CHAPTER

12 ANGLES AND LINES

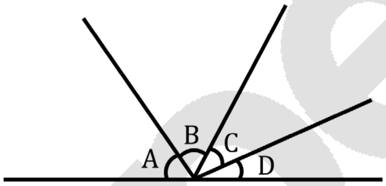
SKILLS TO KNOW

- Straight lines are 180°/Circles of angles sum to 360°
- Supplementary angles sum to 180° and complementary angles sum to 90° (a right angle)
- Vertical angles are congruent
- Parallel lines theorem: the alternate exterior and alternate interior angles of parallel lines intersecting with a transversal are congruent.
- Triangles: angles sum to 180°, exterior angle theorem, isosceles & equilateral triangles*
- Quadrilaterals and angles*
- Angle bisectors
- Angle "hopping": synthesizing all the rules together.

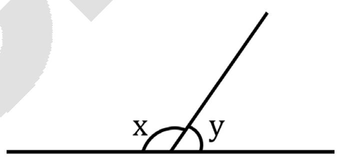
*Angles and lines skills also intersect with skills in our chapters on Triangles, Polygons, and Similar Shapes & Ratios. Also see these chapters for related skills and problems.

STRAIGHT LINES AND CIRCLES

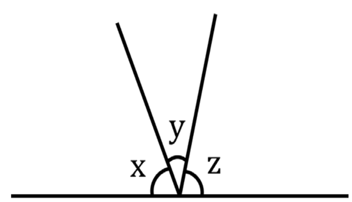
Straight lines represent 180°. Anytime you see two or more angles popping out to one side of a straight line, these angles must sum to 180°.



$A+B+C+D=180°$

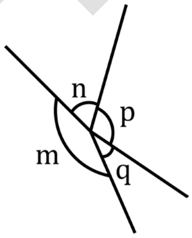


$x+y=180°$

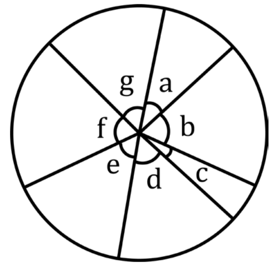


$x+y+z=180°$

The sum of all the angles formed by lines that all converge at a single point (like spokes of a bicycle, or a "circle" of angles) is always 360°. Similarly, the interior angles of a circle always sum to 360°.



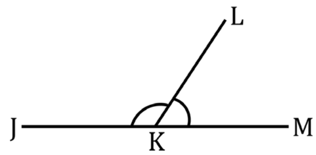
$m+n+p+q=360°$



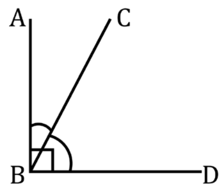
$a+b+c+d+e+f+g=360°$

SUPPLEMENTARY & COMPLEMENTARY ANGLES

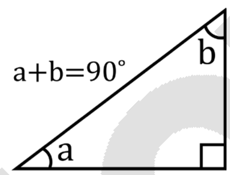
Supplementary angles are any two (and only two) angles that add up to $180°$. An example could look like this:



Complementary angles are any two (and only two) angles that add up to $90°$. An example could look like this:



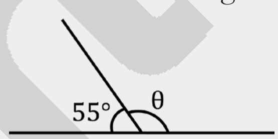
Or this:



The biggest mistake students make on problems with this vocabulary is confusing these words.

To keep these terms straight, remember: "Supplementary" and "Straight" both start with an "s:" **Supplementary** angles sum to a **straight** angle of 180 degrees. Alternatively, imagine how the "**s**" in supplementary looks like the $\mathbf{8}$ in $180°$, and the "**c**" in complementary looks a little like a backwards "$\mathbf{9}$" in "$90°$."

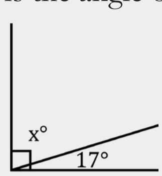
What is the value of θ if the two angles are supplementary?



Because we know that the two angles are supplementary, they must add to $180°$. As a result, we can write out the equation $55° + \theta° = 180°$. Solving for $\theta°$, we get $\theta° = 125°$.

Answer: $125°$.

What is the angle of $x°$?

