CHAPTER

2 SYSTEMS OF EQUATIONS

SKILLS TO KNOW

- Solve a system of equations
- Set up word problems involving systems of equations
- Distinguish between systems with zero, one, or infinitely many solutions
- Solve for a given variable in a system of equations with single, infinite, or no solutions

SOLVE A SYSTEM OF EQUATIONS

Many students taking the ACT® know how to solve a system of equations, but often reviewing these simple problems can help students build speed and accuracy. Likewise, many students get in the habit of only using one method—knowing both will help you solve problems more efficiently. Finally, the ACT® versions of these questions may be trickier to handle than the versions you tackled in math class.

Remember there are two main ways to solve systems of equations: **Substitution** and **Elimination**.

SPEED TIP! Which method you choose to do the problem doesn't matter in terms of accuracy (you won't get a question wrong for choosing one method over another)—but sometimes one method will be faster than another—**knowing which is best in certain situations can help you speed up and finish the test in time.**

Let's take a look at a quick example of each:

Substitution

The phrase to keep in mind when solving systems of equations with substitution is: **"ISOLATE to ELIMINATE."**

The most confusing thing about these problems is how sometimes you only need to solve for one variable or the other, but if you approach the problem haphazardly, you may have to plug the other variable back in at the end—or worse—you'll think you have the answer, but you don't. What if you could make sure you solve for the variable you actually need EVERY TIME? See more about that below.

Likewise, when these problems get more complex, you'll often have three variables and need to reduce those down to two variables. For these problems, knowing how to target and eliminate a specific variable is essential.

Let's take a look at "ISOLATE to ELIMINATE" in action:

What value of y solves the following system of equations?

$$x+6y=32$$
$$5x+y=24+2x$$

As you can see, the first equation has no coefficient in front of the y and the second has no coefficient in front of the x. As such, to substitute, you could really solve for either. BUT the question is specifically asking for y! Now let's think, what variable do you want to ELIMINATE, meaning—which do you want to get rid of? That would be the x, as we want to solve for or keep the y. To isolate the x we simply act as if we are variable assassins. If you want to get rid of an enemy in a video game, you push that enemy into the corner—you isolate it—then eliminate it! That's what we'll do to x. The easiest variables to isolate are those without a coefficient, so we'll use the second equation in which x has no coefficient:

STEP 1: ISOLATE! (Equation 1)

$$x+6y=32$$
$$x=-6y+32$$

Now we "substitute" x into the other equation and the x values will disappear. But before we do, we're going to combine like terms so we only have to substitute once.

STEP 2: SIMPLIFY (Equation 2)

$$5x+y=24+2x$$
$$y=24+2x-5x$$
$$y=24-3x$$

STEP 3: SUBSTITUTE

$x=-6y+32$	
$y=24-3(-6y+32)$	Substitute into equation 2 from equation 1
$y=24+18y-96$	Distribute the -3
$y=18y-72$	Simplify, combining like terms
$-17y=-72$	Subtract $18y$ from both sides
$y=\frac{72}{17}$	Divide both sides by -17

TIP: **Always substitute in with PARENTHESES!** If you don't, you may forget to distribute.

i.e. $y=24-3(-6y+32)$

Don't forget these!

What is the value of n in the solution to the following system of equations?

$$5n-m=26$$
$$3n+2m=39$$

A. 13 **B.** 9 **C.** 7 **D.** 5 **E.** –7

Again, here m in the second equation has no coefficient, and we want n so let's isolate m! Remember, **ISOLATE TO ELIMINATE**!

Equation 1: Isolate m:

$$\begin{aligned} 5n-m&=26 \\ 5n&=26+m \\ 5n-26&=m \\ m&=5n-26 \end{aligned}$$

Substitute m into Equation 2:

$$\begin{aligned} 3n+2m&=39 \\ 3n+2(5n-26)&=39 \\ 3n+10n-52&=39 \\ 13n&=91 \\ n&=7 \end{aligned}$$

Answer: **C**.

Use substitution: when one variable has 1 (or no) coefficient (i.e. m in the question before).

Use elimination: When two coefficients in two equations in front of the same variable match or are multiples of each other.

Elimination

Elimination involves stacking two equations and then adding or subtracting straight down after you've lined up your like terms.

If the following system has a solution, what is the x-coordinate of the solution?

$$2x+2y=58$$
$$3x-2y=27$$

A. 12 **B.** 17 **C.** 31 **D.** 34 **E.** 85

Here we want x, so we want to eliminate y. We're in luck because the coefficients of y match: y and $-y$ are opposites of each other, and if stacked, will "zero" out.

