CHAPTER

16 SOLIDS

SKILLS TO KNOW

- Cubes
- Cones
- Rectangular prisms
- Pyramid
- Triangular prisms
- Spheres
- Surface Area

Instead of flat shapes like circles, squares, and triangles, solid geometry deals with spheres, cubes, and pyramids (along with any other three dimensional shapes). Solid geometry problems most often involve **surface area** and **volume**.

Because Solids build on knowledge of 2D shapes, **I recommend you do this chapter after completing other geometry topics.**

BASIC CONCEPTS:

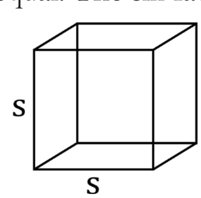
Volume: Volume is a three-dimensional assessment of space filled. We always multiple three dimensions in some way to find volume, with units cubed (cm^3, ft^3, etc.).

Surface Area: Surface area is the sum of all the individual areas of each side of a solid shape. Surface area uses units squared (in^2, m^2, etc.).

NOTE: With tough solids, the equation you need is usually included as part of a question (though not always). You should know cube, prism, and rectangular solid formulas in any case.

CUBES

A cube's height, length, and width are all equal. The six faces of a cube are also all congruent.



Surface Area $= 6s^2$, where s is the side length

Volume $= s^3$, where s is the side length

The volume of a rectangular solid is $v = lwh$. For cubes, because l, w, and h are all equal to s, you can simplify the equation to s^3.

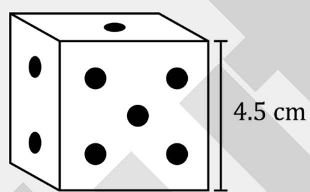
The surface area of a cube is $216\,\text{sq. in.}$. What is the volume of the given solid?

Using algebra and formulas, we can back track and find out the side length:

$$\text{Cube Surface Area} = 6s^2$$
$$6s^2 = 216$$
$$s^2 = 36$$
$$s = 6$$

If each side of a cube is 6 inches in length, we can find the volume of the solid by cubing our side:

$$\text{Cube Volume} = s^3$$
$$6^3 = 216\ \text{ft}^3$$



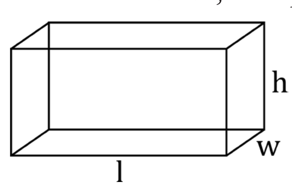
Jil is planning on making a paper model of a 6-sided die. How much paper, in square inches, would Jil need to if he wanted to make the side lengths $4.5\,\text{cm}$?

This problem is asking us to find the surface area of this cube.

$$\text{Cube Surface Area } = 6s^2$$
$$6(4.5)^2 = 121.5\,\text{cm}^2$$

RECTANGULAR SOLIDS (AKA RECTANGULAR PRISMS)

A rectangular solid (or rectangular prism) is essentially a box. It has three pairs of opposite sides that are congruent and parallel. A cube, as mentioned above, is a specific case of a rectangular solid.

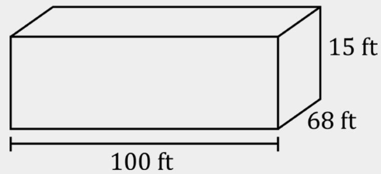


Volume $= lwh$, where l is the length of the figure, w the width, and h the height.
This formula is the same as finding the area of the base of the rectangle (lw) times the height (h).

Surface Area $= 2lw + 2lh + 2wh$

This formula finds the areas for all the flat rectangles on the surface of the figure (the faces) and adds those areas together.

In a rectangular solid, six faces cover the figure: three congruent pairs of opposite sides. To visualize this, I think of a book: the top and bottom, spine and length, and back/front cover. Find the individual areas of each of these three pairs, multiple each of those areas by 2, and sum.



