CHAPTER

15 SIMILAR SHAPES

SKILLS TO KNOW

- Ratios
- Similar shapes
- Similar triangles
- Hinge theorem
- Parallel lines theorem
- Similar solids
- Area and volume

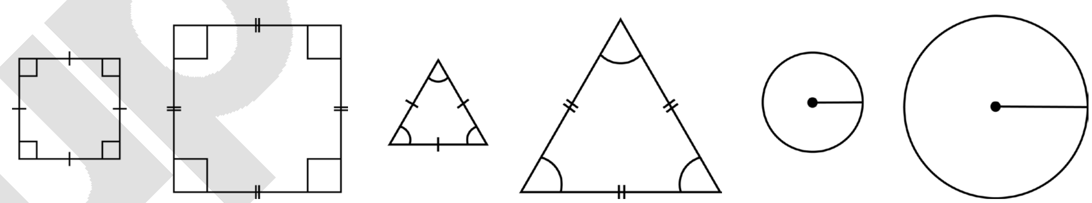
RATIOS

A ratio compares two different values. In the case of geometry, it may compare two areas or even two side lengths of two shapes. We cover how to solve problems using basic ratios in our chapter on Ratios, Rates and Units in Book 1.

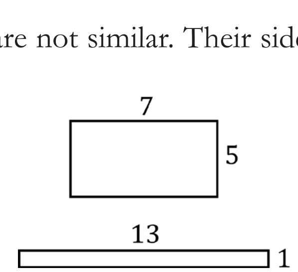
SIMILAR SHAPES

Imagine blowing up a picture of a triangle using a copy machine or even zooming in on a square on your computer screen. That's essentially what similar shapes are: zoomed in or zoomed out versions of the same shape.

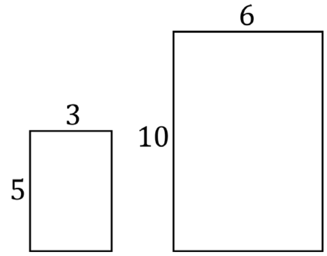
Similar shapes can be different sizes but must have the same proportionate shape. All corresponding angles of similar figures are equal. For example, all regular polygons and circles are similar since they share the same shape. Here are some examples:



On the other hand, these rectangles are not similar. Their sides are not in equal ratios, i.e. 7:5 is not equal to 13:1.

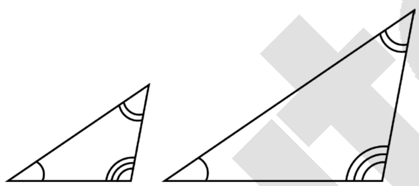


These rectangles, on the other hand, are similar due to their ratios being in harmony. $5:10=1:2$, just as $3:6=1:2$. The sides of the right figure are twice the corresponding sides on the left figure.

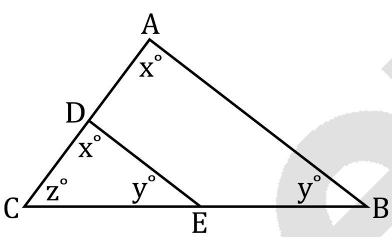


SIMILAR TRIANGLES

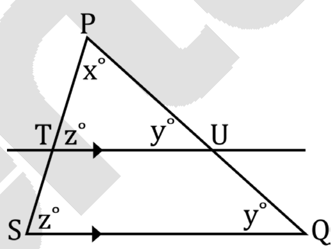
Triangles that have the same shape but different sizes are similar. This means they will share the same corresponding angles and the corresponding side lengths will be proportional.



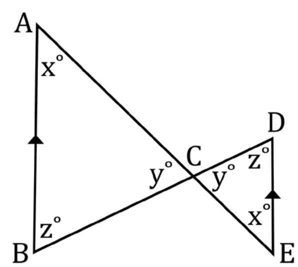
Similar triangles can show up in different ways on the test. Sometimes they will show up side by side, one inside the other, sitting on top of each other via parallel lines, or even sharing sides. When we denote similarity, be careful: the order of the letters in your similarity statement MATTERS.



