CHAPTER

8 AVERAGES

SKILLS TO KNOW

- Mean (Average), Median, Mode, and Range
- Stem-and-leaf plot
- Set transformations (adding members, multiplying/dividing members, increasing/decreasing members of a set) and how they affect mean, median, and mode
- Reading tables that involve these measures of central tendency
- Average rate word problems

MEAN, MEDIAN, MODE, RANGE

Mean (Average)

The words MEAN and the AVERAGE are the same thing: the average value of a set of numbers, found by adding all of the numbers in a set and then dividing by the number of items in the set!

Average of n items in a set:

$$\{x_1, x_2, \ldots x_n\} = \frac{x_1 + x_2 + \ldots + x_n}{n} \text{ or } AVERAGE = \frac{SUM\ OF\ ALL\ ITEMS}{NUMBER\ OF\ ITEMS}$$

The average of $6,\ 9, 15 = \frac{6+9+15}{3} = 10$

Median

If you line up a set of numbers in **numerical** (chronological) order, and you find the number physically in the middle of this list, that number is called the MEDIAN. If there are an even number of items in a list, then average the two middle values.

The median of $2, 6, 9,\ 15,\ 17$ is 9, because 9 is physically in the middle of the list.

The median of $2, 6, 9, 10, 15, 17 = 9.5$, because the average of $9\ \&\ 10$ is 9.5, and $9\ \&\ 10$ are physically in the middle of the list.

Mode

The mode occurs most often in a set. I like to think that **MO**de and **MO**st both start with "**MO**" to remember this one.

If you have more than one number that occurs the most (i.e. if three numbers occur five times each), **you can have MULTIPLE modes, unlike median and mean.**

Mode of $\{x_a, x_b, x_b\} = x_b$

Mode of the set $\{5,6,6,6,7,7,7\} = 6$ ***AND*** 7

Range

The range of a set of values is the difference between the greatest value and the least value.

The range of this set: $2, 6, 9, 10, 15, 17$ is found by subtracting the least number from the greatest one: $17 - 2 = 15$

Average/Mean Word Problems

On these problems, usually the normal equation for mean comes into play, but more often, your problem won't be as simple as this one:

In Los Angeles, the daily high temperatures in degrees Fahrenheit (F) over one week of June were $72°, 72°, 75°, 82°, 88°, 97°,$ and $104°$.

To the nearest degree Fahrenheit, what was the mean daily high temperature for that week?

A. $72°$ **B.** $74°$ **C.** $82°$ **D.** $84°$ **E.** $85°$

All we have to do here is add up all the numbers and divide by the number of temperatures—that's easy:

$$\frac{72+72+75+82+88+97+104}{7} = \frac{590}{7} = 84.285\ldots$$

Answer: **D.**

What's not so easy are problems that actually give you the sum, the number of terms, and maybe a few numbers that contribute to the sum. These kind of "average" problems are much more common.

METHOD 1: USE ALGEBRA

One way to solve them is to plug into your original average equation and then roll up your Algebra sleeves!

Mario has taken 4 of 6 equally weighted tests in his Biology class and has an average score of exactly 87 points. What must he score on the 5th and 6th test, on average to bring his average up to 90 points?

A. 96 **B.** 100 **C.** 93 **D.** 99 **E.** 94

We'll have to plug into the equation and solve for what we don't know. Let's assess what we know. We will build out multiple copies of the equation. Just write down the formula and plug in what you know. Repeat if necessary.

He's averaged 87 points (average) on 4 tests (number of items). Let's plug that into our equation:

$$AVERAGE = \frac{Sum\ of\ All\ Items}{Number\ of\ Items}$$

$$87 = \frac{Sum\ of\ First\ Four\ Tests}{4}$$

As you can see, I use plain words to represent what goes where—these are easier to understand than variables and the technique helps me more clearly set up my equation. Ok, so I can't solve for the individual test score, but I CAN solve for the sum of the first four test scores.

