Functional Analysis

Lecture Notes

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Finite dimensional linear spaces

Definition 1.1

Linear Space $\mathbf{X}$, $\{z_1, z_2, \ldots, z_N\}$ (linearly independent elements) is maximal if $\forall x \in \mathbf{X}, \exists \{\alpha_i\}_{i=1}^N$, $x = \sum_{j=1}^{N} \alpha_i z_i$.

Remark 1.1

If $\{z_1, z_2, \ldots, z_N\}$ and $\{w_1, w_2, \ldots, w_M\}$ are maximal, then $N = M$.

Proof.

Let $w_k = \sum_j a_{kj} z_j$ and $z_j = \sum_t bjt w_t$, then $w_k = \sum_t (ab)_k t w_t$ where a, b are matrices formed with elements $\{a_{kj}\}, \{b_{jt}\}$. Then $ab = \mathbf{I}_d \Rightarrow N \geq M$. By representing z_j in terms of w_k, we have $M \geq N$. Therefore, $N = M$. $\square$

Definition 1.2

The dimension of LS $\mathbf{X}$ is the number of elements of a maximal set of linearly independent elements. $\dim(\mathbf{X}) = \infty$ if $\forall k \geq 1, \exists$ set of linearly independent elements $\{z_i\}_{i=1}^k$.

Finite dimensional NLS

- $\mathbf{X}$ is NLS, $\dim(\mathbf{X}) = N < \infty$ with $\{z_i\}_{i=1}^k$ being a maximal linearly independent set. $x = \sum_{j=1}^{N} \alpha_j(x) z_j$.

Claim 1.1

$\exists c_0 < \infty$ s.t. $\sum_{j=1}^{N} |\alpha_j(x)| \leq c_0 \|x\|$.

Proof.

Let $\{x_p\}_{p\ ge1}$ be a sequence s.t. $\sum_{j=1}^{N} |\alpha_j(x_p)| \geq p\|x_p\| \Rightarrow \|y_p\| \leq \frac{1}{p}$ with $y_p \triangleq \frac{x_p}{\sum_{j=1}^{N} |\alpha_j(x_p)|}$. We have $\alpha_j(y_p) \leq 1, \forall j \geq 1, p \geq 1$. Therefore. $\exists \{p_k\}$ such that $\alpha_j(y_{p_k}) \to \alpha_j, \forall N \geq j \geq 1 \Rightarrow y_{p_k} \to y = \sum_{j=1}^{N} \alpha_j z_j$. Since $\|y_{p_k}\| \leq \frac{1}{p_k} \Rightarrow \|y_{p_k}\| = 0$, we have $\alpha_j = 0, \forall N \geq j \geq 1$, which contradicts the fact that $\sum_{j=1}^{N} |\alpha_j| = 1$. $\square$

Lemma 1.1

$B(0,1) = \{x \in \mathbf{X} \colon \|x\| \le 1\}$ is compact.

Proof.

Let $\{x_p\}_{p\ge1} \subseteq B(0,1)$, then by the above claim we have $\sum_{j=1}^{N} |\alpha_j(x_p)| \le c_0$. Therefore. $\exists \{p_k\}$ such that $\alpha_j(x_{p_k}) \to \alpha_j, \forall N \ge j \ge 1 \Rightarrow x_{p_k} \to x = \sum_{j=1}^{N} \alpha_j z_j$. Also, $\|x\| = \lim_k \|x_{p_k}\| \le 1$, so $x \in B(0,1)$. □

Lemma 1.2

$\mathbf{X}$ is NLS, $\dim(\mathbf{X}) = N < \infty$ with $\{z_i\}_{i=1}^{k}$ being a maximal linearly independent set. $x = \sum_{j=1}^{N} \alpha_j(x) z_j$. Let $\|x\|_0 = \sum_{j=1}^{N} |\alpha_j(x)|$, then $\exists c$ s.t. $\|x\| \le c\|x\|_0$.

Proof.

Set $c = \max\{z_i\}_{i=1}^{N}$. □

Lemma 1.3

$\mathbf{X}$ is NLS, $\dim(\mathbf{X}) = N < \infty$ with $\{z_i\}_{i=1}^k$ being a maximal linearly independent set. Then $\mathbf{X}$ is complete.

Proof.

Let $\{x_p\}_{p\geq 1} \subseteq \mathbf{X}$ be a Cauchy sequence of $\mathbf{X}$, with $x_p = \sum_{j=1}^{N} \alpha_j(x_p) z_j$.

