CHAPTER

16 QUADRATICS AND POLYNOMIALS

SKILLS TO KNOW

- Polynomial Tips & Terminology
- Simplifying Polynomial Expressions
- Quadratic Equations:
  - Standard, Factored Zeros, and Vertex Equation Forms
  - How to find the Vertex of any Quadratic
  - Using the Discriminant to Find the Number of Solutions
- Polynomial Long Division
- How to solve for unknowns (a, k or h) when given a polynomial equation, vertex, and/or point(s)

TIP: Many problems that involve polynomials and quadratics are also coordinate geometry related. Find these problems in the **Graph Behavior** and **Translations** and **Reflections** chapters in this book.

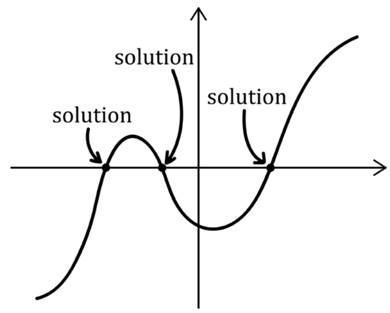
Additionally, this chapter builds on the knowledge presented in the chapter **FOIL and Factoring** in Part One of this book. Think of this chapter as that chapter on overdrive!

POLYNOMIAL TIPS & TERMINOLOGY

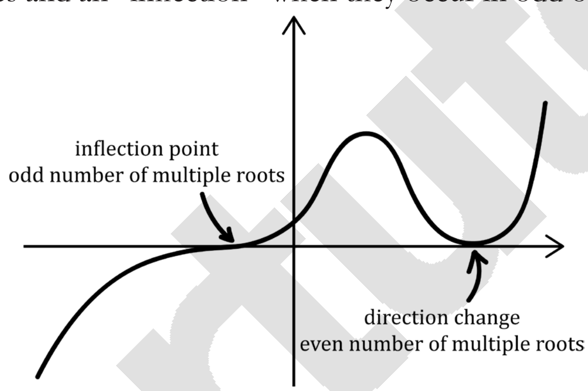
Often with polynomials, a problem looks harder than it is. Many of these problems throw around fancy sounding vocabulary that really isn't very complex, but still intimidates some students. A few things to remember:

1. **Many polynomial "terms" mean the same thing.** If we know a **"solution"** or a **"root"** of a quadratic or polynomial, it is the same as a **"zero."** A solution to a quadratic or polynomial is essentially an **"x-intercept."** Remember an x-intercept occurs when x is a number and y, or the entire expression (or equation in the form $f(x)=$ or $y=$), is equal to zero. In a coordinate plane, these "solutions" are the points at which the line crosses the x-axis and $y=0$.
2. **If we know a "solution" "x-intercept" or "zero," we instantly know one "factor" of the polynomial.** Because of the Zero Product Property, we know that if a solution is n, then $(x-n)$ must be a factor of that polynomial. So if a solution is -3, then $(x+3)$ must be a factor of the polynomial. If a factor is $(x-9)$, then $x=9$ must be a solution.
3. **The signs of solutions are the OPPOSITE of the signs of numbers in factors.**
For example, if a polynomial $f(x)=(x-7)(x+9)$ has a factor of $(x-7)$ the sign is negative. But the related SOLUTION or root would be $x=7$, which is positive. That's because to solve for the value of x, we need to set the factored piece equal to zero and solve. $x-7=0$ simplifies to $x=7$. Many students get confused on this point and jump the gun, selecting an answer choice with the wrong sign before carefully thinking through what they need.

4. If given a graph of a polynomial, **every time a line crosses the horizontal axis (the x-axis)**, it represents a **"distinct, real solution."** For each distinct, real solution, we will see a factor in the factored form of the polynomial. Remember the word **"distinct"** just means unique.



5. If a factor occurs twice in the factorization of a polynomial, it is called a **double root.** Factors that occur more than once (and create equal roots or **"multiple roots"**) are not **"distinct"** factors. Multiple roots, in graphs, create a "bounce" or direction change at the zero when they occur in even quantities and an "inflection" when they occur in odd ones.



SIMPLIFYING EXPRESSIONS

On the ACT®, you'll be expected to simplify expressions that involve polynomials. We cover the basics of simplifying expressions in Part One (Algebra Core) in the chapters **FOIL and Factoring** and **Basic Algebra.** Because this basic skill is necessary for ANY problems involving polynomials, you may want to review those chapters if you find yourself making careless mistakes in the problem set at the end of this chapter.

