THE BEST ACT PREP COURSE EVER

TRIGONOMETRY GRAPHS

ACT Math: Lesson and Problem Set

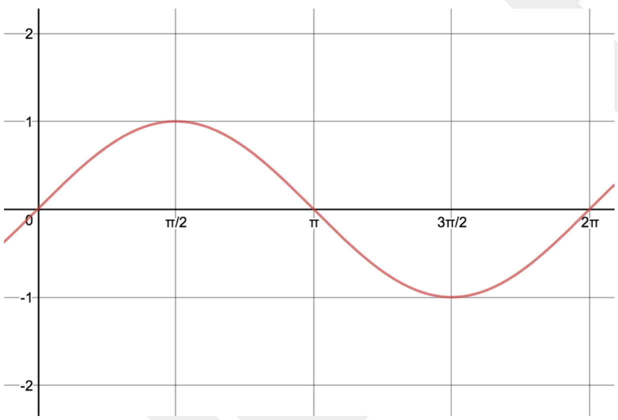
SKILLS TO KNOW

- Parent function forms of basic trigonometric functions
- Effects of basic transformations (scalars, coefficients, and constants)
- Graphs of inverse trigonometric functions
- Interpreting points on graphs of trigonometric functions
- Finding minimum and maximum values

GRAPHING BASIC TRIGONOMETRIC FUNCTIONS:

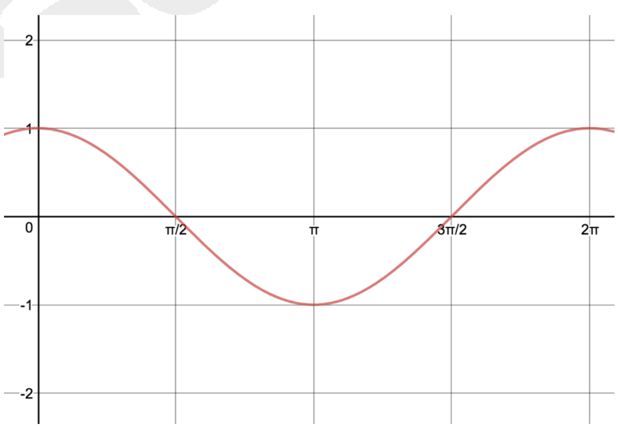
SINE FUNCTION:

The sine function, which is the ratio of the opposite to the hypotenuse, starts at 0 and moves between -1 and 1 in a sinusoidal pattern, with a period of 2π radians or 360°.



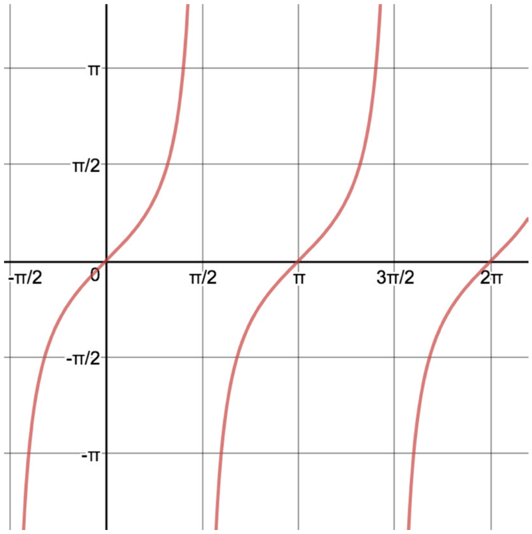
COSINE FUNCTION:

The cosine function, which is the ratio of the adjacent to the hypotenuse, starts at 1 and moves between -1 and 1 in a sinusoidal pattern, with a period of 2π radians or 360°.



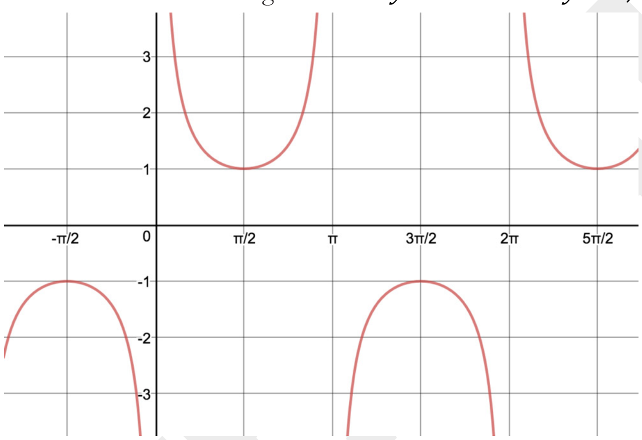
TANGENT FUNCTION:

The tangent function, which is the ratio of the opposite to the adjacent, starts at 0 and has a range of $-\infty$ to ∞, with a period of π radians or 180°.