Then

$\forall \varepsilon > 0, \|x_p - x_q\| \leq \varepsilon \Rightarrow \sum_{j=1}^{N} |\alpha_j(x_p) - \alpha_j(x_q)| \leq c_0 \|x_p - x_q\| \leq c_0 \varepsilon,$

and it follows that $\{\alpha_j(x_p)\}_p$ is a Cauchy sequence which converges to α_j, $\forall j \in [N]$. Let $x = \sum_{j=1}^{N} \alpha_j z_j \in \mathbf{X}$, then

$\|x_p - x\| \leq c \|x_p - x\|_0 \overset{p\to\infty}{\to} 0.$ $\square$

Infinite-dimensional unit ball is not compact

Lemma 1.4

Let $\mathbf{X}$ be NLS, and $\mathbf{Y} \subseteq \mathbf{X}$ is a closed linear subspace of $\mathbf{X}$, and $\mathbf{X} \setminus Y \neq \emptyset$. The $\forall \varepsilon > 0, \exists z, \|z\| = 1, \|z - y\| \geq 1 - \varepsilon, \forall y \in \mathbf{Y}$.

Proof.

Let $w \in \mathbf{X} \setminus \mathbf{Y}, d = \inf_{y \in \mathbf{Y}} \|w - y\|$, then $d > 0$. $\forall \delta > 0, \exists y_0 \in \mathbf{Y}$ s.t. $\|w - y_0\| \leq (1 + \delta)d$. Let $z = \frac{w - y_0}{\|w - y_0\|}$. Then $\|z - y\| = \left\| \frac{w - y_0}{\|w - y_0\|} - y \right\| = \frac{w - y_0 - y\|w - y_0\|}{\|w - y_0\|} \geq \frac{d}{(1+\delta)d} = \frac{1}{1+\delta} \geq 1 - \varepsilon$ by setting δ accordingly. □

Proposition 1.1

Unit ball $B(0, 1) = \{x \in \mathbf{X} \colon \|x\| \leq 1\}$ in infinite-dimensional NLS $\mathbf{X}$ is not compact.

Proof.

We aim to construct a sequence $\{z_j\}_{j \geq 1}$ s.t. $\|z_j\| = 1, \|z_i - z_j\| \geq \frac{1}{2}, \forall i \neq j$. Let $w \in \mathbf{X}, w \neq 0, z_1 = \frac{w}{\|w\|}$. Induction step: suppose we have

$$\{z_j\}_{j=1}^{N}, Y_N = \text{span}\{z_j\}_{j=1}^{N} = \left\{ \sum_{j=1}^{N} \alpha_j z_j, \alpha_j \in \mathbb{K} \right\}$$ is a closed subspace spanned by

$\{z_j\}_{j=1}^{N}$ □

Infinite-dimensional unit ball is not compact

Cont'd.

(note that the finite dimensional subspace Y_N is complete, so a converging sequence is Cauchy and it converges to a point in Y_N). Because $\mathbf{X}$ is infinite-dimensonal, we can find a point $z_{N+1}, \|z_{N+1}\| = 1, \|z_{N+1} - z_j\| \geq \frac{1}{2}, \forall j \in [N]$. Also, $\{z_j\}_{j=1}^{N+1}$ is linearly independent. If $B(0,1)$ is compact, then for the sequence $\{z_j\}_{j\geq 1}$, $\exists \{j_k\}$ s.t. $z_{j_k} \to z \in B(0,1)$. However, this indicates that $\{z_{j_k}\}$ is a Cauchy which is impossible, because $\|z_i - z_j\| \geq \frac{1}{2}$. □

Definition 1.3

A NLS $\mathbf{X}$ is separable if there exists $D \subseteq \mathbf{X}$ which is coutable and dense in $\mathbf{X}$.

Example 1.1

Let $\mathcal{B}$ be the Borel σ-algebra of subsets of $\Omega = [0,1]$, and $\mathcal{M} = \{\text{singed measures on } \mathcal{B}\}$. Let $\|\mu\| = \mu^+(\Omega) + \mu^-(\Omega), \forall \mu \in \mathcal{M}$. Then $\|\cdot\|$ is a norm. Let $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise. Then $\|\delta_x - \delta_y\| = 2$. Then $\mathcal{M}$ is not separable.

Zorn's lemma

Definition 1.4

Suppose the relation $<$ is reflexive $(a < a, \forall a)$, antisymmetric $(a < b, b < a \Rightarrow a = b)$, and transitive $(a < b, b < c \Rightarrow a < b)$. Then a set with a partial order is called a partially ordered set.

Lemma 1.5

(Zorn's lemma) Let $(\mathbf{X}, <)$ be a partial ordered set. Assume that every totalled ordered subset Y of $\mathbf{X}$ admits a upper bound. Then $\mathbf{X}$ has a maximal element.

