CHAPTER

14 COMPLEX AND RATIONAL NUMBERS

SKILLS TO KNOW

- Definition of an imaginary number, complex number, and i
- Solving equations and simplifying expressions which include i
- Adding and subtracting imaginary and complex numbers
- Multiplying by a complex conjugate (rationalization)
- Using the complex plane

DEFINING IMAGINARY NUMBERS

What is i ?

i is simply equal to $\sqrt{-1}$. Because this number has no place on a number line and does not have the same value as "real numbers" it is called an "imaginary number." This does not mean, however, that i does not exist or that it has no use in mathematics.

Because $\sqrt{-1} = i$, radicals of negative numbers can be rewritten without the negative sign by placing an i outside the radical: $\sqrt{-x} = i\sqrt{x}$. Example: $\sqrt{-5} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$. We essentially "factor out" the negative part of the root and replace it with i .

What is a complex number?

A complex number is a number that can be expressed as a combination of a real and an imaginary number in the form of $a+bi$ where a and b are real numbers. But the catch is, **<u>a complex number is not necessarily imaginary</u>**. If b is zero, then the imaginary part of the number does not exist, but the number is still complex as it can be expressed in $a+bi$ form. Essentially, the set of complex numbers includes **<u>all numbers real and imaginary</u>**. All real numbers can be expressed in $a+bi$ notation when $b=0$, but in the case of imaginary complex numbers, $b \neq 0$. For example, $5+2i$ is an example of a complex number with imaginary parts.

A real "part" of a complex number cannot be combined with its imaginary part. $5+2i$, for example, is fully simplified. You cannot combine the 5 with the $2i$ portion.

MULTIPLES OF i

A pattern emerges when we take i to a power. Because of this pattern, we can simplify many expressions that include i to a power.

i^1	$\sqrt{-1}$
i^2	-1
i^3	$-i$
i^4	1
i^5 (equivalent to i^1)	$\sqrt{-1}$
etc.	etc.

Formally, $i^n = i^{n \bmod (4)}$. I realize that equation is a bit confusing, so let's look at an example:

Which of the following is equal to i^{53}?

A. i **B.** i^2 **C.** i^3 **D.** i^4 **E.** -1

i^1	$\sqrt{-1}$
i^2	-1
i^3	$-i$
i^4	1

To solve we first figure out how 53 fits into the pattern:

A way to think of this process is the idea that anything to a power that is a multiple of 4 will equal 1, and we can then count through the chart to find another value a little more than or less than that value accordingly.

53 is not divisible by 4, but we can find the closest number that is, 52. We know that $i^{52} = i^4$, for example, so if we continued this pattern of multiplying i to the 8th, 12th, 52nd and every other exponent divisible by 4, we would get the answer 1. Then we can simply step one more forward in the chart to find the 53rd power of i, which simply equals i^1, or i, $\sqrt{-1}$. The answer is **A**.

We could also solve this problem more algebraically with the same idea in mind; because 4 goes into 52 evenly (13 times), we can rewrite our expression, using the exponent rules to factor the exponent:

	i^{53}
Applying the Product of Powers: $a^b a^c = a^{b+c}$	$= i^{52} * i^1$
Now we apply Power of a Power: $\left(a^b\right)^c = a^{bc}$	$= \left(i^4\right)^{13} * i$
Now we replace i^4 with its value, 1:	$= 1^{13} * i$ $= i$

Answer: **A.**

For the complex number i such that $i^2 = -1$, what is the value of $i^4 + 2i^3 + i$?

A. $-1-i$ **B.** $-1+i$ **C.** $1-i$ **D.** $1+3i$ **E.** $-1+3i$

We can just take this problem one part at a time. In general, our goal is to reduce down any pieces by extracting out even powers of i, as these have real number values since $i^2 = -1$ and $i^4 = 1$. The first monomial in this expression, thus, can be rewritten as $\left(i^2\right)^2 = \left(-1\right)^2 = 1$, or you can memorize that the fourth power of i is positive 1. In any case, we have:

$$1 + 2i^3 + i$$

$2i^3$ is the same as $2i\left(i^2\right) = 2i\left(-1\right) = -2i$, leaving us with:

$$1 - 2i + i$$

We combine the like terms (the "i" terms) to get our final answer:

$$1 - i$$

Answer: **C.**

ADDING COMPLEX NUMBERS

As we mentioned above, real number parts and imaginary number parts cannot be added together. However, complex numbers can be added together by combining all the real terms and all the imaginary terms.

What is the sum of $\sqrt{-27}$ and $\sqrt{-48}$?

