CHAPTER

2 PROPERTIES OF NUMBERS

SKILLS TO KNOW

- Definitions of key math terms
- The "divisibility" rules and how to apply them
- How to solve "digits" problems
- How to approach "could be true" or "must be true" questions

KEY MATH TERMS

To answer this style of problem correctly, you'll need to know several basic math terms and how to apply them.

INTEGER: A number that is a whole number and includes no fraction or decimal parts. Integers can be negative, zero or positive. (Examples: -400, -7, 0, 1, 3, 56, 230).

ZERO: A number that is **even** and **neither positive nor negative**. Also, an integer. Numbers cannot be divided by zero, and zero cannot be the value of the denominator (bottom) of a fraction. Sometimes, this word is also used to indicate an x-intercept of a polynomial (when y is zero and x is a number).

IRRATIONAL NUMBERS: Decimals that are neither terminating nor repeating are called irrational because they cannot be written as fractions (Examples: $\sqrt{2}$, π).

RATIONAL NUMBERS: A number that can be written as a fraction. Rational numbers include fractions and decimals (repeating or terminal) as well as natural numbers, whole numbers, and integers (Examples: $-\frac{1}{2}$, 0.444, $0.\overline{5}$, 0.56, -4).

NATURAL NUMBERS: All positive integers, plus zero (Examples: 0, 1, 2, 3, 4, etc.).

REAL NUMBERS: Numbers that do not have an imaginary component. The set of rational numbers and the set of irrational numbers together make up the set of real numbers.

COMPLEX NUMBERS: A complex number is a number that can be expressed in the form $a+bi$, where a and b are real numbers (note: b *can* equal 0) and i is the imaginary unit which satisfies the equation $i=\sqrt{-1}$. Complex numbers include all imaginary numbers AND all real numbers—the coefficient of "i" in a non-imaginary (AKA real) number is simply zero. For example, 5 could be expressed as $5+0i$. In other words, all numbers in the world are complex numbers, but not all numbers are "imaginary" (Examples: $5i$, $\sqrt{-6}$, 67, 0, $0+0i$).

FACTOR: A factor of a number is an integer which can be multiplied by another integer to produce that number. (Example: 5 is a factor of 10 because 5 multiplied by 2 is equal to 10. This means that 2 is also a factor of 10). Prime numbers have no factors other than 1 and themselves. Remember, FACTORS are like pieces of a number that multiply together—don't confuse them with MULTIPLES which are numbers that any given number is a factor of (i.e., 8 is a multiple of 2, but 2 is a factor of 8).

PRIME FACTORS: Prime factors are factors (not including 1!) that are prime (have no factors other than 1 and themselves) and when multiplied together form a given number (Example: the prime factors of 63 are 7, 3 and 3 because 7, 3, and 3 are all prime, and when you multiply $(7)(3)(3)$ you get 63). We only list positive factors when listing the prime factorization of a number.

PRIME NUMBER: A number for which the only factors are itself and 1. IMPORTANT: 1 (one) is **NOT** a prime number!

PERFECT SQUARE: A perfect square is an integer that has an integer square root (Example: 4 is a perfect square, because its square root is 2, an integer).

DIVISIBILITY RULES

A number is divisible by:	Example	Reasoning
2 if the ones digit is divisible by 2 (if the number is even).	42	"2" is the ones digit.
3 if the sum of its digits is divisible by 3.	39	$3+9=12$ 12 is divisible by 3. 39 is divisible by 3.
4 if the last two digits are divisible by 4.	124	24 is divisible by 4. 124 is divisible by 4.
5 if the ones digit is 0 or 5.	40	"0" is the ones digit. 40 is divisible by 5.
6 if the number is divisible by 2 and 3.	66	$6+6=12$; 12 is divisible by 3. 6 (ones digit) is divisible by 2. 66 is divisible by 6.
9 if the sum of its digits is divisible by 9.	423	$4+2+3=9$ 9 is divisible by 9. 423 is divisible by 9.
10 if the ones digit is 0.	450	"0" is the ones digit.

How many prime numbers are there between 1 and 50?

This question is a hybrid of a question that requires your knowledge of prime numbers and what I'll call a "counting" problem—one in which you must write down a list of numbers of sorts and count up how many are in that list.

First, let's review what prime means: a number that is not divisible by anything except itself and one. Let's also remember, **<u>1 is not prime.</u>**

Start by listing off prime numbers by counting up through each number—you can also write down all the numbers, and slash off any that are not prime or circle those that are. Remember, the only even number that is prime is 2.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

Now if there are any numbers you're not sure about, apply the divisibility rules:

> If divisible by 2: Even number ending in 0, 2, 4, 6, 8
> If divisible by 3: Add digits – is sum divisible by 3?