Definition 1.5

Let $\mathbf{X}$ be a LS. $\{x_\theta\}_{\theta \in I}$ is a base of $\mathbf{X}$ if (1) $\forall N, \{x_j\}_{j=1}^{N} \subseteq \{x_\theta\}_{\theta \in I}$ is linearly independent; (2) $\mathbf{X} = \text{span}\{x_\theta\}_{\theta \in I}$.

Theorem 1.1

Let $\mathbf{X} \neq 0$ be a LS. Then $\exists \{x_\theta\}_{\theta \in I}$ be a base of $\mathbf{X}$.

Zorn's lemma

Cont'd.

Let $\Omega = \left\{ \{x_\theta\}_{\theta \in I} \mid \forall N, \{x_j\}_{j=1}^{N} \subseteq \{x_\theta\}_{\theta \in I} \text{ is linearly independent} \right\}$ which is equipped with the partial order $<$ defined by $\{x_\theta\}_{\theta \in I} < \{y_\theta\}_{\theta \in I'}$ if $\forall \theta \in I, \theta' \in I', x_\theta = y_{\theta'}$. Let $\Omega' = \left\{ \{x_\theta\}_{\theta \in I_\alpha} \mid \alpha \in J \right\}$ be a totally ordered subset of Ω, and let $z = \bigcup_{\alpha \in J} \{x_\theta\}_{\theta \in I_\alpha}$, then z is a upper bound for Ω'. By Zorn's lemma, $(\Omega, <)$ has a maximal element $x = \{x_\theta\}_{\theta \in I}$. If $\mathbf{X} \neq \text{span}(\{x_\theta\}_{\theta \in I})$, then $\exists y \in \mathbf{X} \setminus \text{span}(\{x_\theta\}_{\theta \in I})$. y and $\{x_\theta\}_{\theta \in I}$ are linearly independent, so $\{y\} \cup \{x_\theta\}_{\theta \in I} \in \Omega$ and $\{x_\theta\}_{\theta \in I} < \{y\} \cup \{x_\theta\}_{\theta \in I}$. Therefore, x is not a maximal element of Ω. This contradiction shows that $\mathbf{X} = \text{span}(\{x_\theta\}_{\theta \in I})$. □

The Hahn-Banach theorem

Definition 1.6

Let $\mathbf{X}$ be a LS, $\ell\colon \mathbf{X} \to \mathbb{R}$ is a linear function: $\forall x, y \in \mathbf{X}, f(x+y) = f(x) + f(y),\ f(\alpha x) = \alpha f(x), \forall \alpha \in \mathbb{R}$.

Theorem 1.2

Let $Y \subseteq \mathbf{X}$ be a linear subspace, $\ell\colon Y \to \mathbb{R}$ is linear. $p\colon \mathbf{X} \to \mathbb{R}$ is (a) positive homogeneous, $p(\alpha x) = \alpha p(x), \forall x \in \mathbf{X}, \alpha \geq 0$; (b) subadditive, $p(x+y) \leq p(x) + p(y), \forall x, y \in \mathbf{X}$. If $\ell(x) \leq p(x), \forall x \in Y$, then $\exists L\colon \mathbf{X} \to \mathbb{R}$ s.t. (1) L is linear; (2) $L(y) = \ell(y), \forall y \in Y$; (3) $L(x) \leq p(x), \forall x \in \mathbf{X}$.

Proof.

Part 1. Suppose $Y \neq \mathbf{X} \Rightarrow z \notin Y$. Define $Y_z = \{\alpha z + y\colon \alpha \in \mathbb{R}, y \in Y\}$. Then we can extend ℓ from Y to Y_z by $L(\alpha z + y) = \alpha a + \ell(y), a = L(z)$. We need $L(\alpha z + y) \leq p(\alpha z + y)$, which already holds for $\alpha = 0$. When $\alpha > 0$,

$$L(\alpha z + y) \leq p(\alpha z + y) \iff a \leq p(\frac{y}{\alpha} + z) - \ell(y/\alpha)$$

$$\iff a \leq p(y+z) - \ell(y), \forall y \in Y \iff a \leq \inf_{y \in Y} p(y+z) - \ell(y)$$

The Hahn-Banach theorem

Cont'd.