We'll review one quick concept here, though.

Remember that you can only combine **like terms. Like terms** have the same polynomial variable(s) and respective degree(s).

For example:
x^3, $2x^3$, and $5x^3$ are like terms. Here, we could add all three together to create a single sum.

$$x^3 + 2x^3 + 5x^3 = 8x^3$$

Similarly $2xy + 6xy$ are like terms:

$$2xy + 6xy = 8xy$$

x^3, x^2 and $2x$ are NOT like terms. If we add them together, they do not create a single term.

$$x^3 + x^2 + 2x$$

The "degree" or exponent on the variable(s) must match in order for elements to be like terms.

Because large polynomial expressions with terms such as x^7 or $8x^3y^7$ would never occur in Algebra 1, we have saved the more complex problems that may involve such large terms for the end of this chapter. Still, the idea of simplifying is pretty much the same as what you did back in middle school (and what we covered in Part One): make sure terms have the same variable(s) and same degree(s) before you combine them.

What follows is the idea that every term in a polynomial can only be formed from pieces that are like terms. We can isolate like terms in equivalent equations or expressions to help us solve complex polynomial problems, and we can focus only on a few like terms at a time to help in solving these problems, too.

What polynomial must be added to $4x^3 + x - 12$ so that the sum is $x^3 - 4x - 2$?

A. $-3x^3 - 5x - 10$ **B.** $5x - 3x^3 - 10$ **C.** $3x - 5x^3 + 10$

D. $-3x^3 - 5x + 10$ **E.** $3x^3 - 5x - 10$

This problem requires only basic algebra. We can visualize the problem better by setting up the equation vertically, lining up each term with a "blank." **<u>The vertical columns are all representative of a single like term</u>** (the first column the x^3 terms, the center column the x terms, and the final column the integer or constant terms.)

$$\begin{array}{r} 4x^3 + x - 12 \\ + \_\_\_ + \_ + \_\_ \\ \hline x^3 - 4x - 2 \end{array}$$

We can break this down, now, one piece at a time.

$4x^3$ plus what equals x^3?

I can see that subtracting $3x^3$ would give me that answer, so I begin by writing in $-3x^3$:

$$\begin{array}{r} 4x^3 + x - 12 \\ -3x^3 + \_ + \_\_ \\ \hline x^3 - 4x - 2 \end{array}$$

Now I move to the 2nd term: x plus what equals $-4x$? $-5x$ is clearly what I would add to get to $-4x$, so I place that in the 2nd blank.

$$\begin{array}{r} 4x^3 + x - 12 \\ -3x^3 - 5x + \_ \\ \hline x^3 - 4x - 2 \end{array}$$

Now for the final term: -12 plus what is -2? I can see that I add 10 to -12 to get -2. So I fill that in:

$$\begin{array}{r} 4x^3 + x - 12 \\ -3x^3 - 5x + 10 \\ \hline x^3 - 4x - 2 \end{array}$$

Answer: $-3x^3 - 5x + 10$.

We could also solve this by creating a variable for our answer, n, and making an algebraic equation:

$$\left(4x^3 + x - 12\right) + n = \left(x^3 - 4x - 2\right)$$

After isolating n we get:

$$n = \left(x^3 - 4x - 2\right) - \left(4x^3 + x - 12\right)$$

Then we distribute our negative sign:

$$n = \left(x^3 - 4x - 2\right) - 4x^3 - x + 12$$

We use the commutative property to place like terms together:

$$n = x^3 - 4x^3 - 4x - x - 2 + 12$$

And finally simplify:

$$n = -3x^3 - 5x + 10$$

QUADRATICS

The new SAT® fiercely tests your knowledge of parabolas and quadratic equations. If you are also studying for the SAT®, be sure to spend extra time on this section. These questions appear on the ACT® as well, but less often. Still, you should know all **the basic forms** of the quadratic equation.

VERTEX FORM

The Vertex Form of a parabola is: $f(x) = a(x-h)^2 + k$

- The vertex of the parabola in this form is (h,k).
- When a is positive, the parabola opens upwards, and the minimum is (h,k).
- When a is negative, the parabola opens downwards, and the maximum is (h,k).

Which of the following equations of a parabola has a vertex at $(2,-4)$?