$\Delta CAB \sim \Delta CDE$



$\Delta PTU \sim \Delta PSQ$



$\Delta ACB \sim \Delta ECD$

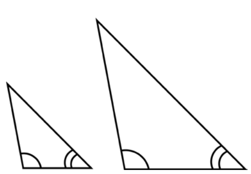
For instance, if $\Delta ACB \sim \Delta ECD$ then $\overline{AC}$ is similar to $\overline{EC}$ and $\overline{CB}$ is similar to $\overline{CD}$ as they are in the same corresponding positions in the name of each triangle. Confusing which sides are similar is a common careless error on similar shape questions.

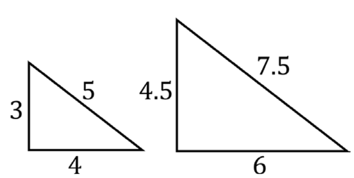
Ways you can prove similarity

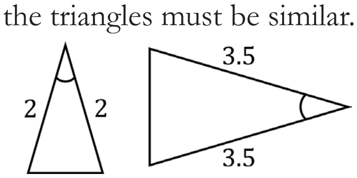
AA: If two angles are congruent to two angles of another triangle, then the triangles must be similar.

SSS: If all the corresponding sides of two triangle are in the same proportion, then the two triangles must be similar.

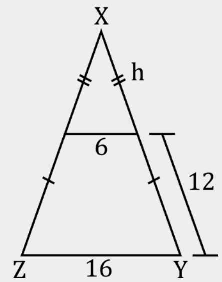
SAS: If one angle of a triangle is congruent to the corresponding angle of another triangle and the lengths of the sides adjacent to the angle are proportional, then the triangles must be similar.







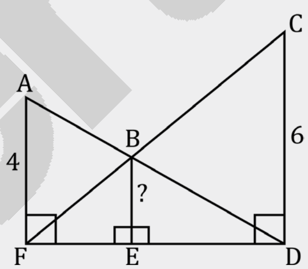
In ΔXYZ, what is the length of side h?



We can set up a ratio because we can see congruent sides of the similar triangle ZXY and the shorter triangle formed with a base of 6. We set the short leg on the right, h, proportional to the long leg on the right, h plus 12 and the short base on the upper, smaller triangle proportional to the larger base, 16: $\frac{h}{h+12}=\frac{6}{16}$.

Be careful! Students will often think $\frac{h}{12}=\frac{6}{16}$. However, we're dealing with triangle bases, so we must consider the WHOLE length of the triangle side. Cross-multiplying, we have $16h=72+6h$, $10h=7.2$, which gives us $h=7.2$ units.

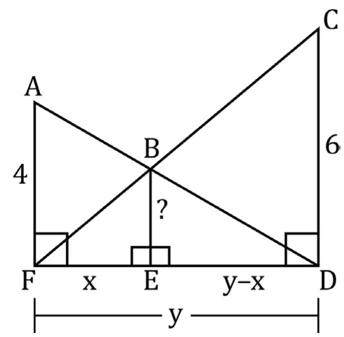
Answer: $h=7.2$.



Lines $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ are all parallel with one another in the diagram above, which shows two right triangles intersecting one another. What is the length of $\overline{BE}$?

First, we know that there are sets of similar triangles within the diagram. Since ΔCFD and ΔBFE share the same angle at F, we can conclude that the two triangles are similar. We can use the same logic to conclude that ΔADF is similar to ΔBDE. With this knowledge, we can start setting up proportions to find the missing sides to the triangles.

