CHAPTER

14 POLYGONS

SKILLS TO KNOW

- Regular polygons
- Names and area formulas for various polygons
- Picture frames and borders

WHAT IS A POLYGON?

A polygon is an enclosed, two-dimensional shape that is made of entirely straight lines. All lines are connected, making vertices. Polygons contain no curved sides.

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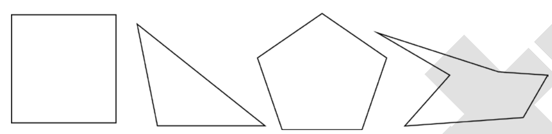
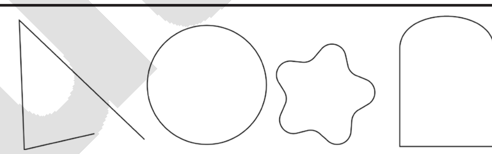
	

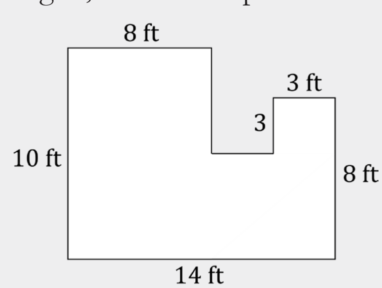
For any given polygon, the number of angles will always be equal to the number of sides. For example, a polygon with 5 sides will also have 5 angles.

We largely classify polygons by the number of sides they have. A **triangle**, for example has 3 sides (tri- being the prefix for three). Other shapes to know include the **quadrilateral** (4 sides), **pentagon** (5 sides), **hexagon** (6 sides), **heptagon** (7 sides, rare), and **octagon** (8 sides).

Perimeter of Polygons

One of the easiest elements of a polygon to solve for is its perimeter, equal to the sum of all the side lengths. One type of perimeter problem that occurs on the ACT®, which involves irregular polygons with lots of right angles, looks something like this:

Jamie received a new floor plan request for a house that she is designing. Assuming all walls meet at ninety degree angles, what is the perimeter of the new floor plan?



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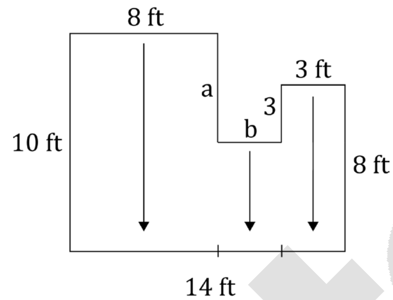
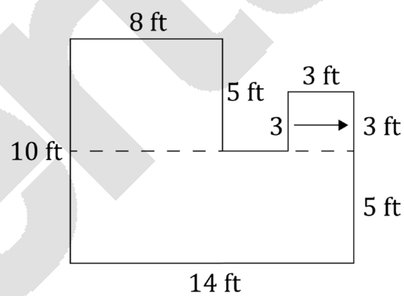
This kind of problem presents a drawing that gives many side lengths, but may not give all of them. However, because we know all the angles are $90°$, we can infer the other side lengths by looking at the "span" of parallel sides.

To find the perimeter, we need to find the sides labeled a and b below. To find b, we know that the horizontal lines all span a width of 14 ft: because the top lines on the figure are all parallel to that 14 ft side, if I imagine all the vertical lines disappearing, and collapse all the horizontal lines down to meet the bottom, they would span the same length as the total length along the bottom. Thus:



$$8+b+3=14$$
$$11+b=14$$
$$b=3$$

Now the vertical line is a bit more tricky. We'll need to "section off" the 3 by 3 area on the right to see that side a begins 5 feet vertically higher than the base of the figure. We get this by subtracting 3, the height of the little "square" pop up on the right from the 8 on the far right of the figure.



To find a, we now subtract this length of 5 from 10, the total span of the vertical figure given by the line on the left:

$$10-5=5$$

So $a=5$.

Now we add all the sides together:

$$10+8+5+3+3+3+8+14=54$$

Answer: 54.

