CHAPTER

13 ABSOLUTE VALUE

SKILLS TO KNOW

- Basic definition of absolute value and how to apply it
- How to solve basic absolute value equations and inequalities
- How to solve "properties of numbers" problems involving absolute value (Could be true, must be true, etc.)
- How to set up word problems using the idea of "open sentence" equations and inequalities

Note: For more absolute value problems, see chapters in this book on Graph Behavior and Translations and Reflections.

THE BASICS

Remember, absolute value means the distance between a number and zero.

So if you are asked to solve $|-4|$ you ask yourself, "how far from zero is -4?" As you can see in the number line below, it's four away from zero.

In practice, when you see those bars, make the value between them positive. That's it. If the value between the bars is positive already, the answer is the same as the number between the bars. $(|7| = 7)$.

If the value is negative, you make it positive. $(|-2| = 2)$. In any case, any absolute value **is always zero or positive—it cannot be negative.**

Absolute value signs are like a form of parentheses—in PEMDAS you treat them on the same level as you would treat parentheses, i.e., complete all work within them first, then simplify once you have a single numeric value between the bars (if possible). For example:

$$|-2-3|-4=?$$

$$|-5|-4=$$

$$5-4=1$$

Answer: **1**.

SOLVING ABSOLUTE VALUE EQUATIONS

To solve absolute value equations, follow these four steps:

STEP 1: ISOLATE

Isolate the part of the equation that has absolute value in it (often it will already be isolated).

How many real solutions exist for the equation: $|x^2-5|-9=0$

$\left\|x^2-5\right\|-9=0$	The absolute value part is not isolated yet!
$\left\|x^2-5\right\|=9$	Add 9 to both sides to isolate the absolute value.

STEP 2: CHECK FOR THE IMPOSSIBLE

Remember <u>absolute values cannot be negative!</u> If your answer is negative at this stage, you're done. The answer would be no solution. For example, $|x^2-4|=-12$ is impossible! No absolute value can ever be negative!

Our problem $|x^2-5|=9$, however, sets an absolute value equal to a positive—so we move on.

STEP 3: SPLIT THE PROBLEM

Split the problem into two cases. **Case 1:** When the element between the bars is positive. Think about it—what you're doing is changing the sign on what the answer would be if there were no bars—as such you need to simply REMOVE the bars, and **Case 2:** Multiply ONE side of the equation by -1. I typically multiply whichever side is simplest (i.e. if a plain integer is on one side, that's the side I'll choose so I don't have to distribute the negative).

Again, our equation is $|x^2-5|=9$.

Case 1: Remove the bars	**Case 2: Multiply one side of the equation by -1**
$x^2-5=9$ $x^2=14$ Add 5 to both sides $x=\pm\sqrt{14}$ Take the square root of both sides We have two real, irrational solutions: $\sqrt{14}$ and $-\sqrt{14}$	**Don't forget to distribute the negative if necessary $x^2-5=(9)(-1)$ Multiply by negative one $x^2-5=-9$ Add 5 to both sides $x^2=-4$ No real numbers squared equal a negative. Given that the question does not want non-real solutions, there are no solutions from this case.

STEP 4. DOUBLE CHECK!

It's always a good idea to double check absolute value questions if you have time—though if you don't forget step 2, you'll probably be fine. In any case, I find that plugging in at the end of these problems helps eliminate extraneous solutions and careless errors, particularly on more challenging problems (i.e. in the last 15 or so questions on the test.) For this example, when I plug in $\sqrt{14}$ and $-\sqrt{14}$, I get $14-5=9$—that's correct so we're good.

The answer to the question is **2**—there are **two real solutions. Remember to always reread the question** before you put an answer. Often you'll be asked not for the numeric solution, but rather the *number* of possible solutions, the sum of the solutions, or some other value! This **"safety net"** step is particularly important on absolute value problems!

ABSOLUTE VALUE INEQUALITIES

Here's the deal—when approaching absolute value inequalities you can usually turn them into equations and then test points after you've solved the thing down. That method works and there's nothing really wrong with it—though testing points is sometimes time consuming. The great thing about this method is that it's easy to learn—and your prep time is limited. These come up less often than other problems in this area, so this is the method I recommend for its simplicity (it's just less likely you'll mess it up!) for anyone aiming for under a 31-32:

Step 1: Make the inequality sign into an equals sign
Step 2: Solve as an equality
Step 3: Take your "hinge points" create regions, then test regions

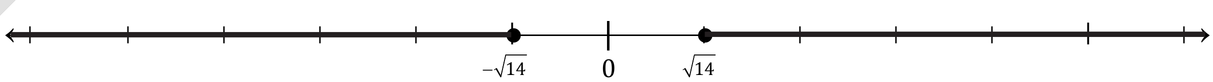
For simplicity's sake, let's take our previous problem and make it an inequality: $\left|x^2-5\right|-9\geq 0$.