A. $f(x) = (x-2)^2 + 4$ **B.** $f(x) = (x+2)^2 - 4$ **C.** $f(x) = (x-2)^2 - 4$

D. $f(x) = (x+2)^2 + 4$ **E.** $f(x) = (2x)^2 - 4$

Because all of the answer choices are in vertex form, our job is easy. $(2,-4)$ is equivalent to (h,k). $h = 2$ and $k = -4$. We're looking for $(x-h)$ or $(x-2)$ in the parentheses, and $+k$ or -4 to the right of the parentheses. This gives us the function $f(x) = (x-2)^2 - 4$, which matches answer choice C.

Answer: **C.**

FACTORED FORM

A polynomial is in factored form when it is expressed as a product of two or more monomials, polynomials or constants.

- The factored form of a polynomial usually takes the form: $f(x) = a(x-n)(x-m)$
- When a is positive, the parabola opens upwards.
- When a is negative, the parabola opens downwards.

For example:

$$f(x) = -2(x-4)(x+9)$$

is in factored form.

Each factored element represents a solution or a "zero" of the polynomial, which can be found by setting any factored element equal to zero. In the example above, $x-4=0$ would produce a solution of $x=4$, while $x+9=0$ produces a solution of $x=-9$.

Note: **when quadratics have no real solutions, they cannot be expressed in this form unless complex numbers are used.** For example: $y = (x-(2i))(x+(2i))$.

In the case of quadratics, **the x-value of the vertex is always the average of the two zeros**, so factored form can be used to easily find the vertex or axis of symmetry in a quadratic. (The ACT® rarely makes use of this fact, but the SAT® tests it regularly).

STANDARD FORM

The Standard Form of a parabola has the general form: $f(x) = ax^2 + bx + c$

- $-\frac{b}{2a}$ is the x-value of the vertex. The y-value of the vertex can be found by plugging in this value for x and solving for y (or $f(x)$).
- The vertex is always either the maximum or the minimum of the graph.*
- When a is positive, the parabola opens upwards.
- When a is negative, the parabola opens downwards.
- The sum of the two roots is $-\frac{b}{a}$ (not necessary to know)
- The product of the two roots is $\frac{c}{a}$ (not necessary to know)

*This is a great tip for those word problems when you throw a ball in the air and want to know when it reaches its highest point. See **Function as a Model** chapter for more of these.

To solve a quadratic equation using the **quadratic formula**, you must first have an equation in Standard form. For this reason, recognizing this form is essential on the ACT®!

THE QUADRATIC FORMULA

Earlier in this book, we covered how to solve a quadratic by factoring in the chapter on **FOIL and Factoring**. Another way to solve quadratic equations is the quadratic formula. Most students learn the basics of this formula early on in algebra, but the ACT® tends to invoke such knowledge in more challenging or creative ways, so we've saved covering it until now. Additionally, factoring tends to work on easier problems, but it doesn't always work on hard ones.

THE GENERAL QUADRATIC EQUATION

For a quadratic equation of the form:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where x represents an unknown, and a, b, and c are constants with a not equal to 0.

You should have this memorized. However, if you have a TI-83 or TI-84, I also recommend programming this into your calculator. You can find more information on how to do so on our website: supertutortv.com/blog-resources/act-calculator-programs

What are the two roots of the equation $-2x^2 + 5x - 1 = 0$?

A. 2 and -1 **B.** -2 and 1 **C.** $\frac{-5 \pm \sqrt{33}}{-4}$ **D.** $\frac{-5 \pm \sqrt{17}}{-4}$ **E.** $\frac{-5 \pm \sqrt{17}}{4}$

Before using the quadratic equation, let's factor out a -1 to make our lives a little easier. All I need to do is multiply each term by negative 1:

$$2x^2 - 5x + 1 = 0$$

From this equation we have $a = 2$, $b = -5$, and $c = 1$. Plugging these values into the quadratic equation, we get:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 - 8}}{4}$$

$$x = \frac{5 \pm \sqrt{17}}{4}$$

Since we factored out a -1 earlier, our answer is in a slightly different form from available answers, so we can simply multiple the numerator and denominator by -1:

$$x = \frac{5 \pm \sqrt{17}}{4}\left(\frac{-1}{-1}\right) = \frac{-5 \pm \sqrt{17}}{-4}$$

Answer: **D.**

For what integer k are both solutions of the equation $x^2 + kx + 13$ negative integers?

