CHAPTER

3 FACTORING BASICS, LCM & GCF

SKILLS TO KNOW

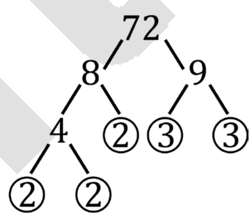
- How to use a Factor Tree
- How to use a Factor Rainbow
- How to find Least Common Multiples
- How to find Greatest Common Factors
- Solving word problems involving LCM and/or GCF

FACTOR TREES

One skill every ACT® math student should know is how to use a "factor tree." Factor trees are tools used to determine the prime factorization of a number.

Remember that prime factorization finds all the prime numbers that when multiplied together equal the given number.

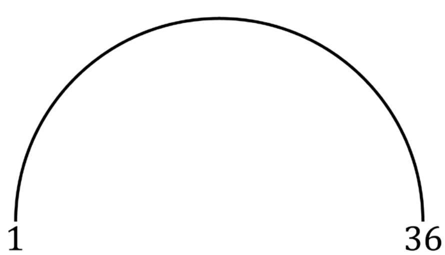
Let's say we want to factor a number like 72:



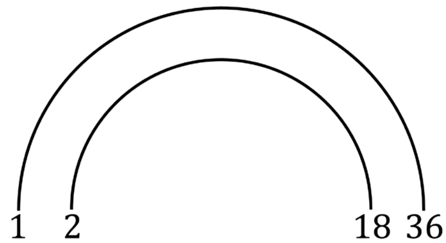
We branch off pairs of factors until we cannot branch off anymore. At this point, we get to the prime factorization (circled above). It doesn't matter which pair of factors you choose to start with, so long as you keep going until no more factors can be divided out. For example, I could have started with 2 and 36 instead of 8 and 9. If you're not sure whether a number is prime, use the divisibility rules (see Chapter 2, page 8).

FACTOR RAINBOWS

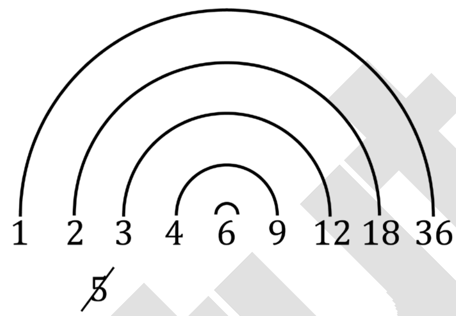
Another valuable tool to know is how to break down a number with a factor rainbow. For this tool, we start by drawing a "rainbow" from the number 1 to the number we are trying to factor. Below we'll use 36 as an example.



Then you count up one number at a time, adding a "rainbow" arch for each pair of factors. For instance, I count from 1 to 2, and ask, does 2 go into 36? Yes, it does, 18 times. So I add 2 and 18 to my rainbow.



Then I count to 3 and ask, does 3 go into 36? If so how many times, and so forth. I continue, adding pairs 3 and 12, 4 and 9, then I get to 5, but 5 does not go into 36, so I write it down below the rainbow and slash it out. Then I get to 6, the square root of 36, and I'm done.



What the factor rainbow gives us is not our PRIME factors, but ALL of the factors of a number. It also gives us all the factor pairs. If you are required by a problem to count the total number of factors, rather than to find the prime factors, this is your best tool.

LEAST COMMON MULTIPLE

The least common multiple of two or more numbers, often abbreviated as LCM, is the smallest whole number that has those two (or more) numbers among its factors. For example, the LCM of 2 and 3 is 6, since it is the smallest whole number that has both 2 and 3 among its factors.

I like to think of LCM as building a "Lego" kit (easily enough, it starts with "L"!). Do you know how Lego kits sometimes have a certain object you can build, say a pirate ship, but then when you open the box there are instructions for two other pirate ships that you could also build? But you can't build all three ships at once? To save money, Lego only includes enough pieces to build one ship at a time, but some pieces may only be used on one ship or another. That's similar to the way LCM works. Each LCM is like an entire pirate ship Lego Kit, each number that you're trying to find the LCM of is like a pirate ship, and the numbers that are prime factors of these "pirate ship" numbers are like Lego pieces—the pieces that multiply together to build each number.

For example, if I wanted the LCM of 18 and 24, the pirate ships are 18 and 24, and the Lego pieces are the prime numbers that make up these "pirate ships:" 3, 3, 2 for 18 and 3, 2, 2, and 2 for 24. To build both "pirate ships" I'll need two 3's (because 18 needs two 3's but 24 only needs one) and three 2's (because 24 needs three 2's but 18 only needs 1). In your LCM or Lego kit you need to have the elements necessary to build all two numbers (or pirate ships) but you can reuse the pieces (or prime factors) that are needed to build each. To then find the LCM, you multiply all the "pieces" you have together, so I'd multiply to get $3\times3\times2\times2\times2$ to get 72.

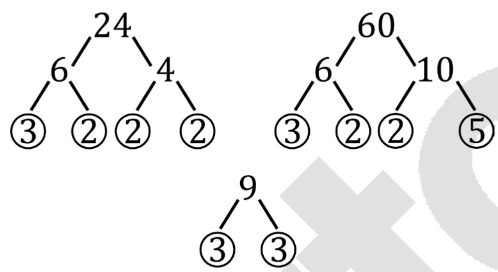
Below, a step-by-step example:

What is the least common multiple of 24, 9, and 60?

