CHAPTER

13 CIRCLES

SKILLS TO KNOW

- Formulas for area, diameter, circumference
- Sectors of circles (area and arc length)
- Problem Solving & Circles: Tangents, Radii, & Drawing More Lines
- Circles, Arc measures, and Angles
- Pie Charts: Circle Angles & Fractions/Percents /Probability
- Area subtraction ("Donut" problems)

NOTE: Circle Equations and coordinate geometry involving circles is covered in the chapter Conic Sections.

BASIC CIRCLE FORMULAS

Most of you likely know these circle formulas, but may need a bit of review. Be sure you have all of these memorized, and if you're prone to careless errors, always double check which element you need when you read the question! Remember pi (π) is approximately equal to 3.14.

Area: $A = \pi r^2$
Diameter: $d = 2r$
Circumference: $C = \pi d$ or $C = 2\pi r$ ($d = 2r$)

If the circumference of a circle is 120π, what is the area of the circle?

A. 60 **B.** 60π **C.** 120 **D.** 3600 **E.** 3600π

For this problem, we'll need to first use what we know to find the radius, and then use that information to find the area. Remember on problems like this, if you find the radius, you can always use that to find anything else you might need.

First, because $C = 2\pi r$ we set $120\pi = 2\pi r$.

Now we solve for r, dividing both sides by 2π:

$$\frac{120\pi}{2\pi} = \frac{2\pi r}{2\pi}$$

$$\frac{120\cancel{\pi}}{2\cancel{\pi}} = \frac{\cancel{2}\cancel{\pi} r}{\cancel{2}\cancel{\pi}}$$

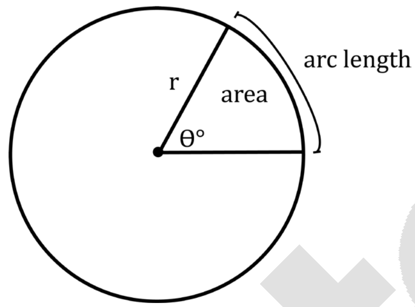
Simplifying, the pi's cancel on both sides, the 2's cancel on the right, leaving::

$$\frac{120}{2} = r$$

$$60 = r$$

CIRCLE SECTORS

Sectors are "slices" of a circle (think like a pizza or pie slice). Arcs are the length of the circumference that the sector involves.



Problems involving sectors and arcs often ask you to find the area of the sector, the length of the arc, or the angle of the sector.

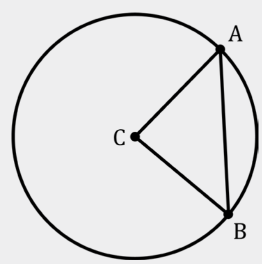
AREA OF A SECTOR FORMULA

$$A = \frac{\theta}{360}\pi r^2$$

ARC LENGTH FORMULA

$$A = \frac{\theta}{360}2\pi r$$

If $\angle ACB = 80°$, and $AC = 5$ cm, what is the arc measure of $\overset{\frown}{AB}$?



A. $\frac{10}{9}\pi$ cm **B.** $\frac{20}{9}\pi$ cm **C.** 5π cm **D.** $5\sin 40°$ cm **E.** $10\sin 40°$ cm

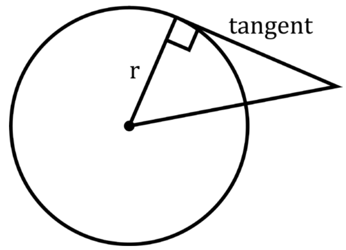
We are given the length of $\overline{AC}$, which is the radius. Let's label it with the variable a. With this, we can find the entire circumference using the equation $C = 2\pi r$. Plugging in r, we get $C = 2\pi 5 \rightarrow 10\pi$. Then, we can set up a ratio $\frac{a}{10\pi} = \frac{80}{360}$. Cross multiplying, we get $800\pi = 360a$. Simplifying, we can get $\frac{20}{9}\pi = a$.

