CHAPTER

9 RATIOS, RATES & UNITS

SKILLS TO KNOW

- How to set up ratios from word problems
- How to set up proportions and cross-multiply to solve
- How to convert units
- Dimensional Analysis
- How to solve distance problems
- How to solve work problems
- How to solve wheel distance problems

Note: Chapter 12, Direct and Inverse Variation, encompasses many proportion problems and builds on this chapter. Ratios, Rates and Units further utilizes knowledge of units.

RATIOS: THE BASICS

While percents and fractions measure part to whole relationships, ratios typically measure part to part. When you see the word RATIO, unless the "whole" or "all" is specifically stated in the sentence, the problem is telling you a relationship between two "parts." For example, the ratio of blue marbles to red marbles is $3:7$ means that for every 3 blue marbles there are 7 red ones.

Beware of FRACTION problems that use the word RATIO.

Tricky problems may use the word ratio to describe a fraction. For example, the ratio of blue marbles to all marbles in the bag is $3:10$ means that for every 3 blue marbles, there are 10 total marbles. In other words, there are 7 non-blue marbles and $\frac{3}{10}$ of all the marbles are blue. When your ratio is "to all," you may have a fraction in disguise.

Most basic ratio problems can be solved in one of two ways:

Method 1: Turn your ratios into fractions
Method 2: Put an x next to it.

For both methods, it's important to understand you'll be dealing with numbers that represent the **ratio** AND numbers that represent the **actual** numbers. Also remember **the whole is always the sum of the parts.**

Let's see an example of these methods in action.

A triangle has interior angles that are in the ratio of $4:5:7$. What is the difference in degrees between the largest and smallest angle measure?

From chapters on triangles and angles and lines (both in Book 2), you should know that these shapes have an interior angle sum of 180 degrees. This is the "actual" whole. We also know the ratio of the parts.

Method 1: Turn your ratios into fractions

One way we can solve this problem is to turn all our information into fractions to solve for the actual angle measures. When we know all the ratio "parts" we can also find the ratio "whole" by adding these parts together. 4, 5, and 7 represent the proportional parts. If I add these together, I get the "whole" of the proportion:

$$4+5+7=16$$

We have sixteen parts in total. This is our "whole," or the bottom of each fraction we will create.

Now I can turn each "part" into a fraction. Remember, a fraction is always "part over whole." Each part of the proportion represents a different fraction, i.e.:

$$4 \rightarrow \frac{4}{16} \qquad 5 \rightarrow \frac{5}{16} \qquad 7 \rightarrow \frac{7}{16}$$

I need the difference of the largest angle and the smallest angle. I now know the largest angle is $\frac{7}{16}$ of the total degrees in the triangle or:

$$\frac{7}{16} \cdot 180 = 78.75$$

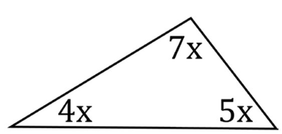
I also know the smallest angle is $\frac{4}{16}$ or $\frac{1}{4}$ of the total degrees in the triangle:

$$\frac{4}{16} \cdot 180 = 45$$

Now I find the difference: $78.75-45=33.75$ degrees.

Method 2: Put an "x" next to it:

A great trick for solving ratio problems is to know that you can always represent the "actual" amount of something that you only know the ratio part of by putting a variable next to all the "ratio" numbers you know. Ratios are like a "reduced fraction" of actual numbers. To get back to those actual number if we were dealing with reduced fractions, we would multiply by a common ratio. Here we do the same, with that common ratio equal to x: Given a ratio $4:5:7$, I can express the **actual measures** of each angle as $4x$, $5x$, and $7x$.



Now that I have an algebraic representation of the actual numbers, I can sum these to be the actual total, 180. Now I just use algebra to simplify:

$$4x+5x+7x=180$$
$$16x=180$$
$$x=11.25$$

Now be careful, x **<u>is not the answer to the question</u>**! I can solve for $7x-4x$, or $3x$, the difference between the largest and smallest angles, by plugging x into the expression for what I need:

$$3(11.25)=33.75$$

Answer: 33.75.

NOTE: This "put an x next to it" method works very well in similar shape problems as well. (See Book 2 for Chapter 15: Similar Shapes.)

SOLVING PROPORTIONS

If a proportion is appropriate, it should represent a linear relationship such that as one value increases, so does the other. Proportions can also be used to solve for unit conversions. Be care you have a proportion situation before assuming you can use a proportion. See the Direct and Inverse Variation chapter for more on determining this.