We set up a giant addition problem to eliminate the y terms:

$$\begin{array}{r} 2x+2y=58 \\ +\ \ 3x-2y=27 \\ \hline 5x+0y=85 \\ x=17 \end{array}$$

As you can see, the y's disappear, and we're left with the answer: **B**.

Problems aren't always so convenient—when coefficients don't match, we can manipulate the situation to make them match:

> Solve for y:
> $$2x+3y=16$$
> $$4x-5y=21$$

Here, no coefficients match, and we have no variables with a coefficient of one, but the 2 and the 4 are multiples of each other. The idea is to get two coefficients that match in number but have opposite signs, so they cancel. As such, we can multiply the first equation by -2 to make the resultant $-4x$ cancel with the $4x$ in the second equation.

$$-2(2x+3y=16)$$
$$\downarrow$$
$$(-2)2x+(-2)3y=(-2)(16)$$
$$\downarrow$$
$$\begin{array}{rl} & -4x-6y=-32 \\ + & 4x-5y=21 \\ \hline & 0x-11y=-11 \\ & y=1 \end{array}$$

WORD PROBLEMS INVOLVING SYSTEMS OF EQUATIONS

Sometimes students get caught up on word problems that require systems of equations. Make sure you engage your mind and think through logically what each value represents. If you get stuck in setting up a word problem, <u>make up numbers to help you understand what is what. In other words, imagine you know what each variable stands for.</u>

> At Montesquieu's Bistro, you can get 5 pastries and 2 espressos for $\$21$ (before tax). The price of a pastry is p dollars. The price of a pastry is equal to the price of 2 espressos. Which of the following systems of equations, when solved, gives the price, n dollars, of an espresso and the price, p dollars, of a pastry at Montesquieu's Bistro?
>
> **A.** $\begin{cases} 5p+2n=21 \\ n=2p \end{cases}$ **B.** $\begin{cases} 5p+2n=21 \\ p=2n \end{cases}$ **C.** $\begin{cases} 2p+5n=21 \\ p=2n \end{cases}$ **D.** $\begin{cases} 5+2pn=21 \\ 2p=n \end{cases}$ **E.** $\begin{cases} 5n+2p=21 \\ n=2p \end{cases}$

Price of pastry = p
Price of espresso = n

To set this up, think about what's happening. Let's say pastries (p) are $\$1.50$ each. If we bought 5, that would be $5\times(\$1.50)$. Substitute p back in for the $\$1.50$ and you have $5p$. Now we need to do the same for espressos (n)—again if we knew the price we would multiply and get $2n$ as the cost of two espressos. We need the sum of these two to equal the total cost, 21:

$$5p+2n=21$$
$$p=2n$$

For the second equation, we simply use translation—taking English and turning it into math. The price of a pastry (p) is equal to ($=$) the price of two espressos ($2n$).

These two equations determine that the answer is **B**.

That was probably easy, but the method works in harder situations too:

Strategy 1: Make up numbers to make the problem more understandable so you can figure out what goes where.
Strategy 2: Turn English words into math equivalents.

ZERO, ONE, OR INFINITE SOLUTIONS

You'll need to know how to figure out whether problems have one, zero, or infinite solutions.

To figure out which condition is met, you have two choices:

METHOD 1: Solve the System

- **Solve by elimination or substitution and evaluate your final answer.**
- **When you finish your problem you'll have one of three conditions:**

One solution	Zero solutions	Infinite solutions
If you get a single x or y value, you have one solution	If you get a statement that is never true, you have no solutions.	If you get two values that always equal each other, you have infinite solutions.
Example: $x=6$ or $y=0$	**Example:** $5=6$ or $7=0$	**Example**: $5=5$ or $y=y$

If the following system of equations has a solution, what is the x – coordinate of the solution?

$$3y=12-x$$

$$3y=x-6$$

A. –9 **B.** –3 **C.** 1 **D.** 9 **E.** No solution exists

Here we can short cut and substitute in for $3y$; both equations are equal to $3y$, so we can make the right side of both of the equations equal to each other. Remember you can substitute whole expressions—you don't have to just isolate the variable, you can isolate anything identical!

$$x-6=12-x$$
$$2x=18$$
$$x=9$$

Answer: **D**.

If you get a single x or y value, you have one solution. Note: if you have nonlinear equations in your system, it's a great idea to double check answers and plug x back in to make sure y exists too. With linear equations that's not necessary, but it can be a good way to check your work quickly.

How many solutions exist in the system of equations below?

$$\frac{y}{2} = 8x + \frac{3}{2} \qquad \frac{16}{3}x = \frac{y-3}{3}$$

A. Zero solutions
B. One real solution
C. One imaginary solution
D. Two real solutions
E. Infinitely many solutions

Put the equations in two columns and simplify.