Let's start by calling $\overline{FD}$ as the variable y, $\overline{FE}$ as the variable x and $\overline{ED}$ as simply $y-x$. This is shown below:



Now let's set up a proportion for ΔCFD compared to ΔBFE:

$$\frac{BE}{6}=\frac{x}{y}$$

And the same for ΔADF to ΔBDE.

$$\frac{BE}{4}=\frac{y-x}{y}$$

For the latter proportion, we can simplify it down to this:

$$\frac{BE}{4}=1-\frac{x}{y}$$

Notice now that we can substitute the proportion for ΔCFD and ΔBFE.

$$\frac{BE}{4}=1-\frac{BE}{6}$$

And solve for $\overline{BE}$:

$$\frac{BE}{4}+\frac{BE}{6}=1$$

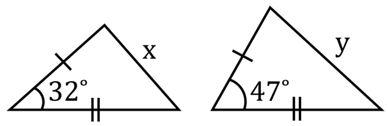
$$\frac{3BE}{12}+\frac{2BE}{12}=1$$

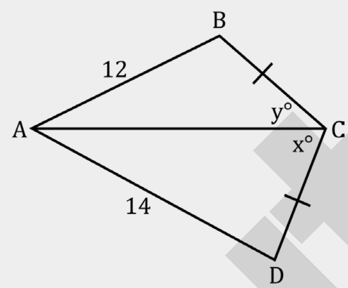
$$\frac{5BE}{12}=1$$

$$BE=\frac{12}{5}$$

HINGE THEOREM

The hinge theorem states that if two triangles have two congruent sides (see picture below) joined by different angles, then the triangle with the larger angle between those sides will have a longer third side (below, $y > x$). This also gives way to the converse of the hinge theorem, which states that if two triangles have two congruent sides, then the triangle with the longer third side will have a larger angle opposite that third side. Below, y must be greater than x because $47° > 32°$:





In the diagram shown above, what can be concluded about angles x and y ?

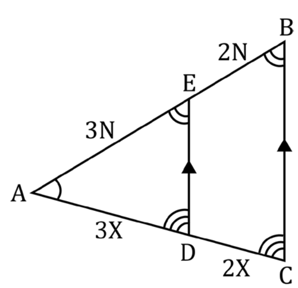
A. $x > y$ **B.** $x < y$ **C.** $x = y$ **D.** $x + y = 90$ **E.** No conclusion can be made.

We see that $\overline{AC}$ is a shared side by both triangles and $\overline{BC} = \overline{CD}$. Thus, we can use the hinge theorem to assume that $x < y$.

Answer: **A**.

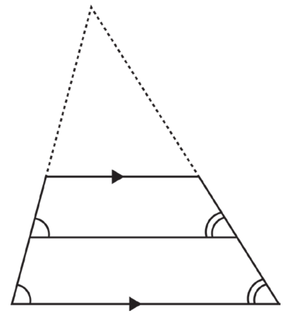
PARALLEL LINES THEOREM

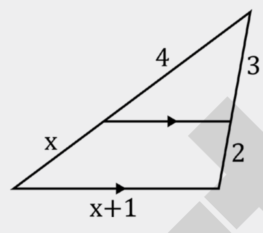
Parallel lines to the triangle's base always divide triangles into two similar triangles. Below, you can see how $\Delta ADE \sim \Delta ACB$.



The **side splitter theorem** states that if a line is parallel to a side of a triangle and intersects the other two sides, then this line divides those two sides proportionally. Above, you can see how if $AD : DC$ is in a ratio of $3:2$ then $\overline{AE}$ to $\overline{EB}$ must also be in a ratio of $3:2$.

This rule also extends to trapezoids—because we can extend the lines of trapezoids, we can see how this proportionality happens when we have trapezoids with parallel "cuts" horizontally, too.





In the diagram above, a scalene triangle is split by a line that is parallel to the base of the triangle. What is the perimeter of the triangle?

