CHAPTER

18 RATIONAL EXPRESSIONS AND EQUATIONS

SKILLS TO KNOW

- Simplifying expressions by factoring/canceling
- Finding a common denominator
- Combining unlike terms
- Simplifying rational expressions containing square roots

WHAT IS A RATIONAL EXPRESSION?

Rational expressions are fractions that have a polynomial either in the numerator, denominator, or both. To make identifying them easier, just look for **fractions that include variables**. Questions on rational expressions usually ask you to simplify or rewrite the expression or to add, subtract, multiply or divide fractions that contain variables.

Examples of rational expressions:

$$\frac{m^2-9}{3m^3+7m-2};\ (x+1)(x-3)^{-3};\ \frac{(x+2)(2x-5)}{(7x-1)(x+4)}$$

WAYS TO SIMPLIFY RATIONAL EXPRESSIONS

1. FACTORING AND CANCELING

We can factor and cancel out factored elements in rational expressions. For example, asked to simplify $\frac{x+2}{(x+2)(x-4)}$, we can simply cancel out the $x+2$ in the numerator and denominator to get $\frac{1}{x-4}$. Be careful to note, however, if you do cancel that term, x *cannot equal* -2. Remember that we can't divide by zero, so any factor in the bottom of a fraction must not be equal to zero. Because negative 2 would make the original expression undefined, it must be excluded from the possible x values in our answer.

When we cancel the pieces of a rational expression, we may need to keep track of the elements that we have removed to ensure we don't find extraneous solutions. That means taking note of any values that could make the denominator equal to zero, even if they cancel.

Simplifying Fractions with Factoring

When $\frac{x^2+6x}{x^3+x^2-30x}$ is defined, it is equivalent to which of the following expressions?

A. $\frac{6}{x-30}$ **B.** $\frac{1}{x^2-5}$ **C.** $\frac{1}{x-5}$ **D.** $\frac{1}{x+6}$ **E.** $-\frac{1}{5}$

First, I take note of the question's wording: **when the function "is defined."** This little phrase is very helpful: it means I don't have to worry about extraneous solutions or keeping track of elements I cancel.

Now, let's completely factor both the numerator and denominator so we can see what cancels out:

$$\frac{x^2+6x}{x^3+x^2-30x}$$

$$\frac{\cancel{x}\cancel{\left(x+6\right)}\ 1}{\cancel{x}\cancel{\left(x+6\right)}\left(x-5\right)}$$

Crossing out an x and an $x+6$, we get:

$$\frac{1}{x-5}$$

Answer: **C.**

2. FORMING COMMON DENOMINATORS

Like regular fractions, rational expressions need to have common denominators before they are added or subtracted.

Which of the following expressions is equivalent to the expression $\frac{3}{n}+\frac{n}{8}-\frac{6}{r}$?

A. $\frac{n-3}{n+8-r}$ **B.** $\frac{24r+n^2r-48n}{8nr}$ **C.** $\frac{n-3}{8nr}$ **D.** $\frac{11n+n^2r-2n}{8nr}$ **E.** $-144n^2r$

To combine these terms we need a common denominator. First, we find the least common multiple (LCM) of the denominators (n, r, and 8) (remember LCM and LCD are essentially the same). Because we have no common factors, we simply multiply all these terms together to find the LCM. This is simply $8nr$. Now, we turn each individual term into an equivalent fraction by multiplying by the missing "pieces" of this LCM on both the top and bottom. Remember, when we multiply a fraction by the same amount on the top and bottom, we are multiplying by one: it doesn't change the "value" of the individual piece. Here, the first term is missing is $8r$, so the top and bottom are multiplied by $8r$. The center term is missing nr, so it is multiplied on the top and bottom by nr, and the final term is missing $8n$, so it is multiplied on the top and bottom by $8n$.

$$\frac{3\left(8r\right)}{n\left(8r\right)}+\frac{n\left(nr\right)}{8\left(nr\right)}-\frac{6\left(8n\right)}{r\left(8n\right)}$$

After expanding we are left with:

$$\frac{24r}{8nr}+\frac{n^2r}{8nr}-\frac{48n}{8nr}$$

Which simplifies to:

$$\frac{24r+n^2r-48n}{8nr}$$

Answer: **B.**

These methods can also be applied to more complex rational expressions.

