CHAPTER

21 COORDINATE GEOMETRY

SKILLS TO KNOW

- Synthesizing physical distance and coordinate points
- Conic sections and polynomials in coordinate geometry
- Finding minimums and maximums in a system
- Representing concepts in graphs
- 3D Graphing Basics

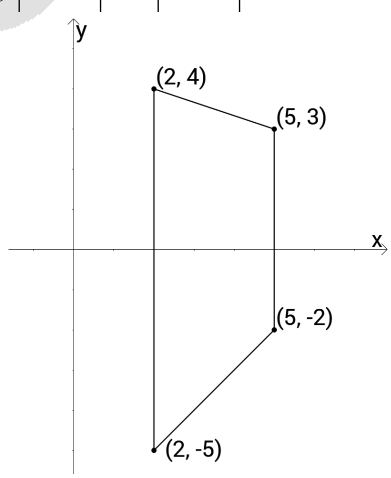
For more coordinate geometry, also see Chapter 5: Intercepts and Slopes, Chapter 6: Distance and Midpoint, Chapter 20: Conics, Chapter 22: Graph Behavior, and Chapter 23: Translations and Reflections.

FINDING PROPERTIES OF LINE SEGMENTS FROM COORDINATES

Oftentimes on the ACT®, you'll be asked to jump between the world of coordinates and the universe of "actual lengths." You'll need to know how to easily convert information from coordinate points into physical distances.

Finding Lengths of Horizontal/Vertical Lines

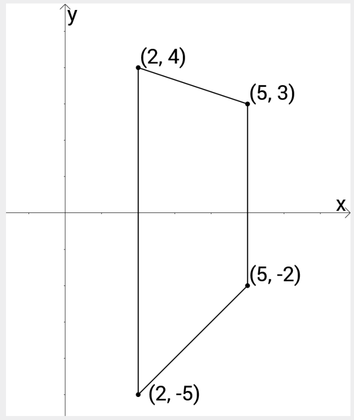
To find these lengths when you have vertical or horizontal lines, you'll subtract the y-value from another y-value or an x-value from another x-value and find the absolute value of the difference. For example, in the picture below, to find the length of the line defining the right side of the trapezoid with endpoints $(5,3)$ and $(5,-2)$, $|3--2|$ or $|-2-3|$.



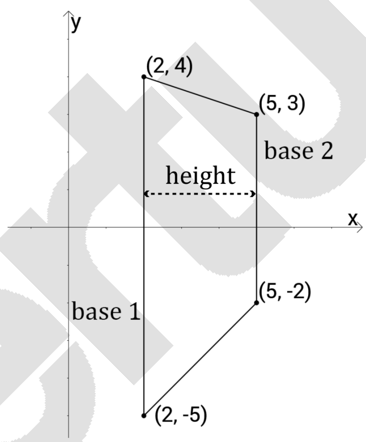
Because we are using absolute value, the order we subtract in does not matter.

We can do the same to find the distance between a point and a line or two parallel lines: simply subtract the difference in x-values or y-values and take the absolute value. This often comes into play when finding the height of a shape in a coordinate plane.

What is the area, in square units, of the trapezoid graphed below?



The area of a trapezoid is one-half the sum of the bases times the height. All of these lengths are vertical or horizontal distances, so we can calculate each amount by taking the absolute value of the difference of the respective x or y values. Here, the bases are our vertical lines and the "height" is the horizontal distance between the parallel vertical lines.



The length of the bases are equal to the absolute value of the difference of the y-values of the endpoints. For the left most vertical line:

$$\left|4-(-5)\right|=\left|9\right|=9$$

And for the right most vertical line:

$$\left|3-(-2)\right|=\left|5\right|=5$$

Now that we've converted these "coordinates" into a physical distance, we use a similar process to find the height. The height of the trapezoid equals the x-distance between the parallel bases, which we find by subtracting the x-values of these lines (5 for the right line, 2 for the left line) and taking the absolute value:

