CHAPTER

7

SEQUENCES AND SERIES

SKILLS TO KNOW

- Sequence vs. Series
- Arithmetic Sequences
- Geometric Sequences
- The Sum of a finite Arithmetic Sequence (Arithmetic Series)
- Miscellaneous Patterns

A **sequence** is a string of numbers, possibly going on infinitely, that follows a strict pattern. The first number in the sequence is the first term, the second is the second term, and so on.

A **series** is the sum of all of the numbers in a sequence.

One kind of sequence is the **arithmetic sequence**. It is a very simple kind of sequence. You can probably figure it out just by looking at a few. In this type of sequence, every term increases by a constant amount. In other words, to move up one value in the sequence, you add a certain number. Then you add that number again to reach the next value in the sequence.

Let's look at a few arithmetic sequences:

$$0, 2, 4, 6$$

Here's another:

$$a_3 = \{6, 12, 18, \ldots\}$$

And one more:

$$\{5, 8, 11, 14, \ldots\}$$

As you can see, in an arithmetic sequence every term increases by a constant amount. The first sequence started with 0 and increased in increments of 2. The second sequence started with 6 and increased in increments of 6. The third sequence started with 5 and increased in increments of 3. We call these increments the **common difference**. Arithmetic sequences can also subtract an equal increment or use increments that are fractions or decimals.

$$1.5, 1, .5, 0, -.5, -1\ldots$$

The arithmetic sequence above, for example, the common difference is -0.5: we subtract 0.5 to find each subsequent term.

Sometimes, arithmetic sequence problems are easy enough to do by simply working forward with the understanding of the pattern at play.

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What is the fifth term in the arithmetic sequence $\{3,8,13,\ldots\}$?

Here, all we need to do is figure out how much we increase each term by, and then work forward two more numbers in the sequence. To get from 3 to 8 we add 5 (since $8-3=5$), so 5 is our common difference. We can now calculate the 4th and 5th terms:

4th term: $13+5=18$

5th term: $18+5=23$

Answer: 23.

However, if you are looking for, say, the 450th term, it would be tedious to add the same number 449 times. For cases such as this, you'll need to either know the formula or extrapolate the pattern of the sequence and apply that pattern.

Remember that the sum of a single number (let's use d for **common difference**) added together x times $\overbrace{\left(d+d+d+d+\ldots+d\right)}^{x\text{ times}}$ is the same thing as dx. Since we are adding the same number (our **common difference**) over and over, we can use multiplication to speed up the process. If the first term is a_1, and we increase by increments of d, we can express the n^{th} **term in the sequence** (a_n) as:

FORMULA FOR THE n^{th} TERM OF AN ARITHMETIC SEQUENCE

$$a_n = a_1 + (n-1)d$$

Where a_n is the n^{th} term in the sequence, a_1 is the first term in the sequence, n is the number of the term in the sequence, and d is the common difference.

We use $n-1$ because we only add "d" for the first time in reaching the second term. We'll add a 2nd "d" when we get to the third term. A 3rd "d" when we get to the fourth term. Therefore, we must subtract one from the number term we are on to figure out how many "d's" we have added.

To see why this works, let's look at an example.

In the arithmetic sequence, $3,7,11,15,\ldots$, what would be the 21st term?

A. 86 **B.** 83 **C.** 80 **D.** 4 **E.** 19

First, I'll say that I really am NOT a huge fan of formulas. I find them difficult to remember and am constantly confusing them. On questions like these, I find the best way to learn the formula is to understand where it comes from so that I can derive it at will. After deriving it enough times, I "see" the formula because of how it works and then apply it correctly. Most of the time, in fact, I'm using the ideas of the formula without actually using the formula itself.

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For this question, I'll start by punching out a chart that goes through each term and tracks the pattern. Remember n is used to simply keep track of which number in the sequence I'm on. I'll break each piece down into what I'm adding:

n	Value of the term	Calculation of the term	What's happening
1	3	3	First term
2	7	$3+4$	First term plus 1×4
3	11	$3+4+4$	First term plus 2×4
4	15	$3+4+4+4$	First term plus 3×4

At this point, I see the pattern: I add the first term to 4 times one less than n.

