CHAPTER

17 TRIANGLES

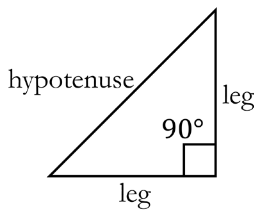
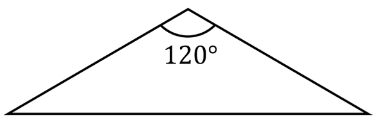
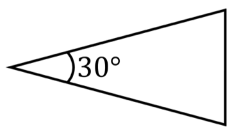
SKILLS TO KNOW

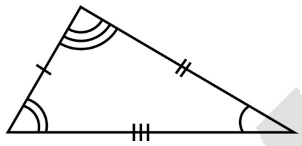
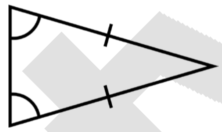
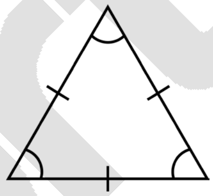
- Triangle types: obtuse, acute, right, equilateral, isosceles, scalene
- Triangle Inequality Theorem
- Triangle Area & Perimeter
- Pythagorean Theorem and its converse
- Pythagorean Triples
- Special Triangles (30-60-90, 45-45-90)

Note: Similar Triangles are covered in this book's chapter on Similar Shapes & Ratios. For more problems that involve triangles, also see this book's chapters on SOHCAHTOA, Trigonometry, Angles and Lines, Translations and Reflections, Circles and Polygons.

TRIANGLE BASICS

Triangles are three-sided figures that can be classified according to their angles and/or side lengths. All angles in a triangle always sum to **180 degrees**. Below, you'll find a description of many different classifications of triangles.

Right Triangle	Triangles are right triangles if one angle is 90 degrees. The longest side of a right triangle is called a hypotenuse. The shorter two sides are called legs. Missing pieces of right triangles can be solved using the Pythagorean Theorem and basic Trigonometry (SOHCAHTOA).	
Obtuse Triangle	Triangles are classified as obtuse if one angle is greater than 90 degrees.	
Acute Triangle	Triangles are classified as acute if all angles are less than 90 degrees.	

Scalene Triangle	Triangles are classified as scalene if all sides are unequal. Therefore, all angles will be unequal as well.	
Isosceles Triangle	Triangles are isosceles if at least two sides are congruent. The angles opposite those sides are also congruent.	
Equilateral Triangle	Triangles are classified as equilateral if all sides are congruent. All angles of an equilateral triangle will be 60 degrees.	

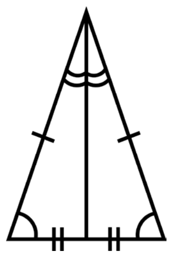
Isosceles Triangles

When two sides of a triangle are congruent, we classify that as an isosceles triangle.

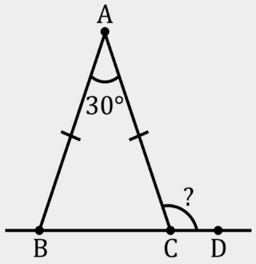
If we know two angles in a triangle are congruent, we also know the sides opposite those angles are congruent. If we know two sides in a triangle are congruent, we also know the angles opposite those sides are congruent. For example, in ΔFGH, if $\overline{FG}$ and $\overline{FH}$ are congruent, then $\angle H$ and $\angle G$ are also congruent. If we knew instead that $\angle H$ and $\angle G$ were congruent, we could also assume that sides $\overline{FG}$ and $\overline{FH}$ are congruent.

Equal angles in an isosceles triangle are called **base angles** ($\angle H$ and $\angle G$ at right). The non-congruent side in an isosceles triangle (when applicable) is typically called the **base** (side $\overline{HG}$), while the other sides are called the **legs** ($\overline{FH}$ and $\overline{FG}$). The angle included by the legs (angle $\angle F$) is called the **vertex angle**.