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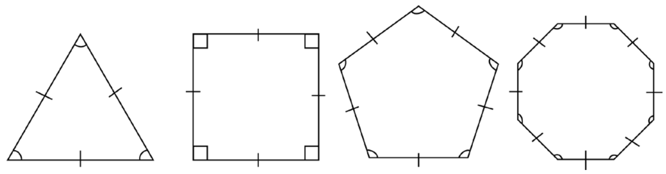
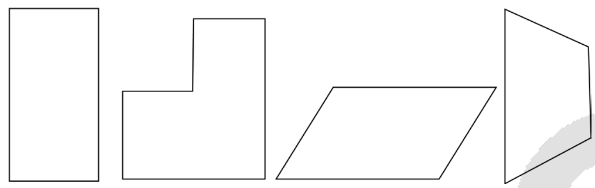
REGULAR POLYGONS

As seen below, **regular polygons** have equal sides and angles all around the shape.

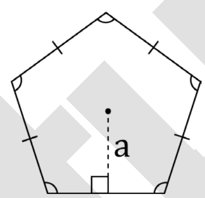


In contrast, **irregular polygons** do not have all equal sides and angles:



We can find the area of any regular polygon using a single formula:

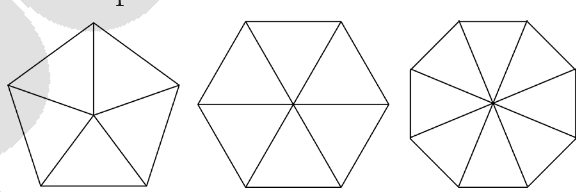
AREA OF A REGULAR POLYGON FORMULA



$$\textit{Area for Regular Polygons} = \frac{\textit{apothem} \times \textit{perimeter}}{2}$$

Where the apothem (a) is the length of the distance from the center of the shape to a side at a right angle.

A simpler way to think of this formula is to cut your shape into multiple tiny isosceles triangles, which all converge at the center of the shape:



We can think of the area of each of these whole regular polygons as the sum of the areas of the smaller triangles. To find the area of any one of these little triangles, we calculate $\frac{1}{2}$ base (b in diagram below) times the height (a in the diagram below, aka the apothem, or distance to the center at a right angle). We then multiply the area of one little triangle by the number of sides.

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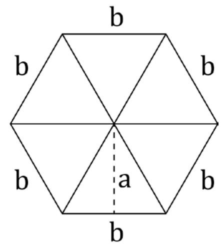


Area of the little triangle: $\frac{1}{2}ba$

Area of total hexagon (6 triangles): $=6\left(\frac{1}{2}ba\right)$

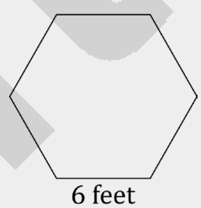
However, we could also simply add all the "bases" together first to find the perimeter before multiplying by the height of the little triangle (The apothem of the shape) and multiplying by $\frac{1}{2}$. The perimeter would be $6b$. We can factor out the b to visualize this algebraically:

$$(6b)(a)(1/2)$$

That is where this area formula comes from.

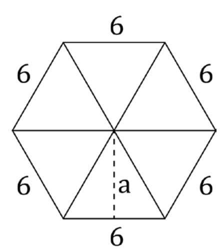
In any case, if asked to find the area of a regular polygon, you can either:

1. Memorize the above formula
   OR
2. Divide the polygon into isosceles triangles that converge at the center, find the area of one triangle, and multiply by the number of sides to find the total area.



A regular hexagon with side lengths of 6 feet is given above. What is the total area of the hexagon?

Let's use the triangle method to solve:



$b=6$. We can find a by figuring out the angles in the little triangle. I see the center of the shape cuts $360°$ into 6 equal slices.

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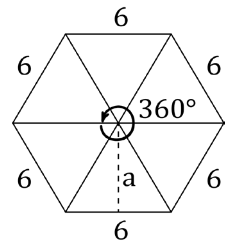
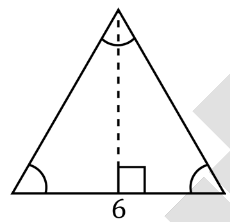
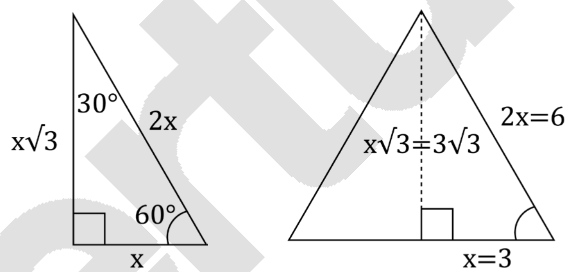


Thus, I can find the top angle of the little triangle by dividing 360 by $6 = 60$ degrees. Now I know the little triangle is isosceles, so I can find the base angles by taking $180 - 60 = 120$ degrees left for the bottom two equal angles. I divide 120 by 2 to get 60 degrees. Aha! This is an equilateral triangle (some of you may have had that memorized already; if so, bravo!). I don't even need SOHCAHTOA or law of sines to find this height, which in an isosceles (or equilateral) triangle is always a perpendicular bisector.