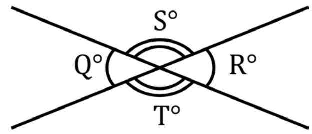


We know that the two angles are complementary and add up to $90°$, so we can write $x° + 17° = 90°$. Solving for $x°$, we get $73°$.

Answer: $73°$.

VERTICAL ANGLES

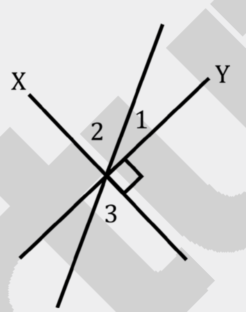
Here's a diagram of "vertical" angles:



Q and R are vertical angles, as are S and T.

The important thing to remember about vertical angles is that they are congruent; that is, angle Q and R equal each other, and angle S and T equal each other. We can see how this relationship holds if we think about the straight angles. We know $Q+S=180$ because these two angles share a straight line. We also know $S+R=180$, because these two angles share a straight line. It follows that $Q=180-S$ and $R=180-S$, and thus $Q=R$. You can see algebraically why this is true.

In the figure below, line x is perpendicular to line y. Another line intersects the two previous lines at the exact same point. The measure of $\angle 2$ is 57 degrees. What is the difference between $\angle 3$ and $\angle 1$?

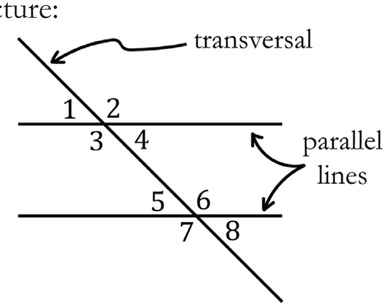


We know that $\angle 2$ and $\angle 1$ must add up to $90°$. This means that if $\angle 2$ is $57°$, $\angle 1$ must be $33°$. Furthermore, $\angle 3$ is the same as $\angle 2$ since they are vertical angles. We thus subtract: $\angle 3-\angle 1=57°-33°=24°$.

Answer: $24°$.

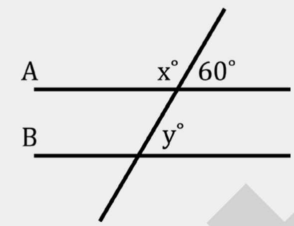
PARALLEL LINES THEOREM

The Parallel Lines Theorem says that alternate exterior or alternate interior angles are equal, corresponding angles are equal, and interior angles are supplementary. Reading that makes my head spin, so let's simplify and use a picture:



The transversal is a line that cuts through the two parallel lines. When it does so (and is not perpendicular to the parallel lines), it creates four equal small angles and four equal big angles. In other words, the big angles are all congruent (2, 3, 6, 7), and the small angles are all congruent (1, 4, 5, 8). The other thing to know is that any small angle in this picture added to any big angle will equal 180 degrees. As long as you understand these principles, you don't really need to worry about what "alternate interior angles" vs. "corresponding interior angles" means.

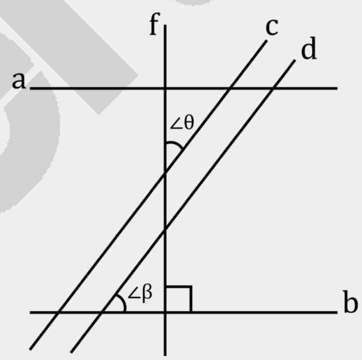
Lines A and B are parallel in the figure below. A transversal intersects the two lines and three angle measures are given in degrees. What is the value of $x - y$?



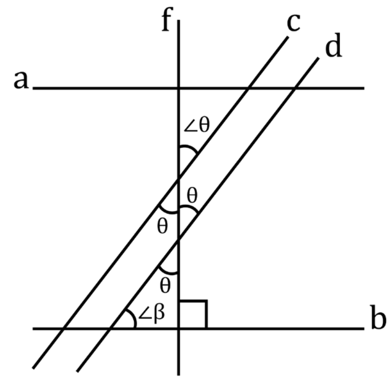
$x°$ is supplementary to 60 degrees so it must be 120 degrees. Furthermore, $y°$ is the same as 60 degrees due to the Parallel Lines Theorem. Therefore, $x - y = 120 - 60 = 60$.