Answer: **B.**

PROBLEM SOLVING & CIRCLES: TANGENTS, RADII, & DRAWING MORE LINES

Circle problems can sometimes overwhelm students. Where do you start? What do you know?

One element that gives you additional information in a circle problem is a **tangent line**. By definition, **tangent lines** that touch circles create **right angles with the radius of the circle**.



When you see the word TANGENT in a problem involving a circle:

- **Draw a radius** from the center to the tangent point(s).
- **Mark the angle** formed by a radius and tangent line as 90 degrees.
- **Look for right triangles** or draw any necessary additional lines to create them.
- **Use the Pythagorean Theorem, SOH CAH TOA, or other rules** you know as appropriate to solve.

RADII are your FRIENDS!

Another element to keep in mind is the **RADIUS** of the circle. Often, general circle problems require that you **draw more radii** than are already drawn for you. Over half the time when my students are stuck on circle problems, they failed to recognize or draw radii.

When you see a circle with lines or intersecting/inscribed shapes:

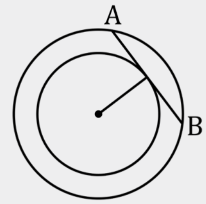
- **Draw all the radii you can** to any points given on the circle (and particularly to any tangents)
- **Look for triangles** in circles or draw lines to make triangles (particularly right triangles or isosceles ones).
- **Radii are always equal**; mark them as such.
- **Mark triangles formed by two radii as isosceles.** The opposite angles will be equal and the altitude will always be a perpendicular bisector the other side.

When in doubt, DRAW MORE LINES!

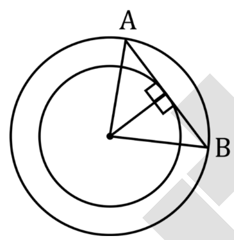
- **Draw additional lines or chords** to form right or isosceles triangles, squares, rectangles or other shapes formed by existing points, if possible.
- **Cut complex shapes into more manageable pieces**, particularly with area or sector length problems.

Let's take a look at an example and apply these ideas!

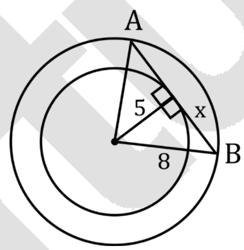
What is the length of the chord $\overline{AB}$ if the radius of the inner circle is 5 and the radius of the larger circle is 8?



Remember our bullet points: because radii are our friends, we draw radii to the points marked, A and B. Because we see the word tangent, we draw a radius to the point of tangency, and mark the angle formed as 90 degrees. These radii create right triangles:

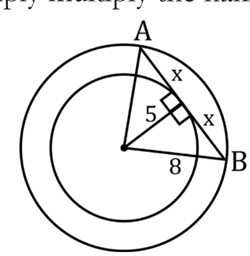


Because we know the radii of the 2 circles, we can solve for the triangle.



$$8^2 = 5^2 + x^2$$
$$64 = 25 + x^2$$
$$39 = x^2$$
$$x = \sqrt{39}$$

But we need the length of the whole chord. We know the isosceles triangle formed by A, B, and the two drawn radii of the large circle will have a perpendicular bisector down the middle. That's because the altitude of an isosceles triangle is always a bisector of the opposite side. Thus x is half of the chord. To find the whole chord, we simply multiply the half chord by two.



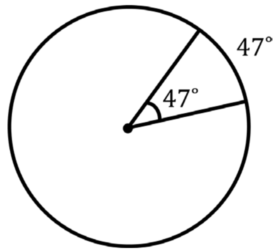
$$2 \cdot \sqrt{39}$$

Answer: $2\sqrt{39}$.

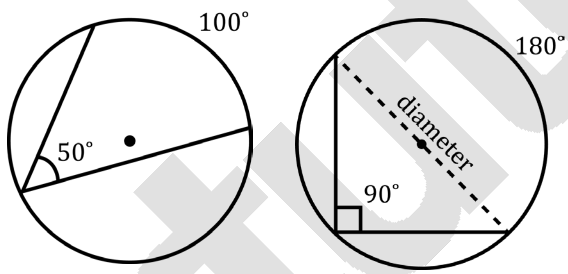
CIRCLE ARC MEASURES AND ANGLES

Angles

Central angles (angles formed by two radii that touch the circle in two places) in a circle are always the same degree measure as their corresponding arcs. For example, in the picture below, the arc is $47°$ and so is the central angle.

