CHAPTER

# 15 FUNCTIONS

## SKILLS TO KNOW

- The definition of a function
- Function Notation
- Compound (nested) functions—plugging in and evaluating or solving for a variable
- Weird symbol problems (symbol-defined operation problems)
- Inverse function problems

## FUNCTION BASICS

A function is a mathematical construct that relates each value of a set of input values to a unique output value. More simply, a function relates each value of $x$ to no more than one value of $y$.

Functions can be written in $(x, y)$ notation, or in function notation ($f(x)$), which we'll cover next. Some common examples of functions are: $y = x^2$, $f(x) = 1$, and $x + y = 7$.

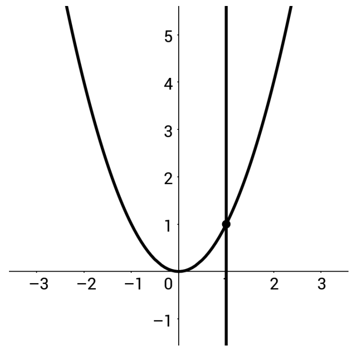
Some examples of equations that are not functions include: $x^2 + y^2 = 49$, $x = y^2$, and $x = 12$.

These are not functions because for some $x$-values, there is more than one corresponding $y$-value. For example, in the first equation, the points $(0,7)$ and $(0,-7)$ are both on the curve given by the equation, meaning that the equation does not relate each $x$ value to a unique $y$ value. With a function, every $x$-value generates a unique $y$-value.

Sometimes I like to think of functions like a vending machine that masks the inside contents of the machine. When you push a button, you expect to get what the button advertises. For example, if the top button says "Coke," pushing it will always give you a Coke. The position of the buttons is like the $x$-value: each one is different. But the output is like a $y$-value. Though you can have more than one button programmed to the same output (say, the top three buttons on a machine say "Coke"), the input (button 1, 2, or 3) will reliably give you the same result every time. You won't randomly get an apple juice if you push button number 1.

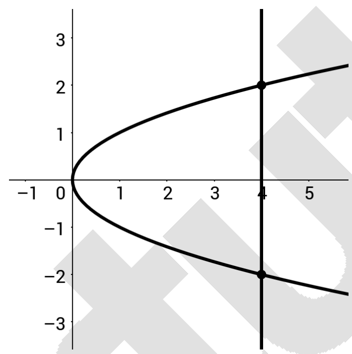
A handy trick for deciding whether an equation is a function is called the "vertical line test." In other words, you can test a relation to see if it is a function by graphing it and running a vertical line through the graph at any $x$ point. At all points, the line, if a function, will only intersect the graph at one point. If it intersects at more than one point, the graph is not a function.

Here, you can see that $y = x^2$ is a function.




We could move the vertical line anywhere and it would only cross the graph once.

The following graph of $x = y^2$, however, is not a function: a vertical line will cross the graph twice in many places, specifically, for any $x > 0$. For example, $x = 3$ at $y = +\sqrt{3}$, AND $-\sqrt{3}$.




To use the previous analogy, this would be like pushing the same button on our vending machine and having two possible outcomes. Thus, the equation is not a function.

## FUNCTION NOTATION

Function notation is a special way of writing the "$y$" value of a function. The most common function notation is $f(x)$ (pronounced $f$ of $x$). This does not mean you multiply variable $f$ times $x$! Writing $f(x)$ is the same as writing a "$y$." Other variables can be used in place of $f$ or even $x$. For example, I could say a function $P$ is defined by $p(r) = r + 7$. Here you see I've swapped the $f$ for "$p$" and the $x$ for "$r$." You might also see that I can refer to the function as "$P$," "$p$ of $r$," or $p(r)$. All of these expressions are identical. Whenever we "define" a function and we have the pattern of a letter followed by a letter in parenthesis, we are likely dealing with function notation. We'll discuss these ideas more in the Function as a Model chapter in this book.

We can think of an ordered pair in function notation:
$(x, f(x))$, where $f(x)$ is equal to $y$.

We can also turn statements in function notation into ordered pairs. For example, if $f(3) = 7$, this equation basically is saying when $x = 3$, $y = 7$. We can rewrite this fact as the ordered pair $(3,7)$.

More commonly, you'll be asked to plug a value into a given equation. When you see $f(5)$, for example, you'll need to let whatever variable is in the parenthesis be replaced by $5$.