Step 1: List all numbers' prime factorizations:

$$24 = 2\times2\times2\times3$$
$$9 = 3\times3$$
$$60 = 2\times2\times3\times5$$

If you can't see these numbers easily, use a factor tree to find these factors:



Step 2: Identify each unique prime factor (your "Lego pieces") and the greatest number of times it occurs in any of the numbers (each "pirate ship"). The idea is that you want to cover your bases and be sure to include all the pieces you need to build out your LCM. This means that even though the factor 2 appears twice for 60, it appears three times for 24, so we need three copies of 2, not two copies. We can easily use two copies to create 60 so long as we have three copies to cover 24:

2 occurs a maximum of 3 times (in 24).
3 occurs a maximum of 2 times (in 9).
5 occurs a maximum of 1 time (in 60).

Step 3: Multiply the unique prime factors by the greatest number of times each occurs, and then multiply all of the numbers together to form your LCM (or "Lego kit"):

$$\overbrace{2\times2\times2}^{3\ \textit{times}}\times\overbrace{3\times3}^{2\ \textit{times}}\times\overbrace{5}^{1\ \textit{time}} = 8\times9\times5 = 360$$

Answer: 360 is the LCM of 24, 9, and 60.

What is the least common denominator for the expression below?

$$\frac{1}{17\cdot29\cdot31}+\frac{1}{17^2\cdot31^2}+\frac{1}{17\cdot23\cdot29}$$

$$17^1\cdot29^1\cdot31^1 = 17^1\cdot23^0\cdot\mathbf{29^1}\cdot\mathbf{31^1}$$
$$17^2\cdot31^2 = \mathbf{17^2}\cdot23^0\cdot29^0\cdot\mathbf{31^2}$$
$$17^1\cdot23^1\cdot29^1 = 17^1\cdot\mathbf{23^1}\cdot\mathbf{29^1}\cdot31^0$$

Bolded are instances of the highest powers of the prime factors.

The least common denominator will be the product of the highest power of each respective prime.

Answer: $17^2\cdot23^1\cdot29^1\cdot31^2$.

GREATEST COMMON FACTOR

The greatest common factor of two or more numbers, often abbreviated as GCF, is the largest whole number that is a factor of all the numbers. For example, the GCF of 18 and 24 is 6 because 6 is the largest whole number that is a factor of 18 and 24. We're looking for what prime factors (or even just factors) overlap, and then we simply multiply these numbers together to get the GCF.

How to find the GCF

What is the greatest common factor of 4620 and 1575?

Step 1: List all number's prime factorizations (use a factor tree if necessary):

$$4620: 2\times2\times3\times5\times7\times11$$
$$1575: 3\times3\times5\times5\times7$$

Step 2: Match all the numbers that appear in both prime factorizations:

3, 5, and 7

(Note: we only list 5 once; if 5 appeared twice in both prime factorizations, we would list it twice.)

Step 3: Multiply all the overlapping numbers:

$$3\times5\times7=105$$

Answer: 105 is the GCF of 4620 and 1575.

NOTE: If there are no matches between the prime factorizations, the GCF is 1. The ACT® is notorious for "trick" questions, so be sure to know this fact!

GCF AND LCM WITH VARIABLES

The same concept can be applied to variables in monomials.

What is the greatest common factor of $21x^3y$ and $63x^2y^2$?

Even though there are variables now, we can go through the same steps as before.

$$21x^3y: 7\times3\times x\times x\times x\times y$$

$$63x^2y^2: 3\times3\times7\times x\times x\times y\times y$$

Matching the common numbers and variables, we find the product of the overlapping numbers:

$3\times7\times x\times x\times y$, which is $21x^2y$.

Answer: $21x^2y$.

SOLVING WORD PROBLEMS WITH GCF/LCM

One thing about word problems, as you'll learn more about in Chapter 17, is that it can be tricky to figure out which technique to apply to the problem.

For word problems involving GCF and/or LCM, it's important to determine whether you're looking for the GCF or the LCM. Sometimes people confuse when to use which.

Here are some tips:

GCF:

If the problem is asking…

- To break something into smaller parts
- For you to find the "greatest" or maximum

In general, GCF will be useful for division problems that are asking for the maximum possible solution. (greatest number of items in a group, etc.)

LCM:

If the problem is asking…

- For you to find the overlap in a pattern that repeats over time (clocks that chime so many minutes apart, lights that blink at certain intervals, etc.)
- For you to find something that is "least" or a minimum
- For you to find a common denominator

> One light blinks every 8 seconds. Another light blinks every 6 seconds. When you see them blink at the same time, how long will it be until the first time the lights blink at the same time again?

First, we need to decide if this problem is looking for the GCF or LCM. Because the problem is asking for the least number of seconds that it will take for the lights to blink together, finding the LCM is the correct technique to apply.

Using the method we wrote about previously, let's list all the prime factorizations:

$$8 : 2\times2\times2$$
$$6 : 2\times3$$

Then, we list each unique prime factorization that appears the most amount of times.

So, for 8, we have $2:3$ times. For 6, we have $3:1$ time.

Then, we multiply them:

$$2\times2\times2\times3=24$$

Answer: 24 seconds.