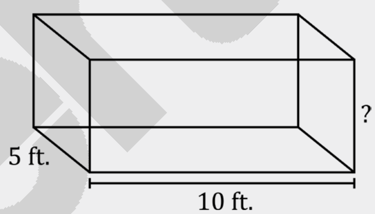
The local grocery store is planning on painting the inside of their newly renovated shop. The shape of the store is a rectangular prism that is 100 feet long, 68 feet wide, and with a ceiling height of 15 feet. Without counting the doors and windows of the store, how much paint would be needed, in square feet, to paint the ceiling and interior walls of the store?

Since we are not painting the floor, we only have to find the area of 5 of the 6 sides. First we know that the larger wall is $15\times100=1500\,\text{ft}^2$ and the smaller wall is $15\times68=1020\,\text{ft}^2$. The ceiling is $68\times100=6800\,\text{ft}^2$.

Next, we multiply by two for each of the wall sizes and add the ceiling:

$$2(1500)+2(1020)+6800=11{,}840\,\text{ft}^2$$

Answer: $11{,}840\ \text{ft}^2$.



A large fish tank in the shape of rectangular prism has base measurements of 10 feet by 5 feet. If the fish tank must contain 350 cubic feet of water, how deep, in feet, is the tank at minimum?

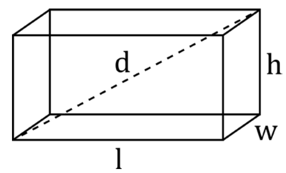
The total volume of the rectangular prism is 350 cubic feet. We can use algebra to figure out the last missing side minimum.

$$5\times10\times h=350$$
$$50h=350$$
$$h=7\,\text{ft.}$$

Answer: $7\,\text{ft.}$

3D DIAGONAL LENGTH (AKA THE SUPER PYTHAGOREAN THEOREM)

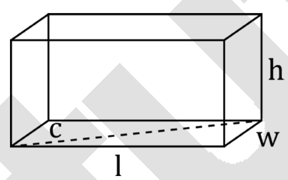
The diagonal of a rectangular solid is the longest interior line of the solid. It touches from the corner of one side of the prism to the opposite corner on the other.



THE SUPER PYTHAGOREAN THEOREM

The diagonal of a rectangular solid with sides l, w, and h is: $d=\sqrt{l^2+w^2+h^2}$.

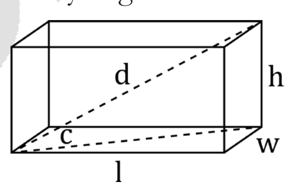
This formula is not given on the exam, but can come in handy on some problems. It's not that hard to remember, because it closely models the original Pythagorean theorem. However, you can also solve this problem without the formula **by breaking up the figure into two flat triangles and using the Pythagorean Theorem twice.**



First, create an equation for the length of the diagonal (hypotenuse) of the base of the solid using the Pythagorean theorem.

$$c^2=l^2+w^2$$

Now, create an equation for the 3D-diagonal. Use h and c as pictured as the "legs" of the triangle shown in 3 dimensions, and again use the Pythagorean Theorem.



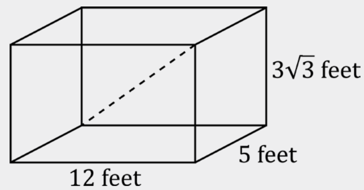
$$d^2=c^2+h^2$$

By substitution, we can now derive the Super Pythagorean Theorem, substituting in for c^2 in the 2nd equation, using the first equation:

$$d^2=\left(l^2+w^2\right)+h^2$$

$$d=\sqrt{l^2+w^2+h^2}$$

Engineers are looking to strengthen a portion of a bridge with a metal beam that will be a diagonal of a rectangular prism frame. How long does the beam have to be to fit the frame section shown below?