The **altitude** to the base of an isosceles triangle is always a **perpendicular bisector** that bisects the vertex angle, forms a perpendicular (right) angle with the base, and cuts that base into two equal pieces.

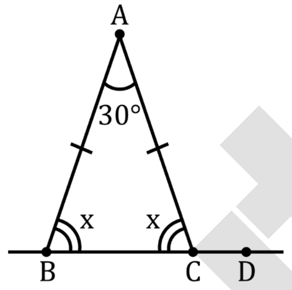


Likewise, if we have an altitude of a triangle that bisects its vertex angle and is a perpendicular bisector of the opposite side, we also know we have an isosceles triangle. We could prove this rule by similar triangles (HL Theorem), though you can also just memorize this instance. On the ACT®, this rule can help you instantly know more information about a triangle.



ΔABC is an isosceles triangle with length $\overline{AB}$ equal to length $\overline{AC}$. $\angle A$ has a measurement of $30°$. What is the measurement of the $\angle ACD$?

Knowing that ΔABC is an isosceles triangle, we know $\angle ACB$ and $\angle ABC$ are congruent. Let's set each of these equal to x:



Now to solve, we can use the fact that a triangle has $180°$ in its interior. First, I create an equation with my variables based on the sum of all angles in the triangle, $180°$:

$$180° = 30° + x + x$$

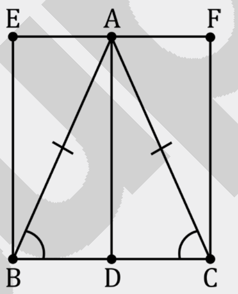
$$150° = 2x$$

$$75° = x$$

$\angle C$ and $\angle B$ are both $75°$. We can now find $\angle ACD$ with our knowledge that angles along a straight line will sum to $180°$. Here I subtract:

$$180° - 75° = 105°$$

Answer: $105°$.



ΔABC is an isosceles triangle within rectangle $BEFC$. $\angle ABD$ measures $50°$. Point D is the midpoint of $\overline{BC}$. What is the measurement of $\angle BAF$?

$\overline{AD}$ is the perpendicular angle bisector of the inner triangle in this figure, because we know that triangle is isosceles. Since equal sides indicate opposite equal angles, we know that $\angle ABD$ and $\angle ACB$ are equal, each $50°$. We can conclude that $\angle BAC$ is $80°$ by subtracting the other two angles from 180: $180° - (50° + 50°) = 80°$. We can then find $\angle BAD$ by cutting 80 degrees in half given the bisector $80° / 2 = 40°$.

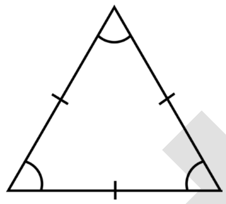
We also know $\angle DAF$ is $90°$, because the bisector of an isosceles triangle meets the base at $90°$ and we know BC is perpendicular to AD, FC, and EB (given the rectangle information in the question). These three vertical lines are thus parallel and form $90°$ angles with the horizontal parts of the rectangle. Then we can add $\angle BAD$ and $\angle DAF$ to find our answer:

$$90° + 40° = 130°$$

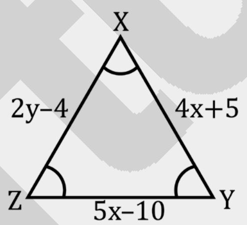
Answer: $130°$.

EQUILATERAL TRIANGLES

An equilateral triangle is a triangle in which all three sides are equal. Equilateral triangles are also equiangular; that is, all three internal angles equal $60°$ (one-third of $180°$).



The dimensions of equilateral triangle ΔXYZ are given in the figure below. What is the value of y in inches?