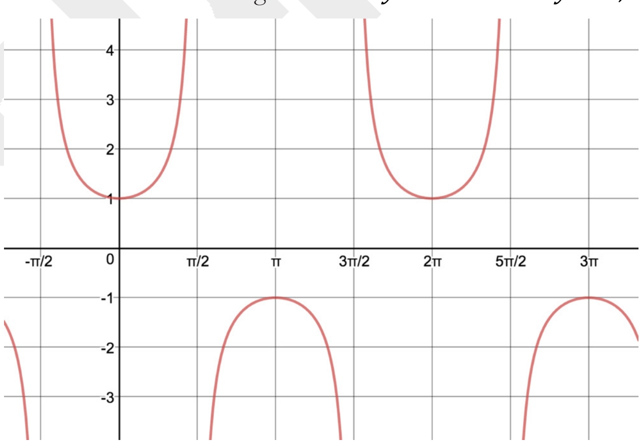
COSECANT FUNCTION:

The cosecant function is the reciprocal of sine, and therefore the ratio between the hypotenuse and the opposite. It starts at ∞ and has a range of $-\infty < y \leq -1$ and $1 \leq y < \infty$, with a period of 2π radians or 360°.



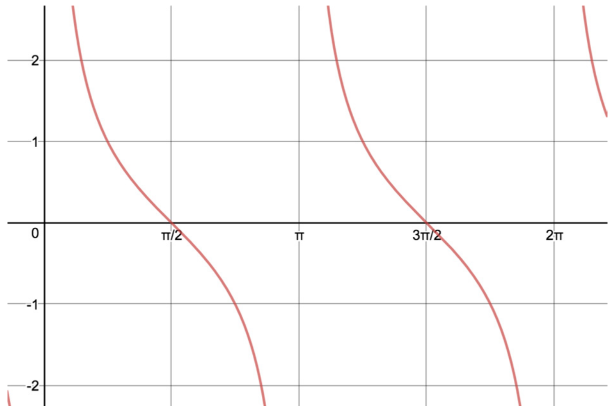
SECANT FUNCTION:

The cosecant function is the reciprocal of cosine, and therefore the ratio between the hypotenuse and the adjacent. It starts at 1 and has a range of $-\infty < y \leq -1$ and $1 \leq y < \infty$, with a period of 2π radians or 360°.



COTANGENT FUNCTION:

The cosecant function is the reciprocal of cosine, and therefore the ratio between the adjacent and the hypotenuse. It starts at ∞ and has a range of $-\infty$ to ∞ with a period of π radians or 180°.



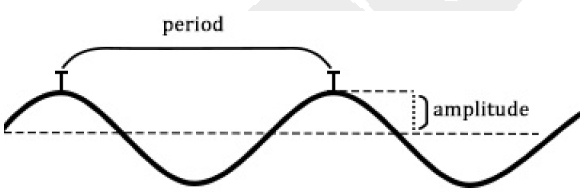
BASIC TRIGONOMETRIC TRANSFORMATIONS:

Trigonometric graphs may seem like a very difficult or confusing concept, but in reality, it is very similar to normal graph translations. However, there are a few different terms to learn and some tricks to ensure your graph and translation is correct.

	sin	cos	tan
Parent Equation	$y = a\sin b\left(x-h\right)+k$	$y = a\cos b\left(x-h\right)+k$	$y = a\tan b\left(x-h\right)+k$
Amplitude	a	a	undefined
Period	$\frac{2\pi}{b}$	$\frac{2\pi}{b}$	$\frac{\pi}{b}$
Phase Shift	h	h	h
Vertical Shift	k	k	k

(Note: the periods for sine and cosine are the same, but NOT for tangent!)

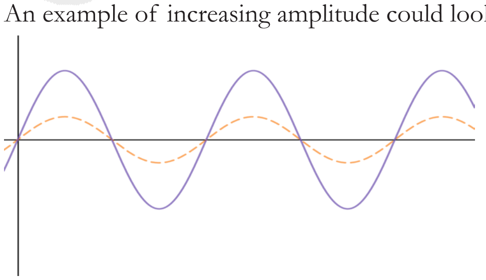
We can compare this to normal transformations: $y = m(x-h)+k$. As you can see, it's extremely similar.
Below is a basic diagram of the important terms for trigonometric graphs:



SCALAR MULTIPLICATION:

One common transformation is the multiplication of a trigonometric function by a scalar quantity. For sinusoidal functions, this has the effects of altering the amplitude of the wave, but not changing the period or overall behavior.