Using the Super Pythagorean Theorem:

$$\sqrt{12^2+5^2+\left(3\sqrt{3}\right)^2}=\sqrt{144+25+27}$$
$$=\sqrt{196}$$
$$c=14$$

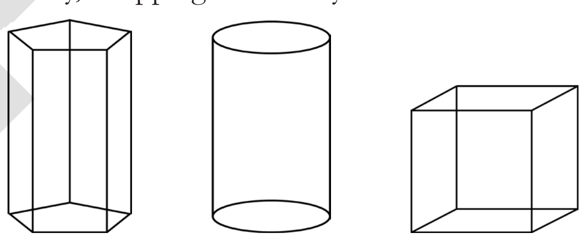
We could also use the regular Pythagorean theorem to find the missing diagonal. First, identify the triangle created by the legs (5 and 12) and diagonal of the rectangular base. Seeing 5 and 12, we can solve for the hypotenuse quickly as we know this is a Pythagorean triple: 5-12-13. Thus, the diagonal of the base is 13 feet. Next, we must deal with the last triangle whose legs are the height of the figure ($3\sqrt{3}$) and the base's diagonal:

$$13^2+3\sqrt{3}^2=c^2$$
$$169+27=c^2$$
$$196=c^2$$
$$14=c$$

Answer: 14 ft.

PRISMS

A prism is a three-dimensional shape that has (at least) two congruent, parallel bases. Basically, you could pick up a prism and carry it with its opposite sides lying flat against your palms, or create one using a Play-Doh™ Fun Factory, chopping off what you extrude at $90°$ angles.

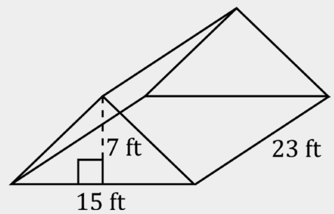


(A few of the many different kinds of prisms.)

The **volume of a prism** (when at a right angle to the ground) $= \boldsymbol{Bh}$, where B is the area of the base and h is the height.

This formula works for rectangles, as described earlier, and when the base is another shape (i.e. a circle, star, triangle, etc.)

The isosceles right triangular prism will be the top cover of a shading structure at the local park. With the given dimensions below, how much material, in square feet, is needed to create the cover's sides, front, and back panels? (Note that the cover will not have the bottom piece.)

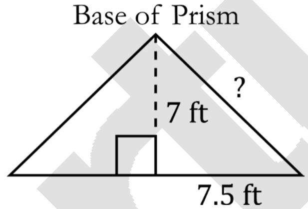


First, let's find the surface area of the individual shapes that make up the solid. The triangle bases have an area of:

$$\frac{15\times7}{2}=52.5\text{ ft}^2$$

Next, we must use the Pythagorean theorem to find the missing sides of the triangle, which are also the missing lengths of the rectangles. Because the base triangle is isosceles, we know 7ft is a perpendicular bisector of the base, 15.

Base of Prism



$$7^2+7.5^2=c^2$$
$$105.25=c^2$$
$$10.26\approx c$$

Now we find the area of the two sides of the shade cover.

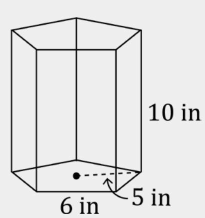
$$2(10.26\times23)=235.98\text{ ft}^2$$

When we add the two areas we will have the surface area we are looking for.

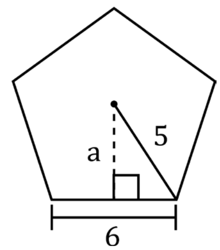
$$235.98+52.5=288.48\text{ ft}^2$$

Answer: 288.48 ft^2.

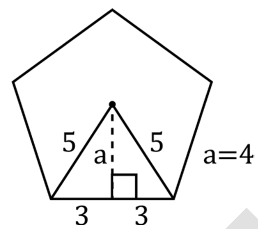
The figure shown below is a pentagonal prism. The base of the prism is a regular pentagon with dimensions shown below. Find the volume of this solid in cubic inches.



We must first find the area of the base pentagon. To do so we want the apothem: the distance from the center to the side at a 90-degree angle. We can draw this line on the pentagon base and see that it forms a right triangle with our given length 5.



We also know it bisects the side that is 6 inches (knowing that the pentagon is regular, we know all lengths will be proportional).