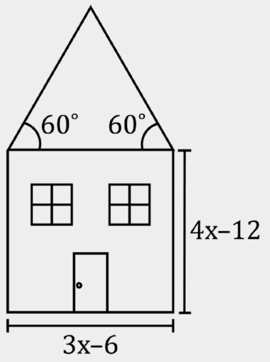
Knowing that ΔXYZ is an equilateral triangle, we know that all the sides are equal to each other. We can find x by setting $\overline{XY}$ equal to $\overline{YZ}$.

$$4x + 5 = 5x - 10$$
$$5 = x - 10$$
$$15 = x$$

Now we need to find the length of one side. We plug 15 into either side expression with $x = 4(15) + 5 = 65$. So our side length is 65. Now we can solve for y.

$$2y - 4 = 65$$
$$2y = 69$$
$$y = \frac{69}{2} = 34.5 \text{ inches}$$

Answer: 34.5 in.



Derek is trying to figure out how much wood he needs to build the frame of his roof. He knows that the front is a square with the side measurements given below. He also knows the angle measurements of the roof as it meets the ceiling of his house. How much wood, in feet, does he need to build the triangular frame?

First, we will find the measurement of the sides of the house by solving for x.

$$3x-6=4x-12$$
$$-6=x-12$$
$$6=x$$

Now we can plug in x to either one of the sides to find the length.

$$3(6)-6=12$$

Since the problem explains that the front of the house is a square, we can conclude that all sides are 12 feet. This also means that the triangular frame on top will also have sides of 12 feet. Note that the $60°$ measurements of the angle tell us that the frame is an equilateral triangle. If two angles of a triangle are $60°$, so is the third. Thus, we can add up the sides to determine the perimeter and the amount of wood needed.

$$12+12+12=36 \text{ feet}$$

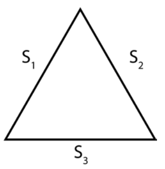
Answer: 36 ft.

TRIANGLE AREA AND PERIMETER

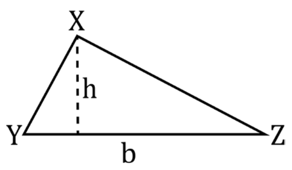
The perimeter of a triangle is the sum of its sides. Perimeter $=s_1+s_2+s_3$

The area of any triangle is:

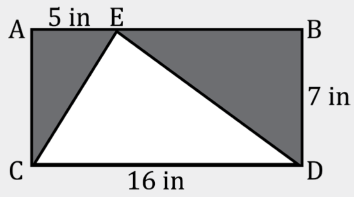
$$A=\frac{1}{2}bh$$



Where b is the measurement of the base of the triangle and h is the measurement of the height of the triangle. Remember that ANY side of a triangle can be a base, and the height is the straight line perpendicular distance from the vertex opposite the base to the base or the plane the base rests upon.



The rectangle below is divided into three triangles, two of which are shaded. What is the total area of the two shaded regions in square inches?



ΔAEC has a height of 7 inches since $\overline{BD}$ and $\overline{AC}$ are the same length. Furthermore, $\overline{EB}$ is 11 inches long, because it is $\overline{CD} - \overline{AE}$. Now we can find the area of both shaded triangles and add them.

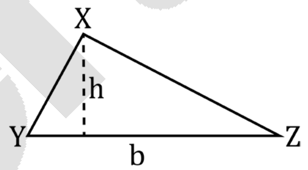
$$\frac{1}{2}(5\times7)+\frac{1}{2}(7\times11)=56 \text{ square inches}$$

Answer: 56 in.^2

PYTHAGOREAN THEOREM/CONVERSE

The Pythagorean theorem, also known as Pythagoras' theorem, relates the lengths of the three sides of a right triangle. If you know two sides of a right triangle, with this theorem, you can always find the third. It states that the square of the hypotenuse is equal to the sum of the squares of the other two sides. The theorem can be written as an equation relating the lengths of the sides a, b and c, where c is the longest side (opposite 90 degrees), or hypotenuse.

$$a^2+b^2=c^2$$



We can also work backwards with this formula. If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle. This is called the converse of the Pythagorean Theorem. With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, knowing only its side lengths.

