CHAPTER

20 CONICS: CIRCLES AND ELLIPSES

SKILLS TO KNOW

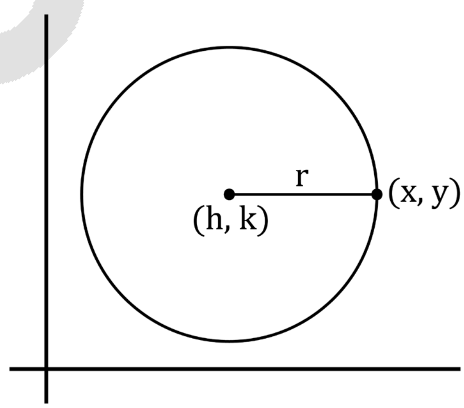
- The Circle Equation and its application
- The Ellipse Equation and its application
- How to approach miscellaneous conic section problems
- Identifying what type of graph (circle, ellipse, hyperbola, polynomial, etc.) an equation corresponds to

Conics problems lie at the intersection of coordinate geometry and geometry itself. If you're looking for more work on circles, we have an entire chapter on that in our 2nd book in this series (which focuses on geometry). In this chapter, however, we will focus on circle equations, and briefly on ellipses and other graph identification problems. Circle equations appear on the ACT® far more often than the other types of problems covered in this chapter.

If you are short on prep time, focus on circles!

THE CIRCLE EQUATION

The formula for a circle is $(x-h)^2+(y-k)^2=r^2$ with the center point at (h,k).
The radius, r, is the distance from the center point to the edge of the circle.
This graphic shows a visual representation of the variables:



A circle in the standard (x, y) coordinate plane has an equation of $(x-3)^2 + (y+7)^2 = 32$. What are the radius of the circle and the coordinate of the center of the circle?

	Radius	Center
A.	32	$(-3,7)$
B.	32	$(3,-7)$
C.	6	$(3,-7)$
D.	$\sqrt{32}$	$(3,-7)$
E.	$\sqrt{32}$	$(-3,-7)$

We know that the formula for a circle is $(x-h)^2 + (y-k)^2 = r^2$.

We can match this with the equation given in the problem:

$$\begin{array}{c} (x-h)^2 + (y-k)^2 = r^2 \\ \downarrow \qquad\quad \downarrow \quad\;\; \downarrow \\ (x-3)^2 + (y+7)^2 = 32 \end{array}$$

Matching the terms, we know that $h = 3$, $k = -7$ and $r^2 = 32$. Now we know the points of the circle's center $(3,-7)$ and that the radius $= \sqrt{32}$. Sometimes a problem like this will simplify the radical, but sometimes it won't. Always check your answers before completing that step.

$\sqrt{32}$ can be simplified to $4\sqrt{2}$—but as you can see that is not an option.

Answer: **D.**

A more difficult problem could look like this:

A circle in the standard (x, y) coordinate plane has center $(9,-5)$ and passes through the point $(1,1)$. Which of the following equations represents this circle?

A. $(x-9)^2 - (y+5)^2 = 100$

B. $(x-9)^2 + (y-5)^2 = 100$

C. $(x-9)^2 - (y+5)^2 = 10$

D. $(x-9)^2 + (y+5)^2 = 100$

E. $(x+9)^2 + (y-5)^2 = 10$

We know that the equation of a circle is:

$$(x-h)^2 + (y-k)^2 = r^2$$

The problem tells us the center of the circle, so:

$$h=9;\ k=-5$$

Now all we need to solve for is the radius. Because we know a point that the circle passes through, we can just plug this into the x and y in the equation to get:

$$(1-9)^2+(1+5)^2=r^2$$

$$(-8)^2+(6)^2=r^2$$

$$64+36=r^2$$

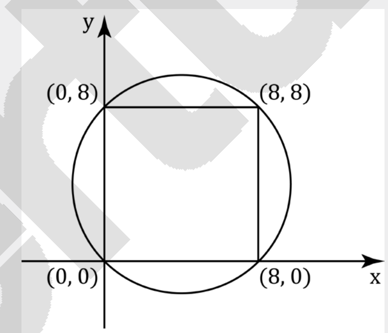
$$100=r^2$$

Thus, our answer is $(x-9)^2+(y+5)^2=100$.

