CHAPTER

23 TRIGONOMETRY GRAPHS

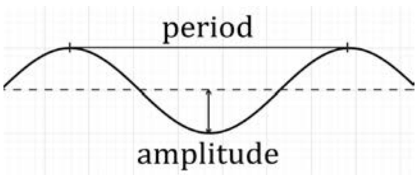
SKILLS TO KNOW

- The shape and orientation of basic trigonometric function graphs on the coordinate plane
- Standard equations for trig graphs and their basic transformations (scalars, coefficients, and constants)
- Graphs of inverse trigonometric functions (goal of 34+ only)
- Finding minimum and maximum values

GRAPHS OF BASIC TRIGONOMETRIC FUNCTIONS

If you're aiming for a top score, I recommend you become familiar with the basic graph forms of trig functions. If you have them memorized, these questions are much easier.

Before we get too far, let's define some terms:

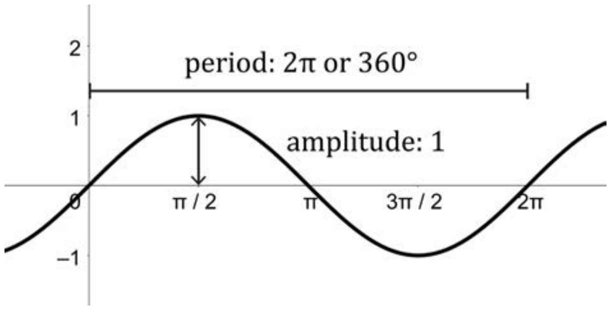


Amplitude: The amplitude is the vertical height from the midline you could draw horizontally through a sin or cos function and the maximum or minimum of the graph. You can also think of it as the half height of the wave. Note: the graph of tan has no amplitude.

Period: The period of a trig graph is horizontal distance it takes the graph to complete one full cycle. For sin and cos: measuring from the midline you'll trace one wave up to the maximum and one wave down to the minimum and then back to the midline. Alternatively, you can measure the distance between two maximum points (shown below) or two minimum points. For tan, it means the portion of the graph between two adjacent asymptotes or one single "squiggly" line that approaches positive and negative infinity. (More on tangent on the next page.)

Sine Function

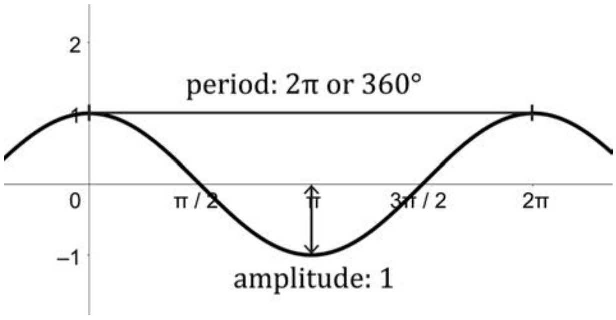
The sine function, which is the ratio of the opposite to the hypotenuse, starts at 0 and moves between y-values of 1 and -1 in a sinusoidal pattern (in other words, it looks like a wave).



$\sin(0)=0$ (passes through the origin)

Cosine Function:

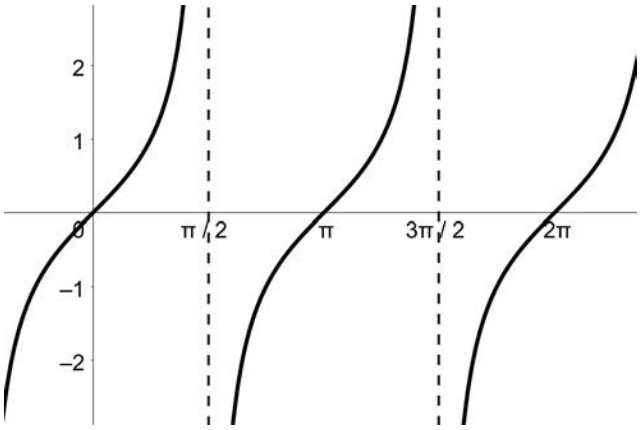
The cosine function, which is the ratio of the adjacent to the hypotenuse, starts at 1 and moves between y-values of -1 and 1 in a sinusoidal pattern (aka in a wave).



$\cos(0)=1$, passes through $(0,1)$, not the origin.

Tangent Function

The tangent function, which is the ratio of the opposite to the adjacent, has a range of $-\infty$ to ∞.



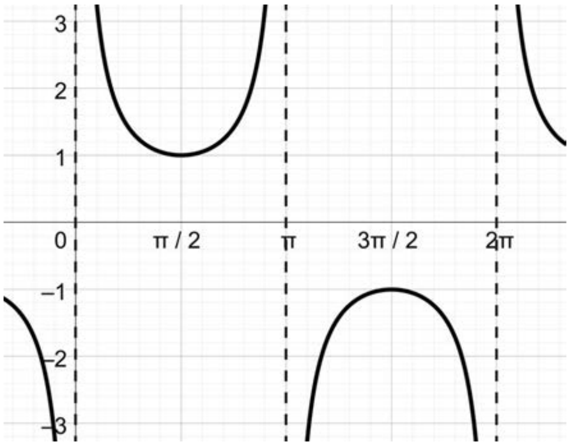
Period: π radians or 180°.
Amplitude: N/A
Passes through origin $(0,0)$ with a complete cycle centered at the origin.