Amplitude: the distance from the maximum and minimum point from the center of the trig graph (vertical stretch for normal graphs).
An example of increasing amplitude could look like this:

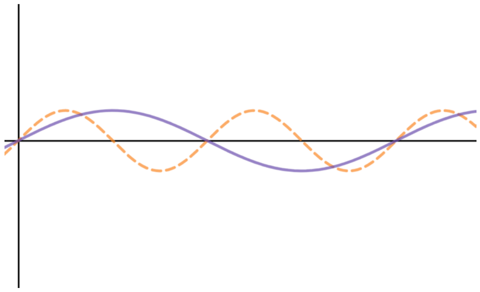


Notice how the solid line graph is "taller"?

The easiest way to finding the amplitude is to count up to the crest. If there has been a vertical shift, check where the "midline" of your function is. That is, if a graph has a transformation from $y = \cos(x)$ to $y = 3\cos(x)$, we can plot the points like we did with the vertical transformation example.

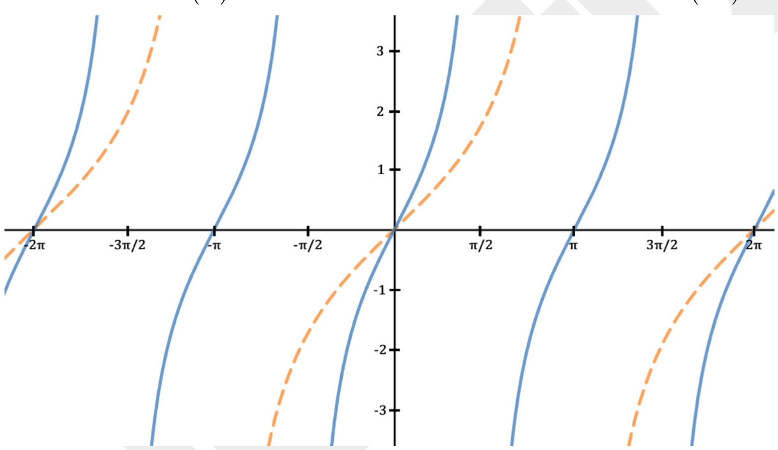
COMPRESSION/EXPANSION THROUGH COEFFICIENTS:
By multiplying the argument of the function by a coefficient, we can compress or stretch the graphs. This alters the period of the function but not the amplitude.

Period: the distance between each peak or crest of the curve (horizontal stretch for normal graphs).



Notice how the solid line graph is "wider"?

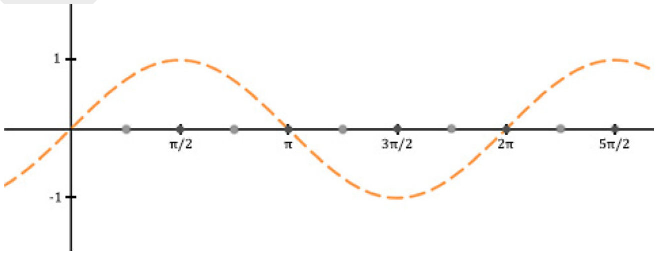
The period is determined by how many time a graph repeats itself within 0 to 2π on the x-axis.
For example, $\tan(x)$ (dashed line) has a period of π, while $\tan(2x)$ (solid line) has a period of $\frac{\pi}{2}$.



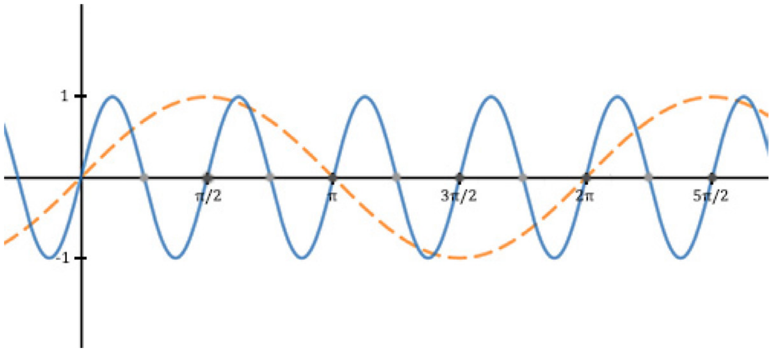
How do you determine period and how do we graph it?
Let's do a transformation from $y = \sin(x)$ to $y = \sin(4x)$.

Having an period of 4, we know that there will be 4 crests within the span of 2π. Let's take a point that is easily visible and divisible by 4 to do our transformation: $(2\pi, 0)$

To be accurate, we can plot the points like we did with the vertical transformation example.



