CHAPTER

11 WORD PROBLEMS

SKILLS TO KNOW

- How to avoid careless errors
- How to identify key words that are equivalent to math symbols (Translating from English to Math!)
- Using made up or concrete numbers to reason the relationship between variables
- How to approach logic word problems
- How to approach Venn Diagram problems
- How to set up and solve linear equations, quadratic and other algebraic word problems
- "Per" means divide (also see Chapter 9: Ratios, Rates and Units)
- How to approach multi-step word problems

**For more word problems, see other chapters such as:
Data Analysis
Speed & Rates
Fractions
Averages
Properties of Numbers

AVOIDING CARELESS ERRORS

Word problems require you to weed out unnecessary information and convert words to numerical expressions. The difficult part is decoding the problem.

Most word problems can be solved in three steps:
STEP 1: Read a question and translate words to numbers and symbols.
STEP 2: Turn it into a solvable equation or system of equations.
STEP 3: Ensure you've answered the right question.

In general, I see more careless errors on word problems than on any other type of math question on the ACT®. **Always re-read the question before you put your final answer.** This is a habit you must practice. Almost half of missed word problems are the result of good math and poor attention to detail.

You can also rework a word problem once you have an answer choice by reading along again with the word problem and plugging in the answer you got. In other words, **double check your answers if you have time.**

Finally, whenever **I have extra time** at the end of a test, if I've already reviewed anything I'm unsure of, **word problems are the first type of question I double check**.

TRANSLATING ENGLISH TO MATH

One skill you'll need for these questions is how to translate key terms that correlate with mathematical symbols.

Sometimes this relationship is pretty obvious:

> The sum of a number and seven is nine. What is four times the number?

SUM means add $(+)$.
A NUMBER means use a variable (i.e. x or n).
IS means equals $(=)$.
TIMES means multiply $(\times$ or $*)$.

First we formulate an equation and an expression:

$$\begin{aligned} n+7&=9 \\ n&=2 \end{aligned} \qquad ?=4n$$

We can solve the equation to get $n=2$, and then plug 2 in the expression:

$$4n=?$$
$$4(2)=8$$

That was easy—but sometimes things get more confusing.

Let's first review a load of English words and what they translate to in math.

Word	Meaning	Notes	Expression
OF	Multiply	Generally used with percents or fractions	half **of** the toys $=0.5*t=0.5t$
LESS THAN*	Subtract	Be sure to put the number **before** the phrase "less than" **after** the subtraction sign. I like to think of less than like a fishin pole—you must "throw the line out" to the far side of the equation.	The apples weigh 2 lbs. **less than** the pears. $a=p-2$
MORE / -ER THAN*	Add	Order doesn't matter with this one as it does with "less than"	Jon is 5 inches **taller than** Sue. $j=5+s$ or $j=s+5$
PRODUCT	Multiply		The **product** of 5 and x is t. $5*x=5x=t$
QUOTIENT	Divide	Be sure to divide the first number after this word by the second word after it.	The **quotient** of 10 and b is c. $10\div b=c$ or $\frac{10}{b}=c$
WHAT (NUMBER/ FRACTION)	Create a variable	Be sure to make up a new variable	**What number** is five less than 7? $\boldsymbol{x}=7-5$
DIFFERENCE	Subtract		The **difference** between x and y is 5. $x-y=5$
IS / EQUALS	Equals		What fraction **is** half of $\frac{3}{4}$? $n=\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)$
THE DISTANCE BETWEEN	Absolute value of the difference	These are called "open sentence" equations and inequalities	The distance between a number and five is less than $.25$. $\lvert x-5\rvert<0.25$

*These represent the MOST COMMON elements students mess up. If you ever are confused, make up REAL NUMBERS to remember what these words men. For example, if apples weigh 2 lbs. LESS than pears, I could have 3 lbs. apples, 5 lbs. pears. I can then see $\text{apples}=\text{pears}-2$ because $3=5-2$. I use real numbers to help prevent confusion and ensure the right step.