With our knowledge of the side splitter theorem, we can conclude that the two triangles are similar. We know if the right side is split in a ratio of $\frac{3}{2}$, then so is the left. Thus:

$$\frac{4}{x}=\frac{3}{2}$$
$$3x=8$$
$$x=\frac{8}{3}$$

We could also have set up our proportion as follows, using similar triangles:

$$\frac{3}{4}=\frac{5}{4+x}$$

Now solve for x via cross multiplication.

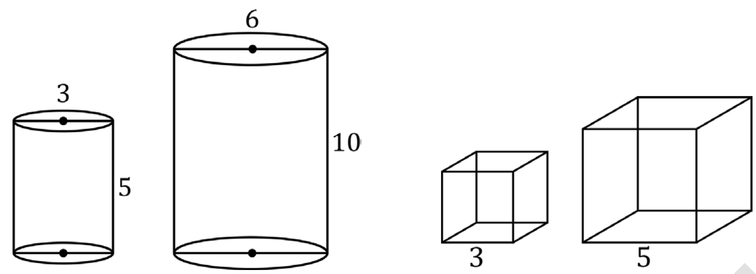
$$12+3x=20$$
$$3x=8$$
$$x=\frac{8}{3}$$

Now we add all the sides together to find the perimeter.

$$4+3+2+1+\frac{8}{3}+\frac{8}{3}=15\frac{1}{3}$$

SIMILAR SOLIDS

Similar solids also operate much like similar shapes, but with an extra dimension.



AREA AND VOLUME OF SIMILAR SOLIDS

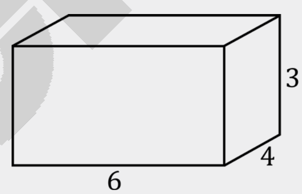
Two shapes are similar if all their corresponding angles are congruent and all their corresponding sides are proportional. **Two solids are similar** if they are the same type of solid and their corresponding radii, heights, base lengths, widths, etc. are proportional.

Surface Areas of Similar Solids

In two dimensions, when two shapes are similar, the ratio of their areas is the square of the ratio of their side lengths.

Surface Area Ratio: If two solids are similar with sides, heights, or other one dimensional attributes in a ratio of $a:b$, (i.e. the scale factor is $a:b$) then the surface areas are in a ratio of $\left(\frac{a}{b}\right)^2$.

The dimensions of the rectangular prism below are tripled. What is the surface area of the new solid?



First, let's find the surface area of the smaller prism.

$$2(4\times6)+2(6\times3)+2(3\times4)=96\,units^2$$

If the dimensions are triple, the ratio of the side lengths of the prism would be $1:3$. This means that the surface area would have a ratio of $1:9$. Now we set up a proportion to find the surface area of the bigger solid, using our ratio, $1:9$.

$$\frac{96}{x}=\frac{1}{9}$$

$$x=864\,units^2$$

Alternatively, we could first upsize each side length: 3 would become 9, 4 would become 12 and 6 would become 18. The surface area would then be:

$$2(9\times12+9\times18+18\times12)=864$$

Answer: 864.

Volumes of Similar Solids

Just like surface area, volumes of similar solids have a relationship that is related to the scale factor (i.e. the ratio of the side lengths).

Volume Ratio: If two solids are similar with single dimension lengths (& thus scale factor) in the ratio of $a:b$, then the volumes are in a ratio of $\left(\frac{a}{b}\right)^3$.

If the volumes of two spheres are in a ratio of $8:27$, what is the ratio of their diameters?

To solve this problem, we know that the three-dimensional ratio (i.e. ratio of the volumes) is the single dimension ratio (or diameter ratio) cubed, i.e.:

$$\frac{8}{27}=\left(\frac{x}{y}\right)^3$$

Where $\frac{x}{y}$ is the ratio of the diameters (or of any specific single dimension part to another single dimension part).

Thus, we solve by taking the cube root of $\frac{8}{27}$:

$$\sqrt[3]{\frac{8}{27}}=\frac{2}{3}$$

Answer: $2:3$.