Jim is looking to buy a ladder so that he can put up lights on his house. The side of Jim's house is perpendicular to the level ground. The ladder would have to be 10 feet away from the building and would have to reach 18 feet up the side of the home. Approximately how long does Jim's ladder have to be so that he is able to put up his lights?

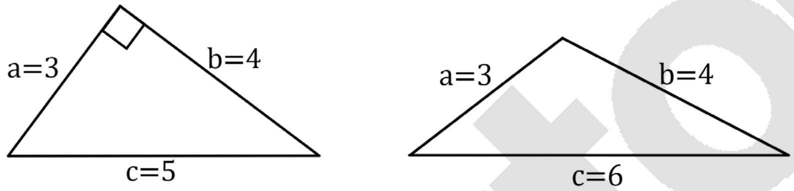
When the ladder rests against the side of a house that is perpendicular to the ground, it creates a right triangle with legs of 10 and 18 feet. To find the length of the ladder, we can use Pythagorean Theorem.

$$10^2 + 18^2 = c^2$$
$$424 = c^2$$
$$20.6 \approx y$$

After some basic algebra, we find that the ladder must be around $20.6\,\text{ft.}$

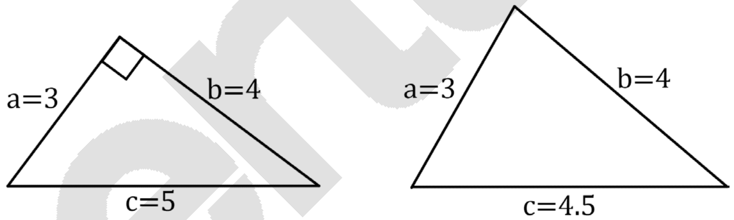
We can also extend the Pythagorean theorem to understand whether a triangle with given side lengths is **acute** or **obtuse**.

Remember how in a right triangle, $a^2 + b^2 = c^2$? Let's imagine what would happen if a and b stay the same, but c becomes longer. The angle opposite c would expand as sides a and b "hinge" open.



The triangle would thus become obtuse. In other words: if $a^2 + b^2 < c^2$, where c is the longest side, then the triangle is obtuse, with an obtuse angle opposite side c.

Now imagine if instead of "growing" our longest side, we shrank it. In that case, sides a and b would "hinge" tighter together, and c would be smaller.



In other words: if $a^2 + b^2 > c^2$, where c is the longest side, then the triangle is acute.

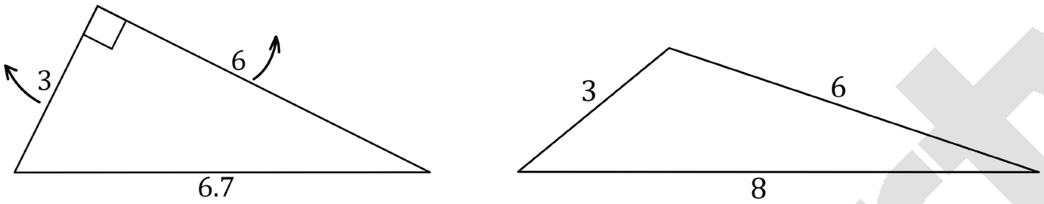
The side lengths of a triangle are 3, 6, and 8. Which of the following best describes this triangle?

- **A.** Scalene right
- **B.** Isosceles obtuse
- **C.** Scalene obtuse
- **D.** Isosceles right
- **E.** Scalene acute

With our knowledge of triangle classification, we can rule out B and D as the side lengths given in the question are all different, indicating that the given triangle is scalene. Next, we must determine the angles to figure out which scalene triangle we have. Through the Pythagorean theorem, we can rule out A since:

$$3^2 + 6^2 = 8^2$$
$$45 \neq 64$$

Now, finding out if the triangle is obtuse or acute is rather easy once we've gotten to this step. Note that when we did Pythagorean theorem above, c would have to be $\sqrt{45} \approx 6.708$. This means that the angle opposite of the hypotenuse would have to be larger than $90°$ to make a triangle with a hypotenuse of 8. Thus, the triangle is obtuse. We can imagine this visually also:



Answer: **C**.