LESS COMMON TRIG GRAPHS

If you don't want a 34-36, skip these!

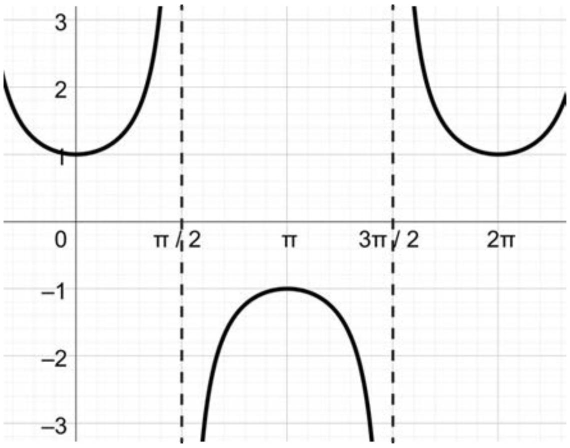
Cosecant Function

The cosecant function is the reciprocal of sine, and therefore the ratio between the hypotenuse and the opposite. It starts at ∞ and has a range of $-\infty < y \leq -1$ and $1 \leq y < \infty$, with a period of 2π radians or 360°. It doesn't have an amplitude.



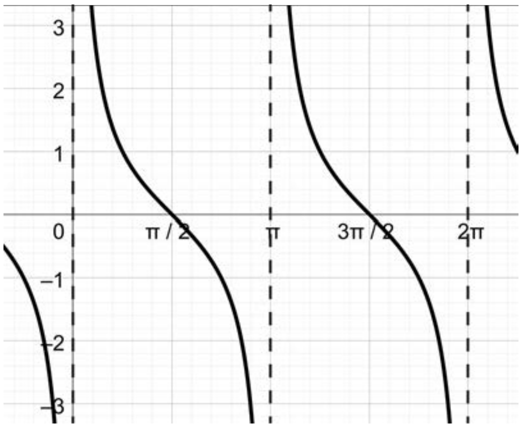
Secant Function

The secant function is the reciprocal of cosine, and therefore the ratio between the hypotenuse and the adjacent. It starts at 1 and has a range of $-\infty < y \leq -1$ and $1 \leq y < \infty$, with a period of 2π radians or 360°. It has no amplitude.



Cotangent Function

The cotangent function is the reciprocal of cosine, and therefore the ratio between the adjacent and the hypotenuse. It starts at ∞ and has a range of $-\infty$ to ∞ with a period of π radians or 180°.

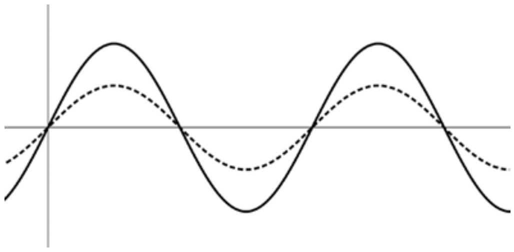


BASIC TRIGONOMETRIC TRANSFORMATIONS

Most trigonometric graph problems require you to know the parent equation of the graph and how each constant affects the graph. As with all translations, remember you can always shift a trig graph to the right using $(x-h)$ or the left using $(x+h)$. Similarly, you can shift it up by replacing y with $(y-k)$ and down by replacing y with $(y+k)$. In the standard forms below, the sign on the k will be flipped as it is on the opposite side of the equation as y. Here's an overview of the variables and constants in the standard equation forms:

	sin	cos	tan
Parent Equation	$y = a\sin b(x-h)+k$	$y = a\cos b(x-h)+k$	$y = a\tan b(x-h)+k$
Amplitude	a	a	undefined
Period	$\frac{2\pi}{b}$	$\frac{2\pi}{b}$	$\frac{\pi}{b}$
Phase Shift (horizontal)*	h to the right	h to the right	h to the right
Vertical Shift*	k up	k up	k up

*If h or k are negative numbers, then the shift is in the opposite direction.

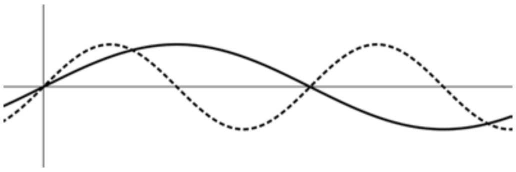


You should be able to look at graphs of different trig functions and see how one is a transformation of the other. Here, for example, we have two sin waves. The solid line is taller, so it has a greater amplitude. Thus its equation should have a larger number in the "a" position of its equation, $y = a\sin b(x-h)+k$. Conversely, the dotted line should have a smaller number in the "a" position of its equations. Since the graphs appear to have the same period, the rest of the values in their equations are likely identical to each other.

CALCULATOR TIP: When in doubt on trig graph translations, use your graphing calculator! Make up numbers that adhere to the parameters of the question and you can gauge how these graphs behave.

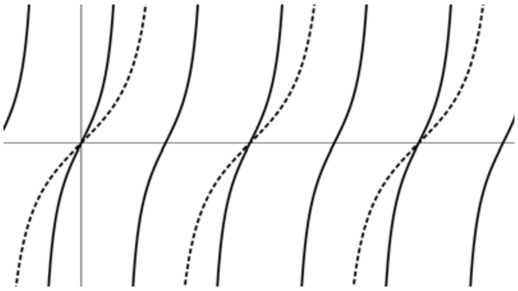
Translations with Period

To affect the period of a trig graph, you'll need to change the value at "b." **Remember, the coefficient b and the value of the period are INVERSELY proportional.** The bigger the value of "b" the smaller the period. The smaller the "b," the larger the period.



