CHAPTER

18 VECTORS

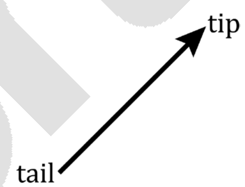
SKILLS TO KNOW

- Vector Terminology: Standard Position, Component Form, Magnitude, Direction
- Adding and subtracting vectors
- Finding the magnitude of a vector
- Scalar multiplication
- Unit vector
- Different Forms of Vectors

Vectors have appeared on the ACT® with slightly greater frequency over the past several years. Though many students learn these in physics class, not math class, for whatever reason, they're on the math portion of the ACT® exam. Still, vectors are far less important than many other areas on the test. If you're aimin for a 32+ on the math section, consider reviewing this chapter.

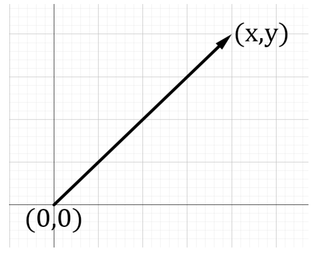
VECTORS DEFINED:

A vector is defined by a magnitude (length) and a direction. We represent vectors visually as line segments at a particular angle:



Direction is essentially the "angle" at which the vector is drawn, but we don't always measure it in degrees or radians. Rather, we think about how far over and how far up our vector extends, i.e. the x-direction movement and the y-direction movement. It's kind of like slope, but we keep the x and y parts separate.

We think of vectors much like we think of coordinate points. To represent a vector with symbols and variables, we write the vector in **component form**. Component form looks like a coordinate pair, but with funky dart shaped parentheses: $\langle x, y \rangle$. If we drew a directional line segment between $(0,0)$ and (x,y), for example, it could represent vector $\langle x, y \rangle$, showing the length of the vector (also called the **magnitude**) and the slope or angle at which that vector is drawn with respect to the axis.



A vector in **standard position** has a starting point at the origin such that the endpoint's coordinates are equal to the vector's component form. The above example, which starts at $(0,0)$ and ends at (x,y) is in **standard position**.

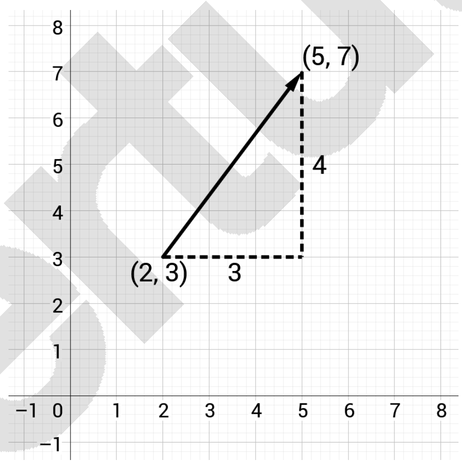
But the vector $\langle x,y\rangle$ does not have to start at $(0,0)$. We can move the vector around anywhere we want, but so long as it maintains the same **direction** (orientation "angle" wise), and has the same **magnitude** (i.e. length) it is the same vector, $\langle x,y\rangle$. I.e. placement on the graph does not define a vector: **length and direction alone define a vector**.

We can calculate the **component form** by finding the difference in x's and difference in y's: we take the endpoint's x and y values and subtract the starting point's x and y values, respectively.

COMPONENT FORM

$$\langle x_2 - x_1, y_2 - y_1\rangle$$

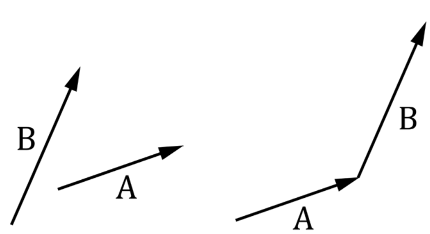
i.e. If a vector began at point $(2,3)$ and ended at point $(5,7)$ we could calculate its component form as follows:



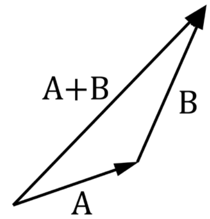
$$\langle 5-2, 7-3\rangle = \langle 3,4\rangle$$

ADDING AND SUBTRACTING VECTORS:

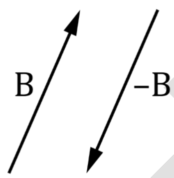
Graphically, we add vectors by connecting them "tip to tail:" with one's ending point as the other's starting point.



