CHAPTER

19 LOGARITHMS

SKILLS TO KNOW

- Definition of a logarithm
- Logarithm rules & their application
- Solving problems with logarithms
- Definition of a natural log

WARNING: Before tackling this chapter, it is important that you have a handle on exponents. You must be familiar with fractional exponents and general exponent rules to approach the more difficult problems here.

LOGARITHM BASICS

Whenever you must solve for an exponent, using logarithms or "logs" is one path to doing so. A logarithm is essentially an exponential expression but expressed in a different way to facilitate finding the exponent when you know the other parts of the expression.

For example, a logarithm could be used to answer the question: "What power do we have to raise 2 to in order to get 128?" Using exponential expression I could turn that sentence into this equation:

$$2^x = 128$$

As you can see, however, using our normal old math skills or even the rules of exponents will not give us an easy answer unless we have this relationship memorized (or we start dividing like mad by two, but that only works with integers…). We can rewrite this expression using logs as follows:

$$log_2 128 = x$$

Doing so makes it easier to solve for x. We'll come back to this problem later in the chapter and show you how.

Re-writing exponents into log form is something like rearranging the numbers according to a secret code. You need to have this basic structure memorized.

Here is a simple exponential expression most people know the answer to, and how it translates to log form:

$$2^3=8$$
$$\log_2 8=3$$

The lower expression reads: "log base 2 of eight equals 3."

Let's look at each piece of these two **mathematically equivalent** equations:

2 is our **base**. I think the base is "low" and it's the part that is raised to a power, and also sits "low" next to the word log.

3 is our **exponent**. The phrase I remember to write in log form is "the exponent is the answer." Remember logs solve for exponents.

8 is the number we are taking our log **"of."** We say "log base 2 of 8 is…" It is the "answer" of the original exponentially expressed equation.

THE DFEINITION OF A LOGARITHM

$$\boxed{\log_c a = b \text{ means that } c^b = a}$$

Here, c is our **base**, a is our **original "answer"** we are taking the log **"of"** and **"b"** our exponent is the answer to the logarithmic expression.

Again, placing numbers in log form makes it easier to solve for the exponent as a calculator has the ability to solve a logarithmic expression.

What is the value of x in the equation $100 = 10^x$?

If we didn't know the answer (2), we could solve it using a log and our calculator: $\log_{10}(100) = x$. Before we do, let's discuss Common Logarithms.

Common Logarithms

$$\log(x) = \log_{10}(x)$$

The word **"log"** without any base next to it **is short for base 10**. In other words, $\log_{10}(100) = x.$ can be expressed as $\log(100) = x$. So the word "log" on your calculator will automatically solve for anyting in base 10, such as the expression above. We call this a **common logarithm**.

Plugging $\log(100)$ into the calculator (hit LOG then 100 then ENTER), we get $x = 2$. This makes sense because 10 raised to the power of 2 is $100.$ Thus the answer to the question above is two.

Answer: 2.

If $x = 50^{(b-c)}$, what is $\log_{50}(x) + c$?

A. $-b$ **B.** b **C.** $\log_{50} b$ **D.** $\log_b 50$ **E.** $\frac{\ln b}{\ln 50}$

Using the definition of logarithms: $\log_c a = b$ means that $c^b = a$

$$x = 50^{(b-c)} \text{ can be rewritten as } \log_{50} x = b - c$$

Since $\log_{50} x = (b-c)$ and we're looking for $\log_{50}(x)+c,$ we can just add c to both sides of the equation:

$$\log_{50} x + c = b - c + c$$

Isolating the second half of the equation we can simplify to get what we need:

$$(b-c)+c=b$$

If you can't see why this works, you can also think of this as substituting into the expression. Remember the pattern of equation-expression (we cover this in Basic Algebra in Part One). Whenever you have an equation and are asked for an expression: substitute! That pattern holds here. "If $\log_{50} x = (b-c)$ what is $\log_{50}(x)+c$? See the big ugly clump that is present in both the equation and the expression? $\log_{50}(x)$? We are essentially group substituting in $b-c$ for that big ugly clump to get $b-c+c$ or b as our answer.

Answer: **B**.

Natural Logarithms

A special kind of logarithm is the **natural log**, which is simply a logarithm with a **base of e.**

$$\ln(x) = \log_e(x)$$

A natural log is denoted as $\ln(\ \ )$. This is the same as writing $\log_e(\ \ )$. The natural log (as well as e in general) is useful in continuous growth problems (think compounding interest), and e is a defined number around 2.7 in value, but all that's not terribly important. What's more important is that you can you solve problems that involve e and $\ln$ should they appear on the ACT®.