Notice how the solid line graph is "wider"? This graph would have a smaller value of b than the dotted line because it has a larger period.

Here's another example: $\tan(x)$ (dashed line) has a period of π, while $\tan(2x)$ (solid line) has a period of $\frac{\pi}{2}$.

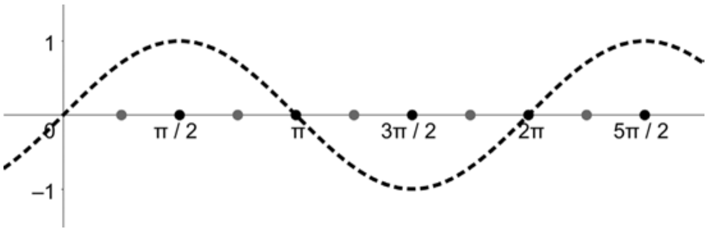


As you can see, the one with the smaller "b" value of 1 ($\tan(x)$) has a larger period.

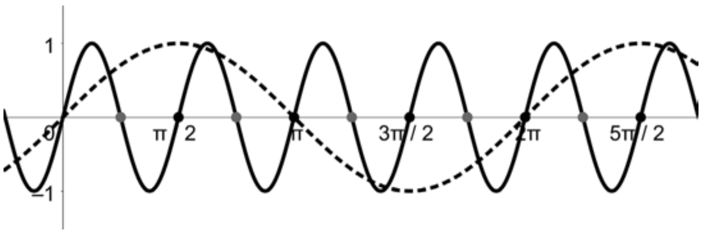
Let's do a transformation from $y = \sin(x)$ to $y = \sin(4x)$.

Because $b = 4$, we know that there will be 4 complete cycles within the span of 2π. Let's take a point that is easily visible and divisible by 4 to do our transformation: $(2\pi, 0)$

To be accurate, we can plot the points.

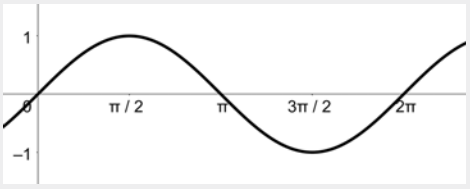


The dashed line is our original equation, $y = \sin(x)$. We should dot $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π first. Each of these is a "stopping" point of a full cycle in the graph. So the new graph will pass through each of these points on the x-axis. The graph will also pass through the points half way between these points. So I dot halfway between each dark dot to label the remaining times the trig function will hit $x = 0$. Then, I can start drawing the graph:

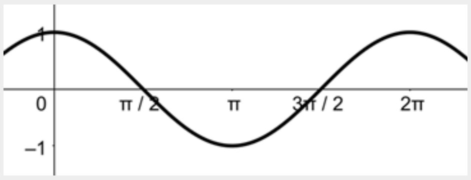


Notice how at every dark dot, the sine graph makes a full wave and restarts? And that there are 4 complete cycles by the time the function hits 2π.

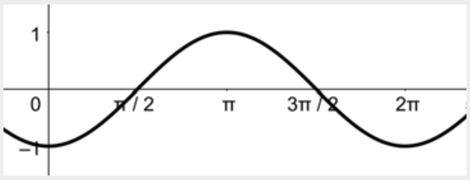
The graph below shows $y = \sin(x)$. Which of the following choice is a graph of $y = \sin\left(x - \frac{\pi}{2}\right)$?



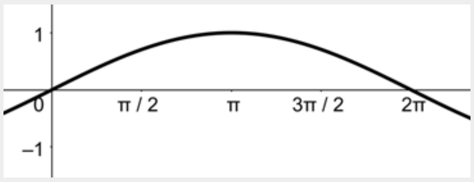
A.



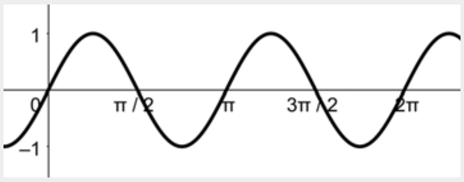
B.



C.



D.

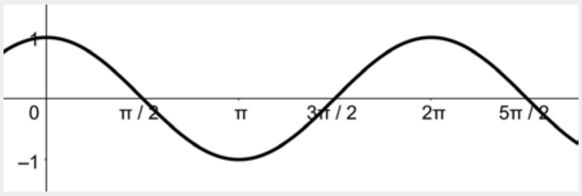


E.

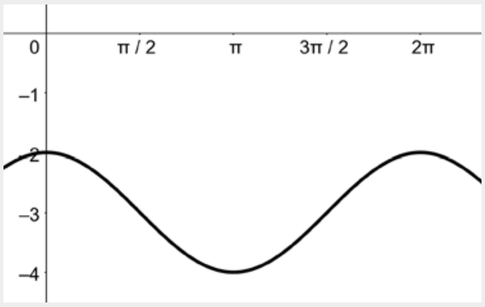
To solve, we know this has to be a phase shift (horizontal x movement), as the transformation is limited to the $(x - h)$ position. Because h equals $\frac{\pi}{2}$, we simply move the original graph $\frac{\pi}{2}$ to the right. Thus the answer is B.