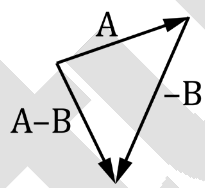
We then play "connect the dots" and draw a vector line segment between the starting coordinate of our first vector and the endpoint of our second vector, creating a triangle. The vector that starts at the first vector's starting point and goes to the second's ending point is the **resultant vector**.



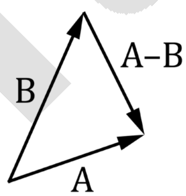
One way to subtract two vectors, visually speaking, is essentially to add the vectors, except first reverse the direction of the 2nd vector before adding (i.e. turn it around 180 degrees). For this problem, let's calculate Vector A minus Vector B. We first find $-B$ by **reversing our vector in the opposite direction**.



Then again we place them tip to tail:



Another way to think of subtraction is to put the lines "tail to tail":



To add or subtract vectors using the written representation is even easier: we just add or subtract the corresponding parts of the vectors. Don't forget to distribute the negative sign when subtracting!

Adding

$$\begin{aligned}\langle 3,5\rangle+\langle 7,6\rangle&=\langle 3+7,5+6\rangle\\&=\langle 10,11\rangle\end{aligned}$$

Subtracting

$$\begin{aligned}\langle 8,9\rangle-\langle 3,6\rangle&=\langle 8-3,9-6\rangle\\&=\langle 5,3\rangle\end{aligned}$$

If $\vec{a} = \langle -3,5 \rangle$ and $\vec{b} = \langle 4,7 \rangle$, what is $\vec{a} - \vec{b}$?

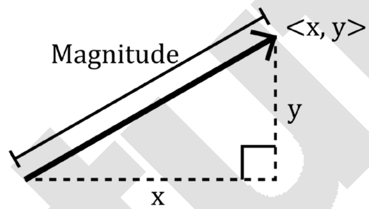
$\vec{a} - \vec{b} = \langle -3,5 \rangle - \langle 4,7 \rangle$ First, substitute in the vectors.

$= \langle -3-4, 5-7 \rangle$ Next, combine the corresponding terms.

$= \langle -7,-2 \rangle$ Finally, simplify.

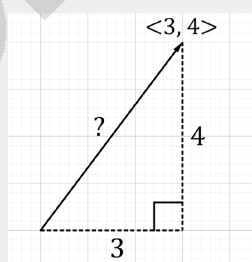
FINDING THE MAGNITUDE OF A VECTOR:

Again the **magnitude** (or **norm**) of a vector is the length of its arrow. Finding the magnitude of a vector is relatively simple. Remember a vector has an x and y value, an amount of horizontal movement and vertical movement. These movements happen at a 90 degree angle to each other. Recognize that a vector is graphically the hypotenuse of this movement, and you'll realize finding the magnitude of vector $\langle x, y \rangle$ simply requires finding the hypotenuse of a right triangle that has legs x and y.



We can find the magnitude of the vector using the Pythagorean theorem on the values in the vector's component form.

What is the magnitude of vector $\boldsymbol{n} = \langle 3,4 \rangle$?



We take the values 3 and 4 from the component form and plug them into the Pythagorean theorem:

$$\sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

TIP: Sometimes magnitude is denoted by what looks like absolute value signs, don't get confused!

For example: $|a|$ or $|\vec{a}|$ or $\|a\|$ or $\|\vec{a}\|$ **all mean the same thing**: the magnitude of vector a.

Also, the **magnitude** is also sometimes called the **norm** of a vector or the **Euclidian norm**.

If a vector is defined such that $\boldsymbol{t} = \langle 5,12 \rangle$, what is its **norm**, $\|t\|$?