CHANGE OF BASE FORMULA

Your calculator knows every value of log (log base 10) and ln (natural log base e), but in reality, many log problems are not base 10 or base e. So what do you do if you don't have a log conveniently in base 10 or base e? True, some advanced calculators may perform logs of any base (though often this function is hidden in a menu somewhere). For all other cases, though, you'll need to use the **Change of Base Formula**. Using this formula, you can calculate logs of any base by converting them to a different base.

CHANGE OF BASE FORMULA

For all positive numbers a, b, and c, where $b \neq 1,$ and $c \neq 1$:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Example: $\log_3 9 = \dfrac{\log_{10} 9}{\log_{10} 3}$

This formula is pretty easy to remember, if you look at the taller number in the original expression first (that goes on top) and the lower number in the original expression (the base) which goes on the bottom. Now let's get back to that problem from the beginning:

What power do we have to raise 2 to in order to get 128?

Again I could start by setting up:

$$2^x = 128$$

Then I rearrange in log form:

$$\log_2 128 = x$$

Now I apply the Change of Base Formula:

$$\log_2 128 = \frac{\log 128}{\log 2}$$

And use my calculator: (LOG 128)/(LOG 2) press ENTER and I get 7.

Answer: 7.

If the number of bacteria in a colony, n, is a function f such that $n = 2^t$, after how many hours, to two decimal places, will there be $100{,}000$ bacteria?

A. 16.27 hours **B.** 16.61 hours **C.** 17.11 hours **D.** 23.46 hours **E.** 50.00 hours

$100{,}000 = 2^t$, which implies $\log_2 100{,}000 = t$.

If you plug this equation into your calculator using the **Change of Base Formula** (Enter (LOG 100000)/(LOG 2) and hit ENTER), you'll find that it equals 16.61. Thus, after 16.61 hours, there will be $100{,}000$ bacteria.

This can be checked by plugging in $t = 16.61$ to see that $2^{16.61} = 100{,}000$.

Answer: **B**.

MORE PROPERTIES OF LOGARITHMS

POWER PROPERTY

$$a \log_b x = \log_b x^a$$

PRODUCT PROPERTY

$$\log_a x + \log_a y = \log_a xy$$

QUOTIENT PROPERTY

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

These rules, in addition to the **definition of a log** and the **Change of Base Formula** discussed earlier, allow you to manipulate and solve many problems with logarithms, so memorize them for the ACT®!

Each of these rules is essentially related to the exponent rules, but each rule is written differently in the same way that a log rewrites the mathematical information contained in an exponential expression.

Additionally, it's good to understand rules that derive from the definition of logs:

DEFINITION OF LOGARITHMS

$$n^{\log_n a} = a$$

$$\log_x x^n = n$$

Logarithm of the Base:

$$\log_x x = 1$$

We assume you've learned these in class (typically Algebra II) and just need a refresher. If this is all new to you, you may need to look up some more videos or information on each of these properties and go back to the basics and then complete at least a worksheet's worth of practice on a**t least the first three rules (Google "properties of logarithms worksheet or practice")** in addition to working through this chapter.

Which of the following expressions is equivalent to $3\log\sqrt[9]{x}$?

A. $\log\sqrt[9]{3x}$ **B.** $\log\sqrt[9]{9x}$ **C.** $\log x^{\frac{9}{3}}$ **D.** $\log x^{\frac{1}{3}}$ **E.** $3\log x$

We can rewrite $3\log\sqrt[9]{x}$ as $3\log x^{\frac{1}{9}}$ since we know roots and fractional exponents are equivalent.

Using the **Power Property of Logarithms**, we can then simplify where $a = 3$, $b = \text{base } 10$ (nothing needed, thus ignore b), and $x = x^{\frac{1}{9}}$:

$$a\log_b x = \log_b x^a$$

$$3\log x^{\frac{1}{9}} = \log\left(x^{\frac{1}{9}}\right)^3$$

Distributing the exponent 3, we get the answer:

$$\log x^{\frac{3}{9}} = \log x^{\frac{1}{3}}$$

Answer: **D**.

What is $\log\sqrt{x} + \log x^3$?