When $n=2$, for example, I have only one 4 to add. When $n=3$, I add two 4's. So for the 21st term, I'll add 20 4's:

$$3+20(4)=83$$

I can also solve this directly using the formula. Again, the more I write out a chart like this, the better I get at remembering that formula, too. If you're aiming for a 32+ on the math, mastering the formula after understanding why it works is a good idea.

We are looking for the **21st** term, so $n=21$. Our **first term** is 3, so $a_1=3$. The **common difference** between each term is 4, so $d=4$. Plugging all these in:

$$a_n=a_1+(n-1)d$$

$$\begin{aligned}3+(21-1)4&=3+(20)4\\&=3+80\\&=83\end{aligned}$$

Answer: **B**.

GEOMETRIC SEQUENCE

A geometric sequence is to the arithmetic sequence what multiplication is to addition. That is, instead of adding by something every term, we multiply by something every term. We call this multiplication increment the **common ratio**, often denoted by the variable r.

Here's an example of a simple geometric sequence:

$$g_1=\{1,2,4,8,16,\ldots\}$$

The first term is 1 and every subsequent term is multiplied by 2. 2 is our **common ratio**.

Let's take a look at another example, and try to find a relationship between our original number and those that follow. If we can find that pattern, we'll see how to create a formula to solve for a term in a geometric sequence.

$$2,6,18,54,162,\ldots$$

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Let's list out each term in a chart of sorts and try to deduce what is going on at each step:

n	Value of the term (a_n)	Calculation of the term	What's happening
1	2	2	First term
2	6	$2(3)$	When $n=2$, multiply first term by 3 once
3	18	$2(3)(3)$	When $n=3$, multiply first term by 3^2 once
4	54	$2(3)(3)(3)$	When $n=4$, multiply first term by 3^3 once
5	162	$2(3)(3)(3)(3)$	When $n=5$, multiply first term by 3^4 once
n	a_n	$a_1\left(3^{n-1}\right)$	Multiply first term, a_1, by 3^{n-1}

Over time, I realize that the exponent of 3 is always one less than my term n. That's where the $(n-1)$ comes from. I could find this pattern and then apply it to a big number, or write out the formula.

FORMULA FOR THE n^{th} TERM OF A GEOMETRIC SEQUENCE

To find the n^{th} term in a geometric sequence, where a_n is the n^{th} term in the sequence, a_1 is the first term in the sequence, and r is the common ratio (what we multiply each subsequent term by):

$$a_n = a_1 r^{n-1}$$

What is the 8th term of the geometric sequence $-1, 3, -9, 27, \ldots$?

A. 33 **B.** –35 **C.** –63 **D.** –81 **E.** 162

Using the formula above, we can plug in numbers and find the answer. We know we're looking for the **5th term**, so $n=5$, and that the **first term** is -1, so $a_1=-1$. The **common ratio**, or amount we multiply by each time is -3, so $r=-3$.

$$a_n = a_1 r^{n-1}$$

$$\begin{aligned} a_n &= -1*(-3)^{5-1} \\ &= -1*(-3)^4 \\ &= -1*81 \\ &= -81 \end{aligned}$$

Answer: **D**.

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ARITHMETIC SERIES (OR THE SUM OF AN ARITHMETIC SEQUENCE)

Sometimes, a question or word problem will ask you to find the sum of a certain number of terms in an arithmetic sequence. This sum is called an **arithmetic series**.

This, too, has a formula, but it's actually one of the easiest ones to remember, because essentially, it's the same as the **average formula**.

> Amelia is saving for a bicycle. She saves $\$5$ the first day. She saves an additional $\$6$ the second day, an additional $\$7$ the third day, and so on, such that each day she increases her daily addition to her savings by $\$1$. If she saves money in this fashion for 25 days, saving an additional $\$29$ on the 25th and final day once she reaches her savings goal, how much money does Amelia save, in total, for her bicycle?

To answer this question, we could try to count out every day, but that would be time consuming:

$$\$5+\$6+\$7+\ldots+\$29$$

I would have to type in 25 different numbers into my calculator.