$$87(4) = Sum\ of\ First\ Four\ Tests = 348$$

Now let's go back to what we need. We need to know the average of the last two tests. We also know he's got 2 tests left (out of a total of 6 tests) to bring his grade up to a 90 (desired average). Let's start by creating a new equation incorporating in the information we know, again using our formula, but this time for the final grade equation, and filling in what we don't know with English. When in doubt, write it all out:

$$90(Desired\ Average) = \frac{348(Sum\ of\ First\ Four\ Tests) + \_\_ + \_\_\_(Sum\ of\ Last\ Two\ Tests)}{6(Number\ of\ Tests)}$$

Now I am going to think about the last two test scores—those two blanks. To make this easy, let's assume they are identical and equal to n—remember if you can't solve for a precise value you can make an assumption like this. Also, if they are identical, then n would also be their average. (If two values are equal, they are also their own average).

$$90 = \frac{348 + 2n}{6}$$

$$90(6) = 348 + 2n$$

$$540 = 348 + 2n$$

$$192 = 2n$$

$$n = 96$$

He needs an average of 96, because two tests of score 96 would get him the points he needs!

Answer: **A.**

METHOD 2: NO ALGEBRA METHOD

Another way to solve many of these problems involves no algebra at all.

The first step I take with this method is to start with some assumptions that could be true. First, I know his average on 4 tests is 87 points. I don't know what he got on each test, but I know that one possibility would be that he actually scored 87 on every single test. That's my first assumption. I write out blanks for each number I "know" and the two that I don't know and fill them in as so:

$$\underline{87}\ \ \underline{87}\ \ \underline{87}\ \ \underline{87}\ \ \_\_\ \ \_\_$$

My goal is to actually make this list look like six 90's in a row-- if he averaged 90 it would be as if he scored 90 on each test. With that in mind, now I add to each score that already exists "3" points—that is what he needs to "catch up" to 90 points for each of those tests, i.e. I'm three points short on each of those four tests, so he needs three more points for each time he got an 87.

Then for the last two tests, he will need the equivalent of 90 points for each. If I add the "catch up" three points for the first four tests, plus 90 points from the last two, that is the total amount of points he will need to have averaged 90:

$87\ 87\ 87\ 87\ \_\_\ \_\_$	This row represents the points he has so far on four tests.
$+3\ \ 3\ \ 3\ \ 3\ \ 90\ \ 90$	This row represents the number of points he still needs.
$90\ 90\ 90\ 90\ \ 90\ \ 90$	This is what I actually want his grade card to look like.

So now I add up all the numbers in that 2nd horizontal row:

$$3+3+3+3+90+90=192$$

I also know he needs to score 192 points in two test sittings—so the number of points, on average, per test he needs to score is 96 points per test (192 divided by 2):

$$192 \div 2 = 96$$

I can also think of this as redistributing the "catch up" points from the four 3's—I need 12 more points and I can distribute 6 of these on the first additional 90 point test and 6 on the 2nd to get 96 points per test remaining. Make sense?

Answer: **A.**

Median Problems

What is the median of the following set?

42, 33, 85, 60, 15, 29

A. 33 **B.** 37.5 **C.** 44 **D.** 60 **E.** 72.5

The first rule of medians: put everything in order!

$$42, 33, 85, 60, 15, 29 \text{ becomes } 15, 29, 33, 42, 60, 85$$

Now cross off equal numbers of numbers on the left and right to find the center values:

$$\cancel{15}, \cancel{29}, 33, 42, \cancel{60}, \cancel{85}$$

Because we have an even number of items in our set, we can now find the average of 33 and 42:

$$\frac{33+42}{2}=\frac{75}{2}=37.5$$

Answer: **B.**

Mode Problems

What is the mode of the following set?

0, 1, 3, 4, 8, 9, 9, 1, 2, 6, 3, 1, 5

A. 1 **B.** 3 **C.** 3.5 **D.** 4 **E.** 9

To identify the mode, we also want to rearrange numbers in order so we can be sure which occur most often—if the mode is very apparent though we can just count in place.

$$0, 1, 1, 1, 2, 3, 3, 4, 5, 6, 8, 9, 9$$

Here 1 occurs three times. It is the mode.