The dashed line is our original equation, $y = \sin(x)$. We should dot $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π first, because that is when there is a full cycle. Then, dot halfway between each dark dot to label every time the trig function will hit x=0. Then start drawing the graph to make:

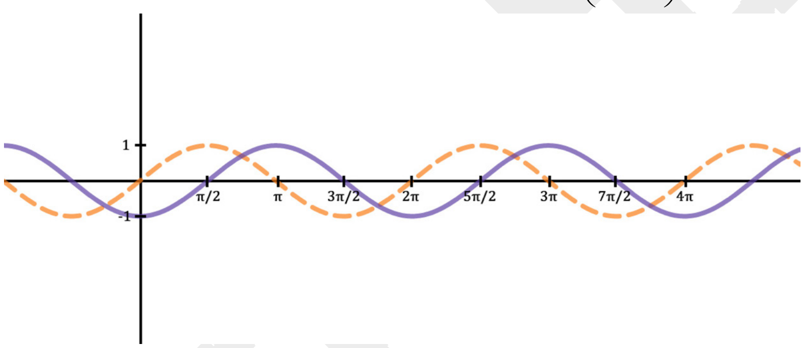


Notice how at every dark dot, the sine graph makes a full wave and restarts? And that there are 4 complete cycles by the time the function is 2π?

LATERAL TRANSFORMATIONS: PHASE SHIFT

By adding a constant within the argument or after the trig function, you can shift the functions both horizontally and vertically, just like with parabolas. The only difference is that it's called a "phase shift" instead of a lateral or horizontal shift. This does not affect the amplitude or period of the function.

For example, this graph shows a phase shift of $y = \sin\left(x - \frac{\pi}{2}\right)$ from $y = \sin(x)$.

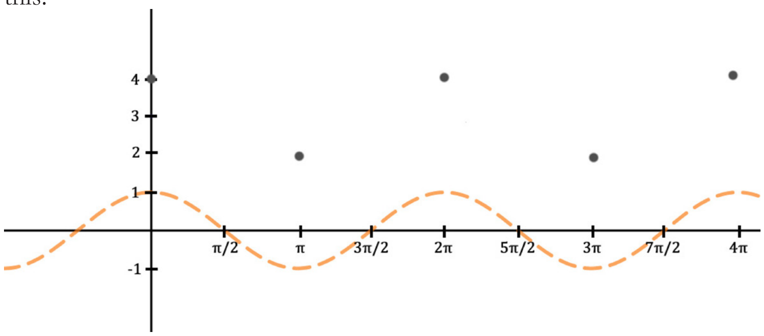


The dashed line is $y = \sin(x)$ and the solid line is $y = \sin\left(x - \frac{\pi}{2}\right)$.

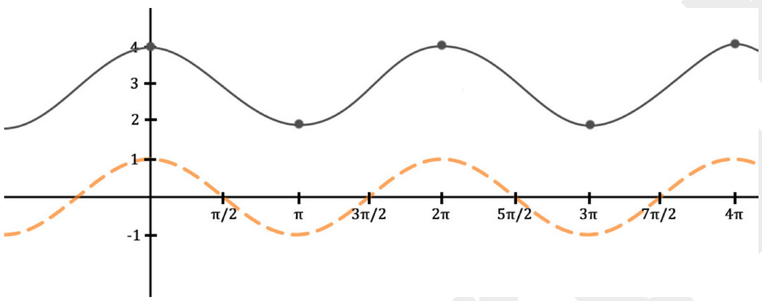
As we can see in the graph, the units for trigonometric graphs are different from the regular graph. This is because as we learned in the unit circle, these trig functions reach whole numbers at intervals of π.

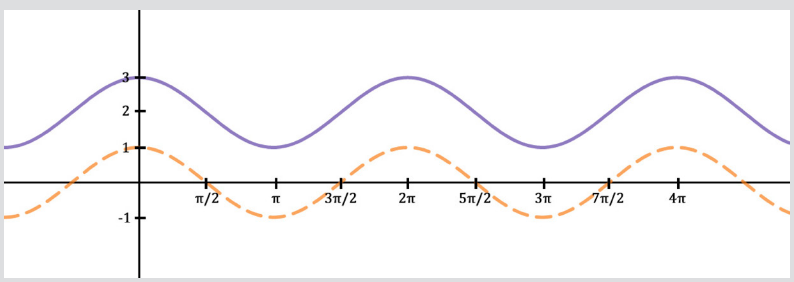
VERTICAL TRANSFORMATIONS: VERTICAL SHIFT:

Like normal graphs, vertical shift is simply up or down; there's no weird units or compressions. A safe way to make sure you don't mess up the curves is to pick points and shift them before drawing the entire line. To illustrate, you can draw the points of all the maximums and minimums of each crest before drawing the entire line. This ensures your shift is correct and your curves will look pretty. If we want to make $y=\cos(x)$ become $y=\cos(x)+3$, it can look like this:



We plot the points of the maximum and minimum of each crest to make sure we will have the right graph. Then we can draw the actual line:





The original equation, $y=\cos(x)$, is seen in the dashed line. What would be the transformed equation?