> Which of the following inequalities is equivalent to the expression below?
>
> 11 is greater than or equal to 22 less than the product of n and $\frac{1}{3}$
>
> **A.** $11>22-\frac{1}{3}n$ **B.** $11\geq 22+\frac{1}{3n}$ **C.** $-11\leq -\frac{1}{3}n$
>
> **D.** $11\geq \frac{1}{3}n-22$ **E.** $11\leq \frac{1}{3}n+22$

Our first number is 11, and it says that 11 **is greater than or equal to**. This means the equation must be $11\geq$ so far. Reading on, we see 22 **less than**, which means "$\_\_\_\_ -22$." The blank is "the **product** of n and $\frac{1}{3}$," so the second part of the equation is $\left(n*\frac{1}{3}\right)-22$. (Note: it's always a good option to use parentheses.) Joining the two parts together, we get $11\geq\left(n*\frac{1}{3}\right)-22$. This expression is also equivalent to $11\geq\frac{1}{3}n-22$.

Answer: **D**.

MAKING IT CONCRETE

A technique I often use to help solve word problems is to make the ideas concrete. What that often entails is making up some numbers so that I understand how the numbers given should work together and so that I can set up an equation. This technique works when I see variables in answer choices and need to know how to set up an equation, as well as when I simply want to solve a word problem and need to set up an algebraic equation to solve for my unknowns.

> For a fundraiser, a club decides to make layered hot cocoa jars and to deliver them the week before winter break begins. They estimate it will cost $\$7$ for ingredients per cocoa jar and $\$65$ to order all the mason jars and decorations. The club decides to sell each jar for $\$10$. Assuming they have no other expenses, which of the following equations represents the profit in P that they will make by selling j jars of hot cocoa?
>
> **A.** $P=10j-65$ **B.** $P=7j+65$ **C.** $P=3j-65$
>
> **D.** $P=-17j+65$ **E.** $P=17j-65$

For this question, we want to know how much money they will make overall, including costs. We know that each jar j costs $\$7$, there is a fixed cost of $\$65$ for the other materials, and the complete jars sell for $\$10$ each. But where do we put the numbers? One way to know is to make up a number of jars and then figure out how you would set up the calculation. Let's pretend we had 10 jars. If we are paying $\$7$ for each jar, that would be $\$7*10$ or $\$70$ that we'd pay for the ingredients. Then if we sold each one for $\$10$ that would be $\$100$ we'd make. That leaves a profit of $\$30$—however, we still have to pay $\$65$ in fees, so we'd be down $\$35$ and our P would be -35. Now I can plug in that number 10 into j and try to figure out which choice gives me $-\$35$. Choice (C) makes the most sense—we're profiting $\$3$ for each of the 10 jars but then subtract $\$65$ to get a loss of $\$35$. I can plug in and back solve and know that it works.

We can also think of this problem algebraically—the overall cost for the jars is $7j+65$. But we also need to account for the profit. We know that each jar will make them $\$10$, so that translates to $10j$ dollars they make before subtracting costs. As for profit P, it will be the amount of money they make minus the cost. This means that $P=10j-(7j+65)$. By distributing the negative sign and simplifying, we get $P=3j-65$.

If the algebra method confuses you, try making up numbers to help!

Answer: **C.**

LOGIC QUESTIONS

Some other word problems are called "logic problems." These problems don't show up as often.

The basic logic "equation" is an "if—then" statement. We call this statement the "positive" or "Modus Ponens" if you're taking a Logic class:

IF A…THEN B

You should memorize the fact that when you have an "if—then" statement as above, the following statement is ALWAYS true (called the "contrapositive" or "Modus Tollens"):

IF NOT B…THEN NOT A

However, you cannot assume that:

IF B…THEN A
or
IF NOT A…THEN NOT B

I know that sounds confusing, so here's my favorite example for logic problems to help make sense of the above.