A. 14 **B.** –14 **C.** –12 **D.** 12 **E.** 1

First, we could approach this by thinking about factors. To have two real solutions, there must be some product $(x+n)(x+m)$ that creates this function. We know both n and m are positive because we want our solutions to be negative (remember the five tips at the beginning of the chapter: signs in factors are not the same as signs of the solutions). Off hand, $x+1$ and $x+13$, two values whose product is 13, but which produce negative integer solutions, appear to work. The middle term of the product of $(x+1)$ and $(x+13)$ would be $x+13x$ or $14x$. That would make $k=14$.

On looking through the answer choices, we see that this must be correct. Though we can likely get this correct by guessing without the quadratic equation, we can prove it is true with the quadratic equation. Also, that explanation is complex, while the quadratic formula is straightforward:

$$= \frac{-14 \pm \sqrt{14^2 - 4(1)13}}{2(1)}$$

$$= \frac{-14 \pm \sqrt{196 - 52}}{2}$$

$$= \frac{-14 \pm \sqrt{144}}{2}$$

Because the square root of 144 is 12, this simplifies to:

$$= \frac{-14 \pm 12}{2}$$

$$= -\frac{2}{2} \text{ and } -\frac{26}{2}$$

$$= -1 \text{ and } -13$$

We could also use the quadratic formula to *disprove* answer choices on this question. In that sense, it offers more flexibility than factoring. From factoring I'm pretty sure $k=14$, but I might not know why k can't be 12 or -12. With the quadratic formula, I have a tool to check:

$$= \frac{-12 \pm \sqrt{12^2 - 4(1)13}}{2(1)}$$

$$= \frac{-14 \pm \sqrt{144 - 52}}{2}$$

$$= \frac{-14 \pm \sqrt{92}}{2}$$

I know root 92 is not an integer, so this cannot create an integer answer.

Did I *need* the quadratic equation here? Maybe not. Do I always have time to double-check my answers? No. But it's a wonderful backup plan when you can't see the factors to an equation or a problem involves a quadratic and you want to be 100% sure. One of the reasons we didn't teach this earlier is that it isn't necessary on most of the problems you could use it on. Still, it's a great technique to understand if you're looking for an elusive near perfect or perfect score on the math. It could be the back-up plan that helps you unpack some of the most challenging problems on the test.

THE DISCRIMINANT

One of the most powerful parts of the quadratic formula is the discriminant, or the portion of the quadratic formula that falls under the radical sign: $b^2 - 4ac$.

We can use the discriminant to determine how many (if any) solutions a quadratic equation has.

FINDING SOLUTIONS USING THE DISCRIMINANT

Given that $f(x) = ax^2 + bx + c$ the discriminant is defined as $b^2 - 4ac$ (the argument of the root in the quadratic equation) :

1. When this value is positive, there are two real roots
2. When this value is 0, there is one real root
3. When this value is negative, there are no real roots (but it does have two imaginary roots)

Remember, **Roots**, **Zeros**, and **Solutions** refer to the places where the graph **intercepts the x-axis**, or $y = 0$.

Let's unpack why this works. For rule #1, remember that the quadratic formula always has plus or minus before the radical. The plus and minus are the source of the two different solutions, right? So it makes sense that when what follows exists (i.e. there is a real, positive number under the radical), then two real solutions are created.

For rule #2, we can calculate the square root of 0, but "plus or minus 0" doesn't mean much. Whether we add zero or subtract it, we get the same number. Clearly, having a determinant equal to zero creates a single solution, or a double root.

For rule #3, if the number under the radical is negative, we cannot calculate a real solution because the square root of a negative number is by nature an imaginary number. As a result, negative discriminants indicate no real solutions.

Though in many cases we could figure out the number of solutions using the entire quadratic formula or in some cases by factoring, and essentially solving down to the solutions themselves, checking the discriminate can save time and also offers a strategy for more complex problems.

For what non-zero whole number k does the quadratic $x^2 + kx + 2k = 0$ have exactly 1 real solution for x ?

For this problem we can use the discriminant to find the solution. We know that **one real solution** occurs when the **discriminant is equal to zero**, so we set $b^2 - 4ac = 0$ and solve.

$$b^2 - 4ac = 0$$
$$k^2 - 4(2k) = 0$$
$$k^2 - 8k = 0$$
$$k(k-8) = 0$$

Now we know $k=0$ or $k-8=0$, but the problem asks for a **non-zero whole number**, so we can disregard $k=0$.

$$k = 8$$

Answer: 8.