I can see above how the altitude cuts this little triangle into two 30-60-90 special triangles, and use this fact to solve for a. (If you don't know special triangles, see the Triangles chapter.



If the "hypotenuse" of the 30-60-90 triangle is 6, the little base, x, (half of the base of our cut out triangle) is 3, then the side a, $x\sqrt{3}$, is opposite 60 degrees and equal to $3\sqrt{3}$.

Now I can find the area of the little triangle: $\frac{1}{2}bh = \frac{1}{2}(6)\left(3\sqrt{3}\right) = 9\sqrt{3}$

I multiply by 6 (the number of little triangles I drew) to get the total area:

$$(6)\left(9\sqrt{3}\right) = 54\sqrt{3}$$

Answer: $54\sqrt{3}$.

True, finding the "apothem" is difficult in some circumstances, and slicing things into triangles can take time. Also, we don't always have regular polygons, and will need the area of non-regular shapes, too. We also use some of the more specific formulas delineated below to solve for polygon areas.

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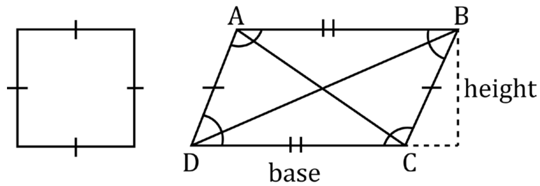
PARALLELOGRAMS

A parallelogram is a quadrilateral in which each set of opposite sides is both parallel and congruent (equal) with one another. The length may be different than the width, but both widths will be equal and both lengths will be equal.



Parallelograms have equal opposite angles (the angles on the diagonals): in the diagram above, $\angle A = \angle C$ and $\angle B = \angle D$ The adjacent angles are supplementary (meaning any two angles that touch the same side of parallelogram will add up to 180 degrees). In the diagram above, $\angle A + \angle B = 180$, $\angle C + \angle B = 180$, $\angle C + \angle D = 180$, $\angle D + \angle A = 180$.

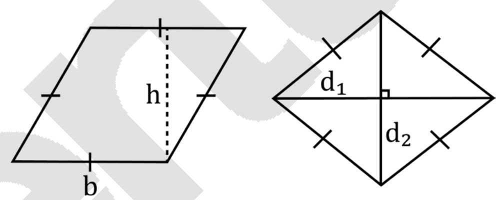
AREA OF A PARALLELOGRAM

$$Area = base \times height$$

FUN FACT: the diagonals of a parallelogram bisect each other.

Rhombus

A rhombus is a parallelogram in which all four sides are equal.



AREA OF A RHOMBUS

$$Area = bh \text{ or } \frac{d_1 \times d_2}{2}$$

Where b is the base, h is the height and d_1 and d_2 are the diagonals.

FUN FACT: The diagonals of a rhombus form a $90°$ angle. A parallelogram is a rhombus if and only if its diagonals form a right angle.

What is the area of a rhombus with diagonals of length 5 and 8?

We can calculate the area of a rhombus as the product of its diagonals divided by two:

$$5 \cdot 8 = 40$$

$$\frac{40}{2} = 20$$

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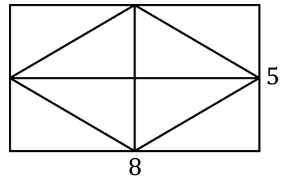
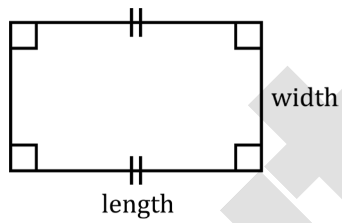
We are essentially finding the area of the "rectangle" we could draw around the rhombus that has a height and width equal to the diagonals. We divide by two, because if we quartered the rhombus, we would see four triangles, each half of a smaller rectangle as shown below:



Answer: 20.

Rectangle

A rectangle is a special kind of parallelogram in which each angle is 90 degrees.



