CHAPTER

12 DIRECT AND INVERSE VARIATION

SKILLS TO KNOW

- Definition of direct and inverse relationships
- How to identify examples of each
- How to set up a direct or inverse relationship from word problems

Direct and Inverse Variation problems are a slightly more complex style of speed/rate/proportion problems. If you struggle with this area, we also recommend you check out our chapter on Speeds & Rates. For now, we will assume you know how to solve a basic proportion and how to cross multiply.

DEFINITIONS

DIRECT VARIATION

A direct variation relationship can be summarized by the equation $y = kx$, where k is some constant. Notice that as x increases, y increases as well according to the constant k (and vice versa). Sometimes we say that "x varies directly as y," or "y varies directly as x." The line of a direct variation relation is a linear graph that goes through the origin $(0,0)$. If you think about it, $y = kx$ is a slope-intercept form of a line with $b = 0$: i.e. $y = kx + 0$ looks a lot like $y = mx + b$.

This kind of a relationship is also the same as a standard proportion. For instance, if 2 cups of flour make 24 cookies, and we wanted to know how many cookies 3 cups could make, we could also use direct variation to solve. This kind of a relationship can be represented by $y = kx$. Anytime a problem could be solved using direct proportions, the problem involves direct variation.

When you have a problem that indicates a **direct variation relationship**, place one variable on the left side of the equation, one on the right side, and then place a **constant of variation** (k) on the right side of the equation.

INVERSE (OR INDIRECT) VARIATION

An inverse relationship can be summarized by the equation $y = \frac{k}{x}$ or $xy = k$, where k is some constant. As x increases, y decreases, since x is in the denominator as opposed to the numerator. Sometimes we say that "x varies inversely as y," or "y varies inversely as x."

Inverse proportions are the same as inverse variations. Problems that involve inverse variation involve one value increasing while the other decreases. For example, the number of hours it takes to do a task and the number of people helping at a constant rate would be inversely related: the more people who help, the less time it takes. As one goes up, the other goes down.

When you have a problem that indicates an **inverse variation relationship**, place one variable on the left side of the equation, one on the right side in a **denominator** (lower half) of a fraction, and a **constant of variation** (k) on the right side of the equation in the **numerator** of the fraction.

CONCEPTS TO KNOW

- When an increase in one variable means an increase in the other (e.g. number of items sold and revenue), and if one variable quals zero, the other is also zero, there is a direct relationship between the two variables
- When an increase in one variable means a decrease in another (e.g. speed and time that it takes to travel a certain distance), there is an inverse relationship between the two variables

JOINT VARIATION

Sometimes you may also see the phrase "x varies jointly as y and z," which simply means that x varies directly as y, directly as z, and $yz = kx$ or $k = \frac{x}{yz}$.

This concept has not appeared on any ACT® tests I have seen, but is the type of information I might expect the ACT® to introduce as a "curve ball" to make a question particularly challenging.

WORD PROBLEM TRANSLATION

To solve word problems that involve inverse or direct variation, you must become familiar with key words and phrases that indicate the two types of relationships, and how to create an equation to guide you through the rest of the problem.

Always be on the lookout for words or phrases like "is proportional to," "varies with," or "directly related to." You must not only recognize these phrases but also understand whether they indicate a direct or indirect relationship. Sometimes you must use common sense to determine what kind of relationship is being described. (e.g. the relationship between hours worked and pay is obviously a direct one: as the numbers of hours worked goes up, so does the pay). Even words like "for each/every" and "per" indicate rates, and sometimes these words are clues you may have a direct or inverse relationship. Unlike word problems best modeled by slope-intercept form linear equations, however, problems that involve direct or inverse variation involve pure proportions: they do not have one time "set up fees," "initial costs," or other one time offsets.

Let's work through some simple example problems to become familiar with this process.

> Jimmy sells each box of fruit for n dollars. Establish an expression that demonstrates the relationship between R, Jimmy's revenue, and f, the number of boxes of fruit that Jimmy sells.

Here, we are not given an explicit relationship, but we can use common sense to determine that the more Jimmy sells, the more money he will make; there is a direct relationship between the number of boxes sold and revenue, and the constant is n the price of a box of fruit. If you're not sure, make up some numbers and play around with them until you see the relationship. For example, if Jimmy sells each box of fruit for $\$5$ (n), and he sells three boxes (f), then he would make $\$15$ in revenue (R).

Thus R is equal to the product of n and f ($R = nf$).

The situation is modeled by $R = nf$.

> x varies directly with the product of a and b and indirectly with the quotient of y and z. Which equation correctly models this relationship?