When $\alpha < 0$, by similar argument we have
$L(\alpha z + y) \le p(\alpha z + y) \iff a \ge \sup_{y\in Y} \ell(y) - p(y - z)$. Therefore, to make a exist we need to show $\sup_{y\in Y} \ell(y) - p(y - z) \le \inf_{y\in Y} p(y + z) - \ell(y)$. $\forall y_1, y_2 \in Y$,

$$\ell(y_1 + y_2) = \ell(y_1) + \ell(y_2) \le p(y_1 + y_2) = p(y_1 - z) + p(y_2 + z)$$
$$\iff \ell(y_1) - p(y_1 - z) \le p(y_2 + z) - \ell(y_2),$$

which shows that $\sup_{y\in Y} \ell(y) - p(y - z) \le \inf_{y\in Y} p(y + z) - \ell(y)$.
part 2. Define

$$\Omega = \{(Z, L_Z) \colon Z \text{ is a subspace of } \mathbf{X}, Y \subseteq Z, L_Z : Z \to \mathbb{R} \text{ is linear; } L_Z(y) = \ell(y), \forall y \in Y;$$
$$L_Z(z) \le p(z), \forall z \in Z\},$$

which is equipped with the partial order $<$ s.t. $(Z_1, L_{Z_1}) < ((Z_2, L_{Z_2}))$ if $Z_1 \subsetneq Z_2$ and $L_{Z_1}(z) = L_{Z_2}(z), \forall z \in Z_1$. Let $\Omega' = \{(Z_\alpha, L_{Z_\alpha}) \colon \alpha \in I\}$ be a totally ordered subset of Ω, then Ω' has a upper bound (Z, L_Z) with $Z = \bigcup_{\alpha\in I} Z_\alpha$. $L_Z(w) = L_{Z_\alpha}(w)$ if $w \in Z_\alpha$. It can be verified that L_Z is well defined and Z is a linear subspace of $\mathbf{X}$. By Zorn's lemma, let (Z, L_Z) be a maximal element of Ω. We will prove that $Z = \mathbf{X}$. Otherwise, if $Z \neq \mathbf{X}$, then $\exists w \notin Z$. We can construct a subspace Z_w and $L_{Z_w} : Z_w \to \mathbb{R}$ using the previous argument, then $(Z, L_Z) < (Z_w, L_{Z_w})$, contradicting the fact that (Z, L_Z) is a maximal element. Therefore, $Z = \mathbf{X}$, and $L_Z = L$ is the extention of ℓ from Y to $\mathbf{X}$ satisfying the three properties in the theorem. $\square$

Convex sets and gauge functions

Definition 1.7

Let $\mathbf{X}$ be a LS over $\mathbb{R}$, and $S \subseteq \mathbf{X}$. $x \in S$ is an interior point of S if $\forall y \in \mathbf{X}, \exists \varepsilon = \varepsilon(y)$, s.t. $x + ty \in S, \forall\, |t| \leq \varepsilon$. Note that this is different from requiring $B(x, \varepsilon) \subseteq S$.

Definition 1.8

$\mathbf{K}$ is a convex subest of $\mathbf{X}$. $x \in \mathbf{K}$ is an interior point. $P\colon \mathbf{X} \to \mathbb{R}^+$, $P \triangleq P_{\mathbf{K},x}$, $P(y) \triangleq \inf \left\{ a > 0 \colon x + \frac{1}{a} y \in \mathbf{K} \right\}$ is the Gauge function.

Proposition 1.2

$P\colon \mathbf{X} \to \mathbb{R}^+$ is positive homegeneous and subadditive:

$$P(\alpha y) = \alpha P(y) \quad (\alpha > 0) \tag{1}$$

$$P(z + y) \leq P(z) + P(y) \tag{2}$$

Convex sets and gauge functions

Proof.

$\exists a_0 > 0, b_0 > 0$ s.t $P(z) \leq a_0 \leq P(z) + \varepsilon, P(y) \leq b_0 \leq P(y) + \varepsilon$, and $x + \frac{1}{a_0} z \in \mathbf{K}, x + \frac{1}{b_0} y \in \mathbf{K}$. By the convexity of $\mathbf{K}$, it can be verified that $x + \frac{1}{a_0 + b_0}(y + z) \in \mathbf{K} \Rightarrow P(z + y) \leq P(z) + P(y)$. □

Proposition 1.3

(1) $x + y \in \mathbf{K} \Rightarrow P(y) \leq 1$
(2) $x + y$ is an interior point of $\mathbf{K}$ iff $P(y) < 1$.

Proof.