(Note: You could also calculate the radius by using the distance formula for the center point and the given point on the circle.)

Answer: **D**.

In the standard (x,y) coordinate plane below, the vertices of a square have coordinates $(0,0)$, $(0,8)$, $(8,8)$ and $(8,0)$. Which of the following is the equation of the circle that circumscribes the square?



A. $(x-4)^2+(y-4)^2=32$ **B.** $(x-4)^2+(y-4)^2=\sqrt{32}$ **C.** $(x+4)^2+(y+4)^2=32$

D. $(x-4)^2+(y-4)^2=16$ **E.** $(x+4)^2+(y+4)^2=4$

Although the center point and radius aren't given explicitly, we know that the center is the midpoint of the two corners of the square (either diagonal works).

The midpoint formula is $\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)=(a,b)$

Let's use points $(0,0)$ and $(8,8)$:

$$\left(\frac{8-0}{2},\frac{8-0}{2}\right)=(4,4)$$

The center of the circle is $(4,4)$. Now, our circle's formula is $(x-4)^2+(y-4)^2=r^2$.

We can plug in any of the 4 points to solve for the radius. We'll plug in $(8,8)$:

$$(8-4)^2+(8-4)^2=r^2$$
$$4^2+4^2=r^2$$
$$16+16=r^2$$
$$32=r^2$$

Therefore, $(x-4)^2+(y-4)^2=32$.

(We could also calculate the radius by finding the length of the diagonal of the square, which would be the diameter of the circle, and the divide by 2 to get the radius.)

Answer: **A**.

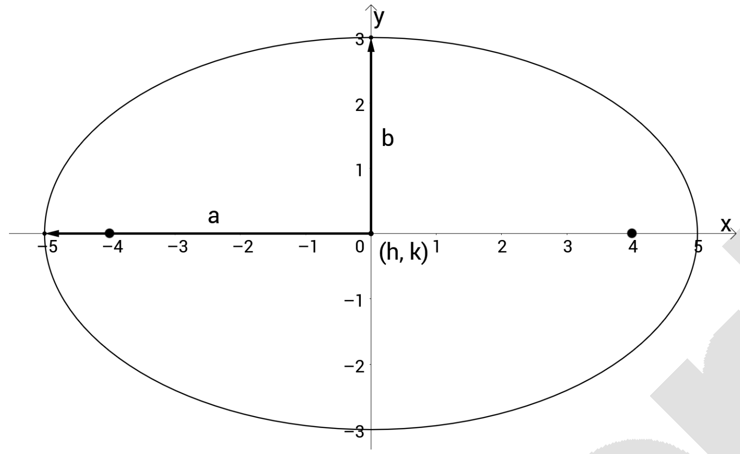
ELLIPSES

An ellipse is defined as a regular oval shape, or an oblong circle. Each ellipse has a horizontal range ("width") and a vertical range ("height") and two focal points, or foci. What is most important is that you know the equation for an ellipse. I usually don't recommend students focus on learning about these until they are comfortably scoring above about a 32 on the section, as these questions tend to be rare.

THE ELIPSE EQUATION

The equation for an ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, with the center labeled as (h,k) and the distance from that center to the end points of the major and minor axes denoted by a and b.

A less confusing way to think of a and b is to imagine that each is a sort of radius, and each corresponds to the axis that its numerator refers to. So in the above equation, if you move a units to the right or left of the center point (h), you'll reach the end points of the x-axis of the ellipse. If you move b units up or down from the center point (k), you'll reach the end points of the y-axis of the ellipse. Whatever number is squared under the numerator with an x in it will always be a horizontal, x-axis move. Whatever number is squared beneath the numerator that includes a y element will always be a vertical, y move.