Answer: **B.**

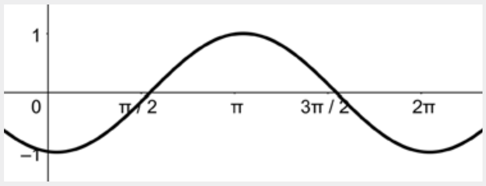
The graph of $y = \cos(x)$ is shown below. Which of the following is a graph of $y = \cos(x) + 3$?



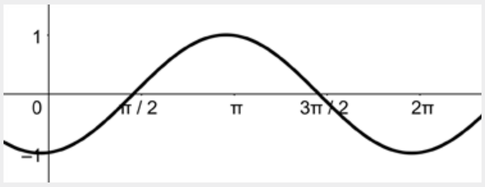
A.



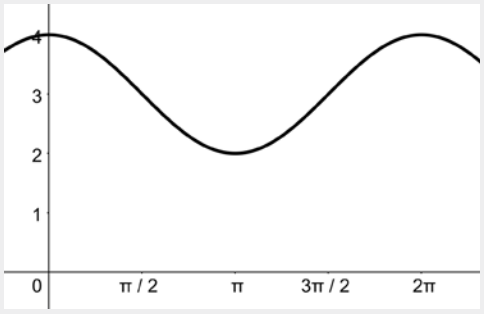
B.



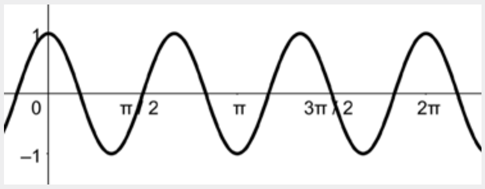
C.



D.

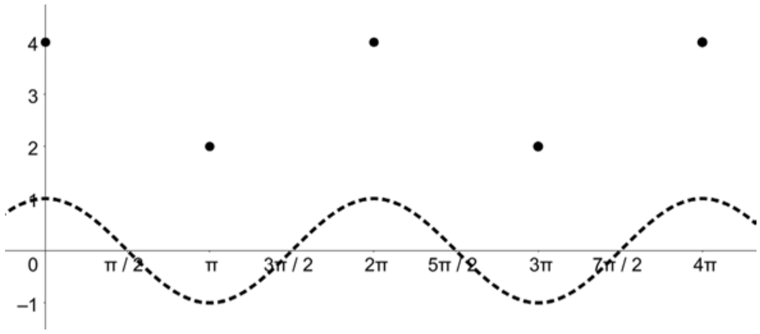


E.

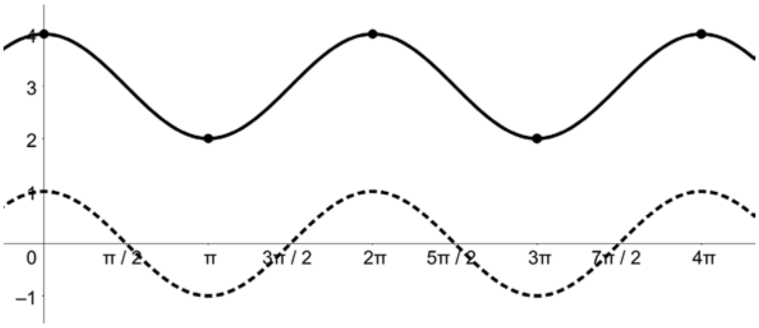


Because the 3 is in the "k" position of our standard equation, we know this is a vertical shift.

A safe way to translate a trig function vertically or horizontally is to pluck points and move them up or sideways as reference points. To illustrate, you can draw the points of all the maximums and minimums of each crest before drawing the entire line. Here we draw:



We plot the points of the maximum and minimum of each crest to make sure we will have the right graph. Then we can draw the actual line:



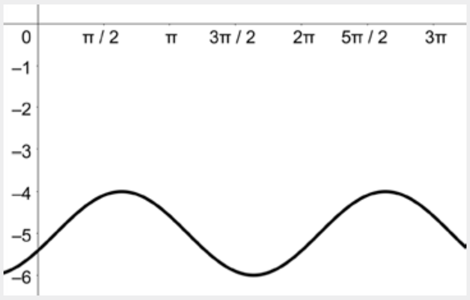
The original equation, $y = \cos(x)$, is seen in the dashed line. So now we can pick the right answer from the graphs given. Clearly the solid line matches answer choice D.

Answer: **D**.

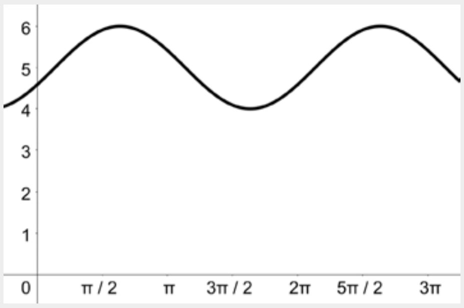
COMBINING VERTICAL AND LATERAL SHIFTS

Which of the following is the graph of $\cos(x+2)-5$?

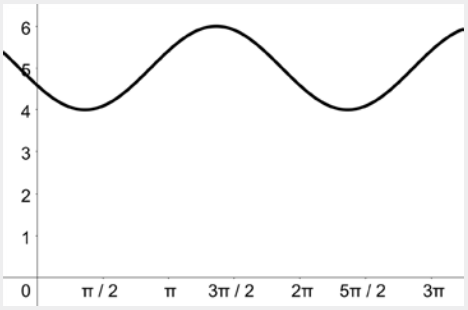
A.