$$\left|5-2\right|=\left|3\right|=3$$

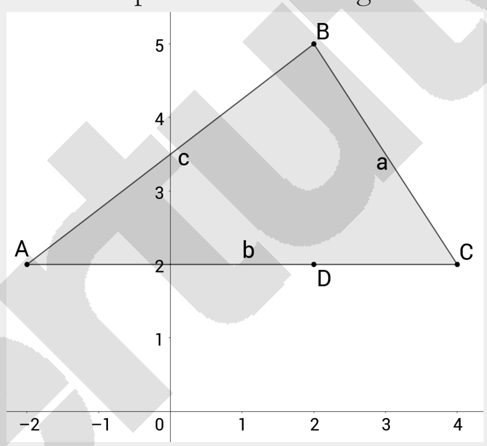
We plug in these values into the formula for the area of a trapezoid:

$$\frac{b_1+b_2}{2}h=\frac{9+5}{2}(3)$$
$$=\frac{14}{2}(3)$$
$$=7(3)$$
$$=21$$

Finding Lengths of Diagonals

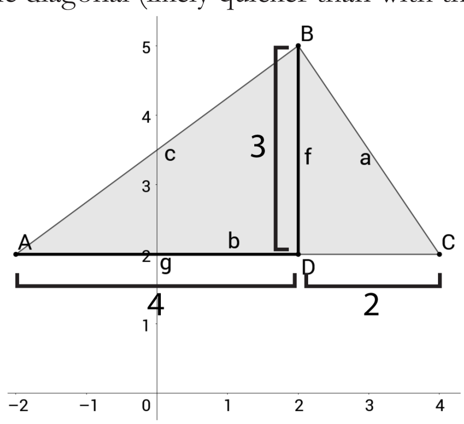
Diagonals lengths are simple to find using the idea of horizontal and vertical distance above, combined with the Pythagorean theorem, or by using the Distance Formula. (Note: The Distance Formula is covered in depth in the chapter on Distance and Midpoint.)

Find the perimeter of triangle ABC:



First, we'll find the coordinates: $A(-2,2)$, $B(2,5)$, and $C(4,2)$.

Now we need to find the length of each. Let's start with side c or AB: I can eyeball the physical lengths of the sides of right triangles formed by a horizontal line (g in the following graph) and vertical line (f in the following graph) that connect the endpoints of the diagonal and use the Pythagorean theorem to find the diagonal (likely quicker than with the distance formula).



Length c (AB): from -2 to 2 (horizontal length, g) is 4 units, from 2 to 5 (vertical length, f) is 3 units. This is a 3-4-5 triangle so:

Length $c = 5$ units

(If you don't have that Pythagorean Triple memorized, you could also run the Pythagorean Theorem on these numbers and find $\sqrt{4^2+3^2} = \sqrt{(16+9)} = \sqrt{25} = 5$. For more on Pythagorean Triples and Pythagorean Theorem, see the Triangles chapter in Book 2.)

Now let's find the other two lengths:

Length b (AC): From -2 to 4 is 6 units, and this is horizontal.

Length $b = 6$ units

Length a (BC): from 2 to 4 is 2 units (horizontal length from D to C), and from 2 to 5 is 3 (vertical length, f). With the Pythagorean Theorem: $a^2+b^2=c^2$ or $\sqrt{a^2+b^2}=c$:

$$\sqrt{2^2+3^2} = \sqrt{(4+9)} = \sqrt{13}$$

Length $a = \sqrt{13}$ units

To find the perimeter, we add the three lengths:

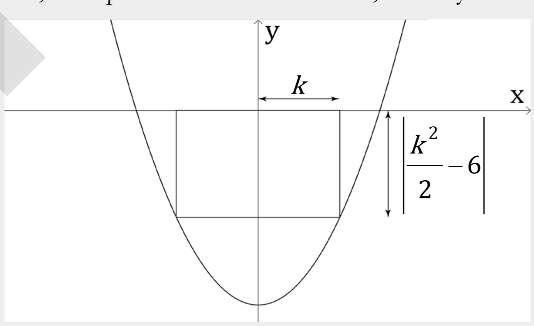
$$5+6+\sqrt{13} = 11+\sqrt{13}$$

Answer: $11+\sqrt{13}$.

CONIC SECTIONS & POLYNOMIALS IN COORDINATE GEOMETRY

Other coordinate geometry problems synthesize skills from Conic Sections (see Chapter 20 in this book) and Quadratic Equations or other polynomials (see Chapter 16 in this book). The principle is the same, however. We want to turn what could be an equation or coordinates into physical distances or vice versa.