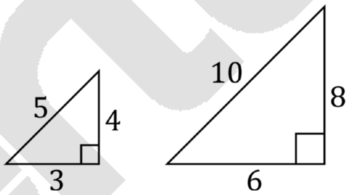
PYTHAGOREAN TRIPLES

A Pythagorean triple consists of three positive integers a, b, and c such that $a^2 + b^2 = c^2$. Such a triple is commonly written (a,b,c), and well-known examples are $(3,4,5)$, $(5,12,13)$, $(8,15,17)$, and $(7,24,25)$.

TIMING TIP: <u>**MEMORIZE THESE!**</u> Knowing these will vastly speed up your math performance!

MULTIPLES OF PYTHAGOREAN TRIPLES

If (a,b,c) is a Pythagorean triple, then so is (ka,kb,kc) for any positive integer k.



BE CAREFUL, though. For the Pythagorean triple to hold, the longest side in the triple pattern must be the hypotenuse. Thus, if I have a triangle marked with sides 5 and 12, and the 12 is the hypotenuse in the picture, the third side is NOT 13! For that, I must run the Pythagorean theorem.

> ΔJKL has sides lengths, in square inches, of 7, 24, and 25. What is the area of the triangle in square inches?

At first this may seem an area problem, and it is. But it's also a Pythagorean triples problem. If you don't know your triples, you won't instantly realize that this is a right triangle and that all you must do to solve is apply the area formula.

The legs (at right angles) form the base and height: 7 and 24.

$$A = \frac{1}{2}bh = \frac{1}{2}(24)(7) = 84$$

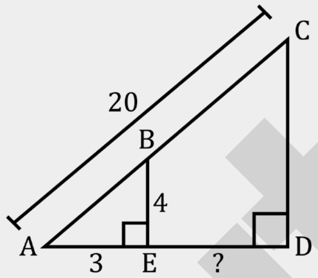
Answer: 84 in.^2

If you didn't have this Pythagorean triple memorized, you could have run the Pythagorean theorem's converse to check if it is a right triangle, validating that

$$7^2 + 24^2 = 25^2$$

Otherwise, you could have used Heron's (or Hero's) formula if you happened to know it (but I'm not teaching that, as it is not necessary to memorize for the ACT).

In right triangle ΔADC, $\overline{BE}$ is parallel to $\overline{CD}$ and $\overline{BE}$ is perpendicular to $\overline{AD}$. $\overline{AC}$ has a length of 20 yards. Furthermore, $\overline{BE}$ has a length of 4 yards and $\overline{AE}$ has a length of 3 yards. What is the length, in yards, of $\overline{ED}$?



You should instantly recognize that ΔAEB is a 3-4-5 triangle. This should indicate we can use the 3-4-5 ratio with ΔADC to determine the length of the rest of the sides. $\overline{AB} = 5$.

We know because of the similar triangles that $\frac{AB}{AC} = \frac{AE}{AD}$. (If the ratio part is confusing, check out our chapter on Similar Shapes in this book). Since $\overline{AB} = 5$, $\overline{AC} = 20$, and $\overline{AE} = 3$, then:

$$\frac{5}{20} = \frac{3}{x}$$

$$5x = 60$$

$$x = 12$$

$\overline{AD}$ is 12 yards in length. Now we can subtract $\overline{AE}$ from $\overline{AD}$ to obtain $\overline{ED}$:

$$\overline{AD} - \overline{AE} = \overline{ED}$$

$$12 - 3 = 9$$

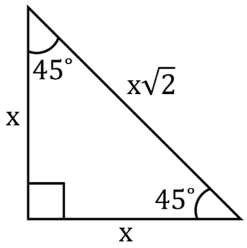
$$\overline{ED} = 9 \text{ yards}$$

Answer: $\overline{ED} = 9$ yards.