LONG DIVISION

Some of you might solve polynomial division problems using Synthetic Division. If so, great! That works for typical polynomial division problems! In this book, we focus on traditional polynomial long division because I find it more intuitive, easier to remember, and more useful for complex problems involving unknown constants.

> If $x=3$ is a zero of the expression $3x^3 - 11x^2 - 2x + 24$, which of the following must also be a factor?

When we're given a "zero" we know we have a factor at $(x-n)$, where n is that zero. In this case, we know that $(x-3)$ must be a factor of this expression. Unlike quadratics, third degree polynomial factors can't usually be deduced efficiently by sight and creative guess and check. Instead, use long division.

We now can set up our long division problem:

$$x-3\overline{\smash{\big)}3x^3 - 11x^2 - 2x + 24}$$

Take the division one step at a time, one term at a time. For the first step we only need to pay attention to the terms of highest degree in both the divisor and dividend, x and $3x^3$, respectively.

How many times does x go into $3x^3$? $3x^2$ times, since $x * 3x^2 = 3x^3$. We write this term on the top line as the first part of our final answer:

$$\begin{array}{r l} & 3x^2 \\ x-3 & \overline{\smash{\big)}3x^3 - 11x^2 - 2x + 24} \end{array}$$

Now, just like in ordinary long division, we multiply this first term by the divisor to get our initial remainder. We subtract the $-9x^2$ from the $-11x^2$ to get $-2x^2$.

$$\begin{array}{r l} & 3x^2 \\ x-3 & \overline{\smash{\big)}3x^3 - 11x^2 - 2x + 24} \\ & \underline{-(3x^3 - 9x^2)} \\ & \qquad\quad -2x^2 \end{array}$$

Now we bring down the next term of lower degree ($-2x$). We ask a similar question: how many times does x go into $-2x^2$? $-2x$ times. We place the $-2x$ above the division bar, multiply $-2x$ times $(x-3)$, and write that product underneath in our work below. We then subtract to simplify (don't forget to change the sign on the 2nd item $+6x$!).

$$
\begin{array}{r|l}
 & 3x^2 - 2x \\
\hline
x-3 & 3x^3 - 11x^2 - 2x + 24 \\
 & \underline{-(3x^3 - 9x^2)} \\
 & \quad -2x^2 - 2x \\
 & \quad \underline{-(-2x^2 + 6x)} \\
 & \qquad -8x
\end{array}
$$

Now we bring down the 24 to meet the $-8x$ and continue. x goes into $-8x$ -8 times, so we write -8 above the division line in the top row. Then we multiply -8 times $(x-3)$ to get $-8x+24$. Now we write $-8x+24$ underneath the rest of our work and once again subtract to at last find our remainder. Because $24-24$ is 0, our remainder is 0.

$$
\begin{array}{r|l}
 & 3x^2 - 2x - 8 \\
\hline
x-3 & 3x^3 - 11x^2 - 2x + 24 \\
 & \underline{-(3x^3 - 9x^2)} \\
 & \quad -2x^2 - 2x \\
 & \quad \underline{-(-2x^2 + 6x)} \\
 & \qquad -8x + 24 \\
 & \qquad \underline{-(-8x + 24)} \\
 & \qquad\qquad 0
\end{array}
$$

If you are left with a constant, do not fret. Simply place the constant remainder over the divisor and you are done!

Answer: $3x^2-2x-8$.

One of the roots of $2x^4+8x^3-2x^2-8x=0$ is -4. What are the other roots?

We are told that -4 is a root, therefore the expression must be divisible by $(x+4)$. If we divide the expression by $(x+4)$ using polynomial long division:

$$
\begin{array}{r|l}
\hline
x+4 & 2x^4 + 8x^3 - 2x^2 - 8x
\end{array}
$$

$$\begin{array}{r} 2x^3 \\ x+4 \enclose{longdiv}{2x^4+8x^3-2x^2-8x} \\ \underline{-2x^4+8x^3} \\ 0 \end{array}$$

$$\begin{array}{r} 2x^3+0x^2-2x \\ x+4 \enclose{longdiv}{2x^4+8x^3-2x^2-8x} \\ -\underline{2x^4+8x^3} \\ 0-2x^2-8x \\ \underline{-0-2x^2-8x} \\ 0 \end{array}$$

We get $\dfrac{2x^4+8x^3-2x^2-8x}{x+4}=2x^3-2x=0$. Now we just have to find the zeros of $2x^3-2x=0$.