If $f(x)=2x+7$, what is $f(3)-f(2)$?

Here when we see $f(3)$, we want the "$y$" value of the entire equation when we plug in $3$ for $x$. Rewrite the equation:

$$f(x)=2x+7$$

Plug in $3$ for $x$, because it is in the parenthesis, so should replace the variable originally in that position:

$$f(3)=2(3)+7$$
$$f(3)=6+7$$
$$f(3)=13$$

Then we can find $f(2)$ by the same principle. Plug in $2$ for $x$:

$$f(x)=2x+7$$
$$f(2)=2(2)+7$$
$$f(2)=4+7$$
$$f(2)=11$$

Just as we can substitute in for $y$ values, we can substitute in for $f(3)$ and $f(2)$ once we solve each down by plugging in the given value for $x$. Though function notation is a bit ugly, it is replaceable in the same way a single variable is. Because $f(3)=13$ and $f(2)=11$, their difference, or $f(3)-f(2)$ is simply $13-11$, or $2$.

Answer: $2$

## COMPOUND (NESTED) FUNCTIONS

Compound functions occur when functions are applied to other functions.

If $f(x)=x^2$ and $g(x)=\sqrt{x}$, what is $g(f(x))$?

We always work nested functions from the inside out when possible. $g(x)$ can substitute in for $f(x)$ by writing $g(x^2)$:

$$g(f(x))\rightarrow g(x^2)$$

Remember, all I do is treat $f(x)$ just as I would a variable, simply replacing it with what I know it equals. But now, to find $g(x^2)$, we need to be careful. It's confusing because our original $g$ equation also uses $x$, but it is not the same $x$ that we have here. What we need to do is take the item in parenthesis and plug it into the variable in parenthesis in our function $g$. In other words, if $g(x)=\sqrt{x}$, and I need to solve for $g(x^2)$, I can plug in $x^2$ for $x$ just as if I wanted to find $g(2)$ I would plug "$2$" in everywhere I see an $x$. Don't get confused by the fact that $x^2$ has an $x$ in it. Substitute, treating the term the same as you would an integer.

$$\begin{array}{ccc} g(x) & = & \sqrt{x} \\ \uparrow & & \uparrow \\ x^2 & & x^2 \end{array}$$

$$g(x^2)=\sqrt{x^2}$$

$$g(x^2)=|x|$$

I know this is a bit confusing with all the $x$'s everywhere. But the more you work on these problems the easier it gets.

Answer: $|x|$.

Using the same equations, I could also find $f(f(x))$. To do so, again, I take the whole of what $f(x)$ equals ($x^2$) and plug it into the original function $f(x)=x^2$, replacing the $x$ in each instance with $x^2$:

$$f(f(x))=(x^2)^2=x^4$$

Given $f(x)=2x+3$ and that $g(x)=x^2-4x$, which of the following is an expression for $g(f(x))$?

**A.** $2x^2-8x+3$ **B.** $4x^2-4x+9$ **C.** $4x^2+4x-3$

**D.** $2x^3-5x^2-12x$ **E.** $x^2-2x+3$

This question is essentially asking us to combine these two equations. Be sure to notice that $f(x)$ is on the inside and $g(x)$ is on the outside, so we will be plugging the whole of what function $f$ equals ($2x+3$) into the "input" value $x$ of the function $g(x)=x^2-4x$. Everywhere we see an "$x$" in the function $g$, we will replace it with $2x+3$:

$$\begin{array}{ccccc} g(x) & = & x^2 & - & 4x \\ \nearrow & & \uparrow & & \nwarrow \\ 2x+3 & & 2x+3 & & 2x+3 \end{array}$$

$$\begin{aligned} g(f(x)) &= g(2x+3) \\ &= (2x+3)^2-4(2x+3) \\ &= 4x^2+4x-3 \end{aligned}$$

Answer: **C**.

## WEIRD SYMBOL PROBLEMS (SYMBOL DEFINED OPERATIONS)

By now, you know what most math symbols mean, whether addition $(+)$, multiplication $(\times)$, or factorial $(!)$ symbols. But did you know that the ACT® can define a new mathematical operation using a weird symbol you've never seen before?! These problems are essentially function problems. The elements on the left tell you what to plug in, and the second half of the equation sets up an expression that incorporates the variable(s) set up in the first half, telling you what to do with the "inputs" you're given in relation to the weird symbol (as denoted in the first half). These problems set up a "secret code" that you must plug values into. It's probably more confusing to describe than to actually do, so let's jump into an example.