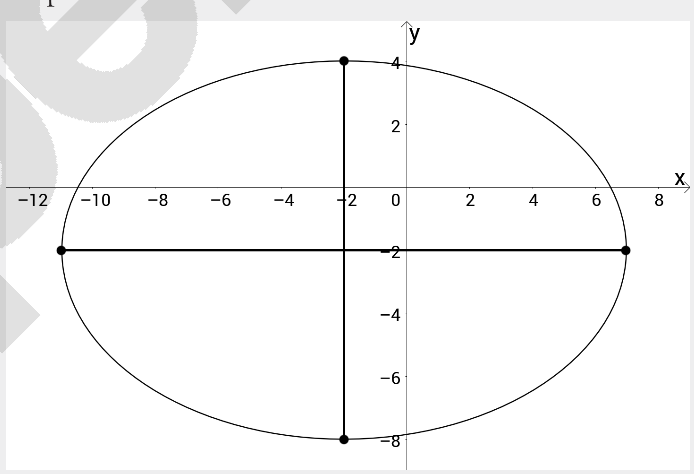


Whichever axis is longer we call the "major axis"—here that is the x-axis. Whichever axis is shorter we call the "minor axis." These terms probably won't be on the ACT®, but I'm mentioning them here to better explain the ideas that might be: how to figure out the equation that matches a particular ellipse graph.

If you think about it, the circle equation is actually a special case of an ellipse, one in which a and b are identical. If we now multiply by r^2 on both sides and clear the fraction, we get $(x-h)^2+(y-k)^2=r^2$, the circle equation.

Though it's unlikely the ACT® will test you on it, the foci of an ellipse are defined such that the distance from any point on the circumference to the two foci added together is equal. For this ellipse, the two foci are at $(-4,0)$ and $(4,0)$. As mentioned above, the summed distance from the two foci to any point on the circumference (denoted by (x,y)), is the same.

Which of the following is the equation of the ellipse that is graphed in the standard (x,y) coordinate plane below?



A. $\frac{(x-2)^2}{81}+\frac{(y-2)^2}{36}=1$ B. $\frac{(x+2)^2}{81}+\frac{(y+2)^2}{36}=1$ C. $\frac{(x+2)^2}{9}+\frac{(y+2)^2}{6}=1$

D. $\frac{(x-2)^2}{9}+\frac{(y-2)^2}{6}=1$ E. $\frac{(x+2)^2}{36}+\frac{(y+2)^2}{81}=1$

First you can calculate out or sketch the center. One way to do this quickly would be to sketch a midline both vertically and horizontally on your test booklet.

TIP: most all pictures on the ACT® are pretty close to scale unless the picture specifically mentions it's not, so you can usually rely on the picture.

Another way to find the center would be to pluck the points on the maximum and minimum points on the ellipse (the end points of the major and minor axes), and then figure out the midpoint of these extremes—that will be your center.

$$\text{Horizontal center: } \frac{-11+7}{2}=-2$$

$$\text{Vertical center: } \frac{-8+4}{2}=-2$$

$$\text{Center: } (-2,-2)$$

Now we find our vertical and horizontal range, and divide by two to find our axes' half lengths (pseudo radii if you will), a and b.

Vertical range: -8 to $4=12$—we divide this by 2 to get our y-axis half length, $b=6$.

$$b=\frac{12}{2}=6$$

Horizontal range: -11 to $7=18$—we divide this by 2 to get our x-axis half length, $a=9$.

$$a=\frac{18}{2}=9$$

Now we square a and b to get our general form: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

$$\frac{(x+2)^2}{81}+\frac{(y+2)^2}{36}=1$$

Answer: **B**.