In the standard (x,y) coordinate plane below, 1 side of a rectangle is on the x-axis, and the vertices of the opposite side of the rectangle are on the graph of the parabola $y=\frac{x^2}{2}-6$. Let k represent any value of x such that $-2\sqrt{3} < x < 2\sqrt{3}$. One potential value of k is depicted in the picture below. Which of the following is an expression in terms of k for the area, in square coordinate units, of any such rectangle?



A. k^3-12k **B.** $-9k$ **C.** $\left|\frac{k^3}{2}-6k\right|$ **D.** $-\left|k^3\right|+12\left|k\right|$ **E.** $k^2+4k-12$

This question is pretty confusing. I suggest unpacking the question until you understand it. You might not need all the explanation here that follows, but I try to be complete because it may take some students more "exploration" than others to have some information click enough so that they can see the solution.

Let's focus on the idea of the rectangle. We have a rectangle with one side on the x-axis and the other side parallel and touching the parabola. To find the area of this rectangle we need the physical distance of each side of this rectangle.

First let's consider the vertical side of the rectangle. From the picture, we know that it could be expressed as $\left|\frac{k^2}{2}-6\right|$. But why? Where did that come from?

From our work previously in this section, we know this vertical rectangle height is the absolute value of the difference between the y-value of the corner that sits on the x-axis (that y-value is clearly zero) and the y-value of the lower right corner of the rectangle (a solution to the equation $y=\frac{x^2}{2}-6$).

Thus that lower corner will be $\frac{x^2}{2}-6$, and we need to subtract zero from that, and take the absolute value to get the length, i.e. $\left|\frac{x^2}{2}-6-0\right|$. Subtracting zero has no effect, so this equals $\left|\frac{x^2}{2}-6\right|$. That looks just like the label on the picture, right? Remember k is "any value of x …" that fulfills a particular parameter, so k is just the same as x: $k=x$.

Thus $\left|\frac{k^2}{2}-6\right|$ is our height, or y-value: the vertical physical distance of the rectangle height for any given value k.

The way the question words this whole situation is mind numbing, yes, but let's keep on.

From the problem, we know k is a number that equals x so long as $-2\sqrt{3}<x<2\sqrt{3}$. But why? This looks like the zeros of the equation, and to test my theory, I can plug these end points into the parabola equation.

First I'll plug in $2\sqrt{3}$:

$$y=\frac{x^2}{2}-6$$

$$y=\frac{\left(2\sqrt{3}\right)^2}{2}-6$$

$$y=\frac{4\times 3}{2}-6$$

$$y=\frac{12}{2}-6$$

$$y=6-6$$

$$y=0$$

Yes—this is a **zero** of the equation (the spot where the parabola crosses the x-axis). If I were to plug in $-2\sqrt{3}$, the solution would also be zero for y (because this term is squared the sign really doesn't matter). The spread of x values is thus the domain of where you could draw a rectangle that has one side between the zeros and the other side touching the parabola in two places below the x-axis. That's all. If I pick an x-value outside of this domain, I wouldn't be below the axis anymore; I would be above it. This parameter keeps our rectangle within the boundaries of the "u" part of the parabola below the x-axis.

Our horizontal distance is already given for "one" value of k on the picture—and it appears that it spans half the width of the rectangle. To get the full width in this case we would need $2k$. In terms of physical distance, this makes sense. The value k only spans from the line of symmetry ($x=0$) to the x coordinate, but we need the whole distance across the parabola, not just the distance from the midline. But let's be careful. Again, this is one possibility according to the question. If k were say 2, $2k$ would be 4, right? That works. But what if k were negative? What if k were -2? If so, we'd need -2 times 2, which would be -4, but then we need to make that a physical distance. The physical distance is the absolute value, which is 4. Because our k can be anything in this range, $-2\sqrt{3} < x < 2\sqrt{3}$, we need a solution that accounts for the possibility that k is negative.

Thus the width is not simply $2k$, but rather $|2k|$.

Now we multiply our width times our height:

$$|2\boldsymbol{k}| \times \left|\frac{\boldsymbol{k}^2}{2} - 6\right|$$

Here's where many students might get stuck. None of our answers are exactly this.

We can quickly eliminate, however, any answers without a k^3 term, as we know, even with absolute value, that there is a third power of k somewhere in the answer. That leaves A, C, and D:

A. $k^3 - 12k$

C. $\left|\frac{k^3}{2} - 6k\right|$

D. $-|k^3| + 12|k|$

You might be tempted to drop the absolute value. If you do this without thinking, and simplify $(2k) \times \left(\frac{k^2}{2} - 6\right)$, you'll get A.