Answer: **A.**

STEM-AND-LEAF PLOT

Tammy surveyed the ages of 30 adults at her most recent family reunion using the stem-and-leaf plot below. What is the median age of these thirty adults surveyed?

Stem	Leaf
2	1, 2, 8, 9, 9
3	1, 2, 2, 2, 3
4	1, 1, 1, 2, 6, 7
5	1, 2, 5, 7, 7, 7, 8, 8, 8
6	4, 5, 7, 9, 9

Key:

Stem	Leaf
2	1

Represents "21"

Now I know what you're thinking—what is a stem and leaf plot!?! Look at the KEY! The ACT® is really nice about giving you clues to obscure ways of displaying information. The 1, 2, 8, 9, 9 in the first row stands for 21, 22, 28, 29, 29—the stem is the tens place, and the leaf is the ones place.

SPEED TIP: If you have an exceedingly long list of numbers in which to find a median and know the number of items in a list, you can count "up" or "down" to find the median. For example, if there are 20 numbers in a set, the median is the average of the 10^{th} and 11^{th} numbers in the set. Rather than count off 18 numbers (9 from the left, 9 from the right), just count in 9 numbers to get to the 10^{th} and 11^{th} terms and save some time.

Here I know there are 30 adults, so I know the median is the average of the 15^{th} and 16^{th} number. I count upwards 14, and circle the next two terms (15 & 16) then find their average.

Stem	Leaf	
2	~~1~~, ~~2~~, ~~8~~, ~~9~~, ~~9~~	
3	~~1~~, ~~2~~, ~~2~~, ~~2~~, ~~3~~	
4	~~1~~, ~~1~~, ~~1~~, ~~2~~, $\boxed{6, 7}$	15^{th} and 16^{th} numbers
5	1, 2, 5, 7, 7, 7, 8, 8, 8	
6	4, 5, 7, 9, 9	

46 and 47 are the 15th and 16th items in the list. Their average is 46.5.

Answer: 46.5.

SET TRANSFORMATIONS

> Each element in a data set is divided by 3, and each resulting quotient is then increased by 8.
> If the median of the final data set is 21, what is the median of the original data set?

With set transformations, think about what happens to a set of numbers—remember a median is often an actual number in a set when there are an odd number of items—and if we divide, multiply, add or subtract to the whole set, what we do to the median is the same as what we do to each number. Now, we might conclude if there is no "cannot be determined" then what is true for sets that include a median must be true for sets that do not. As such, we can do to the median what we do to the set and imagine that it will hold its position.

What we essentially have in a number set is a giant inequality:

$$a < b < c < d < \boldsymbol{n} < f < g < h < k$$

As long as we're dealing with the median, the relationship will hold because even if we had to flip the sign, it's still in the middle. We can multiply, divide, add or subtract to all the elements, and n will remain in the center position. The key to these problems is thinking things through: use logic, make up numbers, and seek to understand what is happening.

As such, we can simply do unto the median as to the rest of the set:

Divide it by 3, add 8, get the new data set version, 21:

$$\frac{n}{3} + 8 = 21$$

$$\frac{n}{3} = 13$$

$$n = 39$$

Answer: 39.

READING TABLES & AVERAGE RATE WORD PROBLEMS

The biggest problem students often have with average problems is the fact that oftentimes, they have to pay attention to information presented in unexpected ways. Be careful, watch your details, and think it through!

The other issue I commonly see students have is with the concept of average rates. An average rate is always the TOTAL of the first amount divided by the TOTAL of the second amount.

If asked to find an average rate:

AVERAGE RATE FORMULA

$$AVERAGE\ (ELEMENT\ A)\ per\ (ELEMENT\ B) = \frac{TOTAL\ OF\ ALL\ ELEMENTS\ A}{TOTAL\ OF\ ALL\ ELEMENTS\ B}$$

For example, if you're asked for Martha's average speed on her way to school in miles per hour, remember that PER means divide, and we want TOTAL miles over TOTAL hours, or TOTAL distance over TOTAL time.