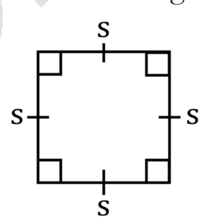
AREA OF A RECTANGLE

$$Area = length \times width$$

The **diagonals of a rectangle** are **congruent** and **bisect** each other.

Square

If a rectangle has an equal length and width, it is called a square. This means that a square is a type of rectangle AND a type of rhombus. But NOT all rectangles are squares, nor are all rhombuses.



AREA AND PERIMETER OF A SQUARE

$$Area = side\ length\ squared \text{ or } s^2$$
$$Perimeter = 4s$$

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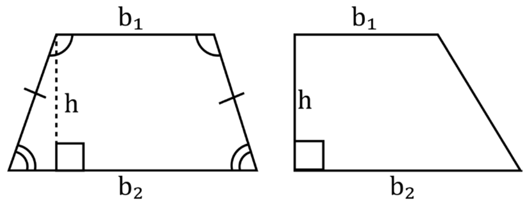
Trapezoid

A trapezoid is a quadrilateral with only one set of parallel sides. The other two sides are non-parallel. We find the area of a trapezoid by averaging the two bases and multiplying that by the height.



AREA OF A TRAPEZOID

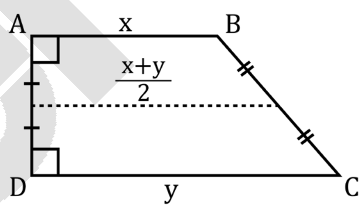
$$Area = h\left(\frac{b_1 + b_2}{2}\right)$$

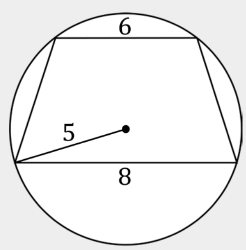
Where h is the height, and b_1 and b_2 are the base lengths.

TIP: If you forget the formula for a trapezoid (or another polygon), cut it into a rectangle and a couple of triangles and you can find the area by adding together the areas of these smaller pieces.

We call a trapezoid with equal side lengths and base angles (see above, left) an "isosceles trapezoid." These trapezoids are symmetric about the line between the midpoints of the top and bottom bases, and have equal diagonals.

TIP: The median of a trapezoid is parallel to the bases as well as half of the sum of the length of the bases as shown below.





An isosceles trapezoid whose bases have lengths 6 and 8 is inscribed in a circle of radius 5 as shown below. The center of the circle lies in the interior of the trapezoid. What is the area of the trapezoid?

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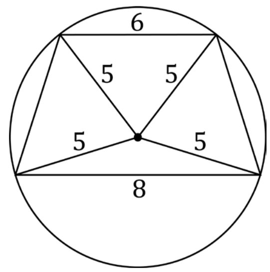
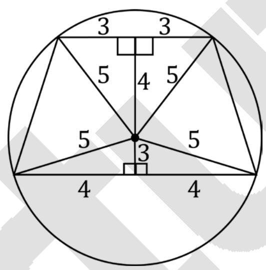
To do this problem, you'll also need to know about circles. One of our biggest tips in that chapter is to **always draw radii** whenever you have any points that don't have radii drawn to them. In this diagram, we'll draw radii from the center to the vertices of the trapezoid.

Remember that isosceles trapezoids are symmetric and have parallel sides. That means we can cut the trapezoid in half and create right angles with this midline. We also know this midline bisects the trapezoid. Again we can do this because it is isosceles.



Here, once we draw radii, look for triangles with right angles.

We can now use the Pythagorean theorem to solve for the missing lengths, 4 and 3.



Now we know the height of the trapezoid is 7, and we use the formula:

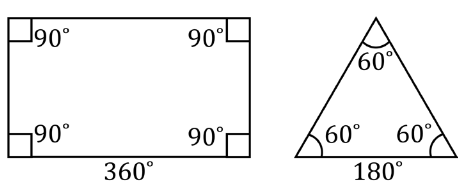
$$\frac{1}{2}(\textit{sum of bases})\cdot \textit{height} = \frac{1}{2}(6+8)\cdot 7 = 49$$

Answer: 49.

Polygon Angles

Whether your polygon is regular or irregular, the sum of its interior degrees will always have a sum determined by the number of sides on the polygon.