All we need to do is find the hypotenuse of a triangle with legs 5 and 12, that's 13. I know this because I have memorized this Pythagorean triple. I can also solve using the Pythagorean theorem:

$$\begin{aligned} 5^2 + 12^2 &= \|t\|^2 \\ 25 + 144 &= \|t\|^2 \\ 169 &= \|t\|^2 \\ 13 &= \|t\| \end{aligned}$$

SCALAR MULTIPLICATION

It sounds intimidating, but scalar multiplication is actually very simple. It *scales* the vector to a different magnitude, keeping the same direction, but changing the length.

For example, $2\langle 5,12 \rangle$ will make a vector with a magnitude twice as large as the original. The component form of this vector will be the original vector's components times the scalar.

$$2\langle 5,12 \rangle = \langle 2(5), 2(12) \rangle = \langle 10,24 \rangle$$

Easy, just distribute the scalar!

If $n = \langle 2,4 \rangle$, what is $3n$?

$$\begin{aligned} 3n &= 3\langle 2,4 \rangle \\ &= \langle 3 \cdot 2, 3 \cdot 4 \rangle \\ &= \langle 6,12 \rangle \end{aligned}$$

UNIT VECTOR

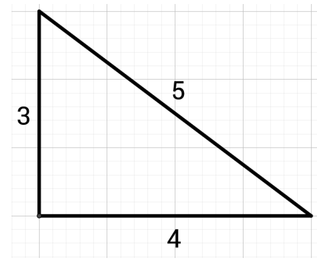
A unit vector is a vector with a magnitude of 1, similar to how the unit circle has a radius of 1. We can find the corresponding unit vector for any vector by scaling that vector down such that the magnitude is one.

To find the corresponding unit vector of any vector, you take two steps:

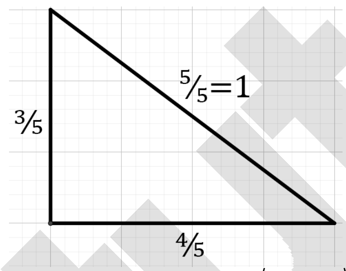
1. Find the magnitude
2. Divide each element in component form by the magnitude.

Why does this work? Essentially the "scalar" we multiply our component form by to scale it to the unit vector is the reciprocal of the magnitude. Remember the component form can form a triangle that shows the magnitude and the two sides.

Take for example this 3, 4, 5 triangle formed by the vector $\langle 3,-4 \rangle$:



If we want the corresponding unit vector, we need to shrink that hypotenuse to 1. What can we multiply it by to do so? $\frac{1}{5}$ (the reciprocal of the magnitude). To create a similar triangle that defines the unit vector, we then multiply both legs by $\frac{1}{5}$ as well:



The vector defined by the "legs" of this triangle is now $\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$. This is our unit vector.

MORE VECTOR NOTATION

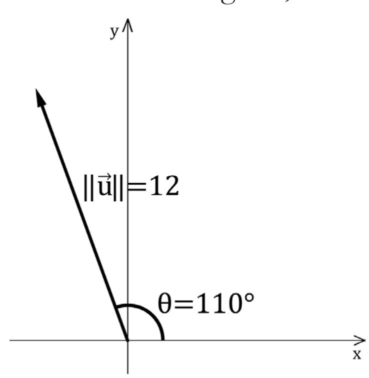
These don't come up very often on the ACT® if ever. Nonetheless, for overachievers, this might be good for a quick read through just in case. You never know what curve balls the ACT® has for you, especially in subjects like Vectors.

Magnitude and Direction Form

In addition to representing a vector using component form, we can also represent a vector using it magnitude and direction, where the magnitude is the geometric length of the arrow, and theta is the angle at which the vector is drawn in relation to the horizontal axis:

$$\|u\|, \theta$$

For example, I can define the angle below as 110 degrees, with a magnitude of 12:



Unit Vector $(\boldsymbol{i}+\boldsymbol{j})$ Form

Sometimes we express component form not with dart shaped parentheses, but **with the letters i and j** and with an addition sign in between them. Sometimes these letters have a little "hat" over them: $\hat{i}, \hat{j}$

For example, $\langle 3,4\rangle$ can be written as $3\boldsymbol{i}+4\boldsymbol{j}$ or $3\hat{i}+4\hat{j}$. This is called **Unit Vector Form**. We can also graph the "i" part on an axis labelled i instead of x and the j part on an axis labelled j instead of y.