AVERAGE SPEED FORMULA

$$Average\ Speed = \frac{TOTAL\ DISTANCE}{TOTAL\ TIME}$$

If you see the word "average" but you're really looking at a rate (anything with the word "per" or "for each/every", or even anything that could be replaced by the word "per"), remember you need the total of each independent element to find that average. Likewise if we want to find the average retail price of a (or PER) jelly donut sold by a particular donut chain, I could find this by adding up the TOTAL retail sales of all the donuts and dividing by the TOTAL number of donuts sold. This is still the SUM over number of ITEMS– it's the same concept as earlier average problems, but sometimes it's useful to understand it as the sum total of the first amount over the sum total of the second.

For each of 3 years, the table below gives the number of auctions held at an auction house, the number of items sold, and the auction house's income.

Year	Auctions	Items sold	Income
2007	45	405	$1,206,903
2008	38	266	$682,024
2009	42	504	$1,568,448

1. To the nearest dollar, what is the average income the auction house made on an item sold in 2007?
2. If, on average, an item between 2007-2009 cost the auction house $1,400 to acquire, process, and sell at auction, disregarding any other expenses, what was the average amount of profit per auction over the three-year period?

1. Here we want the average income per item, NOT the average overall income. DO NOT add up all the income column and divide by 3! PER MEANS DIVIDE!! If you have the word "per" translate that into a division bar! Remember we want the TOTAL income over the TOTAL number of items.

Divide just the 2007 income by the number of items sold to get the average per item.

$$\frac{2007\ income}{\#\ of\ items\ sold\ in\ 2007} = \text{average selling price per item, so } \frac{\$1{,}206{,}903}{405} = \$2{,}980$$

Answer: $\$2{,}980$.

2. Here we have to do more work. Let's figure out what we NEED:

"the average amount of profit per auction:"

Using the pattern I describe above, that will be TOTAL profits divided by TOTAL number of auctions:

$$NEED = \frac{TOTAL\ PROFITS}{TOTAL\ NUMBER\ OF\ AUCTIONS}$$

Total number of auctions is easy— that's three numbers off the chart added together.

Total profits will be more difficult. We know income but profit is different. Profit subtracts costs, so we need the costs in addition to income to figure out total profits. Costs is just a way of saying the "cost of doing business" or "expenses."

$$NEED = \frac{TOTAL\ INCOME - TOTAL\ COSTS}{TOTAL\ NUMBER\ OF\ AUCTIONS}$$

What makes this even more difficult is that we are told an average unit cost, not the total costs. So we must convert that average unit cost to calculate the total costs of doing business. If we know a unit rate, though, we can multiply by the number of units to get to the sum total.

Remember, $\frac{SUM}{\#of\ items} = average$

And by extension: $SUM = (average)(\#\ of\ \ items)$

We need the sum, or total costs (i.e. what it costs to do business), and can find that by using the average cost per item and multiplying by the number of items.

$$Total\ Costs = \frac{Average\ Expense}{Cost\ Per\ Item}\left(Total\ \#\ of\ Items\ at\ Auction\right)$$

The average expense cost per item is given $\$1,400$, and we can find the total number of items at auction by adding each number in the items sold column of the chart:

$$\begin{aligned} Total\ Costs &= (\$1400)(405+266+504) \\ &= (\$1400)(1175) \\ &= \$1,645,000 \end{aligned}$$

Let's head back to our "NEED" equation and fill in what we know. We can find the *total income* by adding all the elements in the income column, we just solved for *total costs*, and again the *total number of auctions* we can find by adding all numbers in that column:

$$NEED = AVG\ PROFIT\ PER\ AUCTION = \frac{TOTAL\ INCOME - TOTAL\ COSTS}{TOTAL\ NUMBER\ OF\ AUCTIONS}$$

$$= \frac{(\$1,206,903 + \$682,024 + \$1,568,448) - \$1,645,000}{(45+38+42)}$$

$$= \frac{\$1,812,375}{135}$$

$$Average\ Profit\ Per\ Auction = \$14,499$$

Answer: $\$14,499$.