But (A) doesn't work because the vertical value $\frac{k^2}{2} - 6$ will never be positive: you cannot ignore the absolute value. Remember back to the idea that this rectangle sits below the x-axis. That means we need this absolute value sign or we'll get a negative value for "y" when we calculate this part of the equation. If this length is negative and the value of k is positive, our area would be negative, and that is not possible.

(C) forgets to account for the idea that k is not the full length of the rectangle, and thus is not our answer.

(D) works. You can figure this out by plugging in numbers into both what you got and this, or by going back to the logic of why A doesn't work: **every y-value that corresponds to the lower corners of the rectangle is always negative, but our physical distance is always positive.** If you realize this fact, you'll know that we can simply multiply $\frac{k^2}{2}-6$ **by negative one** to get the physical distance of the rectangle height. That gives us a vertical height of:

$$\left(\frac{k^2}{2}-6\right)(-1)=\left(-\frac{k^2}{2}+6\right)$$

Now we multiply this new expression by our length $|2k|$ or $2|k|$:

$$\begin{aligned}(2|k|)\left(-\frac{k^2}{2}+6\right)&=\left(-\frac{|k|2k^2}{2}+6\times 2|k|\right)\\&=\left(-|k|k^2+12|k|\right)\\&=-\left|k^3\right|+12|k|\end{aligned}$$

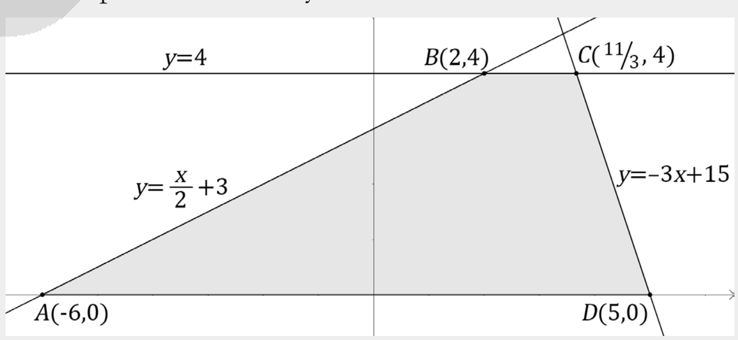
Answer: **D.**

FINDING MINIMUMS/MAXIMUMS IN A SYSTEM

Within a system or a graph, you can find the minimum or maximum by plugging in all the intersection points between any two or more of the inequalities in the system into the function you are trying to maximize. **One of these intersection points will always contain the solution.** This is a concept you simply must memorize.

Consider the set of all points that satisfy all 3 of the conditions below:

$$\begin{Bmatrix} 0<y<6 \\ y<\frac{x}{2}+3 \\ y<-3x+15 \end{Bmatrix}$$



The graph of this set is trapezoid $ABCD$ and its interior, which is shown shaded in the standard (x,y) coordinate plane below. Let this set be the domain of the function $P(x,y)=4y-x$.

What is the minimum value of $P(x,y)$ when x and y satisfy the conditions given?

A. −24 **B.** 6 **C.** −30 **D.** 14 **E.** −5

We want to test all the intersection points of two or more of the inequalities, in other words the "corners" of the bounded region, to see which ordered pair creates the minimum value of $P(x,y)$.

$$P(-6,0)=4(0)-(-6)=6$$

$$P(2,4)=4(4)-2=14$$

$$P\left(\frac{11}{3},4\right)=4(4)-\frac{11}{3}=\frac{37}{3}$$

$$P(5,0)=4(0)-5=-5$$

Because -5 is the smallest value, the answer is **E**.

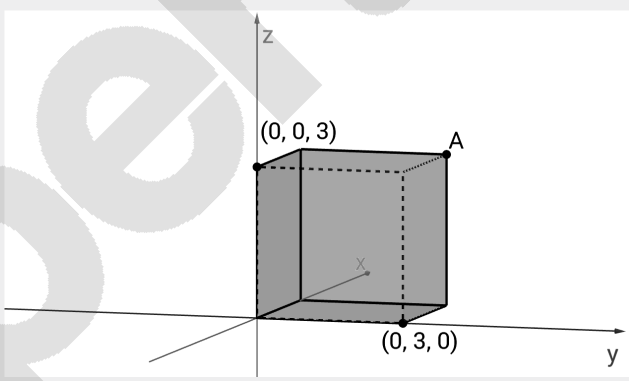
3-DIMENSIONAL COORDINATES

3D Coordinate Geometry on the ACT® is rare. Save this section until you're scoring above a 32 on the math.