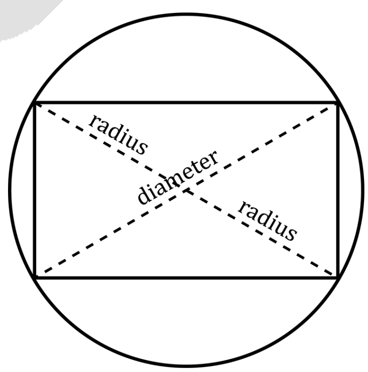


Inscribed angles (formed by two chords) are **half the corresponding arc measure**. For example, a 50 degree angle always opens to a 100 degree arc. A 90 degree angle always opens to a 180 degree arc. In fact, this is the example I use to remember this rule if I forget it, because it's so common to see in problems.



It's also good to note that **any inscribed angles that measure 90 degrees always open to the diameter of the circle**. Thus if you know a triangle is inscribed in a circle, and one side is the diameter, you know it's a right triangle. When you know an inscribed angle measures 90 degrees, it intercepts the diameter.

When a square or rectangle is inscribed in a circle, its angles are inscribed right angles, so its diagonals are diameters, equal in length, bisecting each other at the radius.



Pie Charts: **Circle Angles & Fractions/Percents/Probability**

Some ACT problems will ask you for the **central angle** for a segment of a pie chart or circle graph. Remember that pie charts are composed of segments that proportionally represent data. Also remember that a circle's central angle encompasses $360°$, so you want to figure out a proportional amount **of $360°$.** Remember **<u>of</u>** means **<u>multiply</u>**! To find the central angle of an element in a pie chart, first figure out the fraction of the whole, the percent over 100, or the probability of the element in question, which equals **the part divided by the whole**. Then multiply this fraction by 360 degrees to find the degree measure for the pie chart segment.

DEGREE MEASURE OF PIE CHART SEGMENT

$$\frac{Part}{Whole} \times 360° = Degree\ Measure\ of\ Slice\ in\ a\ Pie\ Chart$$

You can also create a proportion to solve these:

PROPORTION OF PIE CHART SEGMENT

$$\frac{Part}{Whole} = \frac{Degree\ Measure\ of\ Pie\ Piece}{360°}$$

The idea of proportionality also extends to the **area of a pie piece.** For example, whenever a circle represents **probability**, the area of any segment divided by the area of the whole circle equals the probability of the event that segment represents. When a circle represents data, likewise, **the percent or fraction** that represents a part of the data is equal to the area of the pie piece over the area of the whole pie chart.

PROPORTIONALITY USING AREA

$$\frac{Part}{Whole} = \frac{Area\ of\ Pie\ Piece}{Area\ of\ Whole\ Pie\ Chart\left(\pi r^2\right)}$$

$$\frac{Part}{Whole} \times Area\ of\ Whole\ Pie\ Chart\left(\pi r^2\right) = Area\ of\ Pie\ Piece$$

David wants to draw a circle graph showing all the favorite desserts of his friends. When he polled his friends, $\frac{2}{5}$ said ice cream, $\frac{1}{5}$ said cake, $\frac{3}{10}$ said cookies, $\frac{1}{20}$ said candy, and the remaining friends said other desserts. What is the angle measure of the segment of friends who preferred other desserts?

First, we need to know what fraction of friends said other desserts (our "part over whole"). We find this by subtracting all the other fractions from 1 (all these fractions must sum to 1, or one whole "pie", so to find the remaining fraction, we subtract all the parts we know from the whole).