SPECIAL TRIANGLES

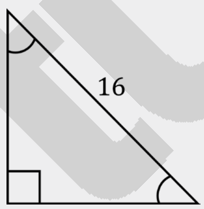
There are two special triangles that you must memorize.

45-45-90

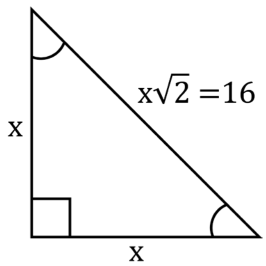


This triangle has particular angle measures that dictate a particular side length ratio. The only isosceles triangle with a right angle, both its base angles are 45 degrees. The hypotenuse of a 45-45-90 triangle is $x\sqrt{2}$, where x is the length of the congruent sides. Using these variable defined ratios, we can easily find side lengths in triangles of this type without the Pythagorean theorem. Given sides of this ratio, we can conversely assume that the triangle's angles are 45-45-90.

Find the perimeter, in centimeters, of the isosceles right triangle shown below, whose hypotenuse is 16.



Here, we have a $45-45-90$ triangle (remember any isosceles right triangle fits this pattern). Let's draw it out, but be careful! A big mistake students make is assuming any side you have without the $\sqrt{2}$ is the leg. Here that's not the case! To find the side lengths, set 16 equal to $x\sqrt{2}$.



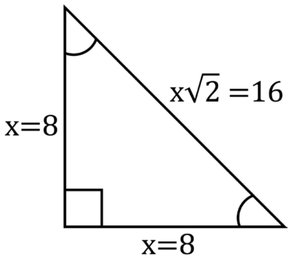
$$16 = x\sqrt{2}$$

$$\frac{16}{\sqrt{2}} = x$$

Now I rationalize the denominator, multiplying by $\sqrt{2}$ on the top and bottom:

$$\frac{16\sqrt{2}}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)} = \frac{16\sqrt{2}}{2} = 8\sqrt{2} = x$$

Now I know the lengths:

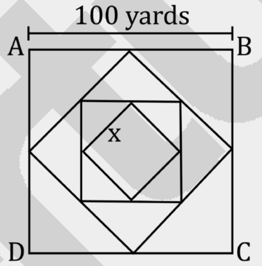


We can add up our sides to find the perimeter:

$$16+8\sqrt{2}+8\sqrt{2}=16+16\sqrt{2}$$

Answer: $16+16\sqrt{2}$.

The following figure was made by beginning with square $ABCD$. The midpoints of the four sides of the square were then joined to form another square. The process was repeated to form a third square and finally once more to form the fourth and smallest square in the middle, which has a side length of x . Find the value of x .



The sides of our outermost square are divided in half by the inner squares. However, what is created is a 45-45-90 triangle that we can use to determine the length of each inner square. We know the sides of the square one step smaller is equal to $\sqrt{2}$ times the length of half of the outer square's side. Thus we can find x by working our way down.

$$\left(100\div 2\right)\sqrt{2}=50\sqrt{2}= \text{ side of second largest square}$$

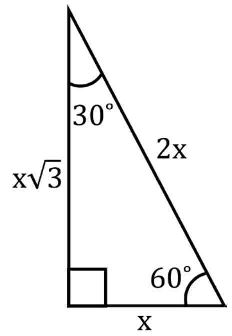
$$\left(50\sqrt{2}\div 2\right)\sqrt{2}=25\sqrt{2}\left(\sqrt{2}\right)=50= \text{ side of third largest square}$$

$$\left(50\div 2\right)\sqrt{2}=25\sqrt{2}= \text{ side of fourth largest square}$$

$$25\sqrt{2}=x$$

Answer: $25\sqrt{2}$.