(1) holds by the defininiton of Gauge function. (2) To see $P(y) < 1 \Rightarrow x + y$ is an interior point of $\mathbf{K}$, note that x is an interior point of $\mathbf{K} \Rightarrow \exists \delta_1 > 0, x + tz \in \mathbf{K}, \forall z \in \mathbf{X}, \forall |t| \leq \delta_1$; $P(y) < 1 \Rightarrow \exists \delta_2 < 1, x + \frac{1}{\delta_2} y \in \mathbf{K}$. By linearly combining $x + tz$ and $x + \frac{1}{\delta_2} y$, i.e. $\delta_2 \left(x + \frac{1}{\delta_2} y \right) + (1 - \delta_2)(x + tz) = x + y + (1 - \delta_2) tz \in \mathbf{K}$, it follows that $x + y + t'z \in \mathbf{K}, \forall |t'| \leq (1 - \delta_2)\delta_1$, so $x + y$ is an interior point of $\mathbf{K}$. □

Convex sets and gauge functions

Proposition 1.4

If $P\colon \mathbf{X} \to \mathbb{R}$ is positive homegeneous and subadditive, then (1) $A = \{x\colon P(x) \leq 1\}$ is convex; (2) $B = \{x\colon P(x) < 1\}$ is convex, and 0 is an interior point of B.

Proof.

It can be verifed by the definition of ocnvex set that A, B are convex sets. Set $\varepsilon = \frac{1}{2}\min\left\{\frac{1}{P(x)}, \frac{1}{P(-x)}\right\}$, then $\forall\, |t| \leq \varepsilon$, $P(tx) < 1 \Rightarrow 0 + tx \in B$, so that 0 is an interior point of B. □

Geometric Hanh-Banach theorems

Theorem 1.3

$\mathbf{K}$ is a convex subset of $\mathbf{X}$, and all points $\mathbf{K}$ are interior points. $\mathbf{K} \neq \emptyset$, $\exists y \notin \mathbf{K}$. Then $\exists L\colon \mathbf{X} \to \mathbb{R}$ which is a linear function, $\exists c \in \mathbb{R}$, such that $\mathbf{K} \subseteq \{x \in \mathbf{X}\colon L(x) < c\}, L(y) = c$.

Proof.

$0 \in \mathbf{K}, P = P_{\mathbf{K},0}, P(z) = \inf\left\{a > 0\colon \frac{1}{a}z \in \mathbf{K}\right\}$. Let $x \in \mathbf{K}$, then x is an interior point and $P(x) < 1$. Because $y \notin \mathbf{K}$, $P(y) \geq 1$. Let $Y = \{\alpha y\colon \alpha \in \mathbb{R}\}$, and $\ell\colon Y \to \mathbb{R}$ is linear. Set $\ell(y) = P(y) \geq 1, \ell(\alpha y) = \alpha\ell(y), \alpha \in \mathbb{R}$. Now we check that $\ell(\alpha y) \leq P(\alpha y), \forall \alpha \in \mathbb{R}$. It holds for $\alpha \geq 0$ by the definition of $\ell(y)$. When $\alpha < 0, \ell(\alpha y) < 0 \leq P(\alpha y)$. By the Hanh-Banach theorem, $\exists L\colon \mathbf{X} \to \mathbb{R}$, $L(y) = \ell(y) = P(y), L(x) \leq P(x), \forall x \in X$. We need to prove that $L(x) < P(y), \forall x \in \mathbf{K}$. We have $L(x) \leq P(x) < 1, \forall x \in \mathbf{K}$ while $P(y) \geq 1$. So $L(x) < P(y), \forall x \in \mathbf{K} \Rightarrow \mathbf{K} \subseteq \{x \in \mathbf{X}\colon L(x) < c\}$ with $c = L(y) = P(y)$. □

Geometric Hanh-Banach theorems

Corollary 1.1

$\mathbf{K}$ is a convex subset of $\mathbf{X}$, $\exists x \in \mathbf{K}$ an interior point, $\exists y \notin \mathbf{K}$. Then $\exists L\colon \mathbf{X} \to \mathbb{R}$ which is a linear function, $\exists c \in \mathbb{R}$, such that $\mathbf{K} \subseteq \{x \in \mathbf{X} \colon L(x) \leq c\}, L(y) = c$.

Proof.

Let $x = 0 \in \mathbf{K}, P = P_{\mathbf{K},0}$. We apply the same proof as that for the Geometric Hanh-Banach theorem. $\exists L\colon \mathbf{X} \to \mathbb{R}$, $L(y) = \ell(y) = P(y), L(x) \leq P(x), \forall x \in X$. $\forall x \in \mathbf{K}$, $L(x) \leq P(x) \leq 1, P(y) \geq 1 \Rightarrow L(x) \leq P(y) = L(y)$. So $\mathbf{K} \subseteq \{x \in \mathbf{X} \colon L(x) \leq c\}$ with $c = L(y) = P(y)$. □

Geometric Hanh-Banach theorems

Theorem 1.4

$\mathbf{K}_1 \neq \emptyset, \mathbf{K}_2 \neq \emptyset$ are two convex subsets, $\exists x \in \mathbf{K}_1$ an interior point, and $\mathbf{K}_1 \cap \mathbf{K}_2 = \emptyset$. Then $\exists L\colon \mathbf{X} \to \mathbb{R}$ which is a linear function, $\exists c \in \mathbb{R}$, such that $\forall x \in \mathbf{K}_1, \forall y \in \mathbf{K}_2, L(x) \leq c \leq L(y)$.