The easiest way to solve simple proportions is cross-multiplying. To do this, we multiply the numerators of each fraction by the denominator of the other.

CROSS MULTIPLICATION

$$\frac{a}{b} \quad \frac{c}{d}$$
$$ad = bc$$

If Maria bought 6 pairs of socks from the department store for 13 dollars, how much did Sheila pay for 9 pairs of the same socks from the same store?

To set up proportions, **create a parallelism with your labels**. For example, flour over cookies equals flour over cookies. Or miles over minutes equals miles over minutes.

Here, we can set up our proportion as dollars over socks equals dollars over socks, keeping Maria on one side and Sheila on the other. We get:

$$\frac{\$13}{6\,\text{socks}}=\frac{\$x}{9\,\text{socks}} \qquad \frac{\leftarrow\text{dollars}}{\leftarrow\text{socks}}$$

The numerators' units are dollars; the denominators' units are socks. x denotes the number of dollars Sheila paid for 9 pairs of socks.

MISTAKE ALERT: Many students don't keep the order the same in proportions (i.e. they'll set dollars over socks equal to socks over dollars). Always keep your proportions in the same pattern! However, you CAN put whatever label you want in the numerator as long as you are consistent (i.e. you could put socks over dollars for both fractions).

Remember that anything "vertical" in a proportion has something in common (either same person/case or units), as does anything on the same horizontal. Here, our horizontal similarity is socks and dollars. Our vertical similarity is the person involved in the purchase. We can set up proportions in several different ways, but there should always be a similarity on the vertical and horizontal regardless.

Cross-multiplying, we get:

$$6x = 9(13)$$
$$6x = 117$$
$$x = \frac{117}{6} \ or \ \$19.50$$

Answer: $\$19.50$.

CONVERTING UNITS

One basic way to convert units is to set up a proportion. In a proportion, as described above, the units must be the same along the vertical or along the horizontal of the two fractions.

How many feet there are in 52 inches?

We can solve this by setting up a proportion: feet over inches equals feet over inches. I place the "actuals" on the left side, and the "conversion factor" on the right side (the number of feet I know contain a certain number of inches).

$$\frac{x \ ft}{52 \ in} = \frac{1 \ ft}{12 \ in}$$

Cross-multiplying, we get $12x = 52$. Solving for x, we have the answer: $4\frac{1}{3}$ feet.

That way is fine for simple problems. But there's a better way for most students aiming to score above a 25 on the math: Dimensional Analysis.

DIMENSIONAL ANALYSIS

If you have struggled with unit conversions, ratios, and rates when encountering word problems, dimensional analysis is the awesome sauce you need to crush most any problems with ratios, rates, and/or units. On the new SAT® it is absolutely vital. On the ACT® it's still an important skill to have.

You may have used this process in chemistry class. It's an intuitive way of converting anything using either a **unit conversion factor** or even a **rate**. Let's talk about both of those.

Unit conversion factors:

A unit conversion factor, such as three feet in a yard, is a ratio of two values that are equal to each other. For example, 12 inches are in a foot, and 60 seconds are in a minute (and by the way, you should have all these memorized for the ACT®). If we divide one of these equivalent amounts by the other, we essential create a "1," i.e.:

$$\frac{60\,seconds}{1\,minute}=1 \text{ or } \frac{1\,minute}{60\,seconds}=1$$

Because these values are the same amount of time, if we divide equal amounts, we get "1", right?

Now, let's take that "1" and multiply by it. When we do so, we won't be changing any amounts, but rather, will be changing the labels.

Let's say I have 300 seconds and I want to convert it to minutes. I multiply by the version of "1" above that will help me "cancel" out the units.

$$300\ \cancel{sec}\times\frac{1\ min}{60\ \cancel{sec}}=\frac{300}{60}\ min=\frac{30}{6}\ min=5\text{ minutes}$$

The one with seconds on the bottom so that the seconds will cancel. Just as I can cancel numbers when I multiply fractions by each other (i.e. $\frac{2}{5}\cdot\frac{1}{2}=\frac{1}{5}$), I can cancel my units when I multiply by this version of "1."Now what's great about dimensional analysis is that you can convert several times in a row very easily. Let's say you want to know how many seconds are in a year. Here, I'll multiply by one conversion factor after another to get my units to cancel until I get to seconds. Whatever I want to get rid of I put on the bottom. Whatever I next need I put on top:

$$1\,year\times\frac{365\ days}{1\,year}\times\frac{24\,hours}{1\,day}\times\frac{60\,min}{1\,hour}\times\frac{60\,sec}{1\,min}$$

$$1\ \cancel{year}\times\frac{365\ \cancel{days}}{1\ \cancel{year}}\times\frac{24\ \cancel{hours}}{1\ \cancel{day}}\times\frac{60\ \cancel{min}}{1\ \cancel{hour}}\times\frac{60\,sec}{1\ \cancel{min}}=31{,}356{,}000\text{ seconds}$$

As you can see, writing this out in one line is much easier with this method than using proportions.