MISCELLANEOUS QUESTIONS ON CONIC SECTIONS

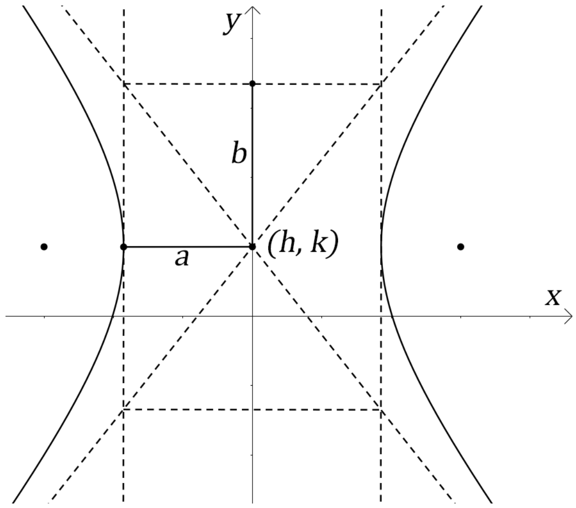
Occasionally, you might see a problem on the ACT® that involves other conic sections, such as the hyperbola. I won't get into to much detail on this graph, but you should know that **it looks like an ellipse equation or circle equation but has a negative sign between the two fractions with x and y squared terms.**

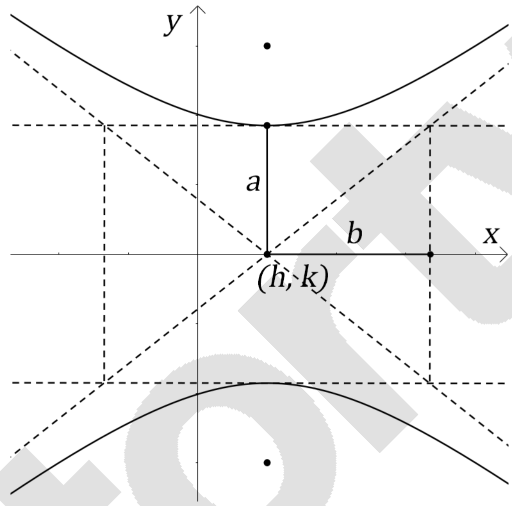
EQUATION OF A HYPERBOLA

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

(Which part is negative only affects the orientation of the hyperbola.)

I'm not going to get into the details of hyperbolas as they are insanely rare on this test, but you should vaguely know what a graph of one looks like:





Which of the following points is a possible y-intercept of the graph of $\frac{y^2}{17}-\frac{x^2}{9}=1$?

A. $(0,9)$ **B.** $(0,3)$ **C.** $(0,\sqrt{17})$ **D.** $(0,17)$ **E.** $(0,-17)$

For this question, you don't actually have to know much at all about hyperboles to solve. But you do need to know what a y-intercept is. Remember y "i"ntercepts occur when y "is" the number (I think of the "i" in intercept matching the "i" in is) and x equals zero. x-intercepts occur when x is the number and y equals zero. Don't confuse the two, and you'll be in fine shape. Here I need to simply plug in zero for x and solve for y.

Because all the answers have 0 for x, we can go ahead and plug it in:

$$\frac{y^2}{17}-\frac{0^2}{9}=1$$

$$\frac{y^2}{17}=1$$

$$y^2=17$$

$$y=\pm\sqrt{17}$$

So our answer is either $(0,-\sqrt{17})$ or $(0,\sqrt{17})$.

Answer: **C**.

In the standard (x, y) coordinate plane, the graph of the equation

$x^2 + y^2 + 18x - 2y = -73$ is a/an:

A. Ellipse
B. Straight line
C. Hyperbola
D. Parabola
E. Circle

Sometimes, the ACT® will ask you to determine what kind of conic an equation is.

When you must do so, the standard way to solve is to complete the square. No, I don't like completing the square (who does?), but here it may be your best option.

TIP: You can sometimes simplify a little with a complex equation and graph on your calculator. However, this is too difficult in this case as it's impossible to isolate the y. The TI-84 CE and similar calculators will graph conics in standard form if you download or have an app from TI, but this equation is not in standard form, so you'd need to complete the square in any case. Additionally, this app may not be approved for use by the ACT®; to us, ACT® calculator rules on this particular matter are unclear. Programs are available that complete the square, but we feel these programs are not allowed according to ACT® calculator policy.