CALCULATOR TIP: If it's easier, convert these to fractions in your calculator to decimals. Alternatively, use lots of parentheses, subtracting one fraction at a time, and let your calculator do the work.

$$1-\left(\frac{2}{5}\right)-\left(\frac{1}{5}\right)-\left(\frac{3}{10}\right)-\left(\frac{1}{20}\right)=\frac{1}{20}$$

Thus $\frac{1}{20}$ of his friends said other desserts. Now, we multiply $\frac{1}{20}\cdot 360$ to find the proportional amount of the central angle that will create an appropriate sized sector for these respondents.

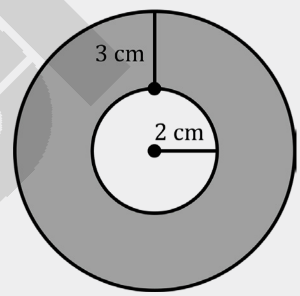
$$\frac{360}{20}=18°$$

Answer: $18°$.

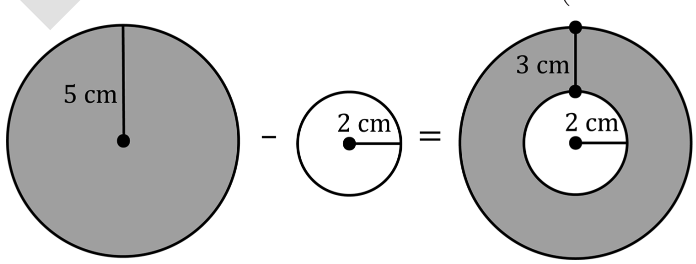
DONUT AND COMPLEX CIRCLE AREA PROBLEMS

Sometimes, you'll be asked to find the area of something that isn't quite a circle or a circle sector, but a part of a circle or a shape that's made from overlapping circles. Whenever you have a funky shaped area to solve for, try to figure out what "shapes" are subtracted from larger shapes to create this resultant shape. Then subtract the area of the elements that are "cut out" from the larger shape's area. You can apply a similar technique to problems involving circumference or perimeter: divide the funky shape into segments, solve for each, and add the results together. Alternatively, cut up the shape into pieces and rearrange the position of elements to find an easier way to solve for the needed area.

The figure below shows a face of a small circular washer. If the inner circle has a radius of 2 centimeters while the width of the washer is 3 cm. What is the total area, in square centimeters, of the shaded area of this washer?



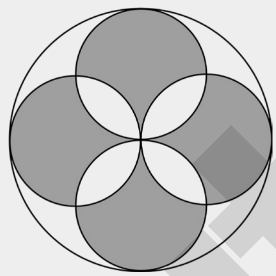
To find the answer, we need the area of the biggest circle (a circle of total radius 5, which we find by adding 2 and 3). Then we'll subtract the area of the smaller circle (with radius 2):



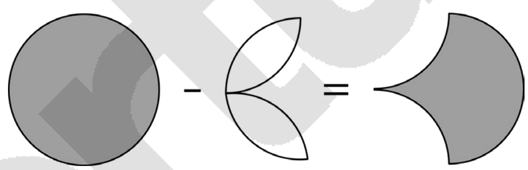
$$\begin{aligned} \textit{Area of Large Circle} - \textit{Area of Small Circle} &= 5^2\pi - 2^2\pi \\ &= 25\pi - 4\pi \\ &= 21\pi \end{aligned}$$

Answer: 21π.

In the figure below, four congruent circles are equally spaced within a single larger circle. Two of the smaller circles are tangent to each other and the sides of the larger circle. The other two smaller circles are also tangent to each other and the larger circle. The largest circle's circumference is 64π. What is the area of the shaded region?



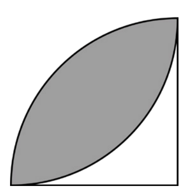
For this problem, we can figure out the area of one small circle and then subtract two "petals" from that area. Then we'll multiply that funky shape by four.