Rates

Now here's the kicker: we can use this SAME technique to find values using rates. While units are simply units of measuring the same thing (such as length or temperature), RATES allow us to jump between two totally different ideas, such as miles and hours or gallons and square feet. Still, the method is the same.

TIP: Rates typically use words such as **"per," for "every," or for "each," which all mean divide.**

For example, if Judy stuffs 4 envelopes **per** hour, I can create a rate that is $\frac{4\,envelopes}{1\,hour}$. Remember, **per** means divide! Now if I know envelopes I can calculate hours or if I know hours, I can calculate envelopes using this method.

> If the recipe for chocolate cornbread calls for $3\frac{1}{4}$ cups of milk for 5 total pieces, how many cups does Ralph need to buy if he wants to make exactly 17 pieces?
>
> **A.** $6\frac{1}{2}$ **B.** $9\frac{3}{4}$ **C.** $10\frac{1}{2}$ **D.** $11\frac{1}{20}$ **E.** 13

The first thing we need to do is set up the ratio. If it takes $3\frac{1}{4}$ cups to make 5 pieces and we are trying to find the number of cups needed, the rate is:

$$\frac{3\frac{1}{4}\,cups}{5\,pieces}$$

Now I ask what I have: pieces. I write that down first. I also think what I need: cups.

I put what I "need" first or on top in my rate, so this rate's orientation above is correct as is (i.e. I don't want its reciprocal).

Now I use dimensional analysis:

$$\underbrace{17\,\cancel{pieces}}_{\text{What I have}} \times \underbrace{\frac{3\frac{1}{4}\,cups}{5\,\cancel{pieces}}}_{\text{Rate}} = \underbrace{11.05\,\text{cups},\ \text{or}\ 11\frac{1}{20}\,\text{cups}}_{\text{What I need}}$$

Answer: **D**.

> At the lumber yard, $10\frac{1}{3}$ feet of wood sells for $\$24.50$. If Sheldon only needs 20 inches of wood for his model train kit, how much will the wood cost in dollars, to the closest dollar?
>
> **A.** 2 **B.** 3 **C.** 4 **D.** 5 **E.** 6

You may have tried to set up a ratio for this one already, but don't be too hasty. Notice how the units are different for the lengths of wood: one is in feet and the other in inches!

First, we have to convert these numbers to uniform values. Either inches or feet will work, but I will use inches. Set up the conversion equation. I put what I have first (feet) and multiply by the unit conversion factor with what I "need" on top (inches) and what I want to eliminate on the bottom (feet):

$$10\frac{1}{3}\ \cancel{feet} \times \frac{12\,inches}{1\ \cancel{foot}} = 124 \text{ inches}$$

Now that our units are the same, we can set up the equation for the price. We could use dimensional analysis OR a ratio. Let's do a ratio this time. Inches over dollars equals inches over dollars:

$$\frac{124\,inches}{\$24\frac{1}{2}} = \frac{20\,inches}{x\ dollars}$$

Cross multiply to get $124x = 490$. Divide both sides by 124 to arrive at the correct solution, $\frac{490}{124}$. We can use our calculators to help round this to the nearest dollar (per the question), 4.

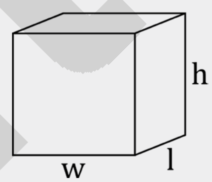
Answer: **C**.

Ratios in Word Problems

Ratios also often appear in various word problems. We can convert our understanding of ratios into algebra to solve these problems. More problems of this nature can be found in the respective chapters on other topics, such as geometry (Book 2), solids (Book 2), or general word problems (this book).