Proof.

Construct $\mathbf{K} = \mathbf{K}_1 = \mathbf{K}_2 = \{x - y\colon x \in \mathbf{K}_1, y \in \mathbf{K}_2\}$, then $\mathbf{K}$ is convex by checking the definition of convexity. It can be verified by the definition of interior point that $x - y$ is an interior point of $\mathbf{K}, \forall y \in \mathbf{K}_2$. Also, $\mathbf{K}_1 \cap \mathbf{K}_2 = \emptyset \Rightarrow 0 \notin \mathbf{K}$. Applying the above corollary to $\mathbf{K}$ with $x - y$ being its interior point and $0 \notin \mathbf{K}$, $\exists L\colon \mathbf{X} \to \mathbb{R}$ which is a linear function such that
$\mathbf{K} \subseteq \left\{x \in \mathbf{X}\colon L(x) \leq c'\right\}$ with $c' = L(y) = 0 \ (y = 0)$. Therefore,
$L(x) \leq L(y), \forall x \in \mathbf{K}_1, \forall y \in \mathbf{K}_2$. Now define $c = \sup_{x \in \mathbf{K}_1} L(x)$, we have
$L(x) \leq c \leq L(y), \forall x \in \mathbf{K}_1, \forall y \in \mathbf{K}_2$. □

Geometric Hanh-Banach theorems

- Applications. 1. NLS $\mathbf{X}$, $\mathbf{Y} \subseteq \mathbf{X}$ is a linear subspace, $\ell\colon \mathbf{Y} \to \mathbb{R}$. $\exists c_0, |\ell(y)| \leq c_0\|y\| \triangleq P(y), \forall y \in \mathbf{Y}$. Then $P(\cdot)$ is positive homegeneous and subadditive. By the Hanh-Banach theorem, $\exists L\colon \mathbf{X} \to \mathbb{R}$ which is a linear function, $L(y) = \ell(y), \forall y \in \mathbf{Y}$, and $L(x) \leq P(x) = c_0\|x\|, \forall x \in \mathbf{X}$. In addition, $L(-x) \leq P(-x) \Rightarrow |L(x)| \leq c_0\|x\|, \|L\|$. Furthermore, $\|L\| = \|\ell\|$.
- Ω is an abstract set, $B(\Omega) = \{x\colon \Omega \to \mathbb{R} \mid \sup_{x\in\Omega} |x(w)| < \infty\}$. Then $B(\Omega)$ is a LS. x is non-negative if $x(t) \geq 0, \forall t \in \Omega$. $x \leq y$ if $x(t) \leq y(t), \forall t \in \Omega$. $\ell\colon \mathbf{Y} \subseteq B(\Omega) \to \mathbb{R}$ positive if $\ell(x) \geq 0, \forall x \geq 0$. $\ell\colon \mathbf{Y} \to \mathbb{R}$ is positive if $x_1 \leq x_2 \Rightarrow \ell(x_1) \leq \ell(x_2)$.

Theorem 1.5

$\mathbf{Y} \subseteq B(\Omega)$ is a linear subspace, $\ell\colon \mathbf{Y} \to \mathbb{R}$ is linear and positive. $\exists y_0 \in \mathbf{Y}, y_0 \geq 1$. Then $\exists L\colon B(\Omega) \to \mathbb{R}$ which is a linear function, $L(y) = \ell(y), \forall y \in \mathbf{Y}$, and L is positive.

Geometric Hanh-Banach theorems

Proof.

Define $p(x) = \inf\{\ell(y) \colon x \le y, y \in \mathbf{Y}\}$. Due to the existence of y_0, $|x(t)| \le c, \forall t \in \Omega \Rightarrow -c\ell(y_0) \le p(x) \le c\ell(y_0)$ for all $x \in B(\Omega)$. It can be verified that (1) P is positive homegeneous; (2)subadditive; (3) $x \le 0 \Rightarrow p(x) \le 0$; (4) $p(x) = \ell(x), \forall x \in \mathbf{Y}$. By the Hanh-Banach theorem, $\exists L \colon B(\Omega) \to \mathbb{R}$ which is a linear function, $L(y) = \ell(y), \forall y \in \mathbf{Y}$, $L(x) \le p(x), \forall x \in B(\Omega)$. It remains to be proved that L is positive. Note that for $x \ge 0$, $-x \le 0$, so $L(-x) \le p(-x) \Rightarrow L(x) \ge -p(-x) \ge 0$. □

Dual of a normed linear space

$\mathbf{X}$ is a LS over $\mathbb{K}$, $\mathbf{X}'$ is the dual of $\mathbf{X}$.