Another way we could do this would be by stacking our list and finding a common sum:

$$\begin{array}{cccccc} 5 & 6 & 7 & 8 & & 16 & 17 \\ \underline{+29} & \underline{+28} & \underline{+27} & \underline{+26} & \ldots & \underline{+18} & \underline{+(0)} \\ 34 & 34 & 34 & 34 & & 34 & 17 \end{array}$$

As you can see, I've stacked the highest term with the lowest term, and counted down on my bottom list as I count up on my top list. I have to figure out then how I "turn the corner," which I do by dividing 34 by 2, getting 17. Because 17 is a whole number, I know I'll turn the corner with the pair $16/18$, have 17 by "itself" and then continue the pattern. If I had a midpoint that was 0.5 (such as 17.5) I know I'd be splitting 17 & 18 as I "turn" the corner and move from top to bottom as I count.

Seeing this pattern, I know there will be 12 pairs (to find the term from the $$ amount, I simply subtract 4 given the original pattern, or I cut 25 in half, and know there are 12 pairs and 1 outlier at the corner turn), plus the outlier, 17.

So I calculate as follows:

$$34(12)+17=425$$

From this I have our answer.

But we can move much faster than we did here if we understand that the principle behind this "stacking" is the idea of averages. Each "pair" we chose was equidistant from our mean/median of the data set. Because the two values are the same "distance" apart (remember we add the common difference to each term, so the spacing is always identical), when we add them together, they create a common sum. They also create a common average, that also happens to be the average of the whole

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set: if we divide by two with any of the pairs in the above we get 17: that's the mean and the median of our data. We can see that as 17 is the number we hit when we "round the corner" from the top list to the bottom list.

Fun fact: in an arithmetic sequence, the **mean** and the **median** are the same amount.

Essentially, we know that the average of the first and the last term is also the average of the set as a whole. Armed with that information, we can find the sum much more quickly. All we need is the first term and the last term. Then we can average those two to find the average of the entire set. From there, we can use the average equation.

$$Average = \frac{Sum}{Number\ of\ items}$$

Now we can solve for the sum, clearing the fraction to get what we want in terms of the sequence:

$$Average\left(Number\ of\ items\right) = Sum$$

The average is the first term plus the last term divided by 2, so we can substitute that in:

$$\left(\frac{\overbrace{first\ term + last\ term}^{Average\ of\ first\ and\ last\ terms}}{2}\right)\left(Number\ of\ items\right) = Sum$$

We can rearrange this to form the formal equation (though memorizing the above is probably easier and works just fine!)

THE SUM OF THE FIRST n TERMS OF AN ARITHMETIC SEQUENCE

$$\boxed{S_n = \frac{n\left(a_{1+}a_n\right)}{2}}$$

Where S_n = sum of the first n terms of arithmetic sequence, n = number of terms in the sequence, a_1 = first term in the sequence, and a_n equals the n^{th} term in the sequence.

And now we have a formula for the sum of an arithmetic sequence! We plug in the first and last terms, 5 and 29 and the number of items in the list, 25:

$$\left(\frac{5+29}{2}\right)\left(25\right) = Sum$$

$$\left(17\right)\left(25\right) = 425$$

As you can see, this is the same answer we got above.

Answer: 425

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The second term in an arithmetic sequence is 63 and the seven term is 78. What is the common difference of the sequence?

For this problem, we first need to know vocabulary, including arithmetic sequence and common difference. You should know these terms as we reviewed them above. The common difference (we'll call this d) is the amount we add each time to move up one term in the sequence. For each additional term, we add d. We can think of this as follows:

		+d	+d	+d	+d	+d
	63	___	___	___	___	78
term:	2nd	3rd	4th	5th	6th	7th

As you can see, we'll add d five times to get to the 7th term from the 2nd one.

Common mistakes people make here:

- They look at the four blanks and assume that means they add d four times (you need to add it FIVE times though, look at the "hops" on top, not the blanks).
- They assume the pattern starts at the first not the 2nd term
- They don't draw it out, and in doing so, get confused or are off by one term (again, adding d four times or perhaps six times). Drawing the problem out helps prevent carelessness.

From the picture, we create an algebraic formula and solve:

$$63+5d=78$$
$$5d=15$$
$$d=3$$

Answer: 3.