We make the greater than sign an equals sign, solve it down in the same way—and get our "hinge" points as $\sqrt{14}$ *and* $-\sqrt{14}$ (see above example if you want to see how I got these numbers).

At this point we need to test three regions, find a test point in each, and plug it into $\left|x^2-5\right|\geq 9$.

Region		**Test Point**		**Plug into**	
Less than $-\sqrt{14}$	$\rightarrow$	test point of $-\sqrt{16}=-4$	$\rightarrow$	$\left|(-4)^2-5\right|=\left|16-5\right|=11$	$\rightarrow$ Is $11>9$? YES
Between $\sqrt{14}$ and $-\sqrt{14}$	$\rightarrow$	test point of 0	$\rightarrow$	$\left|0^2-5\right|=\left|-5\right|=5$	$\rightarrow$ Is $5>9$? NO
More than $\sqrt{14}$	$\rightarrow$	test point of $\sqrt{16}=4$	$\rightarrow$	$\left|4^2-5\right|=\left|16-5\right|=11$	$\rightarrow$ Is $11>9$? YES

As you can see, the first and third test regions are greater than 9, so these adhere to the inequality and make it true. As such the answer is $x\leq-\sqrt{14}$ or $x\geq\sqrt{14}$. We can also write this as $x\leq-\sqrt{14}\cup x\geq\sqrt{14}$.



Sometimes, that little method won't work. Occasionally, on very tough problems, you're stuck using your brain, using logic (or memorization) to keep the inequality sign in—but that takes more study time to learn. Aim to do these if your goal is a 32+ on the math section.

The solution set for the inequality $|3x+a|<5$ is $\left\{x \middle| -\frac{1}{3}<x<3\right\}$. What is the value of a?

Here, I can set this equal to 5 and then solve down my two cases:

$3x+a=5$ or $3x+a=-5$ but now with that last method I get stuck—I could try to plug in the two x "hinge points" but I'm not sure which equation each goes with and the whole process could get time consuming if I have to test each of the two values in each of the two equations—(that's four tests!) not to mention we don't know the sign direction. Instead I'll keep the inequality sign in and instead think about what absolute value means.

$|3x+a|<5$ *means that* $|SOMETHING|<5$—in other words, that SOMETHING is "less than 5 away from zero"—the distance between that something and zero is less than five. With that meaning in mind, I know that that SOMETHING is either five away to the left (at negative five) or five away to the right (at five) or somewhere in that zone even closer to zero. I get to all this through logical reasoning, and now I have something to solve down—with the goal of isolating the x in the middle.

$$-5<3x+a<5$$

$$-5-a<3x<5-a$$

$$\frac{-5-a}{3}<x<\frac{5-a}{3}$$ This looks familiar—let's match it up with our condition.

$$\downarrow \qquad \downarrow$$

$$-\frac{1}{3}<x<3$$ Now we'll turn this coincidence into a couple of equations.

$$\frac{-5-a}{3}=-\frac{1}{3} \qquad \frac{5-a}{3}=3$$

$$-5-a=-1 \qquad 5-a=9$$

$$-a=4 \qquad -a=4$$

$$a=-4 \qquad a=-4$$

NOTE: it's very important that you solve out BOTH equations—the two a values might not be equal and in that case there would be no solution (or you made a mistake)!

In any case, the answer here is $a=-4$.

PROPERTIES OF NUMBERS ABSOLUTE VALUE QUESTIONS

These problems are the kind that you don't likely see much in math at school, and that is what makes them a challenge.

In these problems you often have a parameter—such as "$For\ all\ x>0$" that limits the input of what values we're talking about. When you have that limitation, you can't simply "solve" the problem algebraically—you can either make up a number or use a combination of algebra and logic.

TECHNIQUE #1: Make up a number
Make up a number when you have a parameter, have variables in the answer choice, and <u>have no algebraic way to simplify.</u>

Though in other parts of the book, I may warn you that making up numbers is a slow way to go, with absolute value problems, there is often no straight algebraic way out—only a logical one—and logic can strain your brain. If you're aiming for a 33+ I recommend you learn to also solve problems algebraically and logically, otherwise you can likely get away with making up numbers.

If $x \geq 7$, then $|7-x| = ?$

A. $7-x$ **B.** $x-7$ **C.** $x+7$ **D.** 0 **E.** $-7-x$

Here, the easiest way to the answer is likely to plug in 8. First, we plug into the expression $|7-x|$ in the question stem: $|7-x|=|7-8|=|-1|=1$.