We now see that the triangle has a leg of 3 and hypotenuse of 5. Using our Pythagorean triples, we notice it is a 3-4-5 triangle, so the missing length, the apothem, must be 4. Next, we use this apothem in our formula for the area of a regular polygon:

$$\frac{1}{2}(apothem)(perimeter) = \frac{1}{2}(4)((5)(6)) = 60 \text{ in}^2$$

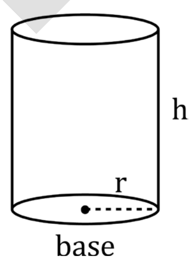
Next we multiply the base's area by the solid's height to find its volume:

$$60 \times 10 = 600 \text{ in}^3$$

Answer: 600 in^3.

CYLINDERS

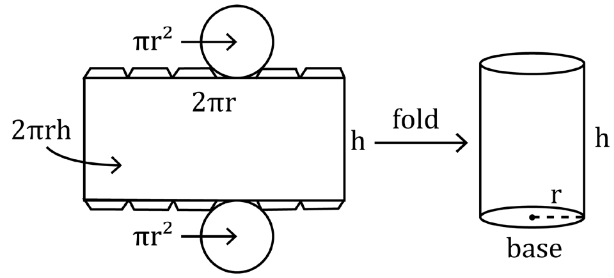
A cylinder is a prism with two circular bases on its opposite sides:



Volume $= \pi r^2 h$, where π is the universal constant (≈ 3.14), r is the radius of the circular base, and h is the height of the cylinder (the straight line drawn connecting the two circular bases).

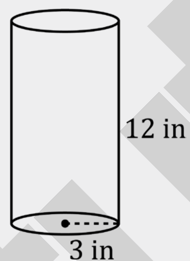
Surface Area $= 2(\pi r^2) + 2\pi r(h)$, where π is the universal constant (≈ 3.14), r is the radius of the base, and h is the height of the cylinder. This is equal to the areas of the two circular bases, $2(\pi r^2)$, added to the area of the side, which if unrolled, looks like a rectangle whose height (h) is the same as the cylinder's height, and whose base is the same as the circumference of the circle ($2\pi r$).

Imagine tearing apart a cylindrical oatmeal box at the seam. You can see how the side is derived:



Total Area$=2\pi r^2+2\pi rh$

A company is shipping t-shirts inside tube packages. The dimensions of the cylindrical package are shown in the figure below. If each t-shirt requires 75 cubic inches of space, what is the maximum number of t-shirts that could fit inside the packaging.



First, let's find the volume of the cylinder.

$$\text{Volume} = \pi r^2 h$$

$$\pi 3^2 \times 12 = 108\pi \approx 339.29 \text{ in}^2$$

Next, we divide our cylinder's volume by 75 to find out how many shirts can fit in our tube.

$$339.29 \div 75 \approx 4.52$$

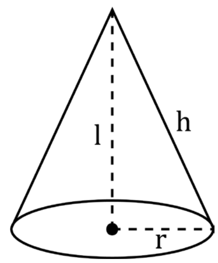
Remember: be careful with rounding on word problems asking for "whole" amounts. Here we don't want to round up, because we can't fit 5 shirts if there is only room for 4.5!

We can fit a maximum of 4 whole shirts into our tube packaging.

Answer: 4.

CONES

A cone is similar to a cylinder, but has only one circular base instead of two. Its opposite end terminates in a point (called an apex), rather than a circle. There are two kinds of cones—right cones and oblique cones. For the purposes of the ACT, focus on right cones. Oblique cones are unlikely to appear on the ACT. In a right cone, when a height (h) is dropped from the apex to the center of the circle, it makes a right angle with the circular base.

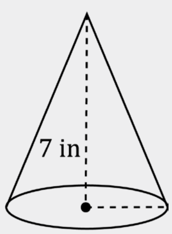


Volume $=\left(\frac{1}{3}\right)\pi r^2 h$, where r is the radius of the base, and h is the height at a right angle to the base.

The volume of a cone is $\frac{1}{3}$ the volume of a cylinder. This makes sense logically, as a cone is basically a cylinder with one base collapsed into a point.