We can factor out a two.

$$2\left(x^3-x\right)=0$$

Then divide both sides by two.

$$x^3-x=0$$

We can now factor out an x:

$$x\left(x^2-1\right)$$

And then apply the pattern from the Difference of Squares (See FOIL and Factoring in Part One).

$$=x\left(x+1\right)\left(x-1\right)=0$$

Using the Zero Product Property, we have three equations we can form:

$$x=0 \text{ or } x+1=0 \text{ or } x-1=0$$

Solving out these equations we find the solution set for $x=\left\{0,-1,1\right\}$.

SOLVE FOR A, K, OR H (PLUGGING IN)

The most basic algebraic skills you'll need on the ACT® is plugging in, which is covered in our Algebra Core (Part One) chapter on Basic Algebra.

But sometimes the same principle plays out in more complex quadratic and polynomial problems. Remember the steps you take when finding a linear equation using slope intercept form, given the slope and a point. To solve those problems, you plug in the $\left(x,y\right)$ point you have and the slope and solve for "b" to find the slope-intercept form of the equation. Here we will do the same thing: **plug in what we know and solve for what we don't know**. Often what we don't know will be some random letter (such as a, h, or k), and what we know is an $\left(x,y\right)$ pair or even a factor $\left(x-n\right)$, which in turn implies a solution $\left(n,0\right)$.

This principle can sometimes save a huge amount of time over long division. I.e., sometimes a problem could be solved either way, but this way is often faster.

> The solution set for x of the equation $x^2 - \frac{k}{2}x + 9 = 0$ is $\{3\}$. What does k equal?
>
> **A.** 9 **B.** 12 **C.** −12 **D.** 3 **E.** −3

This question is simply asking you to solve for a variable, k. We know x. Don't be thrown by fancy wording: "solution set" just means that is what x equals.

Here, we'll plug in what we know (the given value of x) to most quickly find the answer:

$$(3)^2 - \frac{k}{2}(3) + 9 = 0$$

Now, simplify:

$$9 - \frac{3}{2}k + 9 = 0$$

$$18 - \frac{3}{2}k = 0$$

$$18 = \frac{3}{2}k$$

$$k = 18\left(\frac{2}{3}\right)$$

$$= 12$$

We alternatively could solve this problem with long division, knowing that $(x-3)$ is a factor of the polynomial given the solution $x = 3$, but doing so would be time consuming. Likewise, we could set up $(x-3)(x-n)$ and solve for n, setting that FOIL product equal to what we know, and matching up pieces to form equations as necessary. Again, time consuming, not to mention confusing. Our best bet here is plugging in.

Answer: **B**.

> What is the value of k if $(x-5)$ is a factor of $x^3 - 3x^2 - 3kx - 5$?

To solve this problem we could make use of polynomial long division to solve for k. As we are given a factor, this is probably the first method many students would turn to. However, I'm going to first show you a faster way: plugging in.

Because we know $x - 5$ is a factor, we also know $x = 5$ is a solution (remember the tips at the beginning of the chapter! A solution gives you a factor, and a factor give you a solution!). Knowing this, I can set this expression equal to zero, plug in 5 (the solution we know given the factor we know), and solve for k.

$$x^3 - 3x^2 - 3kx - 5 = 0$$

$$5^3 - 3(5)^2 - 3k(5) - 5 = 0$$

$$125 - 3(25) - 15k - 5 = 0$$

$$125 - 75 - 5 = 15k$$

$$45 = 15k$$

$$3 = k$$

Now back to polynomial division. What if you wanted to use that method?

We could divide the long polynomial by the given factor to get:

$$\frac{x^3 - 3x^2 - 3kx - 5}{x-5} = x^2 + 2x + \frac{(-3k+10)x - 5}{x-5}$$

In order for $(x-5)$ to be a perfect factor, it needs to divide evenly into the remainder of the expression, so the numerator of the fraction $\frac{(-3k+10)x-5}{x-5}$ must be divisible evenly by the denominator $(x-5)$. The only numerator that we can easily attain that is divisible by $(x-5)$ is $x-5$. In other words, the x must have a coefficient of 1. Try any other value of $-3k+10$ and you'll see it makes nothing cancel. We set the expression $-3k+10$ equal to 1 and solve for k to get $k=3$. As you can see, the first method may not be as obvious, but it is faster, not to mention less confusing!