A. $y=\cos(x)-1$

B. $y=\cos(x)-2$

C. $y=\cos(x+1)$

D. $y=\cos(x)+2$

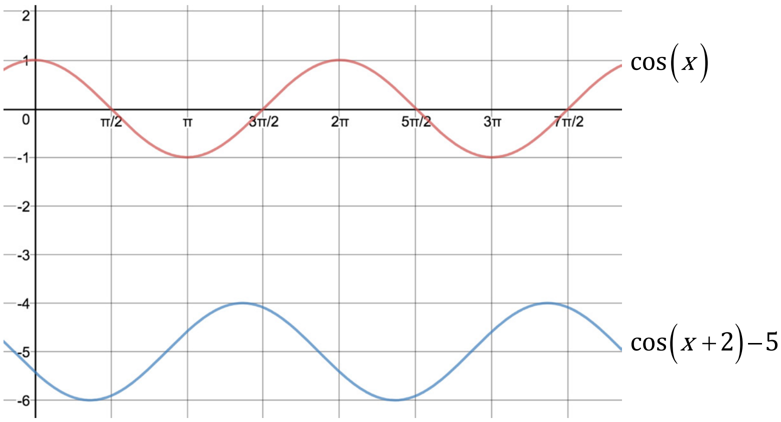
E. $y=\cos(x+2)$

Answer: **D.**

We can count that the graph moved up 2 units, so the new equation would be $y=\cos(x)+2$.

COMBINING VERTICAL AND LATERAL SHIFTS:

For example, when comparing $\cos(x)$ with $\cos(x+2)-5$, the second function has been shifted 2 units left and 5 units down.

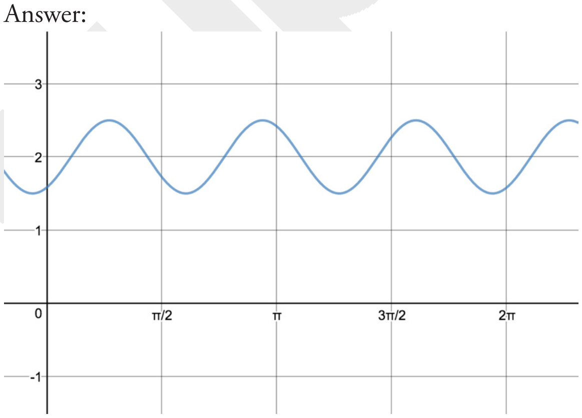


PUTTING IT ALL TOGETHER:

For the ACT, the easiest way to do trigonometric transformations is with your calculator. You can simply plug in answer choices or given information into your calculator to find the answer.
Let's try an example with every possible transformation:

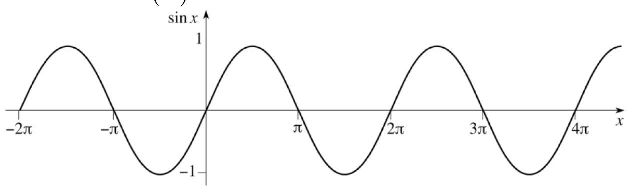
What is the transformation from $y=\sin(x)$ to $y=0.5\sin(3x-1)+2$?

Answer:



Calculator tip: Be sure to make the most efficient use of your calculator so you don't run out of time!

1. The following graph of $y=\sin(x)$ is shown in the standard (x,y) coordinate plane below. What is the period of $\sin(x)$?



A. $\frac{\pi}{4}$
B. $\frac{\pi}{2}$
C. π
D. $\frac{3\pi}{2}$
E. 2π

2. If x, y, z, t, and w represent positive real numbers what is the minimum value of the function $f(x)=w\ [\sin t(x-y)]+z$?

A. $wx-wy+z$
B. $w\sin(tx)-w\sin(ty)+z$
C. z
D. $w+z$
E. $z-w$

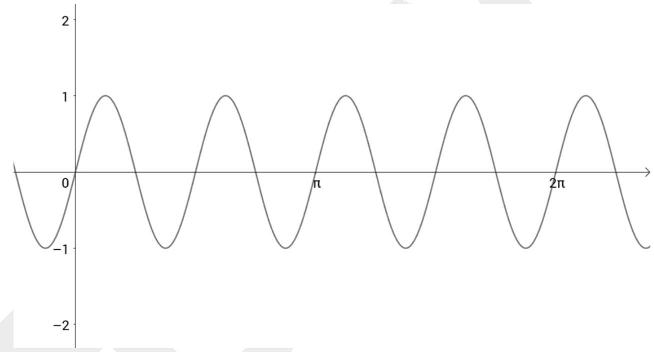
3. In the standard (x,y) coordinate plane, what is the range of the function $f(a)=-7\cos 3(a+4)+5$?