***Surface Area** $=\pi r^2+\pi rl$, where l is the slant height extending from the apex to the circumference of the circular base The surface area is the combination of the area of the circular base (πr^2) and the lateral surface area (πrl).

*(This is rarely tested on the ACT®, and the equation will likely be given to you should you need it)



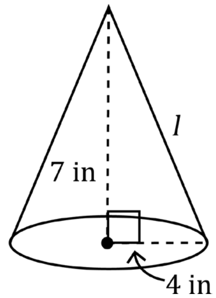
Raylene is creating party hats that are in the shape of right circular cones, but open along the bottom. She wants the height of the cone to be 7 inches tall with a circumference of the base circle 8π inches. With this information, approximately how much material, rounding up to the nearest square inch, would Raylene need to make 5 party hats of this size?

(Note: The total surface area of a right cone is equal to $\pi r^2+\pi rl$, where π is a constant(≈ 3.14), r is the radius, and l is the slant height extending from the apex to the circumference of the circular base, and πr^2 is the area of the circular base.)

We need the lateral surface area that covers the side of the cone, but we are given the formula for the **total** surface area of a cone. Thus, we must subtract off the base (area of a circle) from this formula to isolate the lateral surface area.

Total surface area of a cone (including its base): $\pi r^2+\pi rl$
Surface area of the "party hat" portion of a cone: $\pi r^2+\pi rl-\pi r^2=\pi rl$

First we must find the hypotenuse of the triangle represented by the dotted lines and cone side (a.k.a. l, the slant height of the cone). If the circumference of the circular base is $8\pi=2\pi r$, then $r=4$. Use the Pythagorean theorem to find l, using the radius, $r=4$ in. and the height 7 in. as shown:



$$4^2 + 7^2 = l^2$$
$$65 = l^2$$
$$8.06 \approx l$$

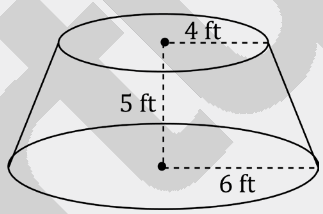
Next, we use the formula $\pi r l$:

$$8.06 \times 4\pi = 32.34\pi \text{ in}^2 = 101.5 \text{ in}^2$$

Per the instructions in the question, this rounds up to 102 in^2.

Answer: 102 in^2.

The city is erecting a statue and needs to know how big the base of the statue will be. The shape of the base is a right circular cone with its top removed with dimensions given below. What is the volume of the base of the statue in cubic feet? (Note: the formula for a cone with its top removed is $V = \frac{1}{3}\pi h\left(R^2 + r^2 + Rr\right)$ in which R is the larger radius and r is the shorter radius.)



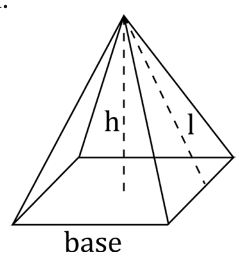
We have all the necessary dimensions, all we have to do is plug in our values:

$$\frac{1}{3}\pi(5)\left(6^2 + 4^2 + 24\right) = 126.67\pi \text{ ft}^3$$

Answer: $126.67\pi \text{ ft}^3$.

PYRAMIDS

Pyramids are geometric solids that are similar to cones, except that they have a polygon for a base and flat, triangular sides that meet at an apex.



There are many types of pyramids, defined by the shape of their base and the angle of their apex, but for the sake of the ACT®, square pyramids occur most often.

A right, square pyramid has a square base (each side has an equal length) and an apex directly above the center of the base. The height (h), drawn from the apex to the center of the base, makes a right angle with the base.

If you encounter a pyramid, you'll likely be given the formulas you need in the question.

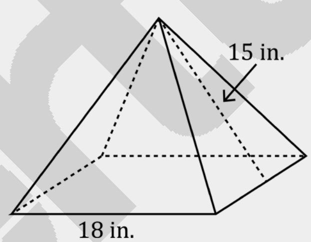
$$\text{Volume} = \frac{1}{3}(\text{area of base})h$$

To find the volume of a square pyramid, you could also say $\frac{1}{3}lwh$ or $\frac{1}{3}s^2h$, as the base is a square, so each side length is the same.