Let's let "A" equal **it is snowing** and "B" equal **it is below 32 degrees outside.**

A **B**
GIVEN: IF **it is snowing**…THEN **it is below 32 degrees outside.**

By extension we can assume the following:

NOT B **NOT A**
CORRECT: IF **it is NOT below 32 degrees outside**…THEN **it cannot be snowing.**

Logic will tell you this is true, and it makes sense. You can't have snow if it's not freezing out. At the same time we can't assume the following:

INCORRECT: IF **it is below 32 degrees outside** THEN **it is snowing.**

Just because it's really cold doesn't mean snow is in the air, right?

INCORRECT: IF **it is NOT snowing** THEN **it is NOT below 32 degrees outside.**

Yet another statement that simply can't be assumed—just because it's not snowing doesn't mean the weather isn't cold.

If the idea $\text{If } A \rightarrow B = \text{If NOT } B \rightarrow A$ confuses you, replace your statement with the temperature/snow example above.

"If Hannah is home, then her shoes are in the hallway."

If the previous sentence is true, then which of the following sentences MUST also be true?

A. If Hannah's shoes are not in the hallway, then she is not home.
B. If Hannah is not home, then her shoes are not in the hallway.
C. If Hannah's shoes are not in the hallway, then she is home.
D. If Hannah's shoes are in the hallway, then she is home.
E. If Hannah is not home, then her shoes are in the hallway.

TIP: For "If-then" statements, the only other statement that must necessarily be true is the contrapositive of that statement. The contrapositive means that you flip the two terms in the sentence and add a "not." For example, "If A, then B"'s contrapositive would be "If **not** B, then **not** A."

For this problem, the contrapositive would be that if Hannah's shoes are **not** in the hallway, then Hannah is **not** home. If this confuses me, I can remember the pattern with my earlier snow example.

Answer: **A.**

VENN DIAGRAMS

Another type of word problem you'll occasionally need to solve is the Venn Diagram problem. These problems ask you to divide elements into groups and deduce information based on groups that overlap.

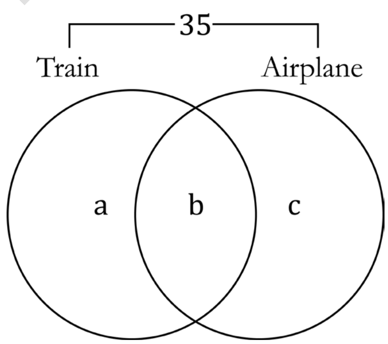
Sam asks 50 students questions about traveling.

Question	Yes	No
1. Have you ever traveled on a train or airplane?	35	15
2. If you answered Yes to Question 1, did you travel on an airplane?	29	6
3. If you answered Yes to Question 1, did you travel on a train?	18	17

How many students have been on both a train and an airplane?

A. 35 **B.** 37 **C.** 2 **D.** 12 **E.** 13

The best way to solve a problem like this is to sort through the information in a Venn Diagram, or diagram of overlapping loops (see example below). You could then label each region a, b and c, and create equations:



$$a = 6$$
$$a + b = 18$$
$$b + c = 29$$
$$c = 17$$

We want to solve for b. Let's take this equation $b+c=29$ and plug in c to solve for b:

$$b+17=29$$
$$b=12$$

There are many other ways we could have solved. For example, we could have used the $a+b=18$ equation and substituted for a. I know some students memorize a formula that askes them to add the two together and subtract (i.e. $35-(18+29)$), but drawing it out works for ALL cases and helps prevent careless errors.

Answer: **D.**

ALGEBRAIC WORD PROBLEMS

The most traditional word problems require you to set up a system of equations or a linear equation and solve for an unknown variable. Often these word problems involve rates (also known as slope) and may also involve an initial value (typically the y-intercept of the corresponding graph of the function). For some problems, you may even have more variables or more unique situations. In any case, the technique is the same: create variables for your unknowns, set up equations, and solve. If you struggle with these questions, I also encourage you to check out our sections on Systems of Equations and Distance and Rates—both of those chapters involve similar issues.