Again, we could have also used the formula for this, but I am not a huge fan of formulas unless I need them. In this case, the formula method actually takes a bit longer. Also, formulas are too easy to remember incorrectly! Still, if you prefer to use the formula, the solution that way looks something like this:

$$a_n=a_1+(n-1)d$$

We know that when $n=2$, $a_n=63$. Plugging in we get:

$$63=a_1+(2-1)d$$
$$63=a_1+d$$

We also know that when $n=7$, $a_n=78$. Plugging in we get:

$$78=a_1+(7-1)d$$
$$78=a_1+6d$$

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Now we have two equations and two unknowns (a_1, d). We can solve this system of equations by elimination or substitution. Because I know I want d, I will solve for a_1 to eliminate it (remember, isolate the variable you want to eliminate.)

$$63 = a_1 + d$$
$$63 - d = a_1$$

Now I substitute this value in for a_1 in my other equation:

$$78 = a_1 + 6d$$
$$78 = (63 - d) + 6d$$
$$78 = 63 + 5d$$
$$15 = 5d$$
$$3 = d$$

Again, I get the answer of 3 (but hopefully you're convinced that this is the long way!).

REMAINDER PROBLEMS

Another type of problem that occurs is a sometimes called a remainder problem. For these problems, a pattern will be established and you'll need to find a specific term (for example, perhaps the 100th term) or which element of the pattern occurs at some particular point in the future.

> Jemal is making a friendship bracelet that follows the pattern of one white bead, one red bead, one yellow bead, one green bead, one orange bead, and then one blue bead. If he follows this pattern in this order, what will be the 35th bead in the sequence?

Here we have a pattern that repeats every six terms. I can write out the pattern using variables for each bead:

1	2	3	4	5	6
W	R	Y	G	O	B

When I get to the 7th term I go back to white:

7	8	9	10	11	12
W	R	Y	G	O	B

As you can see, I am always going to end my row of a pattern with a number that is divisible by 6, because these are in groups of 6. So we call this a "Remainder" problem, because I can solve by dividing the number of the term by 6 and finding the remainder.

35 divided by 6 is 5 remainder 5.

I can also think of this as hitting a multiple of 6 at the number 30:

					30
W	R	Y	G	O	B

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At this point I can extend my pattern and write out the remaining elements from there, or count in according the remainder (5 elements) from the first element, W. Here we see O, or orange, is the color of the 35th bead.

31	32	33	34	35	
W	R	Y	G	O	B

Answer: **Orange**.

MISCELLANEOUS PATTERNS

Sometimes you're given a sequence that doesn't follow the above patterns. In this case, you can't rely on memorized formulas. You'll need to trace out the pattern, figure out the rule at play and apply that rule to find the value you need. Often you'll need to step these out one piece at a time, by hand.

Sometimes these are called **recursive sequences**, because you use a formula to move from one term to another. Don't let that word throw you off.

> Each term, such that $n > 2$, in a sequence is found by using a recursive formula: doubling the previous term and adding some number x. If the first term in the sequence is 2, and the 6th term is -29, what is x?

Here our best bet is to work backwards. This problem sounds confusing, but just follow the pattern. Remember n just refers to what number term we're on. By saying the formula is used from the 2nd term on, we allow the first term to just be stated or defined.

Let's work our way up to the 6th term from the first one, one term at a time, using x for itself.

$$
\begin{aligned}
\text{Each Term } &= 2(\textbf{previous term}) + x \\
\text{First Term } &= 2 \\
&= 2(2) + x \\
\text{Second Term } & \\
&= 4 + x \\
&= 2(4 + x) + x \\
\text{Third Term } &= 8 + 2x + x \\
&= 8 + 3x \\
&= 2(8 + 3x) + x \\
\text{Fourth Term } &= 16 + 6x + x \\
&= 16 + 7x \\
&= 2(16 + 7x) + x \\
\text{Fifth Term } &= 32 + 14x + x \\
&= 32 + 15x \\
&= 2(32 + 15x) + x \\
\text{Sixth Term } &= 64 + 30x + x \\
&= 64 + 31x
\end{aligned}
$$

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Now we simply set that 6th term equal to -29:

$$64+31x=-29$$
$$31x=-93$$
$$x=-3$$

Answer: -3.

Note: should you encounter an atypical sequence on a late problem (after number 50-55), the problem could take some time to complete. You may want to prioritize other problem types that are not as time consuming.

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