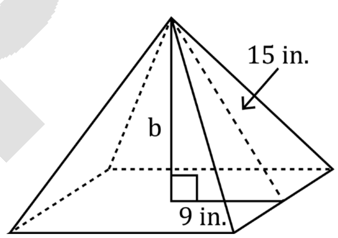
SURFACE AREA OF A PYRAMID

Surface Area = Area of Base + No. of Lateral Triangles($\frac{1}{2}$(Slant length × Base Length))

A regular pyramid with a square base is shown below with a slant height of 15 inches and base side length of 18 inches. What is the volume of the pyramid, in cubic inches?



We must find the height of the pyramid. We can do this by drawing a line from the top vertex of the pyramid to the center of the square base. This creates a right triangle with a hypotenuse of 15 inches and one leg measurement of 9 inches. We use the Pythagorean theorem to determine the length of the other leg.



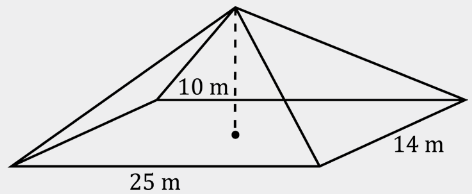
$$9^2 + b^2 = 15^2$$
$$81 + b^2 = 225$$
$$b^2 = 144$$
$$b = 12$$

Now that we know the height of the pyramid, we can find the volume of this solid.

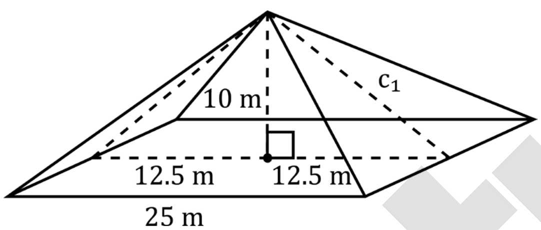
$$\frac{1}{3}(18)(18)(12) = 1296 \text{ in}^3$$

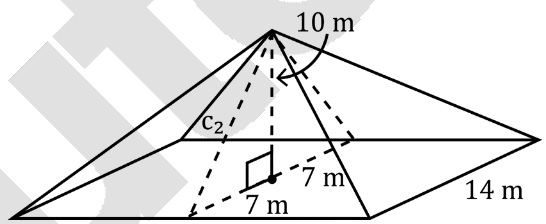
Answer: 1296 in^3.

A homeowner plans to add solar panels onto his house's roof. The roof is shaped as a rectangular pyramid with the dimensions below. He plans to add solar panels on all four sides of his roof. What is the approximate surface area of the regions of the roof that will have solar panels on them?



Since the homeowner is only going to put solar panels on the triangular sides of his roof, we only need to find the surface area of the four triangles that make up the pyramid. First, we must find the two slant lengths of the pyramid. We do this by using the Pythagorean theorem and finding a triangle parallel to the sides of the base through the center:

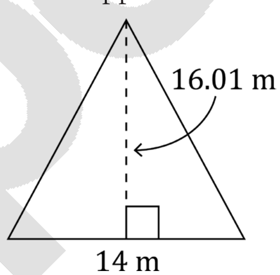


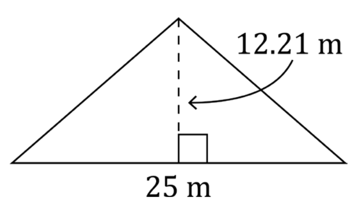


$$10^2 + 12.5^2 = c_1^{\,2}$$
$$256.25 = c_1^{\,2}$$
$$16.01 \approx c_1$$

$$10^2 + 7^2 = c_2^{\,2}$$
$$149 = c_2^{\,2}$$
$$12.21 \approx c_2$$

After we have the slant lengths, we know that these are the heights of our individual triangle surfaces. We can find the area of the triangles, and since we know this is a rectangular pyramid, there are two of the same triangles on the opposite side. So we double the area of each triangle face.