> Isme, Una, and Elliot are selling one type of candy bar for their school fundraiser. If Isme sells 50 bars, Una sells 80 bars and Elliot sells 110 bars, and the gross sales total of all three students' sales is $\$300$, what is the value of Elliot's gross sales?

Remember "gross" means the total profit before any deductions, so we won't need to worry about how much profit students made or what the wholesale cost is, etc.

For this problem, we can first try to figure out the price per bar, and then multiply that price times the number that Elliot sold to find out how much he sold.

If this is confusing, start with what you:
- **<u>Have</u>** (given elements in the problem)
- **<u>Know</u>** (any formulas or facts) and
- **<u>Need</u>** (what the question asks you to solve for)

Write each of these down on your paper to help anchor your problem solving process.

NEED: The total dollar amount Elliot sold.

Think—if Elliot sold 10 bars for $\$2$ each he would have sold $\$20$. How did I get that $\$20$? I multiplied the number of bars times the cost per bar. So that's what I write down:

$$NEED = Cost\ per\ bar * Number\ of\ Bars\ Elliot\ Sold$$

Don't be afraid to make up numbers as I did above to make headway in a problem, and don't be afraid to write things out in English before filling in the numbers you know.

I know that he sold 110 bars, so I need:

$$NEED = Cost\ \ per\ bar * 110$$

Ok. How can I find the "cost per bar"?

Think about what you **KNOW**: The word PER means DIVIDE! So "cost PER bar" translates to TOTAL COST divided by TOTAL BARS.

We call this the unit rate (cost per bar). I want to divide all the money made by all the bars sold, so I take $\$300$ and divide by the sum of all the bars sold:

$$\frac{Total\ Sales\ of\ \ All\ Bars}{Total\ Number\ of\ \ Bars} = \frac{\#300}{\#Una\ Sold + \#\ Isme\ Sold + \#\ Elliot\ Sold}$$

Now I plug in what I **HAVE** (the given numbers): The bottom of this fraction is the sum of the three kids' bars, and the top is the total sales they had. The top number is a given number ($\$300$) and the bottom I can easily calculate:

$$\frac{\$300}{50+80+110} = \frac{\$300}{240} = \$1.25$$

I go back to my NEED equation:

$$NEED = Cost\ \ per\ bar * (110)$$

And plug in: $\$1.25(110) = \$137.50$

Answer: $\$137.50$

Remember when you are trying to solve word problems, if you feel lost:
1. Make up numbers here or there to understand relationships.
2. Remember what you HAVE, KNOW, and NEED.
3. Write out equations using English to help you know where to put what.
4. Remember PER means DIVIDE.

> A fabric store sells cotton and velvet fabrics. Myra pays $\$50$ for 3 yards of cotton and 2 yards of velvet. Jaime pays $\$22$ for 1 yard of cotton and 1 yard of velvet. How much does 1 yard of velvet cost?

For this problem let's make up some variables and use algebra. Let's start with what we NEED to figure out the first variable to make up (this is always a good starting place). We need the cost per one yard of velvet. Let's call that v. We likely also want the cost per one yard of cotton—as that would be the corresponding variable for our other fabric, cotton. Let's call that c.

Let's say cotton costs $\$15/\,yard$: $c = \$15$. We would figure out how much someone paid for two yards by multiplying $2(\$15)$ or $2c$, so that's how we set up the equation. Again I just make up numbers for a moment to make sure I'm putting everything in the right place. We'd do the same for velvet. $2v$ would be the total cost someone spent on velvet if they purchased two yards. So let's create an equation for Myra:

$$\$50 = 3c + 2v$$

Jaime's equation is even easier:

$$\$22 = c + v$$

Now let's figure out what we need. We need to solve for v. Because I want to KEEP v, I'll **isolate** c **to eliminate** it. (remember, isolate whatever variable you want to get rid of and then substitute.) I try to remember the phrase "**isolate to eliminate**" to help me more efficiently work word problems. You could also isolate v, but then you'd need to plug in at the end of the question to solve for v. Let's work off the bottom equation—it has no coefficients so will be easy to use substitution with.