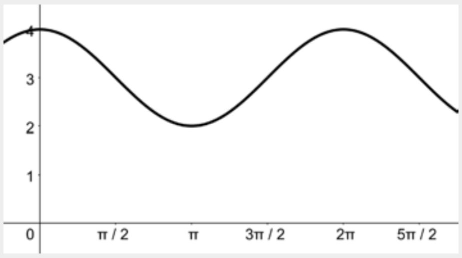
B.



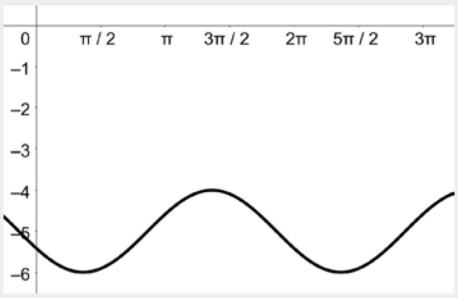
C.



D.

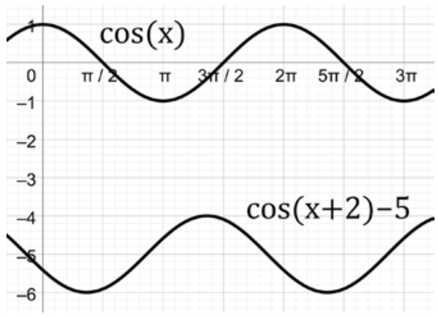


E.



Here we not only need to know the cosine graph, we also have to modify it. This is harder than most trig problems on the ACT®, but it could happen!

First I'll sketch cosine and then shift the max/min points left by 2 and down by 5 (using h and k of $h=-2$ and $k=-5$—remember when h and k are negative, we move left and down).



For the ACT®, if all this overwhelms you, you can always use your calculator. You can simply plug in answer choices or given information into your calculator to find the answer.

HOW TO GRAPH TRIG FUNCTIONS ON YOUR CALCULATOR

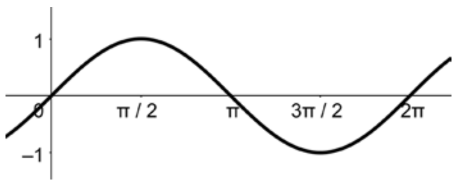
If memorizing is too much, you CAN graph these on your calculator if necessary. To do so on a TI-84:

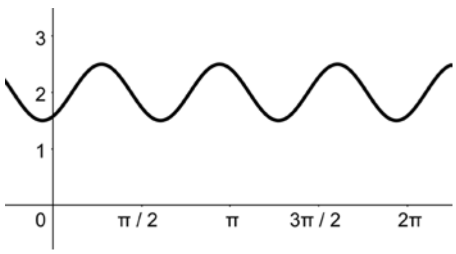
1. Put your calculator in "FUNCTION" and "RADIAN" mode.
2. Use $y=$, enter your equation.
3. Press "ZOOM" then 7.

If you have a different calculator, search online for instructions.

Let's try an example with every possible transformation:

What is the transformation from $y=\sin(x)$ to $y=0.5\sin(3x-1)+2$?



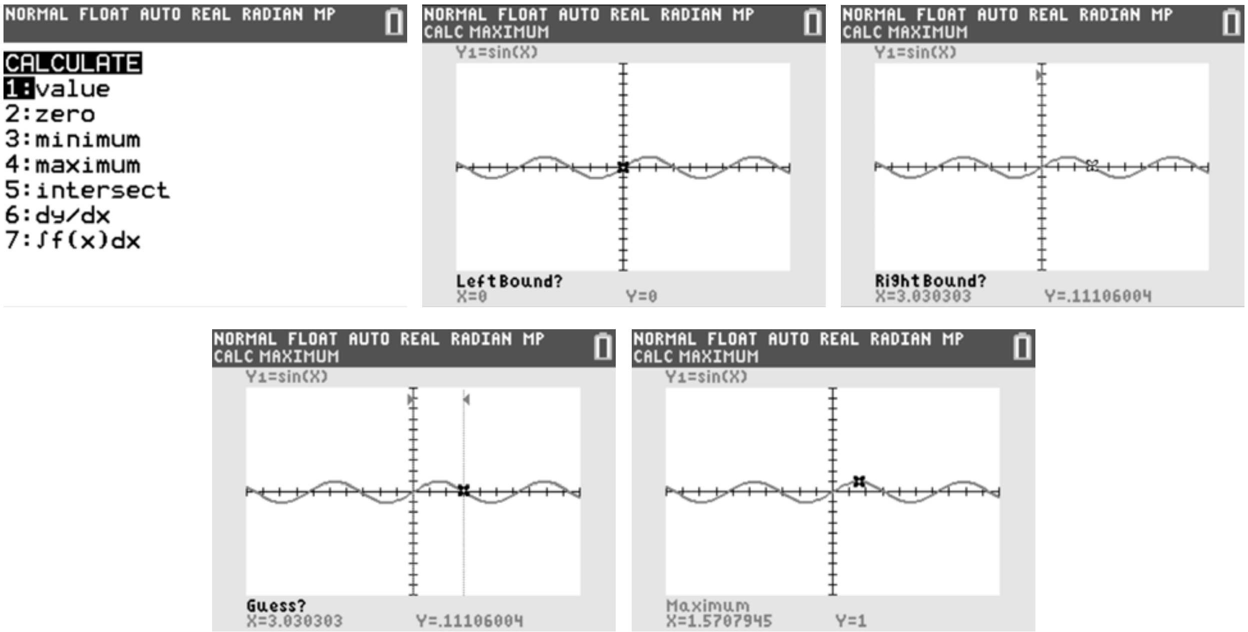


CALCULATOR TIP: Be sure to make the most efficient use of your calculator so you don't run out of time! Graphs take time. Move on while your calculator is rendering the picture and come back once it's done!