We should already know this is not a straight line because of the x^2 term, and that it is not a parabola because of the y^2 term in addition to the x^2 term. A parabola will only have one squared variable. Therefore, we know the graph must be a conic section—and ellipse, parabola, or circle.

For this problem, we'll simplify to a recognizable conic section form by completing the square. Once we complete the square, we'll analyze the form of the equation and knowing what the standard form of all these equation types are (look earlier in this chapter if you need to review those standard forms), will be able to deduce which answer is correct.

First, we combine like variables: $\left(x^2 + 18x\right) + \left(y^2 - 2y\right) = -73$.
Then we complete the square by taking the x terms first.

Completing the Square

STEP 1: Check your coefficient of the squared term, a, and if necessary, divide all elements of the expression or equation by a to make the coefficient of the squared term 1.

$$ax^2 + bx$$
$$x^2 + 18x$$
$$a = 1 \quad b = 18$$

Because the coefficient of x is 1, we do not need to divide all terms by "a." If this were not the case, this problem would be much more difficult, and we would need to divide every term in the entire expression by this number and then factor that number out to get:

$$a\left(x^2+\left(\frac{b}{a}\right)x\right)$$

We would then complete the square off of the piece in parenthesis but that can get confusing. Luckily, we don't have to do that here.

STEP 2: Find the number to complete the square.

We now take half of the coefficient of the second term, which in our case is simply $b=18$, and square it. The goal is to create a perfect square, so we're working off the idea of the square of a sum special pattern.

$$\left(a^2+2ab+b^2\right)=\left(a+b\right)^2$$

$$x^2+18x+?$$

What we're trying to do is engineer a b squared term that "completes" the square.

Matching terms, we see $2b=18$. 18 divided by 2 is 9. $b=9;$ so $b^2=81$.

STEP 3: Then you add this amount in, and subtract it at the same time, as so, to create an equivalent expression:

$$\left(x^2+18x\right)=\left(x^2+18x+81\right)-81$$

STEP 4: Factor

This gives us a perfect square—remember the special pattern called the "Square of a Sum":

$$\left(x^2+2xy+y^2\right)=\left(x+y\right)^2$$

Now we can factor that perfect square piece according to the pattern:

$$\left(x^2+18x+81\right)=\left(x+9\right)^2$$

And now we substitute this in to our expression:

$$\left(x^2+18x\right)=\left(x+9\right)^2-81$$

We now apply the same process to the y-terms:

$$\left(y^2-2y\right)$$

Take -2, divide it by 2 to get -1, and square it to get 1.
Now add one, and subtract one as so:

$$\left(y^2-2y\right)=\left(y^2-2y+1\right)-1$$

Now we use the pattern from the "Square of a Sum" to make this:

$$\left(y^2-2y\right)=\left(y-1\right)^2-1$$

At this point, we can plug both the x portion and y portion back into our original equation, and simplify:

$$\left(x^2+18x\right)+\left(y^2-2y\right)=-73$$

$$\left(x+9\right)^2-81+\left(y-1\right)^2-1=-73$$

$$\left(x+9\right)^2+\left(y-1\right)^2=-73+81+1$$

$$\left(x+9\right)^2+\left(y-1\right)^2=9$$

Because this follows the standard equation form of a circle, we know it is a circle:

$$\left(x-h\right)^2+\left(y-k\right)^2=r^2$$

Answer: **E.**

There is also an alternative, incredibly quick method to solve this problem, but it is hyper-specific to this type of problem and requires memorization. I have it here if you're in the mood for such memory tricks, but completing the square is a more useful skill that will apply to more types of questions and thus I focus on that method above.

CONIC EQUATION TYPES

A conic section of the form $Ax^2+By^2+Cx+Dy+E=0$, in which A and B are both not zero is:

A **circle** if $A=B$.
A **parabola** if $AB=0$.
An **ellipse** if $A\neq B$ and $AB>0$.
A **hyperbola** if $AB<0$.

Because A and B in our equation are 1 and 1, we can see that the equation is a circle.