$$\$22 - v = c + v - v$$
$$\$22 - v = c$$

Now we plug in:

$$\$50 = 3(\$22 - v) + 2v$$
$$\$50 = \$66 - 3v + 2v$$
$$\$50 - \$66 = -v$$
$$-\$16 = -v$$
$$\$16 = v$$

Because we need the cost of velvet, and we solved for v, we are done.

Remember to always **RE-READ** the question before you put your answer to avoid careless mistakes on problems like this one!

> A fitness apparel store sells t-shirts for $\$32$ each. At this price, 50 t-shirts are sold per week. For every $\$1$ decrease in the price, the store will sell five more t-shirts per week. If one week the store wants to maximize revenue, and prices its t-shirts accordingly, how many t-shirts will it sell that week? (Note: revenue is the total dollar amount of retail sales before any costs are deducted and not including any taxes).

This one is tricky. Let's start with what we need.

We need to maximize revenue. How do we calculate revenue? Well, for the first week it's $\$32$ times 50—that would be the cost per shirt times the number of shirts.

$$(cost\ per\ shirt)(\#\ of\ shirts) = (\quad)(\quad)$$

Let's say we decrease the cost per shirt by $\$1$, what happens? The number of shirts goes up by 5:

$$\underset{(COST)}{(\$32-1)}\underset{(\#\,OF\ SHIRTS)}{(50+5)}$$

Let's say we decrease the cost of the shirt by 2:

$$(\$32-2)(50+5(2))$$

Let's say we decrease it by 3:

$$(\$32-3)(50+5(3))$$

I'm starting to see a pattern here, and now I can see where my variable will go. I will let the cost decrease be $\$x$ and the sales increase is going to be $5x$.

$$(\$32-x)(50+5(x))$$

As you can see, I used baby steps to work out the problem one example at a time until the relationship "clicked." When you're not sure, just dive in! Then look for patterns and try to figure out your variables. In this case, and in all cases when we have "maximums" with a single equation, we're dealing with a quadratic or polynomial. Inevitably figuring those out is often best done by writing down one example at a time, stepping forward, and looking at what you have. The variable I made up represents "dollars off" not "number of shirts" (what I need)—but once I figure out the dollars off, however, I can figure out the number of shirts by plugging x into the second expression in the product above: $(50+5(x))$. Remember this expression is the number of shirts.

Now, let's find the maximum of this function:

$$f(x)=(\$32-x)(50+5(x))$$

Remember, "maximum" always refers to the highest "y-value" of an $f(x)$ value. In a quadratic equation, this is the y-value of the vertex. Here we don't need the maximum itself, we need a piece of information given that we are at the max y-value. Bottom line is, we need to find the vertex.

To find the vertex, we could plug this function into our calculator (not a bad idea—and not much more time consuming than my method below), or we could waste time expanding this and then trying to figure out the vertex. Or we could waste even more time completing the square after we expand. But I want the fastest path to the answer.

Whenever you have a quadratic *in factored form*, the fastest path to the vertex or maximum is to average the zeros to find the x value of the vertex, as that is where the maximum will be. The x-value of the vertex is always the average of the two zeros,* which can be easily found by using the Zero Product Property:

**Zeroes are the x-values when* $y=0$ *(x-intercepts of the graph).*

$$(\$32-x)=0 \text{ or } (50+5(x))$$

The first equation gives us:

$$x=32$$

The second:

$$50=-5(x)$$
$$x=-10$$

Now remember because parabolas have symmetry, the x-value of the vertex will be the average of these two:

$$\frac{32-10}{2}=11$$

Thus, the maximum will occur when we reduce the price of the t-shirts by $\$11$. But we're not done yet! Oftentimes, problems want you to find the maximum, or y-value if we plug in 11. But here we need something else.