Answer: $60°$.

In the diagram below, lines a and b are parallel with each other and are perpendicular to the horizontal line. Lines d and c are also parallel. Line f is perpendicular to line b. The measure of $\angle\beta$ is $48°$. What is the measure of $\angle\theta$?



Let's angle hop! First, θ is equal to its vertical angle, so we can label that with a θ. Next, we know d and c are parallel, so we know the corresponding angle next to line d is also equal to θ because of the Parallel Lines Theorem. Now we have θ and β in the same triangle together.



Next, we know the other angle in the triangle with θ and β must be 90 degrees, because the problem states line b is perpendicular to line f. A triangle's angle measure always add up to $180°$ so now we can use algebra to find $\angle 1$:

$$90+48+\theta=180$$
$$138+\theta=180$$
$$\theta=42$$

TRIANGLE ANGLES

The three angles of any triangle must equal $180°$.

> Triangle ABC has angles 37 degrees and 61 degrees. What is the third angle in the triangle?

To solve this problem, we can let x equal our unknown angle. We know the sum of all three angles, 37, 61, and x, is 180:

$$37+61+x=180$$

Now we simply solve:

$$98+x=180$$
$$x=82$$

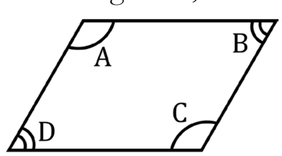
You might want to skim our chapter on Triangles as well before tackling the problem set in this chapter. **Isosceles triangles** are the most common element in problems that deal with angles, followed by the **Exterior Angle Theorem**.

POLYGON ANGLES

We'll go over Polygon Angles more in depth in the Polygons chapter.

For now: **quadrilaterals** have interior angles that sum to 360.

In **parallelograms**, the opposite angles are congruent, and interior angles are supplementary.



Here, $A+B=180$. $A+B+C+D=360$.

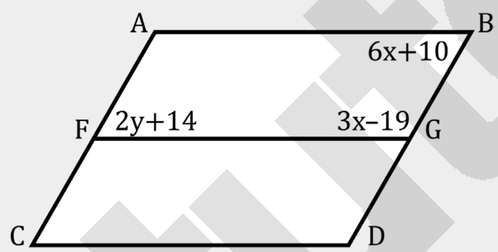
Also, because **parallelograms** are essentially **pairs of parallel lines**, each line is a transversal of sorts, so we can also imagine a parallelogram in the same way we do parallel lines.

Other polygons may also contain parallel lines. Remember that any transversals through these shapes (or even sides that intersect two parallel sides) can also be analyzed using rules for parallel lines as described earlier.

TIP: When in doubt, **extend the lines** of complex shapes or drawings with parallel lines so you can better see the transversal and the relationship between the angles.

In the **trapezoid** below, angles 1 and 2 are supplementary, as are angles 3 and 4. Extending the lines as shown helps us see that relationship.

Parallelogram $ABCD$ is shown below. If G bisects BD and F bisects AC, $\angle BGF = 3x - 19$, $\angle GBA = 6x + 10$, and $\angle AFG = 2y + 14$, what is the value of y?



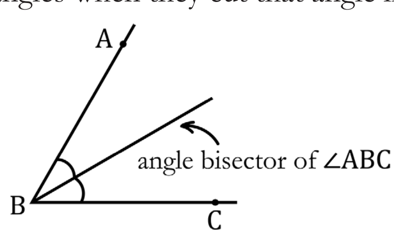
When we have bisectors of lines in a parallelogram, those bisectors create a line that is also parallel to the rest of the shape. We know this because of the ratios of the sides, a principle we'll discuss more in the chapter on similar shapes and ratios. For now, know that segment FG is parallel to AB and CD. We thus know that angles B and BGF are complementary, because of the parallel lines theorem. Thus $6x + 10 + 3x - 19 = 180$. We can solve this and get $9x - 9 = 180$, or $9x = 189$. Thus $x = 21$. We also know that angle ABG must equal angle AFG because opposite angles in a parallelogram are congruent, and side FG forms another parallelogram with side AB. Thus $6(21) + 10 = 2y + 14$. We can solve to get:

$$126 + 10 = 2y + 14$$
$$122 = 2y$$
$$y = 61$$

Answer: 61.