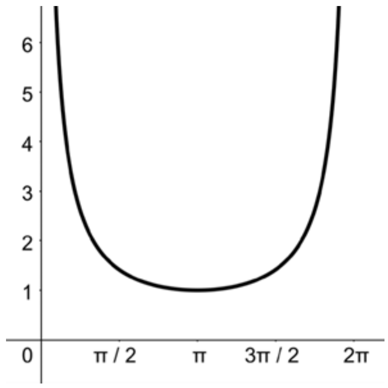
FINDING MAXIMUMS AND MINIMUMS

Like full transformations, finding the (local) maximum or minimum can easily be done on your graphing calculator.

On the TI-series, after you graph your function, if you press "2ND" and then "TRACE" (to get "CALC"), there will be a drop down menu. You can choose "MINIMUM" or "MAXIMUM" and then you'll be brought back to the graph screen. It will ask for "LEFT BOUND," so move your marker to the left side of the local max/min and press "ENTER." Then do the same but on the right for "RIGHT BOUND". After you press enter, the calculator will automatically give you the local min/max (depending on what you chose to look for). For example, look at the screencaps below:



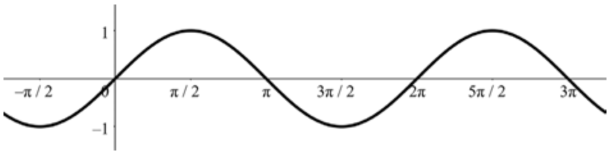
For example, if our equation is $y = \csc(0.5x)$, and we're looking for the local minimum between 0 and 2π, we can first graph the equation on our calculator:



Then, following the instructions above, we can move the "left bound" line to 0 and the "right bound" line to 2π. Then we can let the calculator work for us and see that the local minimum between 0 and 2π is π.

You can also use the trace function to more quickly estimate a minimum or maximum value.

1. The following graph of $y=\sin\left(x\right)$ is shown in the standard $\left(x,y\right)$ coordinate plane below. What is the period of $\sin\left(x\right)$?



A. $\dfrac{\pi}{4}$

B. $\dfrac{\pi}{2}$

C. π

D. $\dfrac{3\pi}{2}$

E. 2π

2. If x, y, z, t, and w represent positive real numbers what is the minimum value of the function $f\left(x\right)=w\ [\sin t\left(x-y\right)]+z$?

A. $wx-wy+z$
B. $w\sin(tx)-w\sin(ty)+z$
C. z
D. $w+z$
E. $z-w$

3. In the standard $\left(x,y\right)$ coordinate plane, what is the range of the function $f\left(a\right)=-7\cos 3\left(a+4\right)+5$?

A. $-12\le f\left(a\right)\le 2$

B. $-2\le f\left(a\right)\le 12$

C. $5-14\pi\le f\left(a\right)\le 5$

D. $0\le f\left(a\right)\le 2\pi$

E. $-2\le f\left(a\right)\le 5$

4. What is the amplitude of the graph of the equation $y-4=\dfrac{1}{2}\cos\dfrac{7\alpha}{9}$?

A. $\dfrac{7a}{9}$

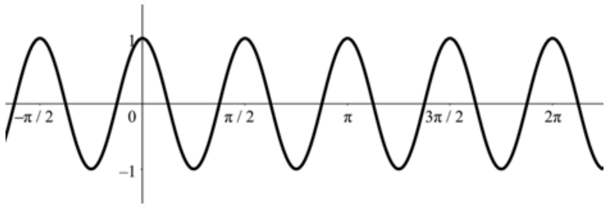
B. 2

C. 1

D. $\dfrac{1}{2}$

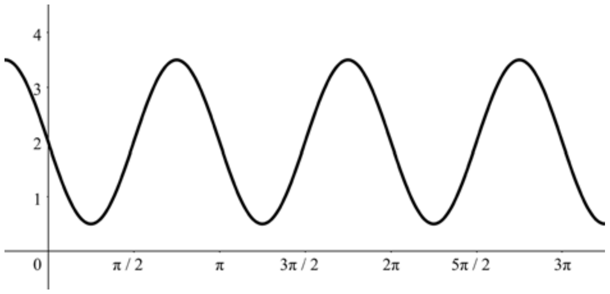
E. 4

5. A trigonometric function with equation $y=\cos\left(bx+c\right)$ where b and c are real numbers, is graphed in the standard $\left(x,y\right)$ coordinate plane below. The period of this function $f\left(x\right)$ is the smallest possible number p such that $f\left(x+p\right)=f\left(x\right)$ for every real number x. One of the following is the period of the function. Which is it?



A. $\dfrac{\pi}{2}$
B. π
C. 2π
D. 1
E. 2

6. The graph of $y=-a\sin(bx)+c$ is shown below for certain positive values of a, b, and c. One of the following value is equal to a. Which is it?