Now I plug in 8 to all the answers and look for a result of "1"—I have to try EVERY answer though in case I randomly chose a number that works twice (it happens—and is a good reason to avoid choosing any number already in the problem).

A. $7-x=7-8=-1$ *NO*
B. $x-7=8-7=1$ *YES*
C. $x+7=8+7=15$ *NO*
D. 0 *NO*
E. $-7-x=-7-8=-15$ *NO*

B is the correct answer, as it equals 1.

TECHNIQUE #2:
Solve with a combination of algebra & logic—it's sometimes faster, but only if your brain is practiced enough to see the logical connections quickly.

Let's try splitting the problem into two cases—

$|7-x|$ is either going to be equal to $7-x$ or $-(7-x)$.

When I split this into two cases, I don't have the luxury of an equation—I only have an expression, and that means I must distribute the negative in the second case: $-(7-x)=7+x=x-7$

Now I have my two cases:

$$7-x \text{ and } x-7$$

Instantly we know the answer is either (A) $7-x$ or (B) $x-7$.

But here I'm not done and I need to use logic, considering the restriction on x. First, I consider the idea that in BOTH of these two cases, if these are what this expression equals, each MUST BE non-negative. That means for the first case, x has to be smaller than 7 to create a non-negative answer. Once x grows to a larger number, the expression would be negative, and an absolute value can't be negative.

In the second case, I can imagine how x MUST BE positive and at least 7—else subtracting 7 is going to send the number into the region of negatives. Since I know the question states that $x \geq 7$, that means the second case is the correct case.

As you can see, all this thinking is a bit confusing—true, I get an algebraic answer—true, if you're aiming for a 36 it's good to understand these problems this well, but for many this method is overkill. The bottom line with doing these the logical way is that you're looking to see the general **behavior of numbers and expressions.**

ABSOLUTE VALUE INEQUALITIES: WORD PROBLEMS

NOTE: These don't show up all that often on the test—I would recommend practicing them if you're aiming for above a 30 and prepping for at least a few weeks.

TIME SAVER TIP: The other thing to remember about absolute value is that it can also be used to represent the phrase: **"the difference between x and a number is"** or in coordinate geometry, "**the distance between a and b is**." For example:

$$|x-3| \leq 7$$

Can be translated as:

"The distance between x and 3 is no more than 7."

OR

"x and 3 are no more than 7 apart."

Think about it—when you subtract 3 from x, you do what you do when you calculated slope. You find the difference in two numbers—one of the y's and the other y, which is the same as the distance. Remember rise over run? You can use the slope formula, or you can measure it out with little boxes on the page as a distance. **When you take the absolute value of a difference, you find the physical distance between two points.** Slope formulas still use the sign (you're figuring out a rate, really) but the concept of distance is the same.

You can also think of these as a way to represent margin of error. Let's say you're building a bookshelf and it's supposed to be $36''$ high, but can have a margin of error of up to a quarter inch. **The difference between** the <u>ideal</u>, 36, and the <u>actual</u> height, d, can be **no more than** a quarter inch.

$$|36-d| \leq .25$$

Because absolute value subtracts one thing from another thing, but then gets rid of the "negative" sign, **it's the same as finding the DISTANCE between two values or numbers**. This idea can come in handy on the occasional word problem. I recommend you commit to memory that phrase—**"the distance/difference between"** —to think of absolute value subtraction problems!

The diameter, d, of the plastic pipes that a hardware stores sells must satisfy the inequality $|4-d| \leq .005$. What is the maximum diameter, in centimeters, that a plastics pipe may have?

Let's talk about what all this represents—

The part in the absolute value sign means that the **distance between 4 and d is no greater than $.005$. 4 and d are no more than $.005$ apart.**

That means we can be $.005$ more than 4, $.005$ less than 4 or in between hanging out even closer to 4 and get d.

If we need the greatest diameter, that would mean we'd add that miniscule margin of error to our "ideal" of 4—

$$4+.005=4.005$$

Answer: 4.005

Now you could back-solve these type of problems, meaning you could plug in all the answer choices and see what works—but that method is slow. These are called "open sentence" inequalities. They're found in most Algebra I books (I know!) so in all likelihood you learned these things back in 8th grade. Seriously, if you want more practice on these, find your younger sister's math book… The best advice is just to memorize the phrase—

THE DISTANCE BETWEEN ( ) and ( ) is (no more than/less than/greater than/at least).

If you can translate these equations to that idea, you'll be able to solve these problems extremely quickly.