A. $\log\left(x^3 + \sqrt{x}\right)$ **B.** $\log\left(x^3 - \sqrt{x}\right)$ **C.** $\log x^{\frac{7}{2}}$ **D.** $\log x^{\frac{5}{2}}$ **E.** $\log x^{\frac{3}{2}}$

This question is testing our knowledge of general logarithm rules. In this case, we need to know the Product Property of Logarithms:

$$\log_a x + \log_a y = \log_a xy$$

Here we have:

$$\log\sqrt{x} + \log x^3$$

Which we can rewrite as:

$$\log x^{\frac{1}{2}} + \log x^3$$

to make things a bit easier.

Now we can see the pattern matches the property rule above. Our base is 10 (because no base is written, we assume 10), so we can effectively ignore the "a" value, and simply multiply the elements:

$$\log x^{\frac{1}{2}} + \log x^3 = \log\left(x^{\frac{1}{2}} \bullet x^3\right)$$

Now we apply an exponent rule, the Product of Powers ($a^b a^c = a^{b+c}$) (see Exponents chapter):

$$\log\left(x^{\frac{1}{2}} \bullet x^3\right) = \log x^{3+\frac{1}{2}}$$

$$= \log x^{\frac{7}{2}}$$

Answer: **C**.

What is $\log_3 53 - \log_3 159$?

A. -1 **B.** $\frac{1}{3}$ **C.** $\log\left(-106\right)$ **D.** $\log_3 8427$ **E.** 3

We can simplify $\log_3 53 - \log_3 159$ using the Quotient Property of Logarithms:

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_3 53 - \log_3 159 = \log_3 \frac{53}{159}$$

Simplifying the fraction $\frac{53}{159}$, we get $\log_3 \frac{1}{3}$.

We can then use our calculator to solve ((LOG 1/3) / (LOG 3) ENTER) to get our answer: -1. Alternatively, we could rewrite this, pulling the negative one outside using the Power Property:

$$\log_b x^a = a \log_b x$$

$$\log_3 3^{-1} = -1(\log_3 3)$$

Then we apply Logarithm of the Base:

$$\log_x x = 1$$

$$-1(\log_3 3) = -1(1)$$

And our answer again is -1.

Answer: **A**.

What is $10^{\frac{3}{2}\log 16}$?

A. $\frac{32}{3}$ **B.** $16^{\frac{2}{3}}$ **C.** 24 **D.** 64 **E.** 10^{24}

One way to solve this is to let your calculator do the work. Solve for $\log 16$, multiply by 1.5, and then store that in your calculator. Then take 10 to that power. **The answer is 64.**

But we can also do this without a calculator. I'll make up a variable "x" to help:

$$10^{\frac{3}{2}\log 16} = x$$

Now let's put this in log form. Remember, the exponent is the answer, and 10 is the base so we can use the common logarithm (i.e. we don't have to write "10" next to log).

$$\log x = \frac{3}{2}\log 16$$

Now we can move the $\frac{3}{2}$ into the "$\log 16$" portion by applying the Power Property of Logarithms.

$$a \log_b x = \log_b x^a$$

$$\frac{3}{2}\log 16 = \log 16^{\frac{3}{2}}$$

Back to our equation, substituting this in we get:

$$\log x = \log 16^{\frac{3}{2}}$$

Now we can drop the word "log":

$$x = 16^{\frac{3}{2}}$$

Now I split the root off from the fractional exponent, factoring out the $\frac{1}{2}$ power:

$$x = \left(16^{\frac{1}{2}}\right)^3$$
$$x = \left(\sqrt{16}\right)^3$$
$$x = 4^3$$
$$x = 64$$

A third way to approach this would be to rewrite $10^{\frac{3}{2}\log 16}$ into $10^{\log 16^{\frac{3}{2}}}$ and then use the rule $n^{\log_n a} = a$. Because a log's base is 10 (unless otherwise indicated), we know that $10^{\log 16^{\frac{3}{2}}}$ can be simplified to $16^{\frac{3}{2}}$, which then equals 64.

Answer: **D**.

What is $\log_{100} 548$?

A. $\log 5.48$ **B.** $\log 548 - \log 100$ **C.** $\log 548^2$ **D.** $\frac{\log 548}{2}$ **E.** $\frac{\log 548}{\log 10}$

For this problem, we can plug this into our calculator, and then back solve each answer choice by plugging each one into our calculator to find the matching answer.

Alternatively, we can use the **Change of Base Formula** $\log_b a = \frac{\log_c a}{\log_c b}$ to convert $\log_{100} 548$ to $\frac{\log 548}{\log 100}$, and because the we know that a log without an explicit base is in base 10, we can solve for the bottom knowing that 10 squared is 100, so the exponent 2 is equivalent to $\log 100$. Thus, we can simplify to $\frac{\log 548}{2}$.

Answer: **D**.