A. $\frac{1}{2}$

B. $\frac{\pi}{2}$

C. $\frac{3}{2}$

D. 2

E. $\frac{7}{2}$

7. Which of the following trigonometric functions has an amplitude of $\frac{1}{3}$?

A. $f(x)=3\tan\left(x+\frac{1}{3}\right)$

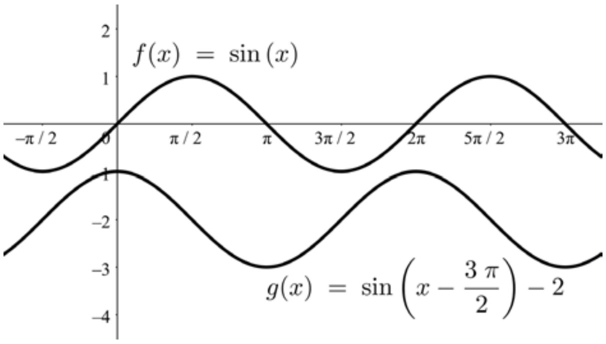
B. $f(x)=\frac{1}{3}\tan(x)$

C. $f(x)=\frac{1}{3}\cos(x)$

D. $f(x)=3\sin(x)$

E. $f(x)=\sin\left(\frac{1}{3}x\right)$

8. The graph of $f(x)=\sin x$ and $g(x)=\sin\left(x-\frac{3\pi}{2}\right)-2$ are shown in the standard (x,y) coordinate plane below. After one of the following pairs of transformations is applied to the graph of $f(x)$, the image of the graph of $f(x)$ is the graph of $g(x)$. Which pair is it?



A. Phase shift $\frac{3\pi}{2}$ units to the right, and vertical translation 2 units down.

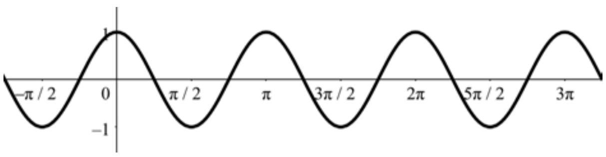
B. Phase shift $\frac{3\pi}{2}$ units to the left, and vertical translation 2 units up.

C. Phase shift 2 units to the right, and vertical translation $\frac{3\pi}{2}$ units down.

D. Phase shift 2 units to the left, and vertical translation $\frac{3\pi}{2}$ units down.

E. Phase shift $\frac{3\pi}{2}$ units to the right, and vertical translation 2 units up.

9. For the function graphed below, the x-axis can be partitioned into intervals, each of length p radians, and the curve over any one interval is a repetition of the curve over each of the other intervals. What is the least possible value for p, the period of the function?



A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. π

D. 2π

E. 4π

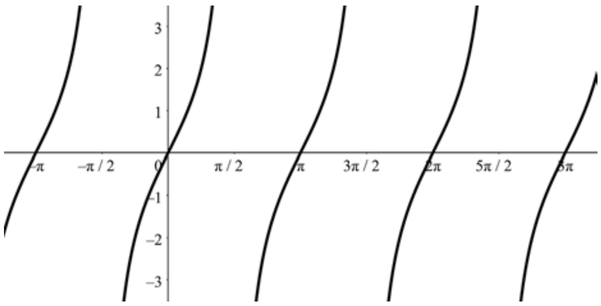
10. If a, b, c, d, and x represent positive real numbers, what is the maximum value of the function $f(x)=a\left[\sin(b(x+c))\right]+d$?

A. $a-d$
B. $d-a$
C. $a+d$
D. d
E. a

11. The domain of function $y(x)=\dfrac{\cos(3x+1)}{2}+4$ is all real numbers. Which of the following is the range of function $y(x)$.

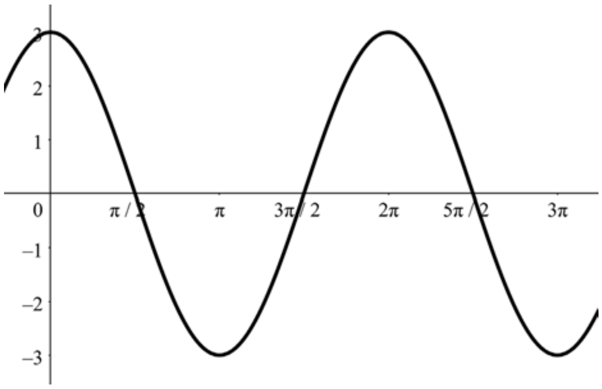
A. $0<y<4$
B. $\frac{7}{2}<y<\frac{9}{2}$
C. $2<y<4$
D. $-\frac{1}{2}<y<\frac{1}{2}$
E. $2<y<6$

12. The graph of $y=2\tan x$ is shown in the standard coordinate plane below. What is the period of $2\tan x$?



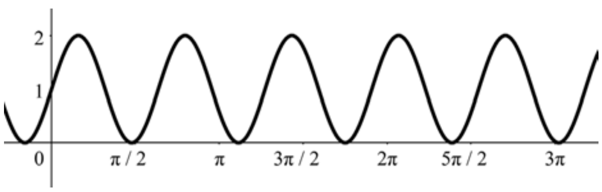
A. $\frac{1}{2}$
B. $\frac{\pi}{2}$
C. π
D. 2
E. 2π

13. A trigonometric function with equation $y=a\cos bx+c$, where a, b, and c are real numbers, is graphed in the standard (x,y) coordinate plane below. The period of this function is the smallest positive number p such that $f(x+p)=f(x)$ for all real numbers x. What is the period of the following function?