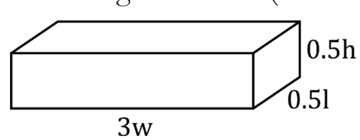
In the cube below, if width was multiplied by 3, and both length and height were halved, which of the following would equal the volume of the cube in terms of the original volume, V?

$$V = w \times l \times h$$



A. $1.25V$ **B.** $0.75V$ **C.** $0.6V$ **D.** $0.5V$ **E.** $0.25V$

We can write our new volume's equation out by modifying each of the original dimensions based on the wording of the problem. The new width is $3w$ (three times the old width) while the new length is $0.5l$ (half the old length) and the new height is $0.5h$ (half the old height). Thus:



$$V_1 = 3w \times 0.5l \times 0.5h$$

Using the commutative property of multiplication, we rearrange to isolate the constant:

$$V_1 = 0.75\left(w \times l \times h\right)$$

Since the original volume, V, was wlh, we know this new volume is 0.75 times that. Substituting in V for wlh, we get $0.75V$.

Answer: **B**.

We also find ratios in speed and distance problems. These can also be approached by memorizing the distance formula. Still, dimensional analysis works, too. **Remember that your rate is an equation in disguise. <u>You can always turn it into a ratio equation or setup a series of elements to cancel out using a rate.</u>**

<u>DISTANCE PROBLEMS</u>

THE DISTANCE FORMULA

$$\boxed{d = rt}$$

Memorize this. When in doubt set up as many equations as necessary using sub-numbers or letters (subscript, i.e. $d_a = r_a t_a$) to denote who/what you're talking about. Then start a chain of substitution until you get to the answer you need.

> Nathaniel drove 360 miles in 9 hours of driving time. How much faster would he have to drive than his average speed to cut his driving time by 3 hours?
>
> **A.** 20 *mph* **B.** 30 *mph* **C.** 40 *mph* **D.** 50 *mph* **E.** 60 *mph*

Use the distance formula (I use subscript to keep each case straight): $d_a = r_a t_a$

For the actual trip (trip a), we let $d_a = 360$ and $t_a = 9hr$, and plug in to the distance formula:

$$360\,mi = r_a(9\,hr)$$

Then solve for r_a:

$$\frac{360\,mi}{9\,hr} = r_a$$

$$40\frac{mi}{hr} = r_a \text{ is his speed for the actual trip}$$

Now make a formula for the hypothetical trip (trip h) and fill in the pieces we know: $d_h = r_h t_h$

We know we want to cut the time by three hours—so $9 - 3 = 6$; $t_h = 6$

In terms of the distance, he's trying to take the same trip, so $d_h = d_a = 360$

$$360\,mi = r_h(6\,hr)$$

$$\frac{360\,mi}{6\,hr} = r_h$$

$$60\,mph = r_h$$

Now read the question—we need how much faster this answer is than our speed for the actual trip, or the difference between the hypothetical faster trip and the actual one:

$$r_h - r_a$$

$$60\,mph - 40\,mph = 20\,mph$$

Answer: **A.**

WORK PROBLEMS

Work problems are another kind of rate problem. We can use formulas or dimensional analysis and ratios to solve these too.

WORK FORMULA

$$w = rt$$

Where w is the amount of work (ex. number of pages), r is the rate of work (ex. pages per hour) and t is the time (ex: hours).

COMBINED WORK FORMULA

$$w = (r_1 + r_2)t$$

Where r_1 is the rate of one worker or machine, and r_2 is the rate of another worker or machine working for the same amount of time.

A whatsit machine makes 35 whatsits per minute. A second whatsit machine makes 50 whatsits per minute. The second whatsit machine starts making whatsits 5 minutes after the first whatsit machine starts. Both machines stop 10 minutes after the first machine starts. How many whatsits were produced by the two machines together?

A. 212.5 **B.** 425 **C.** 600 **D.** 675 **E.** 850

If work equals rate times time $w = rt$, and the number of whatsits produced represents the "work" we now need to figure out rate and time and solve for w. We can also think in terms of ratios, or dimensional analysis, described above.

Remember PER means DIVIDE. So 35 whatsits per minute means divide 35 by 1 minute! 50 whatsits per minute means divide 50 by 1 minute! "Per" also indicates a rate.

Thus here are our two rates:

First machine: $\dfrac{35\ whatsits}{1\ min}$

Second machine: $\dfrac{50\ whatsits}{1\,min}$

From the problem we know the first machine stops after 10 minutes. The second machine works 5 minutes less as it starts 5 minutes later, but ends at the same time. To find the total time that the second machine runs, we subtract:

$$10\ min - 5\,min = 5\ min$$

Now we can either analyze the rate and know that multiplying it by minutes will make the minutes cancel, and give us what we need (whatsits) or we can know to multiply rate times time from the formula.

Remember, $w = rt$!