A. $-12\le f(a)\le 2$
B. $-2\le f(a)\le 12$
C. $5-14\pi\le f(a)\le 5$
D. $0\le f(a)\le 2\pi$
E. $-2\le f(a)\le 5$

4. What is the amplitude of the graph of the equation $y-4=\frac{1}{2}\cos\frac{7\alpha}{9}$?

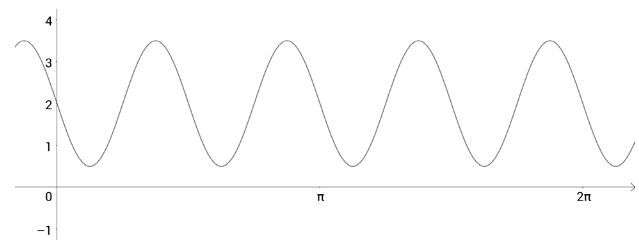
A. $\frac{7a}{9}$
B. 2
C. 1
D. $\frac{1}{2}$
E. 4

5. A trigonometric function with equation $y=\cos(bx+c)$ where b and c are real numbers, is graphed in the standard (x,y) coordinate plane below. The period of this function $f(x)$ is the smallest possible number p such that $f(x+p)=f(x)$ for every real number x. One of the following is the period of the function. Which is it?



A. $\frac{\pi}{2}$
B. π
C. 2π
D. 1
E. 2

6. The graph of $y=-a\sin(bx)+c$ is shown below for certain positive values of a, b, and c. One of the following value is equal to a. Which is it?



A. $\frac{1}{2}$
B. $\frac{\pi}{2}$
C. $\frac{3}{2}$
D. 2
E. $\frac{7}{2}$

7. Which of the following trigonometric functions has an amplitude of $\frac{1}{3}$?

A. $f(x) = 3\tan\left(x + \frac{1}{3}\right)$

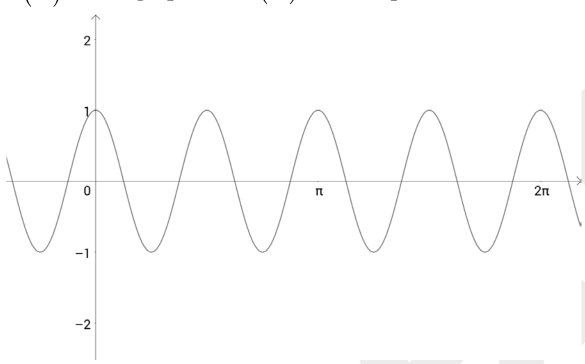
B. $f(x) = \frac{1}{3}\tan(x)$

C. $f(x) = \frac{1}{3}\cos(x)$

D. $f(x) = 3\sin(x)$

E. $f(x) = \sin\left(\frac{1}{3}x\right)$

8. The graph of $f(x) = \sin x$ and $g(x) = \sin\left(x - \frac{3\pi}{2}\right) - 2$ are shown in the standard (x, y) coordinate plane below. After one of the following pairs of transformations is applied to the graph of $f(x)$, the image of the graph of $f(x)$ is the graph of $g(x)$. Which pair is it?



A. Phase shift $\frac{3\pi}{2}$ units to the right, and vertical translation 2 units down.

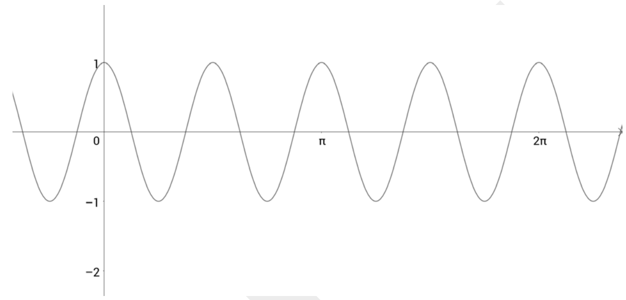
B. Phase shift $\frac{3\pi}{2}$ units to the left, and vertical translation 2 units up.

