×2 matrix:

$$\begin{bmatrix} \text{Row 1, Column 1} & \text{Row 1, Column 2} \\ \text{Row 2, Column 1} & \text{Row 2, Column 2} \end{bmatrix}$$

For two matrices to be equal, they must not only have the same dimensions, but also have the same values in every position.

ADDING AND SUBTRACTING MATRICES

To be able to ADD or SUBTRACT matrices they must have the same dimensions. Then, you simply add together (or subtract) the values that occupy the corresponding positions.

If $M = \begin{bmatrix} -6 & 3 \\ 5 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$, find $M+N$ and $M-N$.

1. Find $M+N$:
   Simply add the values in the corresponding positions:
   $$\begin{bmatrix} -6 & 3 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -6+0 & 3+-3 \\ 5+4 & 2+1 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 9 & 3 \end{bmatrix}$$
2. Find $M-N$:
   Subtraction is similar. Just make sure you don't reverse the order—in subtraction order matters. Simply subtract the values in the corresponding positions:
   $$\begin{bmatrix} -6 & 3 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -6-0 & 3--3 \\ 5-4 & 2-1 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 1 & 1 \end{bmatrix}$$

MULTIPLYING MATRICES BY A SCALAR

A scalar is a single value, number, or expression. When multiplying a matrix by a scalar, the scalar is multiplied by every individual value in a matrix. Unlike matrices, which are generally denoted by capital letters, a scalar is a lower case letter such a k or m. It can also be placed to the left of a matrix (just as a number to the left of parenthesis means multiply by each item in the parenthesis).

Here's what a scalar problem looks like:

If $k=7$ and $M = \begin{bmatrix} 4 & 5 \\ 3 & 2 \\ 9 & 1 \end{bmatrix}$ find kM.

To solve this problem, we'll multiply each item in M by 7:

$$7\begin{bmatrix} 4 & 5 \\ 3 & 2 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} 28 & 35 \\ 21 & 14 \\ 63 & 7 \end{bmatrix}$$

EQUIVALENT MATRICES

When two matrices are equivalent, match up each corresponding position to solve for unknowns and create separate equations.

Solve for a if: $\begin{bmatrix} 4 & 2n & a+3 \end{bmatrix} = \begin{bmatrix} 4 & a+6 & 3n \end{bmatrix}$

Match up the middle terms and set them equal:

$$\begin{bmatrix} 4 & 2n & a+3 \end{bmatrix} = \begin{bmatrix} 4 & a+6 & 3n \end{bmatrix}$$
$$a+6=2n$$

Match up the last terms and set them equal:

$$\begin{bmatrix} 4 & 2n & a+3 \end{bmatrix} = \begin{bmatrix} 4 & a+6 & 3n \end{bmatrix}$$

$$a+3=3n$$

Solve by elimination (you can also solve by substitution):

Subtract:
$$\begin{array}{r} a+3=3n \\ -(a+6=2n) \\ \hline 0-3=n \\ -3=n \end{array}$$

Don't forget to distribute the negative!

Substitute:
$$\begin{aligned} a+3&=3(-3) \\ a+3&=-9 \\ a&=-12 \end{aligned}$$

Answer: -12.

MULTIPLYING MATRICES

Though multiplying matrices perhaps has historically appeared on only 10-20% of ACT exams, you still need to know how to perform this pesky task.

First, understand that matrix multiplication does NOT follow the same, simple rules of matrix addition and subtraction. If you assume it does, you will likely find a nice wrong answer to select. Instead, it's a bit more complicated. Let's break it into steps:

STEP 1: IDENTIFY YOUR MATRIX SIZES

Remember matrices are measured RISE by RUN (or ROWS by COLUMNS):

Here we have a 1×2 matrix: $\begin{bmatrix} 5 & 1 \end{bmatrix}$ and here we have a 2×1 matrix: $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

STEP 2: TEST TO MAKE SURE YOU CAN MULTIPLY

Before multiplying, you need to make sure you *can* multiply the matrices. Only matrices in which the COLUMN NUMBER of the first matrix matches the ROW NUMBER of the 2nd matrix can be multiplied. That sounds a bit confusing, but in practice, it is useful to think that matrices must "handshake" in order to be multipliable.