Simplify $\dfrac{2-\dfrac{3}{x+1}}{\dfrac{2}{x-1}+1}$.

First, we find a common denominator for all the terms in the numerator, and then the same for the denominator. In other words, we want to turn the numerator into a single fraction, and the denominator into a single fraction:

$$\frac{2*\dfrac{(x+1)}{(x+1)}-\dfrac{3}{x+1}}{\dfrac{2}{x-1}+1*\dfrac{(x-1)}{(x-1)}}$$

$$\frac{\dfrac{2(x+1)-3}{x+1}}{\dfrac{2+(x-1)}{x-1}}$$

$$\frac{\dfrac{2x+2-3}{x+1}}{\dfrac{2+x-1}{x-1}}$$

$$\frac{\dfrac{2x-1}{x+1}}{\dfrac{x+1}{x-1}}$$

Now we can use the principal of dividing fractions to further simplify. The "fraction" in the numerator is being divided by the "fraction" in the denominator.

$$\frac{2x-1}{x+1}\div\frac{x+1}{x-1}$$

Therefore, we can multiply by the reciprocal of $\dfrac{x+1}{x-1}$ to calculate this division:

$$\frac{2x-1}{x+1}\times\frac{x-1}{x+1}=\frac{(2x-1)(x-1)}{(x+1)^2}$$

3. SIMPLIFYING RADICAL EXPRESSIONS INVOLVING SQUARE ROOTS

To simplify fractions containing square roots, we must eliminate radicals in the denominator. This is usually achieved by multiplying the original expression by the **conjugate** divided by itself, i.e. we use the idea of equivalent fractions to multiply by an amount equal to 1 that will eliminate the square root in the denominator. If we wanted to simplify $\frac{1}{5+9\sqrt{3}}$ we would need to figure out the conjugate of its denominator. For example, the conjugate of $5+9\sqrt{3}$ is $5-9\sqrt{3}$. Now we would multiply the original expression by the conjugate over the conjugate. In this case, it would be $\frac{5-9\sqrt{3}}{5-9\sqrt{3}}$.

Here we could proceed by multiplying the original expression by the **conjugate over the conjugate**:

$$\frac{1}{5+9\sqrt{3}} * \frac{5-9\sqrt{3}}{5-9\sqrt{3}}$$

The idea of conjugates is based on the special product, **The Difference of Squares** (or the Product of a Sum and a Difference as some may call it). (We cover special products in the FOIL and Factoring Chapter of this book).

The Difference of Squares

$$a^2 - b^2 = (a-b)(a+b)$$

Knowing and applying this pattern, we can see how the square roots in the denominator disappear as we continue to simplify:

$$\frac{5-9\sqrt{3}}{5^2-81(3)}$$

$$\frac{5-9\sqrt{3}}{(25-243)}$$

$$\frac{5-9\sqrt{3}}{(-218)}$$

In the denominator, our middle terms cancel, and we end up squaring both the a and b terms. The root "shifts" to the numerator, and our expression is simplified. I realize this is a bit confusing. My best advice is to study the examples below and practice!

The expression $\frac{11}{10-\sqrt{7}}$ is equivalent to:

A. $\frac{110+11\sqrt{7}}{93}$ **B.** $\frac{110+\sqrt{7}}{107}$ **C.** $\frac{100+\sqrt{7}}{100}$ **D.** $\frac{100+\sqrt{7}}{93+20\sqrt{7}}$ **E.** $\frac{110+11\sqrt{7}}{10\sqrt{7}}$

$\frac{11}{10-\sqrt{7}}$ can be simplified by multiplying the fraction by the denominator's conjugate, $\frac{10+\sqrt{7}}{10+\sqrt{7}}$, to get $\frac{11}{10-\sqrt{7}} * \frac{10+\sqrt{7}}{10+\sqrt{7}}$.

This simplifies to $\frac{11\left(10+\sqrt{7}\right)}{\left(10-\sqrt{7}\right)\left(10+\sqrt{7}\right)}$.

Now, let's use FOIL to expand the bottom (you can also use knowledge of the Difference of Squares pattern if you prefer).

$$\frac{110+11\sqrt{7}}{100-10\sqrt{7}+10\sqrt{7}-7}$$

$$\frac{110+11\sqrt{7}}{100-7}$$

$$\frac{110+11\sqrt{7}}{93}$$

Answer: **A.**