C. Phase shift 2 units to the right, and vertical translation $\frac{3\pi}{2}$ units down.

D. Phase shift 2 units to the left, and vertical translation $\frac{3\pi}{2}$ units down.

E. Phase shift $\frac{3\pi}{2}$ units to the right, and vertical translation 2 units up.

9. For the function graphed below, the x-axis can be partitioned into intervals, each of length p radians, and the curve over any one interval is is a repetition of the curve over each of the other intervals. What is is the least possible value for p, the period of the function?



A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. π

D. 2π

E. 4π

10. If a, b, c, d, and x represent positive real numbers, what is the maximum value of the function $f(x) = a\left[\sin(b(x + c)\right] + d$?

A. $a - d$

B. $d - a$

C. $a + d$

D. d

E. a

11. The domain of function $y(x) = \frac{\cos(3x+1)}{2} + 4$ is all real numbers. Which of the following is the range of function $y(x)$.

A. $0 < y < 4$

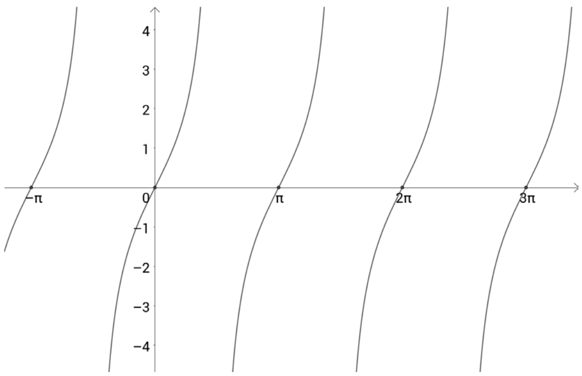
B. $\frac{7}{2} < y < \frac{9}{2}$

C. $2 < y < 4$

D. $-\frac{1}{2} < y < \frac{1}{2}$

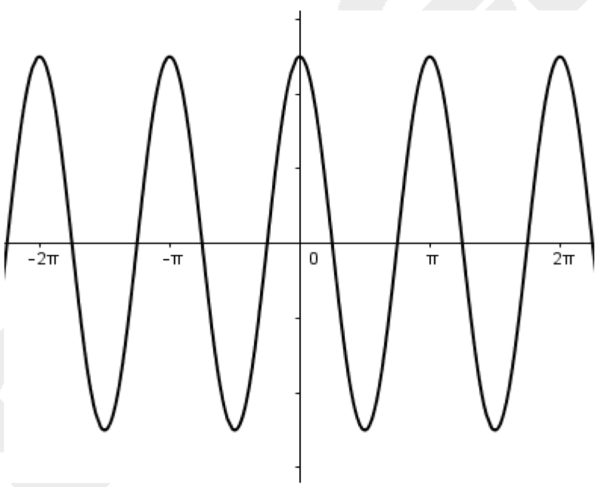
E. $2 < y < 6$

12. The graph of $y = 2\tan x$ is shown in the standard coordinate plane below. What is the period of $2\tan x$?



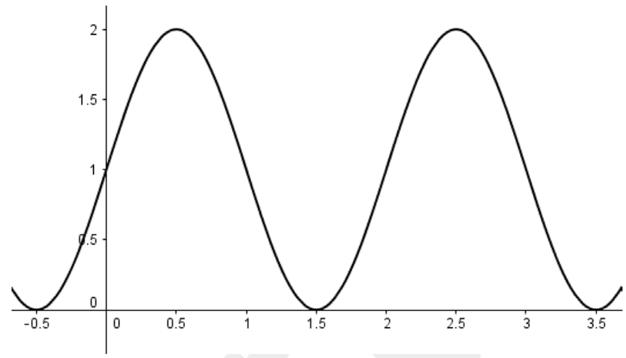
A. $\frac{1}{2}$
B. $\frac{\pi}{2}$
C. π
D. 2
E. 2π

13. A trigonometric function with equation $y = a\cos bx + c$, where a, b, and c are real numbers, is graphed in the standard (x, y) coordinate plane below. The period of this function is the smallest positive number p such that $f(x+p) = f(x)$ for all real numbers x. What is the period of the following function?



A. 1
B. $\frac{\pi}{2}$
C. π
D. 2π
E. 2

14. The graph below could represent which of the following equations?



A. $\sin(\pi x + 1)$
B. $\pi \sin(x+1)$
C. $\sin(x)+1$
D. $\sin(2\pi x)+1$
E. $\sin(\pi x)+1$

15. If $0 \le x \le \pi$, what is the maximum value of the function $f(x) = -2\sin x$?

A. -2π
B. -2
C. 0
D. 2
E. 2π

ANSWER KEY

1. E **2. E** **3. B** **4. D** **5. A** **6. C** **7. C** **8. A** **9. B** **10. C** **11. A** **12. C** **13. E** **14. E** **15. C**

ANSWER EXPLANATIONS

1. **E.** The period of a graph is the length of one cycle of the curve. Since $\sin 0 = 0$ and $\sin 2\pi = 0$, the cycle starts at 0 and ends at 2π. Thus the period is $2\pi - 0 = 2\pi$.