We can separate these square roots into their factors. Remember that to simplify radicals, separate the root expression into the product of two (or more) factors, with each factor now under its own square root symbol.

First we simplify $\sqrt{-27}$:

$$\begin{aligned}\sqrt{-27} &= \sqrt{-1*9*3}\\ &= \sqrt{9}*\sqrt{-1}*\sqrt{3}\\ &= 3i\sqrt{3}\end{aligned}$$

Next we simplify $\sqrt{-48}$:

$$\begin{aligned}\sqrt{-48} &= \sqrt{-1*16*3}\\ &= \sqrt{16}*\sqrt{-1}*\sqrt{3}\\ &= 4i\sqrt{3}\end{aligned}$$

Adding these together:

$$3i\sqrt{3} + 4i\sqrt{3} = 7i\sqrt{3}$$

MULTIPLYING BY A COMPLEX CONJUGATE

Answers with imaginary numbers in the denominator are not "simplified." To simplify a number with the imaginary parts in the denominator, multiply both the numerator and denominator by **"the complex conjugate."** Sometimes this is called **"rationalizing"** the denominator. Multiplying a complex number by its complex conjugate will leave you with a real number in the denominator.

RATIONALIZING A COMPLEX NUMBER

For any complex number $(a+bi)$, its complex conjugate is $(a-bi)$. To rationalize any complex number, multiply by its conjugate to clear the imaginary part:

$$(a+bi)(a-bi)=\mathrm{a}^2+b^2$$

How do we find the complex conjugate? If our denominator is $a+bi$, with a being the real part and bi being the complex part, then our complex conjugate is simply $a-bi$. Just flip the sign! When we multiply the numerator and denominator by this term and FOIL the original term and its complex conjugate, we eliminate the i 's in the denominator.

Remember the special product called "the difference of squares" or the "product of a sum and a difference"?

$$(a+b)(a-b)=a^2-b^2$$

That is the pattern the above process derives from. The goal is to "square" the i term so that it becomes real. Conjugates eliminate the i:

$$\begin{aligned}(a+bi)(a-bi)&=a^2+abi-abi-b^2i^2\\&=a^2-b^2i^2\\&=a^2+b^2\end{aligned}$$

Given the expression $\frac{1}{x+yi}$, rationalize the denominator.

For the example above, our denominator is $x+yi$, so our complex conjugate is simply $x-yi$. Now let's multiply and FOIL:

$$\frac{1}{x+yi}\left(\frac{x-yi}{x-yi}\right)=\frac{x-yi}{x^2-xyi+xyi-y^2i^2}$$

Notice how the two middle terms in the denominator cancel out, leaving us with $\frac{x-yi}{x^2-y^2i^2}$.

But wait! There are still i 's in the denominator! Remember that $i^2=-1$, so $(yi)^2=y^2i^2=-y^2$. We are left with two negative signs that become a positive:

$$\frac{x-yi}{x^2--y^2}=\frac{x-yi}{x^2+y^2}$$

Notice that we are left with only real terms in the denominator.

Using complex numbers, where $i^2=-1$, $\frac{i+1}{i+5}\times\frac{i-5}{i-5}=?$

MISTAKE ALERT: For this problem, we must use FOIL on both the top and bottom, as the pattern is not exactly the same as what we have in our equation box earlier (i.e. we can't square the i and the 5 and add together, our signs would then be wrong). Many students miss this problem as they blindly apply the "pattern" that does not exactly fit this circumstance.

Taking the numerator and denominator separately, we first expand the numerator by using FOIL.

$$\frac{(i+1)(i-5)}{(i+5)(i-5)} = \frac{i^2+i-5i-5}{(i+5)(i-5)}$$

$$= \frac{-1-4i-5}{(i+5)(i-5)}$$

$$= \frac{-6-4i}{(i+5)(i-5)}$$

Now moving onto the denominator, we get:

$$\frac{-6-4i}{(i+5)(i-5)} = \frac{-6-4i}{i^2-25}$$

$$= \frac{-6-4i}{-1-25}$$

$$= \frac{-6-4i}{-26}$$

$$= \frac{3+2i}{13}$$

Answer: **D.**

USING THE COMPLEX PLANE

Problems will sometimes ask you to visualize complex numbers on a two-dimensional plane, where the x-axis is the real axis and the y-axis is the imaginary axis. It looks a lot like an ordinary two-dimensional plane, except points on this plane are denoted with the format $(a+bi)$ instead of (x,y). Let's graph the point $(6-3i)$ for practice.

The 6 corresponds to the "x" or horizontal value, while the 3 corresponds to the "y" or vertical value.