Proposition 1.5

$\mathbf{X}'$ is a complete NLS.

Proof.

Let $\{\ell_n\}_{n\geq 1}$ be a Cauchy sequence, then we can define a linear fucntion $\ell(x) = \lim_n \ell_n(x)$. Then $|\ell(x)| = \lim_n |\ell_n(x)| \leq \limsup_n \|\ell_n\| \|x\|$. Because $\limsup_n \|\ell_n\|$ is bounded, ℓ is a BLF and $\ell \in \mathbf{X}'$. It remains to prove that $\ell_n \to \ell$ in the operator norm. □

Extension of Bounded Linear Functional

Theorem 1.6

$\mathbf{Y} \subseteq \mathbf{X}$. Then $\forall x \in \mathbf{X}$,

$$\inf_{y \in \mathbf{Y}} \|x - y\|_2 = \sup_{\|\ell\|=1, \ell(y)=0, \forall y \in \mathbf{Y}} |\ell(x)| .$$

Proof.

$$|\ell(x) - \ell(y)| \leq |x - y| , \forall x \in \mathbf{X}, y \in \mathbf{Y}$$
$$\Rightarrow \sup_{\|\ell\|=1, \ell(y)=0, \forall y \in \mathbf{Y}} |\ell(x)| \leq \inf_{y \in \mathbf{Y}} \|x - y\|_2 .$$

On the other hand, without loss of generality, let $x \notin \mathbf{Y}$. Define $d \triangleq \inf_{y \in \mathbf{Y}} \|x - y\|_2$ and the subspace $\mathbf{Y}_0 = \{\alpha x + y\}$. Define the BLF f on $\mathbf{Y}_0$ by $f(z) = \alpha d$ for $z \in \mathbf{Y}_0$. Then $\|f\| \leq 1$. Extend f from $\mathbf{Y}_0$ to F on $\mathbf{X}$ by the Hahn-Banach Theorem. Then $F(z) = f(z)$ for $z \in \mathbf{Y}_0$ and $\|F\| = \|f\| \leq 1$. Let $F' = \frac{F}{\|F\|}$, then $\left\|F'\right\| = 1$.
$\left|F'(x)\right| \geq F(x) = f(x) = d = \inf_{y \in \mathbf{Y}} \|x - y\|_2$. It follows that $\sup_{\|\ell\|=1, \ell(y)=0, \forall y \in \mathbf{Y}} |\ell(x)| \geq \inf_{y \in \mathbf{Y}} \|x - y\|_2$. $\square$

Extension of Bounded Linear Functional

Definition 1.9

Let $\mathbf{Y} \subseteq \mathbf{X}$. Define $\mathbf{Y}^{\perp} = \{\ell \in \mathbf{X}' : \ell(y) = 0, \forall y \in \mathbf{Y}\}$.
$\|\ell\|_{\mathbf{Y}} = \sup_{y \in \mathbf{Y}, y \neq 0} \frac{|\ell(y)|}{\|y\|}$.

Theorem 1.7

Let $\ell : \mathbf{X} \to \mathbb{R}$. Then $\|\ell\|_{\mathbf{Y}} = \inf_{m \in \mathbf{Y}^{\perp}} \|l - m\|$.

Proof.

$$
\begin{aligned}
&|\ell(y)| = |\ell(y) - m(y)| \leq \|l - m\| \, \|y\|, \forall m \in \mathbf{Y}^{\perp}, y \in \mathbf{Y} \\
&\Rightarrow \|\ell\|_{\mathbf{Y}} \leq \inf_{m \in \mathbf{Y}^{\perp}} \|l - m\|.
\end{aligned}
$$

On the other hand, $\ell_{\mathbf{Y}}(z) \leq \|\ell\|_{\mathbf{Y}} \|z\| \triangleq p(z)$ for $z \in \mathbf{Y}$. Extend $\ell_{\mathbf{Y}}$ from $\mathbf{Y}$ to L on $\mathbf{X}$ by the Hahn-Banach Theorem. Then $L(z) = \ell(z)$ for $z \in \mathbf{Y}$, and $|L(x)| \leq p(x) = \|\ell\|_{\mathbf{Y}} \|\mathbf{x}\|$ for $x \in \mathbf{X}$. So that $\|L\| \leq \|\ell\|_{\mathbf{Y}}$. In fact, $\|L\| = \|\ell\|_{\mathbf{Y}}$. Let $m = \ell - L$. Then $m \in \mathbf{Y}^{\perp}$ and $\|\ell\|_{\mathbf{Y}} = \|l - m\| = \|L\|$. □

Extension of Bounded Linear Functional

Definition 1.10

Closed Linear Span (CLS) of $\mathbf{A}$: smallest closed linear set which contains $\mathbf{A}$, i.e. $\bigcap_{\beta\in J} F_J$ and each F_J is a linear and closed set containing $\mathbf{A}$.