2. **E.** We know that the range of $\sin x$ is from -1 to 1. In order to minimize the value of the entire function, we want that part of the equation, the sine term, to be its minimum possible value. $\sin t(x-y)$ at its very minimum is equal to -1. It doesn't matter what our actual angle in, because we know that $\sin x$ equals -1 many times, but never is any lower. Substituting, this leaves us with $f(x) = w(-1) + z$. So at it's minimum, $f(x) = z - w$.

3. **B.** We know that the range of $\cos x$ is from -1 to 1. So, we can find the range of the function by plugging in the maximum and minimum possible values of $\cos 3(a+4)$, which regardless of the angle will have those values as it's minimum and maximum. Plugging in $\cos 3(a+4) = -1$, we get $-7(-1)+5 = 12$. Plugging in $\cos 3(a+4) = 1$, we get $-7(1)+5 = -2$. So the range is $-2 \le f(a) \le 12$.

4. **D.** A function represented in the form $y = a\cos(bx - c) + d$ has an amplitude of a. In this problem, the function $y - 4 = \frac{1}{2}\cos\frac{7\alpha}{9}$ has amplitude $\frac{1}{2}$.

5. **A.** The question background is complicated and honestly not worth untangling. What you do know is that you have a graph and can analyze it visually. If you trace one full cycle of the graph, you see that it completes a cycle in $\frac{\pi}{2}$ units, so that is the answer.

6. **C.** The negative sign does not affect the answer, because the question is asking for the amplitude. The amplitude is the the difference between the maximum and the minimum point the graph reaches, divided by two. It can also be though of as how far above/below the graph reaches from the central axis. In this case, the graph reaches a maximum at what looks like $\frac{7}{2}$ and a minimum at $\frac{1}{2}$, so the amplitude is $\frac{\frac{7}{2}-\frac{1}{2}}{2} = \frac{\frac{6}{2}}{2} = \frac{3}{2}$.

7. **C.** In a trigonometric equation, the amplitude is the coefficient of the cosine or sine function. Only C has $\frac{1}{3}$ in front of a cosine or sine function.

8. **A.** In this case it's better to look at the equations given rather than the graph. Trigonometric translations are actually the same as regular graph translations. The difference between $f(x)$ and $g(x)$ is that $\frac{3\pi}{2}$ is subtracted from the x term of $g(x)$, so we get a translation (or phase shift) $\frac{3\pi}{2}$ to the right, and in $g(x)$, 2 is subtracted from the entire equation, which yields a vertical translation 2 units down.

9. **B.** Starting from the relative maximum where $x = 0$, the curve repeats 2 times, finished it's second interval at $x = \pi$. If it repeats twice in the interval $[0, \pi]$, then the interval must be $\frac{\pi}{2}$.

10. **C.** a is the amplitude constant, and d is essentially our central axis, so the maximum is found by adding the amplitude to the central axis. This gives $d + a$, which is answer C.

11. **B.** Rewrite the function in your head like this: $y(x)=\frac{1}{2}\cos(3x+1)+4$. The minimum value a sine or cosine function can equal is the central axis, which in this graph is 4 minus the amplitude, which in this graph is $\frac{1}{2}$, and the maximum of a sine or cosine function is the sum of these two values. The range is $minimum < y < maximum$, so for this function the range is $4-\frac{1}{2}<y<4+\frac{1}{2}$, which is $\frac{7}{2}<y<\frac{9}{2}$.

12. **C.** The period is only affected by a coefficient to the x term, not the entire equation, so this graph has the same period as the parent tangent graph, which is π.

13. **C.** The period of a function is how far a graph travels along the y-axis before it repeats itself. In this case, the cosine graph reaches its peak every π units in a recurring pattern. Thus, its period is also π.

14. **E.** The most obvious change to the graph is that it has been shifted upwards. We can tell it has been shifted up by 1 since its new maximum is 2, while the maximum of the parent function, $\sin x$, is 1. However, note that the period of this function is 2, not 2π. The period of $\sin(ax)$ is equal to $\frac{2\pi}{a}$. Since here $\frac{2\pi}{a}=2$, $a=\pi$. Thus, our function is $\sin(\pi x)+1$.

15. **C.** The parent graph $\sin x$ is flipped upside down by the negative sign in front of -2, and we only go to $x=\pi$, which means that we stop before the graph would go above the x-axis. Thus, our maximum value is 0.