We can multiply a 1×2 matrix times a 2×1 matrix because the two "2's" in the center match or "handshake." However, just because two matrices $A\times B$ can be multiplied does not mean $B\times A$ can be as well.

On the other hand, we can also reverse the order and multiply the 2×1 matrix with the 1×2 because the 1's are the same, or "handshake" in the middle of the orientation.

For example, a 1×4 matrix can be multiplied by a 4×5 matrix (the fours match), but the 4×5 matrix cannot be multiplied by a 1×4 matrix (5 and 1 are not equal).

STEP 3: SET UP YOUR DESTINATION MATRIX

When you set up your multiplication problem, you find the dimensions of the resulting matrix by taking the FIRST and LAST values from the matrix dimensions. So for a 1×2 by 2×1 we get a 1×1 matrix result:

$$\begin{bmatrix}5 & 1\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\ldots\end{bmatrix}$$

For a 1×4 by 4×3, we get a 1×3 matrix result:

$$\begin{bmatrix}1 & 3 & 4 & 5\end{bmatrix}\times\begin{bmatrix}1 & 2 & 3\\4 & 5 & 3\\7 & 1 & 2\\4 & 5 & 3\end{bmatrix}=\begin{bmatrix}\ldots & \ldots & \ldots\end{bmatrix}$$

STEP 4: MULTIPLY!

Multiply each row value times a single corresponding column value and then add the products of these products together. Remember you are multiplying ROWS times COLUMNS. You only take TWO elements at a time to multiply together—those in corresponding order—in order horizontally from the first matrix to those in order vertically in the second matrix—i.e. you take the first item horizontally times the first item vertically, then add the second item horizontally times the second item vertically. If that was confusing for you, just pay attention to the matrix multiplication examples below:

Row 1 item 1 times Column 1 item 1, then Row 1 item 2 times Column 1 item 2:

$$\begin{bmatrix}5 & 1\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}5\times4+1\times3\end{bmatrix}=\begin{bmatrix}20+3\end{bmatrix}=\begin{bmatrix}23\end{bmatrix}$$

Row 1 item 1 times Column 1 item 1, then Row 1 item 2 times Column 1 item 2, then Row 1 item 3 times Column 1 item 3…etc. Add together the items for the 1st row and 1st column into a single blank that bears both features (row 1 column 1 position).

$$\begin{bmatrix}a & b & c & d\end{bmatrix}\times\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\\10 & 11 & 12\end{bmatrix}=\begin{bmatrix}1a+4b+7c+10d & 2a+5b+8c+11d & 3a+6b+9c+12d\end{bmatrix}$$

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\times\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}1a+3b & 2a+4b\\1c+3d & 2c+4d\end{bmatrix}$$

Matrix multiplication is a bit confusing–once you figure out the pattern it's fine, but that pattern can be tough to depict in a book. That's why video teaching is the backbone of Supertutor's teaching elements. Check out supertutortv.com/booksupport for more information on supporting videos.

FINDING A BASIC DETERMINANT

Finally, you may be asked to find or use the definition of a determinant.

A determinant is usually written as a matrix with "straight" brackets as so: $\begin{vmatrix}a & b\\c & d\end{vmatrix}$

Or with standard brackets: $\det\begin{bmatrix}a & b\\c & d\end{bmatrix}$

The great news on the ACT is that you will never (at least not if the test remains consistent with past tests) be required to memorize the formula for a determinant. Rather, if you have a determinant question, the question will include the formula itself. **<u>In such a case, all you have to do is use that formula! Plug in numbers that are in the same positions as the variables in the given expression</u>.**

These might "look" like matrix problems, and that's why they have been put here.

The determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

If the determinant of $\begin{bmatrix} 6 & 4 \\ x & x \end{bmatrix}$ is 24, what is the value of x?

Assume $a = 6$, $b = 4$, $c = x$, $d = x$ (matching items in same position) then plug into the expression $ad - bc$ and set equal to 24:

$$6x - 4x = 24$$
$$2x = 24$$
$$x = 12$$

Answer: 12.