Let's take this one part at a time. We are told explicitly that x varies directly with ab, which means that ab will be in the numerator of the right side of the equation. In other works, $x = k(ab)$ (just as $y = kx$ according to our model for direct variation). The next part is a little more tricky. We are also told explicitly that x varies indirectly with $\frac{y}{z}$. The word "indirectly" tells us that this expression must go in the denominator of the right side of the equation. So:

$$x = \frac{k(ab)}{\frac{y}{z}}$$

Just as our model equation tells us $y = \frac{k}{x}$. When we divide by a fraction, we flip the fraction and multiply, so:

$$x = \frac{k(ab)}{1} * \frac{z}{y}$$

$$x = \frac{kabz}{y}$$

> The time that it takes a car to travel a certain distance, d, is given by T and the car's speed is given by s. Write an expression that expresses T in terms of d and s.

Once again, we are going to use common sense to determine the relationship between variables. There is a direct relationship between distance and time, since more distance means more time, and an indirect relationship between speed and time, since more speed means less time.

These relationships are modeled by: $T = \frac{d}{s}$

Are there other ways to think about a problem like this? Yes. You could memorize the distance equation. Alternatively, you could observe or imagine the units and use dimensional analysis detective work (i.e. if a typical speed is miles PER hour and PER means divide then we can write: $s\,\text{miles per hour} = \frac{miles}{hours} = \frac{d}{T}$, where $d =$ a distance in miles and $T =$ a time in hours). At this point you could do further algebraic manipulation to get the solution.

THE CONSTANT OF PROPORTIONALITY, "K"

Sometimes, direct and inverse variation problems are two-step processes. It is very common for the ACT® to ask students to interpret a series of relationships given inside of a word problem, devise a general formula that models the situation, and then plug values into your derived equation to find an answer.

The first step is often to solve for the constant k using the formula patterns defined above for direct and inverse variation. The second step is to use the equation you've found that includes a numeric value for k, plug into the other values of the equation (such as x or y), and then solve for the remaining variable. These problems will almost always provide a "given case" which includes numbers that can be used to find k.

The perceived intensity of light (lux) produced by a light source varies inversely with the square of the distance, d, from the source, and directly with the square root of the power output, w of the source. If the perceived intensity of light produced 3 feet from light source outputting 400 watts is 169 lux, which of the following is closest to the perceived intensity of light produced 5 feet from the same source running at 900 watts.

A. 76 lux **B.** 91 lux **C.** 451 lux **D.** 456 lux **E.** 2737 lux

The first step in the problem is to translate the word problem into an equation that accurately describes the relationship between our three variables (perceived intensity, distance, and power). Since perceived intensity is directly related to the square root of power, we can place $\sqrt{w}$ in our numerator along with our constant of proportionality k. And since we are told that the intensity varies inversely with the square of distance, we can place d^2 in the denominator, leaving us with:

$$I = \frac{k\sqrt{w}}{d^2}$$

Now to find the constant of proportionality k, we can plug in the first set of values we were given:

$$169 = \frac{k\sqrt{400}}{3^2}$$

$$k = 76.05$$

Now that we have all the information we need, we can find the perceived intensity for the second set of given information. Plugging in 5 feet and 900 watts into the equation we get:

$$I = \frac{76.05\sqrt{900}}{5^2} \approx 91 \text{ lux}$$

Answer: **B.**

The rate of energy consumption of an electric circuit varies proportionally with the square of its voltage, V, and inversely with its resistance, R. If the constant of proportionality is 1, which of the following expressions represents the rate of energy consumption of a circuit?

A. V^2R **B.** V^2R+1 **C.** $\frac{V^2}{R}$ **D.** $\frac{V^2}{R}+1$ **E.** $\frac{V^2}{R^2}$

There are three things that you must understand in this problem: what it means to vary proportionally, what it means to vary inversely, and how to implement a constant of proportionality.

Let's start by assigning a variable, C, to the idea of the "rate of energy consumption." The problem does not name this variable, but it asks for an expression equal to it. I like to have a variable, though, as having one makes the problem easier to set up.

Now, we know that C is proportionally variable with V^2, or in other words, varies directly as V^2. Thus, per the pattern we discussed earlier, we can write on variable on the left side, one on the right, and then place "k" next to the element on the right:

$$C = V^2k$$

Now we know that C varies inversely with its resistance, R. So R must go in the denominator on the opposite side of the equation as C.

$$C = \frac{V^2k}{R}$$

The constant of proportionality refers to the term (here k) that gives the specific amount by which the voltage varies due to changes in voltage or resistance.

Here, the constant of proportionality equal to 1, or $k=1$. Substituting 1 in for k:

$$C = \frac{V^2k}{R}$$

$$C = \frac{V^2(1)}{R}$$

$$C = \frac{V^2}{R}$$

Because we are asked for C, or "the rate of energy consumption" in terms of these variable, the answer is simply: $\frac{V^2}{R}$

Answer: **C.**