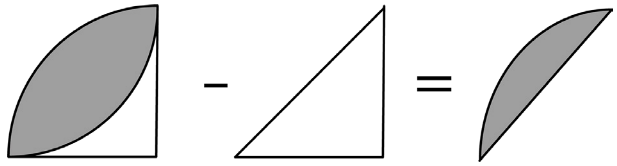
First, let's find the area of the small circle. We know the large circle's circumference is 64π. Thus the diameter must be 64. Thus the diameter of one smaller circle is 32, and its radius is 16.

Now the area of a circle is πr^2 so $16^2\pi = \text{area} = 256\pi$.

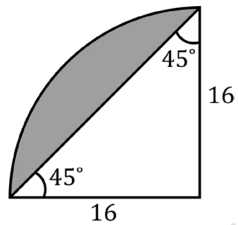
Now we must find the area of one petal. Because we know the shape is even, we can assume each intersecting arc forms a $90°$ span. There are $360°$ in the circles in total, and the "petals" are evenly spaced about that around the very center of the picture, each at $90°$. Thus the petal is formed by two identical curved "parts" of a quarter circle sector:



We can find the area of half of this "petal" by first finding the area of $\frac{1}{4}$ of one of these smaller circles, and then subtracting off the triangular part:

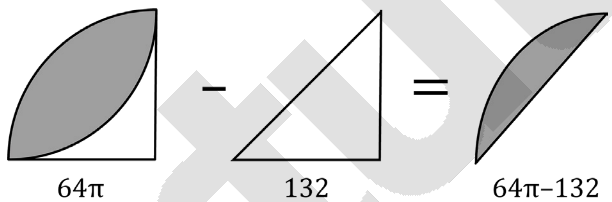


$\frac{256\pi}{4} = 64\pi$. So the $\frac{1}{4}$ circle area is 64π. Now I want to subtract off the triangle in this "pie piece" or sector. I know the triangle has two legs that are part of a quarter circle, so this triangle is a right isosceles triangle (two sides are identical radii; the other side is the hypotenuse opposite $90°$).

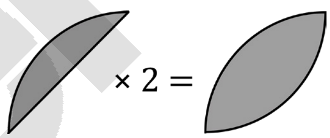


I can find the triangle's area by multiplying the radius times itself and dividing by 2. Since the radius is 16, I take $16^2 = 256$ and divide by 2 to get 132.

Now we subtract: $64\pi - 132 =$ area of half a petal.

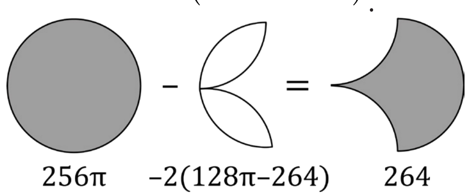


Now we multiply by 2 to get a whole petal area: $2(64\pi - 132) = 128\pi - 264$.



Now we subtract two whole petals from one smaller circle to get that funky shape we have four of:

$$256\pi 2(128\pi - 132).$$



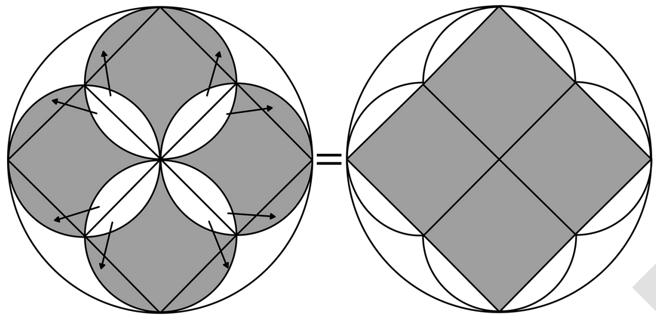
I distribute the negative two to get:

$$256\pi - 256\pi + 264 = 264$$

Finally, we multiply the area of this "funky shape" by 4:

$$4(264) = 1056$$

We could have solved this problem in other ways, too. For instance, we could have rearranged the two petals as cutouts on the small circles to form a square in the middle and realized the shaded region is equivalent to the area of an inscribed square.



Then we could have found the side length of the square, using what we know about the circle diameters and 45-45-90 triangles, and found the area of all four squares and added them together. This way may have been faster, but it's less intuitive to "see."

Answer: 1056.