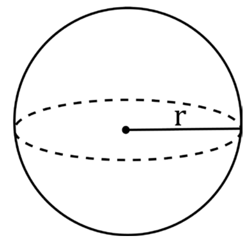


$$2\left(\frac{16.01 \times 14}{2}\right) + 2\left(\frac{12.21 \times 25}{2}\right) = 529.39\ \text{m}^2$$

Answer: $529.39\ \text{m}^2$ is approximately $530\,\text{m}^2$.

SPHERES

A sphere is essentially a 3D circle.



$$\textbf{Volume} = \frac{4}{3}\pi r^3$$

$$\textbf{Surface Area} = 4\pi\left(r\right)^2$$

> How much paint coverage, in square units, is needed to completely paint a spherical ball with a circumference of 36π units at its widest point? (Note: the Surface Area of a Sphere equals $4\pi\left(r\right)^2$.)

We can use the circumference to find out the radius of our sphere. If 36π is the circumference, $C = 2\pi r$, so we plug in:

$$36\pi = 2\pi r$$

$$18 = r$$

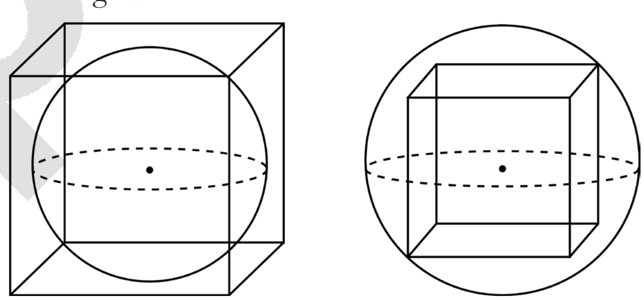
Now we just plug that into our given surface area formula:

$$4\pi\left(r\right)^2 = 4\pi\left(18\right)^2 = 1296\pi \text{ sq. units}$$

Answer: 1296π sq. units.

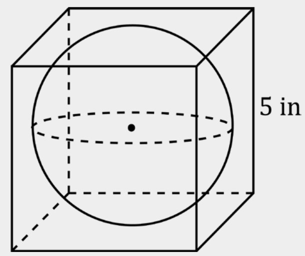
INSCRIBED SOLIDS

Sometimes, solid shapes are placed inside other solids, such as the sphere in the cube on the left, or the cube in the sphere on the right.



When dealing with inscribed shapes, draw on the diagram they give you, and **figure out what lengths are equal when surfaces touch.** For instance, on the left, **the diameter of the circle is the length of the cube side**. On the right, **the diagonal of the cube is the diameter of the sphere**. Always draw out inscribed shapes if no picture is given.

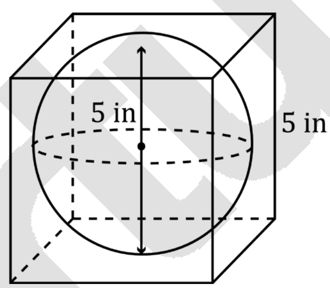
In the figure below, a sphere is inscribed within a cube with side measurements of 5 inches. Find the total volume of the empty space between the sphere and cube in cubic inches.



For this problem, we need to find the volume of the outer cube and the volume of the sphere and find the difference of the two to find out the volume of the void space. First we find the volume of the cube, s^3:

$$5^3 = 125 \text{ in}^3$$

Next, we must find the measurements of the sphere to find its volume. Since the sphere is inscribed, we know that its diameter is also the square's side length, 5.



$$V = \frac{4}{3}\pi\left(2.5\right)^3$$

$$V = \frac{62.5}{3} \times \pi$$

$$V \approx \frac{62.5}{3}\pi \ in^3$$

This means that the radius of the sphere is 2.5 inches, or half the diameter. Now we can find the volume using the formula:

$$125 - \frac{62.5}{3}\pi = 125 - \frac{125}{6}\pi \ in^3$$

Now just subtract the sphere's volume with that of the cube.

Answer: $125 - \frac{125}{6}\pi \ in^3$.