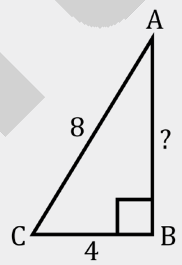
30-60-90



The second special right triangle is the 30-60-90 triangle. Just like the 45-45-90 triangle, this triangle is named after the measurements of the angles. Since 30 degrees is the smallest measurement, the side opposite of that angle will be the smallest size, x. The hypotenuse of a 30-60-90 triangle is opposite the largest angle and equal to $2x$, while the side opposite from the angle of 60 degrees is $x\sqrt{3}$.

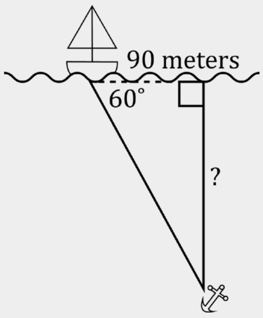
Remember that side length size corresponds to opposite angle size, i.e. the longest side is always opposite 90 degrees, and shortest opposite 30 degrees. If you get confused as to which ratio expression goes on which side, remember $\sqrt{1} < x\sqrt{3} < x\sqrt{4}$ so $1x < x\sqrt{3} < 2x$ and $30° < 60° < 90°$.

The figure below is a right triangle. The length of $\overline{AC}$ is 8 units and the length of $\overline{CB}$ is 4 units. What is the length of $\overline{AB}$?



You should notice that the hypotenuse is twice the length of the shortest leg. This is a key indicator that ΔABC is a 30-60-90 triangle. We know that $x = 4$ so we also know that $\overline{AB} = 4\sqrt{3}$.

Answer: $4\sqrt{3}$ units



You are on a small boat that has just dropped anchor in the middle of the ocean. After you dropped your anchor, the boat drifted 90 meters away from the original drop site. You measure that the angle of the chain attaching the anchor to your boat is 60 degrees from the water surface as show in the diagram below. How deep, in meters, is the anchor from the surface of the water?

We can use our knowledge of 30-60-90 triangles to solve this problem. Since we know our shortest side is 90 meters we can determine that the hypotenuse of the triangle is 180 meters and the other leg is $90\sqrt{3}$ meters.

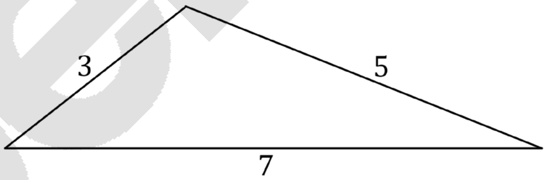
Answer: $90\sqrt{3}$ meters.

TRIANGLE INEQUALITY THEOREM

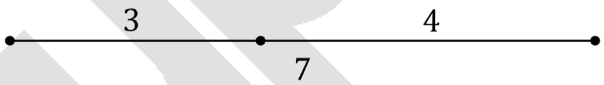
The Triangle Inequality Theorem can be used to determine if the sides of a triangle are long enough to fully create a triangle. The rule for triangle lengths is that the sum of any two sides must be greater than the third side.

$$a+b>c$$
$$a+c>b$$
$$b+c>a$$

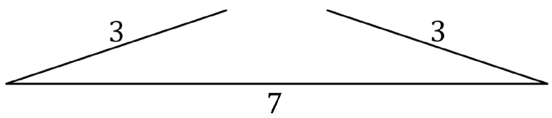
For example, the side lengths 5, 3, and 7 are able to create a triangle because $5+3<7$. However, the side lengths of 3, 3, and 7 are not able to create a triangle because $3+3$ is not greater than 7.



$$3+5>7$$



$$3+4=7$$



$$3+3<7$$

To visualize this, line up your short sides above your longest side. If the sum is equal, you have a two straight lines on top of a straight line. If you try to “pop” up the short legs at an angle, you get a drawbridge not a triangle. If the sum is less than the long side, you don’t even have a drawbridge, you have two helpless legs that can’t touch each other! However, if the two shortest sides are longer than the third they can “pop up” to form a triangle.