Thus, we can say that there are 15 primes numbers between 1 and 50.

FINDING A NUMBER(S) OR PARTICULAR DIGIT(S) BASED ON ITS PROPERTIES

The next thing you need to know is how to break down problems into digits.

TIP: remember that you can always express a number based on its digits as follows:

> Let's say abc is a three-digit number, where a represents the hundreds place, b the tens place, and c the ones place. Alternatively, you can think of it as: $100a+10b+c=$ the numeric value of a number whose digits are $\underline{a}\ \underline{b}\ \underline{c}$. Note: each place value is multiplied by a power of 10, e.g. $10^2 a\times10^1 b\times10^0 c$.

> Let a, b and c stand for digits between 0 and 9, and suppose that:
>
> $$\begin{array}{r} 38a \\ \underline{+b37} \\ 11c5 \end{array}$$
>
> What does the expression $\frac{ab}{c}$ equal?

First, we can solve for the value of a. When we add the ones places in the addition problem, we see that the result of adding a and 7 is equal to 5, but we are not sure what the value of the tens place is because that value is carried into the next column. However, because a must be between 0 and 9, we can determine that $a=8$, because no other value between 0 and 9 would result in a 5 in the ones place. Finally, because $7+8=15$, we must carry a 1 into the tens column.

Now that we know what will be carried into the next column, we can solve for the value of c. We know the value of $8+3+1$ (including the carry value from the addition in the ones place). $8+3+1=12$, but since the 1 is carried into the next column, the value of c is 2.

Next, we can solve for the value of b. Because this is the last column, we don't have to worry about carrying the value of our addition; we know that $3+b+1$ (the carry value from the addition in the tens place) $=11$. Algebraic manipulation shows that $b=7$.

Finally, we can solve for the value of the expression $\frac{ab}{c}$. Plugging in our values of 8, 7 and 2 and then simplifying gives an answer of 28.

> What is the largest integer value that satisfies the inequality $\frac{x}{12}<\frac{17}{33}$?

Don't be intimidated by the less than sign! Most of the algebraic rules that you are familiar with using are still valid here. Just like in an equation, you can add or subtract whatever you want, and can multiply or divide by any positive number. Multiplying both sides by 12 gives:

$$x<12*\frac{17}{33}$$

Plugging in the expression $12*17\div 33$ to your calculator shows a result of ≈ 6.181. Because the problem asks for the largest integer that satisfies this expression, we simply find the largest integer less than 6.181, which is 6.

> Let a be a negative, even integer. The expression $\frac{a^2}{b}$ is a negative, odd integer when b is what type of number?
>
> **A.** Positive even integer
> **B.** Positive odd integer
> **C.** Negative odd integer
> **D.** Negative even integer less than -3
> **E.** Negative even integer greater than -3

In many problems, you'll need to know how certain numbers behave:

Even exponents, for example, always create non-negative solutions. Even numbers multiplied by even numbers create more even numbers.

We thus know a^2 must be an even positive integer, as it is both the product of two even integers, and a negative number squared (a negative times a negative equals a positive). Furthermore, we know it must be divisible by four (two even numbers must each have 2 as a factor—when you multiply them together these two "2's" make 4).

In order to get an odd quotient when starting with an even number (a^2) in the numerator, we must eliminate all the even factors, or divide out all the even "parts" of the number—i.e. all the 2's. If we must divide by a number that has the twos in it, then that number must be even, so we can eliminate answers (B) and (C).

We also know that we need our quotient to be negative. Remember—the numerator, because it has an even exponent, must be a non-negative number. The only way to make this quotient positive would be to divide the numerator by a negative denominator. We can eliminate (A).

Now left with (D) and (E), we have to be careful—when we deal with negative numbers "greater than" means to the right on the number line, or towards zero. So answer (E) is basically the same as saying -2, but -2 won't work. Because the top of the fraction has at least two even factors, it must have at least two "2's" in its factorization and be divisible by at least 4—or some number whose absolute value is at least 4. I know it's confusing because of the negative sign—but a single negative two won't divide out both even pieces hidden in a^2.

Thus, (D) is the answer.

Now if that whole explanation confuses you, then the best thing to do is make up numbers—often this will work just as quickly as reasoning—but sometimes it won't. In this particular instance, making up numbers works fine.