If $x \,\&\, y = 3x + 4y^3$, what is $4 \,\&\, 3$?

The new operation "$\&$" has now been defined, and it indicates that you simply add $3$ times the quantity on the left of the ampersand to $4$ times the cube of the quantity on the right of the ampersand.

The question is a bit confusing because the numbers given for $x$ and $y$ are the same as the numbers in the problem, but just plug in one at a time and you'll be fine. First, let's figure out what $4 \,\&\, 3$ tells us:

$x = 4$ because $4$ is to the left of the ampersand (just as $x$ is in the example equation), and $y = 3$ because $3$ is to the right of the ampersand (just as $y$ is in the first given equation). Now we simply plug these values in for $x$ and $y$:

$$x \,\&\, y = 3x + 4y^3$$

$$4 \,\&\, 3 = 3(4) + 4(3)^3$$

Thus:

$$\begin{aligned} 4 \,\&\, 3 &= 3(4) + 4(3^3) \\ &= 12 + 108 \\ &= 120 \end{aligned}$$

If $\circledast$ represents the operation defined by $a \circledast b = a + \frac{b}{a}$, then $\left(2 \circledast (3 \circledast 9)\right) = ?$

**A.** $\frac{70}{2}$ **B.** $\frac{13}{5}$ **C.** 6 **D.** 5 **E.** 2

When solving these types of problems, start from the innermost function and then work your way out. Take it one step at a time.

Plugging in the numbers of the innermost function:

$$\begin{aligned}(3 \circledast 9) &= 3+\frac{9}{3}\\ &= 3+3\\ &= 6\end{aligned}$$

Plugging this into the outer function gives us:

$$\begin{aligned}(2 \circledast 6) &= 2+\frac{6}{2}\\ &= 2+3\\ &= 5\end{aligned}$$

Answer: **D**.

## INVERSE FUNCTIONS

Inverse functions are exactly what they sound like: functions that undo the effects of other functions. The inverse of a function, $f(x)$, is written $f^{-1}(x)$, and is more formally defined $f^{-1}(f(x))=x$, or $f^{-1}(y)=x$. But let's be honest, that's confusing. To find an inverse, rewrite your equation, **swapping your $x$ for your $y$ and your $y$ for your $x$**. For example, take

$$f(x)=2x$$

To make the problem easier to look at, I can rewrite this to

$$y=2x$$

To solve for $f^{-1}(x)$, simply **switch positions of your $x$ and your $y$** (or your $x$ and your $f(x)$). What was $x$ becomes $y$, what was $y$ becomes $x$:

$$x=2y$$

and manipulate the equation to be in "$y=$" form if necessary. Here, that gives us:

$$y=\frac{x}{2}$$

We can then return to function notation, making our $y$ value $f^{-1}(x)$:

$$f^{-1}(x)=\frac{x}{2}$$

We can check this by seeing what happens when we apply the inverse to the original:

$$\begin{aligned}f^{-1}(f(x)) &= f^{-1}(2x)\\ &= \frac{2x}{2}\\ &= x\end{aligned}$$

This matches our original definition of an inverse function, as it "undoes" the function, giving us $x$, the value we originally plugged into the function.

Let's look at a more complicated example:

Given $f(x)=\dfrac{3x-1}{7-2x}$, which of the following expressions is equal to $f^{-1}(x)$ for all real numbers $x$?

All this question wants is for us to find the inverse function for $f(x)$. The easiest way to do this is simple switch $y$, or $f(x)$, with $x$, and then isolate $y$ to get back into standard function notation.

Here, we've swapped out $f(x)$ with $x$ and $x$ with $y$ $$x=\frac{3y-1}{7-2y}$$

Now we multiply both sides to eliminate the denominator $$(7-2y)x=3y-1$$

Simplify $$7x-2yx=3y-1$$

Place all terms with $y$ on one side $$7x+1=3y+2yx$$

Factor out $y$ $$7x+1=(3+2x)y$$

Divide to isolate the $y$ $$\frac{7x+1}{2x+3}=y$$

Don't forget to replace $y$ with $f^{-1}(x)$! $$f^{-1}(x)=\frac{7x+1}{2x+3}$$

Answer: $f^{-1}(x)=\dfrac{7x+1}{2x+3}$