Re-read the question! The question asks for the number of shirts sold—we plug $x=11$ into the expression that describes the number of shirts sold:

$$50+5x$$
$$50+55=110$$

Answer: 110

Our answer is 110 shirts will sell if the sales are at a maximum.

MULTI-STEP WORD PROBLEMS

These are word problems that do not fit nicely into a few algebraic formulas or are better solved by simply churning through one step at a time. Oftentimes with harder multi-step problems, it can be difficult to see how to get to the end of the problem.

The best approach for these is first to not panic! And second, to simply digest the information one little piece at a time. Write down notes AS YOU READ to help you!

> The pep club is organizing a bake sale and is planning to purchase baking supplies. Each dozen cookies require 1.5 cups of flour and $7/8$ cup of sugar. Originally, the club planned to purchase enough flour and sugar to make exactly 200 cookies, adjusting the recipe scale as necessary, but found that ingredients were cheaper when purchased in specific sized quantities: 14-cup bags of flour and 8-cup bags of sugar. How many extra whole cookies was the pep club able to bake assuming it purchased at least enough flour and sugar to bake 200 cookies?

For this problem we want to take things one at a time.

NEED: First we need to figure out how much flour and sugar we need to make 200 cookies.

KNOW: We know ratios according to each dozen cookies, so let's figure out how many dozen cookies are in 200 cookies.

Step 1: Find out how many dozen cookies are in 200 cookies.

Divide 200 by 12: $16\frac{2}{3}$ dozen cookies

Now we need $16\frac{2}{3}$ times each portion of baking product necessary for a dozen cookies.

Step 2: Find out how much total flour and how much total sugar is necessary to make $16\frac{2}{3}$ dozen cookies.

Multiply $16\frac{2}{3}$ times the amount of flour needed per dozen.

$\left(16\tfrac{2}{3}\,\cancel{dozen}\right)\left(1.5\frac{cups}{\cancel{dozen}}\right)$ (see how the "dozen" label cancels?) $=25$ cups of flour

Multiply $16\tfrac{2}{3}$ times the amount of sugar needed per dozen.

$\left(16\tfrac{2}{3}\,\cancel{dozen}\right)\left(\tfrac{7}{8}\frac{cups}{\cancel{dozen}}\right)$ (see how the "dozen" label cancels?) $=14.58333$ or $14\frac{7}{12}$ cups of sugar

Step 3: Figure out how much flour / sugar would be purchased.

To cover 25 cups of flour, we'd need 2 bags that have 14 cups each (28 cups total).
To cover 14.583 cups of sugar, we'd need 2 bags of sugar with 8 cups each, totaling 16 cups sugar.

Step 4: Figure out how much excess flour sugar exists.

28 cups flour minus 25 cups flour would give us 3 **leftover cups of flour.**
16 cups sugar minus 14.58333 would give us $1\frac{5}{12}$ **or 14.16667 cups of sugar.**

Step 5: Calculate how many dozen cookies you can make with the leftovers.

Flour: This one is easy. 3 cups can make two dozen cookies as each dozen requires 1.5 cups. But flour may not be our "limiting factor."

Sugar: This one is harder. How did we do the flour one? We divided 3 cups by 1.5 cups for the dozen. Let's do the same with the $1\frac{5}{12}$—let's divide that by $\tfrac{7}{8}$: 1.619. We can make 1.619 dozen cookies with the sugar we have left. Because this number is smaller than the flour number, this is our limiting factor that will determine how many cookies we can make.

Step 6: Calculate the extra cookies.

Let's multiply that by 12 now to turn this "dozen" number from our limiting factor of sugar into number of cookies:

$$1.619*12=19.42 \text{ cookies.}$$

We won't be baking partial cookies, so that means we can make 19 **extra cookies.**

As you can see, we need to just take this one step at a time and apply all the ideas we used in other types of word problems.