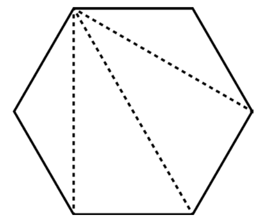
For example, the interior angles of a quadrilateral—whether kite, square, trapezoid, or other—will always add up to 360 degrees.



By that same notion, the interior angles of a triangle sum to 180 degrees, whether the triangle is equilateral (a regular polygon), isosceles, acute, or obtuse.

We determine the sum of the interior angles in one of two ways:

1. Draw all the diagonals possible from a single vertex, count the number of distinct "triangles" you've created, and multiply that number of "triangles" by 180 (as triangles have an interior angle sum of 180):



In the hexagon above, we've created four triangles from this method, so we calculate $4 \times 180 = 720°$. Thus the hexagon's interior angles sum to $720°$.

2. Use the formula for the sum of the interior angles.

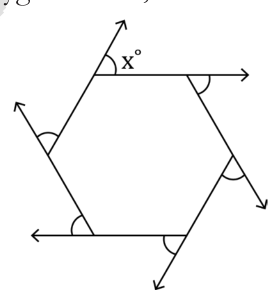
SUM OF INTERIOR ANGLES

> The sum of the interior angles in a polygon $= (n-2)180$, where n is the number of sides on the polygon.

Remember, if you know the sum of the interior angles, you can find each angle of a regular polygon simply by dividing by the number of angles/sides. For example, a regular hexagon would have individual interior angles that measure $\frac{720}{6}$ or 120 degrees each.

Exterior Angles

Exterior angles for EVERY polygon always sum to 360 degrees. Exterior angles are the angles that are supplementary to interior angles. I like to teach this idea because it is SO EASY to remember, even easier than the sum of the interior angles. To draw exterior angles, extend the lines of a polygon to make it look like a pinwheel. In the polygon below, $\angle x$ is an exterior angle:



Thus:

EXTERIOR ANGLE OF REGULAR POLYGONS

> The measure of a single exterior angle of **regular polygon** $= \frac{360}{n}$, where the shape has n-sides.

Remember, if you know an exterior angle, you can easily calculate the associated interior angle as well by subtracting from 180.

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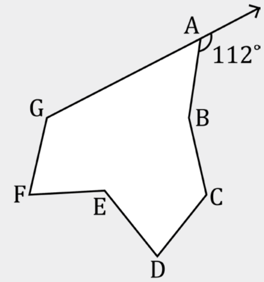
In heptagon $ABCDEFG$ below, exterior angle at A of $\angle\theta = 112°$. What is the sum of the remaining exterior angles?



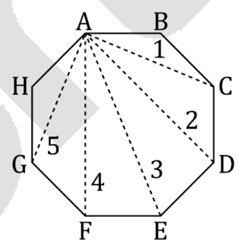
In this problem, we essentially have one exterior angle and we need the rest. All we need to do is subtract $112°$ from $360°$, because we know the sum of the exterior angles of ANY polygon is always $360°$: $360° - 112° = 248°$.

I realize this is a concave polygon. In truth, finding EACH exterior angle would involve some negative angles (yes that's confusing). But if you stick to the rules and keep it simple, this is not a hard problem.

Answer: 248.

Number of Diagonals

Sometimes the ACT may ask the number of distinct diagonals in a polygon. Again, you can find this information using the formula or by drawing it out (or a combination of the two). You can also solve these using combinations (8 vertices taken 2 at a time, or ${}_8C_2$).



To draw this out, first calculate the number of diagonals you can form from a single vertex. In this octagon, that's five.

Then multiply by 8 (the number of vertices) and you'll have the total number of diagonals you could draw BUT your calculation involves a lot of repeats. I.e., you drew a line from A to C but also from C to A. To divide out these repeats, simply divide the product you got by 2 (how many times you've counted each combination of two letters). (See Arrangements Chapter for more on this topic.)

$$\frac{8 \cdot 5}{2} = 20 \text{ diagonals}$$

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Alternatively, memorize this formula or program it into your calculator to solve:

NUMBER OF DIAGONALS OF A POLYGON

$$\boxed{\textit{Number of Diagonals in a polygon of } n-\textit{sides} = \frac{n(n-3)}{2}}$$

$$\frac{8(8-3)}{2}=20$$

Answer: 20.