Multiply everything by 2 $\quad \frac{y}{2} = 8x + \frac{3}{2} \qquad \frac{16}{3}x = \frac{y-3}{3}$ $\quad$ Multiply everything by 3

$$y = 16x + 3 \qquad 16x = y - 3$$

Now we substitute the left equation into the right one:

$$16x = y - 3$$
$$16x = (16x + 3) - 3$$
$$16x = 16x$$
$$x = x$$

Here we have something that equals itself: $x = x$. That statement is always true, so no matter what x or y equal, the equations overlap—that is, they are the same equation. If you get two equal quantities (i.e. $0 = 0$ or $5 = 5$), you have **infinite solutions.**

Answer: **E.**

What is the $x-$coordinate of the solution to the problem set below?

$$\frac{y}{4} = 2x + 3 \qquad \frac{y}{2} = 4x - 1$$

A. -4 **B.** -2 **C.** 2 **D.** 4 **E.** No solution exists

First, multiply the left equation by 4 and the right equation by 2 so the equations are in y-intercept form.

$$y = 8x + 12 \qquad y = 8x - 2$$

Then, set the equations equal to each other to solve for x:

$$8x + 12 = 8x - 2$$
$$12 = -2$$

Here we get two answers that are never equal. If you get something not true (i.e. $5 = 0$, $7 = 2$) you have no solutions (they are parallel lines). Parallel lines will never intersect, so we have **no real solutions.**

Answer: **E.**

METHOD 2: Put EVERYTHING in SLOPE-INTERCEPT FORM

- **Get both equations into slope-intercept form and compare!**

We can quickly look at the slopes and y-intercepts and determine the relationship according to the chart below:

One solution	Zero solutions	Infinite solutions
Different slopes (intercepts don't matter)	Same slopes (parallel lines), different y-intercepts	Same slope, same intercept
Example: $y=2x+1$ and $y=3x+7$	**Example:** $y=3x+2$ and $y=3x+4$	**Example**: $y=3x+2$ and $y=3x+2$

How many solutions exist in the system?

$$y=3x+\frac{4}{3} \qquad 3x=\frac{3y-4}{3}$$

Because the first equation is already in slope-intercept form ($y=3x+\frac{4}{3}$), we'll put the second one in that form too:

$$3x=\frac{3y-4}{3}$$
$$9x=3y-4$$
$$3y=9x+4$$
$$y=3x+\frac{4}{3}$$

Because these are the exact same equation, this set has infinite solutions! Every x value in one equation gives the same y in the other. Because a line is an infinite number of ordered pairs, the solution set is infinite: there are infinite points of intersection because the lines overlap and continue to infinity in both directions.

How many solutions exist for the system of equations?

$$\frac{y}{2}=4x-2 \qquad 2x=\frac{3y-9}{12}$$

Again, we make two columns and manipulate each equation to slope-intercept form.

$$\frac{y}{2}=4x-2$$
$$y=8x-4$$

$$2x=\frac{3y-9}{12}$$
$$24x=3y-9$$
$$8x=y-3$$
$$8x+3=y$$
$$y=8x+3$$

Same slope (8) and **different y-intercepts (-4** and **3)** indicate **parallel lines (no solution)**.

Answer: 0.

How many solutions exist for the following set of equations:

$$y = 3x + 1$$
$$y = 2x + 1$$

As you can see, these have different slopes. Thus, there will be one real solution.

Answer: 1.

Here's one more:

For what value of a, if any, would the following system of equations have an infinite number of solutions?

$$a(y-6) = x \qquad \frac{y}{a} = x + 2$$

For both equations, isolate the y value to put them into slope-intercept form.

$$y - 6 = \frac{x}{a} \qquad \frac{y}{a} = x + 2$$
$$y = \frac{x}{a} + 6 \qquad y = ax + 2a$$
$$y = \frac{1}{a}x + 6$$

Now it's time for a technique called "matchy matchy." Line up matching portions of the equations and set them equal to each other. Remember for infinite solutions **every term must be identical in slope-intercept form!**

$$y = ax + 2a$$
$$y = \frac{1}{a}x + 6$$

We don't care much about the y's—line up the x terms and you'll see that the slopes need to be the same:

$$a = \frac{1}{a}$$
$$a^2 = 1$$
$$a = 1$$

Line up the last term and you'll see that $2a$ must be equal to 6.

$$2a = 6$$
$$a = 3$$

So we got that $a = 3$ and $a = 1$. That's not possible! So thus there are NO values of a that would make these equations have infinite solutions.

Answer: **No such value exists.**