Let $a=-2$:

$$\frac{a^2}{b}=\frac{(-2)^2}{b}=\frac{4}{b}$$

Now we can read through the choices and try to make up a number that works for b, such as -4, which would create a quotient of -1. Clearly, (D) is still the answer.

However, being able to understand the logic behind why (D) works can really help on tougher versions of this problem. If you're aiming for a score above a 32, I recommend you try to understand the logic behind questions like this, too.

Answer: **D.**

> How many numbers between 1 and 100 have a tens digits that is the square of the ones digit?

I know what you're thinking: this is going to take forever!

Don't freak out, these aren't as bad as they look. Still, counting problems, on average, can take a bit more time than some other types of questions. If you have a lot of trouble with time, you can skip these and come back to them later.

The best way to do the problem is simply to start figuring it out—start counting up from one and ask which numbers work:

Does 1 work? No—it doesn't even have a tens digit—let's skip to 10.
Does 10 work? No—1 is not a perfect square of 0.
But 11 does work—1 (tens digit) is a perfect square of 1 (ones digit).

Now let’s start to think—what are we doing here? I want to get to the point that I have a “2” in the ones digit and the tens digit equals “2” squared. 2 squared is 4—so that number is 42.

Now we can go to 3 in the ones digit—three squared is 9, so that number is 93.

I could also think of the set up like this—

_1
_2
_3
_4
…

and I can fill in the tens digits by squaring the ones digit.

By the time I get to four, however, four squared is 16—that’s not a single digit.

As you can see, there are only three numbers that work for this parameter—not nearly as many as you may have originally thought! You might also notice that I tend to “dive in” before I overthink the problem. Sure, staring with “1” may seem a waste of time to some. True, I could try to think out all the ideas behind the problem, but often simply jumping into the “sandbox” and starting to play around with numbers helps you come to an understanding of how the problem works more quickly. That way, even if you don’t see the secret reasoning right away, you are at least making progress. Once you see the problem’s logic, speed up and apply what you know. Your first guess doesn’t have to be right, but it does need to get your brain cells moving so that you have hope of finding the pattern or way that the problem works.

Answer: 3.

Is the product of the two largest primes under $1{,}000$ evenly divisible by the largest prime under $10{,}000$?

At first glance, this problem might look intimidating and time consuming to solve; it looks like it involves solving for 3 large prime numbers and then plugging them into your calculator. However, understanding properties of primes makes this question extremely simple to solve without any algebra or division at all!

First, let's focus on the product of the two largest primes under $1{,}000$. Let's call them a and b. Instead of actually finding them, let's focus on the properties of their product, $a \times b$. We want to know if the largest prime under $10{,}000$, which we will call c, is a factor of ab. If it is, then the answer to the problem is yes, because the definition of a factor is a number that evenly divides another. If the answer is no, then the answer to the problem is also no.

The definition of primes is that they have only 2 factors: one and themselves. Therefore, the prime factorization of the product $a \times b$ is simply a, b and $a \times b$; they are its only factors! Finally, we also know that $a \times b$ cannot be equal to c without even a ballpark estimate, because c is also prime. Therefore, the answer must be **no.**

COULD BE TRUE / MUST BE TRUE

These are questions that ask which of the following "could be true" or "must be true" or some form of similar language.

For these problems, you can reason algebraically or plug in numbers. If you can't see an algebraic solution, I typically recommend plugging in a variety of numbers from the following list:

- Tiny number ($-10{,}000$)
- A negative number (-15, -4, -6)
- Negative one (-1)
- Negative fraction ($-\frac{1}{2}$, $-\frac{1}{9}$)
- Zero (0)
- Fraction ($\frac{1}{4}$, $\frac{1}{7}$)
- One (1)
- Two (if "odd" / "even" problem)
- Positive number (7)
- Giant number ($10{,}000$)

If you've tried all of these, chances are your conclusions will hold.

Two numbers are reciprocals if their product is equal to 1. If a and b are reciprocals and $a>1$, then b must be:

A. Greater than 1
B. Between 0 and 1
C. Equal to 0
D. Between 0 and -1
E. Less than -1

Plug in some numbers and see what happens!

Let's say $a=4$. Then $b=\frac{1}{4}$, a's reciprocal. Thus, the answer is **B.**

TIP: Because this is a "must" be true problem, we can rely on a single example. With "could" be true, we may have to make up several numbers before finding one that matches any given conditions in the answer choices. However, if an answer choice works with a "could be true" question, when you plug in a suitable number, you can be confident it is correct.