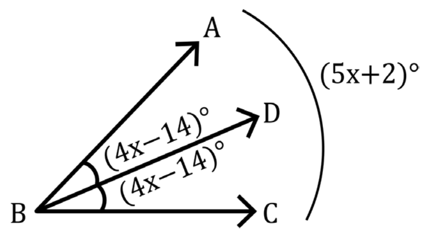
ANGLE BISECTORS

Lines or rays are said to "bisect" angles when they cut that angle in half.



Ray $\overrightarrow{BD}$ bisects $\angle ABC$, which measures $(5x+2)°$. The measure of $\angle ABD$ is $(4x-14)°$. What is the measurement of $\angle DBC$?

We can draw a picture to visualize this problem:



Since ray $\overrightarrow{BD}$ bisects the larger angle, we know that the two smaller angles are equal to each other. We can then set up an equation to solve for x .

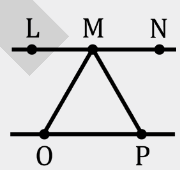
$$2(4x-14)=5x+2$$
$$8x-28=5x+2$$
$$x=10$$

Next, we plug in our x value into the angle expression for $\angle DBC$.

$$4(10)-14=26$$

Answer: 26.

In the figure below, M lies on $\overline{LN}$, $\overline{OM}$ bisects $\angle LMP$, and $\overline{MP}$ bisects $\angle NMO$. What is the measure of $\angle NMO$?



Line $\overline{LN}$ has an angle measure of $180°$. Since $\overline{OM}$ bisects $\angle LMP$ we can label $\angle LMO$ as x and $\angle OMP$ as x as well. Now, when we look at how $\overline{MP}$ bisects $\angle NMO$, we know that $\angle PMN$ must also equal x . We can now use algebra to solve for x , as we know $3x=180$ as the three x 's are along a straight angle; dividing 180 by 3, we get $60°$ as the value of x . However, x is not what we need. We actually need angle $\angle NMO$, which equals $2x$, or $120°$.

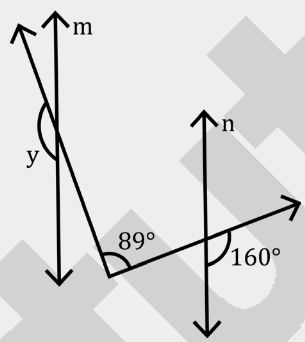
Answer: $120°$.

ANGLE HOPPING

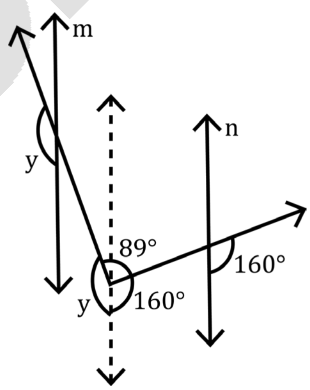
Angles and Lines problems often will require you to think creatively and use several of these rules at once. When in doubt:

1. Go through your list of ways you know how angles relate and check to see if any of those conditions apply (vertical angles, straight lines, circle central angles, right angles, parallel lines, triangles, polygons, exterior angle theorem, bisectors, similar shapes, etc.).
2. Draw additional lines to create transversals, additional parallel lines, triangles or polygons.
3. Extend lines that are parallel to better visualize them.
4. Redraw small portions of the diagram if too many lines exist and you're having trouble seeing the relationship between angles.

Lines m and n are parallel. Given the angles in the diagram below, find the measure of $\angle y$.



To solve this problem, we'll need to hop from line m to line n . The problem is we don't actually have a complete transversal here. To solve, we thus need to draw more lines, or at least one more line. One way we could do this would be to draw a parallel line through the point that is the vertex of the $89°$ angle as so:



When we do this, we see we get a "spindle" of three angles, one of which is $160°$, one of which is y degrees (both because of parallel lines and transversals, and the other is $89°$. Because these three angles must sum to $360°$, we know:

$$89+160+y=360$$
$$y=200-89$$
$$y=111$$

Answer: $111°$.