The first machine will produce:

$$\frac{35 \textit{whatsits}}{1\ \cancel{min}}\left(10\ \cancel{min}\right) = 350\ \textit{whatsits}$$

rate (time)= work

The second machine will produce:

$$\frac{50\ \textit{whatsits}}{1\ \cancel{min}}\left(10-5\right)\cancel{min} = 250\ \textit{whatsits}$$

rate (time) = work

We can then add together the sum of each machine's production:

$$350\ \textit{whatsits} + 250\ \textit{whatsits} = 600\ \textit{whatsits}$$

Answer: **C.**

Notice how the "minutes" on the bottom of the rate cancel with the time in "minutes"—you can always use labels to help you set up rate word problems correctly.

WHEEL PROBLEMS

Sarah rode a bike with wheels 36 inches in diameter. During 4 minutes of her ride, the wheels made 180 revolutions. At what average speed, in *feet per second*, did Sarah travel, rounded to the nearest *foot per second*?

A. 2 **B.** 7 **C.** 15 **D.** 135 **E.** 424

With problems like this, we first figure out what we NEED. **We need FEET PER SECOND. Here we use the fact that the word PER means DIVIDE. So we need** $\frac{\textbf{\textit{FEET}}}{\textbf{\textit{SECONDS}}}$.

To find that, we need to find the TOTAL FEET she traveled and the TOTAL SECONDS she traveled.

TIP: Average Rate is **always** TOTAL DISTANCE over TOTAL TIME.

Let's start with the feet. We have a 36" diameter wheel, but that's not the feet we need. We need the distance traveled. When a wheel travels, the distance it completes is equal to the circumference of the wheel times the number of rotations. With each rotation, the entire circumference touches the ground and equals the amount traveled. Imagine if your wheel was made of a piece of string. Cut it and lay it flat—you'll realize that's also the distance your wheel has traveled.

Again we need $C \times \left(180\,\textit{revolutions}\right)$.

To find C (circumference): $C = \pi d$ where d is the diameter. Let's first make our $36''$ diameter into feet: $36\,in = 3\,ft$.

Now we'll find the circumference:

$$C = \pi d$$

$$C = \pi\left(3\right) = 3\pi$$

This is the number of **feet** we travel **per revolution** (remember PER means DIVIDE!) Take that idea and plug into our equation $C \times (180\, revolutions)$:

$$3\pi \frac{ft}{\cancel{rev}} * 180\, \cancel{rev} \approx 1696\, ft$$

See how the revolutions "cancel" just as variables would? We call that cancellation "dimensional analysis"—making sure your units cancel is a great way to ensure you've set up the problem right.

Let's go back to our original need: $\frac{FEET}{SECONDS}$. We can now fill in the feet: $\frac{1696\, ft}{SECONDS}$.

To find the seconds, we convert 4 minutes to seconds using **dimensional analysis (if you've never done it, Google it!** In short, it's converting between different units):

First come up with the conversion factor: $\frac{60\, sec}{1\, min}$.

Because there are 60 seconds in 1 minute, these are equivalent values. We can multiply ANY number by this conversion rate and doing so won't change the "value" of what we find—i.e. it stands for the same amount, but with different labels. It's similar to multiplying by a giant "1" —when you multiply by one you don't change the value of something. If necessary, flip your conversion rate upside down—but make sure the denominator (bottom) has the same label as the number you're multiplying by (here "minutes" is that label, so we want minutes on the bottom so it will cancel):

$$(4\, min)\frac{60\, sec}{1\, min} = 240\, sec$$

Now we complete our original need item by filling in the missing seconds**:** $\frac{1696\, ft}{SECONDS}$ is $\frac{1696\, ft}{240\, sec} \approx 7$.

Answer: **B.**

TIPS IF YOU'RE STUCK:

1. Remember that oftentimes you have hidden equal quantities—for example a round trip flight is the same distance both directions; if two people start at the same time, their time is the same, etc.

2. Check your units—sometimes these problems will mix measurements (i.e. inches/feet, seconds/minutes). Use **dimensional analysis!**

3. Re-read the question—there may be one more detail you've missed. Don't forget these sometimes show up in three part questions. Look up if you're lost to find more information.

4. Seeing the word "distance" doesn't mean it's actually a distance/rate problem. It may be a Function as a Model problem—see Chapter 10 for identifying that question type. If you have a given equation, forget about the distance formula and use what the test gives you. If you have a given chart of information, forget about the formulas and use the given information first! Then turn to memorized formulas if you need more.