For 3D coordinate planes, there's an additional axis, known as the z-axis.

Points in three dimensions are given in (x,y,z) form, where x and y function in the same way as in 2 dimensions, and z simply adds a third "depth" dimension. When any value in a coordinate of this form has a zero, it is an intercept on that plane.

As shown in the (x,y,z) coordinate space below, a cube has vertices at $(0,3,0)$ and $(0,0,3)$. What are the coordinates of vertex A?



The point lies on the plane perpendicular to the y-axis at 3, so its y value is 3. Similarly, it has a z value of 3. Since a cube's sides are of equal length, it extends in the x direction for 3 as well, so its coordinates are $(3,3,3)$. We could also do each coordinate one at a time, visualizing moving back 3 on the x-axis, up 3 on the z-axis or plane, and over 3 on the y-axis or plane.

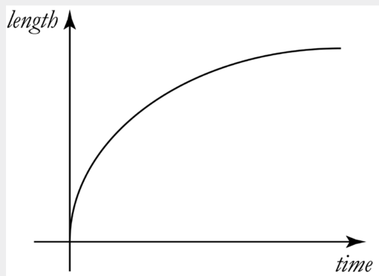
Answer: $(3,3,3)$.

REPRESENTING CONCEPTS IN GRAPHS

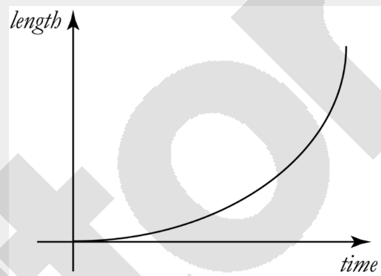
These questions usually don't give exact numbers, and if they do, the numbers probably don't matter much. They generally ask for overall trends and want to test if you know what the graph is supposed to look like.

When a tree sapling began to grow, its roots grew quickly for several months before they hit a stone wall, which slowed down growth until the roots pushed through and resumed growing quickly. Which of the following graphs could represent the length of the sapling's roots as a function of time, in months?

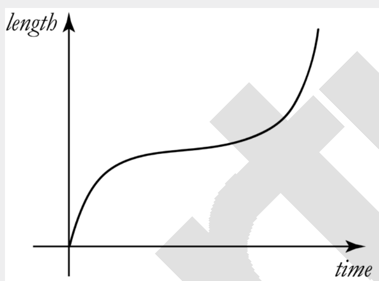
A.



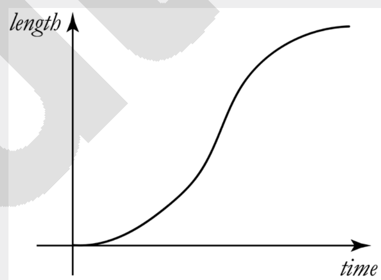
B.



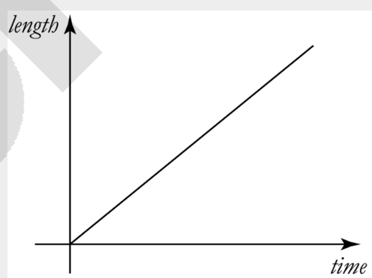
C.



D.



E.



We know there should be some funky looking "slow down" point, and another "speed up" point, so E, linear growth, doesn't make sense, and A and B, different types of exponential growth also don't make sense.

We're left with C and D. What we want is high growth in a short span of time. Because time is on the bottom, we want a steep line at the start and at the end. A steep line would minimize time and maximize growth. Likewise we want a flatter line in the middle. A flatter line would extend time with little growth.

The key to solving this problem is to recognize what slowing down looks like in a graph like this. When the line tangent to the curve at a certain point becomes more horizontal as it moves to the right along the line, it is slowing down.

Thus the best choice is C.

D has a funky change in rate, but it is growing rapidly in the middle, not slowing down.

Answer: **C.**