AREA WORD PROBLEMS

Word problems involving area can be tackled in the same way that traditional word problems are: take things one step at a time, make up variables if necessary, make up numbers as you go to decipher relationships, and convert your units as needed. Often, these problems are culmination of many skills, including polygons, units, and percents. At other times, you'll have weird shapes. **Remember you can cut shapes into easier to manage pieces and add the areas together, or subtract out cutouts to solve for areas of shapes with missing pieces or holes.**

Marina is tiling the area around her bathtub in a herringbone pattern using rectangular subway tile. She has to cover two wall segments that measure 2.5×6 feet and another that measures 5×6 feet. She wants to order at least 15% coverage in tile to ensure she has enough given that she will be cutting tiles on a diagonal to achieve this pattern. Each tile sits flush to each other tile, and measures 2 in by 8 in. If tiles come in a box of 100, how many boxes of tile should she order at minimum?

To solve this problem, we first need the area in total of the walls. Let's convert EVERYTHING to inches first. When you do conversions with area, you'll need to account for a different conversion factor than if you were in a single dimension. I.e. if I find the area of each wall first and THEN convert to inches, there are 144 square inches in one square foot, not 12 square inches in one square foot. We square the ratio of the two "single dimension" measures when working with area, so a $1:12$ ratio turns into a $1:144$ ratio. I realize that can be confusing. Translating everything first in this problem will help me avoid this complication. I use dimensional analysis to keep my units straight.

$$5\text{ft}=\frac{5\ \cancel{\text{feet}}\times12\ \text{inches}}{1\ \cancel{\text{foot}}}=60\ \text{inches}$$

$$6\text{ft}=\frac{6\ \cancel{\text{feet}}\times12\ \text{inches}}{1\ \cancel{\text{foot}}}=72\ \text{inches}$$

$$2.5\text{ft}=\frac{2.5\ \cancel{\text{feet}}\times12\ \text{inches}}{1\ \cancel{\text{foot}}}=30\ \text{inches}$$

Now we need to find the total area to cover with tile. Be sure to get ALL THREE walls!

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Short walls: $2.5\text{ft.} \times 6\text{ft.} = 30\,\text{in.} \times 72\,\text{in.} = 2160\,\text{sq. in.}$ PER WALL (there are two!!)
Now I multiply the short wall amount by 2: $2 \times 2160 = \mathbf{4320}$
Long walls: $60\,\text{in.} \times 72\,\text{in.} = 4320\,\text{sq. in.}$
I add the TWO short walls to the long wall: $4320 + 4320 = 8640\,\text{sq. in.}$

Cool. Now we can take this total square footage and divide by the square foot per tile to see how many tiles we need:

Each tile is $2 \times 8 = 16\,\text{sq. in.}$ per tile

$$8640\,\cancel{\text{in.}^2} \cdot \frac{1\text{ tile}}{16\,\cancel{\text{in.}^2}} = \frac{8640}{16} \cdot 1\text{ tile} = 540\text{ tiles}$$

Now, I need to calculate my 15% overage. To do this, I multiply 540 times 1.15, which is a shortcut to give me 115% of 540 or 540 plus 15% of 540. That gives me 621 tiles.

With 100 tiles per box, Marina will need to order 7 boxes (700 tiles total) to give her a minimum 15% of overage.

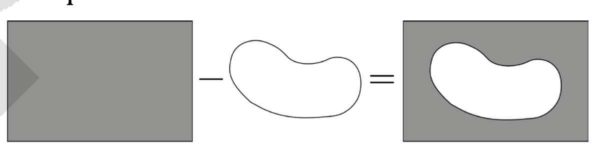
Answer: 7.

Pamela is building a pool in her backyard, which measures 14×20 feet. Her backyard will be covered in grass except for the area taken up by the pool. If the pool is a kidney bean shape, has a uniform depth of 10 feet, and a volume of 1000 cubic feet, what is the area of her grass lawn, to the nearest square meter?

Remember, ANY 3-D shape with uniform depth is like a prism, and the volume of a prism is the area of the base times the height. Here, the pool's surface area, or "footprint," equals the area of the base of such a "prism."

Here, we can subtract the surface area of the pool, which we find by dividing its volume by the depth.

$\frac{1000}{10} = 100$ (see our chapter on "Solids" if you need help with this step). Now we know the surface area of the pool is 100 sq ft, so we can subtract this from the area of her yard: 14×20 or 280 square feet. $280 - 100 = 180\,\text{sq. ft.}$



Thus her lawn will take up 180 square feet.

Answer: 180.

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