A. 1
B. $\frac{\pi}{2}$
C. π
D. 2π
E. 2

14. The graph below could represent which of the following equations?



A. $\sin(\pi x+1)$
B. $\pi\sin(x+1)$
C. $\sin(x)+1$
D. $\sin(2\pi x)+1$
E. $\sin(\pi x)+1$

15. If $0\le x\le\pi$, what is the maximum value of the function $f(x)=-2\sin x$?

A. -2π
B. -2
C. 0
D. 2
E. 2π

ANSWER KEY

1. E **2. E** **3. B** **4. D** **5. A** **6. C** **7. C** **8. A** **9. B** **10. C** **11. A** **12. C** **13. E** **14. E** **15. C**

ANSWER EXPLANATIONS

1. **E.** The period of a graph is the length of one cycle of the curve. Since $\sin 0 = 0$ and $\sin 2\pi = 0$, the cycle starts at 0 and ends at 2π. Thus the period is $2\pi - 0 = 2\pi$.

2. **E.** We know that the range of $\sin x$ is from -1 to 1. In order to minimize the value of the entire function, we want that part of the equation, the sine term, to be its minimum possible value. $\sin t(x-y)$ at its very minimum is equal to -1. It doesn't matter what our actual angle in, because we know that $\sin x$ equals -1 many times, but never is any lower. Substituting, this leaves us with $f(x) = w(-1) + z$. So at it's minimum, $f(x) = z - w$.

3. **B.** We know that the range of $\cos x$ is from -1 to 1. So, we can find the range of the function by plugging in the maximum and minimum possible values of $\cos 3(a+4)$, which regardless of the angle will have those values as it's minimum and maximum. Plugging in $\cos 3(a+4) = -1$, we get $-7(-1) + 5 = 12$. Plugging in $\cos 3(a+4) = 1$, we get $-7(1) + 5 = -2$. So the range is $-2 \le f(a) \le 12$.

4. **D.** A function represented in the form $y = a\cos(bx - c) + d$ has an amplitude of a. In this problem, the function $y - 4 = \frac{1}{2}\cos\frac{7\alpha}{9}$ has amplitude $\frac{1}{2}$.

5. **A.** The question background is complicated and honestly not worth untangling. What you do know is that you have a graph and can analyze it visually. If you trace one full cycle of the graph, you see that it completes a cycle in $\frac{\pi}{2}$ units, so that is the answer.

6. **C.** The negative sign does not affect the answer, because the question is asking for the amplitude. The amplitude is the difference between the maximum and the minimum point the graph reaches, divided by two. It can also be though of as how far above/below the graph reaches from the central axis. In this case, the graph reaches a maximum at what looks like $\frac{7}{2}$ and a minimum at $\frac{1}{2}$, so the amplitude is $\frac{\frac{7}{2}-\frac{1}{2}}{2} = \frac{\frac{6}{2}}{2} = \frac{3}{2}$.

7. **C.** In a trigonometric equation, the amplitude is the coefficient of the cosine or sine function. Only C has $\frac{1}{3}$ in front of a cosine or sine function.

8. **A.** In this case it's better to look at the equations given rather than the graph. Trigonometric translations are actually the same as regular graph translations. The difference between $f(x)$ and $g(x)$ is that $\frac{3\pi}{2}$ is subtracted from the x term of $g(x)$, so we get a translation (or phase shift) $\frac{3\pi}{2}$ to the right, and in $g(x)$, 2 is subtracted from the entire equation, which yields a vertical translation 2 units down.

9. **B.** Starting from the relative maximum where $x = 0$, the curve repeats 2 times, finished it's second interval at $x = \pi$. If it repeats twice in the interval $[0, \pi]$, then the interval must be $\frac{\pi}{2}$.

10. **C.** a is the amplitude constant, and d is essentially our central axis, so the maximum is found by adding the amplitude to the central axis. This gives $d + a$, which is answer C.

11. **B.** Rewrite the function in your head like this: $y(x)=\frac{1}{2}\cos(3x+1)+4$. The minimum value a sine or cosine function can equal is the central axis, which in this graph is 4 minus the amplitude, which in this graph is $\frac{1}{2}$, and the maximum of a sine or cosine function is the sum of these two values. The range is $minimum < y < maximum$, so for this function the range is $4-\frac{1}{2}<y<4+\frac{1}{2}$, which is $\frac{7}{2}<y<\frac{9}{2}$.

12. **C.** The period is only affected by a coefficient to the x term, not the entire equation, so this graph has the same period as the parent tangent graph, which is π.

13. **C.** The period of a function is how far a graph travels along the y-axis before it repeats itself. In this case, the cosine graph reaches its peak every π units in a recurring pattern. Thus, its period is also π.

14. **E.** The most obvious change to the graph is that it has been shifted upwards. We can tell it has been shifted up by 1 since its new maximum is 2, while the maximum of the parent function, $\sin x$, is 1. However, note that the period of this function is 2, not 2π. The period of $\sin(ax)$ is equal to $\frac{2\pi}{a}$. Since here $\frac{2\pi}{a}=2$, $a=\pi$. Thus, our function is $\sin(\pi x)+1$.

15. **C.** The parent graph $\sin x$ is flipped upside down by the negative sign in front of -2, and we only go to $x=\pi$, which means that we stop before the graph would go above the x-axis. Thus, our maximum value is 0.