Answer: **D.**

DEGREES OF POLYNOMIALS

The degree of the polynomial refers to the highest power that exists within in the equation.

- Each factor, each zero, and each time a line touches the x-axis of a graph indicates a possible solution. For each possible solution, we can count at least one "degree" in our polynomial, because the maximum number of solutions for any polynomial is the same as its highest degree. For example, a 2nd degree (quadratic) polynomial, such as $x^2 + 2x - 7$, can have at most 2 distinct, real solutions. A 5th degree polynomial, such as $x^5 - 5x^4 + 3x^2 - 625$, can have at most 5 real solutions.
- We can always have a higher degree linear polynomial than the number of solutions because of imaginary solutions and multiple roots.
- **For more information on this type of polynomial problem see Graph Behavior in this book, as we cover these problems more extensively there.**

What is the minimum degree possible for a polynomial function with factors $(x-3)$, $(x-5)$ and a double root at $x=-1$?

A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5

Because each factor indicates at least one degree, our two factors imply two degrees. Our "double root" implies that we have $(x+1)^2$ as part of the factorization of this function. Thus for those two factors we add another two degrees for a total of 4. At minimum, this polynomial would be a 4th degree polynomial.

Answer: **D**.

A polynomial $f(n)$ has p non-zero terms. If $f(n)$ is the product of function $g(n)$, which has r non-zero terms, and function $h(n)$, which has q non-zero terms, $p>r$ and $p>q$, which of the following could be equal to q?

A. 1 **B.** $p+r$ **C.** $\frac{p}{r}$ **D.** $\frac{r}{p}$ **E.** $p+1$

HARD! Skip this problem if you're not aiming for a 34+.

This problem sounds confusing. Break it down one step at a time. We are dealing with three polynomials. $f(n)=g(n)$ times $h(n)$. We thus would multiply every term in $g(n)$ times every term in $h(n)$ to get every term in $f(n)$.

Let's analyze each answer:

Answer A: I know that neither r nor q can equal 1. If $r=1$ than q would equal p, because there would be the same number of terms in $h(n)$ as in $f(n)$. Think about it—let k equal the one term that r represents:

$$k(x+1)=kx+k$$

A single term does not create any more terms when you multiply by it. It's just like multiplying by a scalar. Thus we couldn't get $f(n)$ with its p terms to have any more terms than $h(n)$.

Answer B: I know that answer (B) cannot be correct, because given the parameters (r is non-zero and $p>r$), if I add r to p, I know I'm adding at least 1 to p. I know $p>q$, so if I add one (or another number) to p, there is no way it is equal to q, rather, it's just larger than q than it started off being. We can't have negative terms, so adding something to p isn't going to work in any case.

By extension, answer E also doesn't work. We can't add to p and expect it to equal something smaller than it.

Answer C: Let's think about this. We'll make up the idea that $r=$ two terms and $q=$ three terms:

$$\left(x^2+5\right)\left(x^2+x+1\right)$$

Let's expand and see what happens:

$$x^4 + x^3 + x^2 + 5x^2 + 5x + 5$$

My x^2 terms combine and I get 5 terms in total, so p would be 5.

However, I can see that if the x^2 terms didn't combine, I could have had 6 terms, that would make p 6. Let's make up another example and see if we can get it to be true. My goal is to have zero overlapping terms so nothing "disappears" as I simplify.

$$\left(x^{72} + 5x^{13}\right)\left(x^2 + x + 1\right)$$

I can see that:

$$x^{144} + x^{73} + x^{72} + 5x^{26} + 5x^{14} + 5x^{13}$$

has six terms, and that's 2×3. Thus I found an example that makes C true: $\frac{6}{2}=3$.

Answer D doesn't make sense. Because we know these values are all positive integers, if I take the larger one and put it on the bottom and a smaller one and put it on the top, we get a fraction less than one. That can't be the number of terms.

This question is a hybrid of a polynomials problem and a "Properties of Numbers" problem. If you struggled with this, check out our chapter on "Properties of Numbers" in Book 2 (Numbers, Trig, Stats and Geometry).

Answer: **C.**