Claim 1.2

CLS $\{\mathbf{X}_\theta : \theta \in I\} = \overline{\left\{\sum_{j=1}^{N} \alpha_j \mathbf{X}_{\theta_j} : N \geq 1, \{\alpha_j\} \subseteq \mathbb{R}, \{\theta_j\} \subseteq I\right\}}.$

Proof.

RHS $\subseteq$ LHS (LHS is closed). In addition, RHS is one F_J which is a linear closed set containing $\mathbf{A} = \{\mathbf{X}_\theta : \theta \in J\}$. □

Extension of Bounded Linear Functional

Definition 1.11

Closed L. Span of $\mathbf{A}$: smallese closed set which contains $\mathbf{A}$, i.e. $\bigcap_{\beta \in J} F_J$ and each F_J is a linear and closed set containing $\mathbf{A}$.

Theorem 1.8

Let CLS

$$\{\mathbf{X}_\theta \colon \theta \in I\} = \overline{\left\{ \sum_{j=1}^{N} \alpha_j \mathbf{X}_{\theta_j} : N \geq 1, \{\alpha_j\} \subseteq \mathbb{R}, \{\theta_j\} \subseteq I \right\}} = \mathbf{A}.$$ Then

$z \in \mathbf{A} \iff \forall \ell \in \mathbf{X}', \ell(\mathbf{X}_\theta) = 0 \ \forall \theta \in J \Rightarrow \ell(z) = 0.$

Proof.

We prove that $z \notin \mathbf{A} \Rightarrow \exists \ell \in \mathbf{X}', \ell(\mathbf{X}_\theta) = 0, \ \forall \theta \in I, \ell(z) \neq 0$. Define $d \triangleq \inf_{y \in \mathbf{A}} \|z - y\|$, $A_0 = \{\alpha z + w \colon w \in \mathbf{A}, \alpha \in \mathbb{R}\}$. Define $\ell(\alpha z + w) = \alpha d$. Then $\ell(\alpha z + w) \leq \|\alpha z + w\|$. By the Hahn-Banach Theorem, extend the funcitonal $\ell \colon A_0 \to \mathbb{R}$ with $|\ell(z)| \leq \|z\|$ for $z \in A_0$ to $L \colon \mathbf{X} \to \mathbb{R}$ with $|L(x)| \leq \|x\|$ and $L(w) = 0$ for $w \in \mathbf{A}$. Then $L(\mathbf{X}_\theta) = 0$ for all $\theta \in J$ and $L(z) = d \neq 0$. □

Reflexive Space

- $\mathbf{X}'$: dual space of $\mathbf{X}$, i.e. the bounded (continuous) linear functional defined on normed linear space $\mathbf{X}$.
- $L_x : \mathbf{X}' \to \mathbb{R}, L_x(\ell) = \ell(x) \leq \|\ell\| \|x\|$, so that $\|L_x\| \leq \|x\|$.
- By the Hahn-Banach Theorem, $\|x\| = \sup_{\|\ell\|=1} |\ell(x)|$. So that $\|x\| = \sup_{\|\ell\|=1} |\ell(x)| = \sup_{\|\ell\|=1} |L_x(\ell)| = \|L_x\|$.
- Define $\mathcal{L}: \mathbf{X} \to \mathbf{X}'', \mathcal{L}(x) = L_x$.

Definition 1.12

$\mathbf{X}$ NLS is reflexive if $\mathbf{X}'' = \mathbf{X}$.

Example 1.2

If $\mathbf{X}$ is finite dimensional, then $\mathbf{X}$ is reflexive.
With $(\Omega, \mathcal{B}, \mu)$, for $1 < p < \infty$, $L^p = \left\{ f : \Omega \to \mathbb{R}, \int_\Omega f \mathrm{d}\mu < \infty \right\}$. Then $(L^p)' = L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. So that $(L^q)' = L^p = (L^p)''$, and L^p is reflexive.

Reflexive Space

Example 1.3

$C[-1,1]$, $\|\|_\infty$, $\|x\|_\infty = \sup_{t\in[-1,1]} |x(t)|$.

Claim 1.3

$C[-1,1]$ is not reflexive.

Proof.

Recall that $\|x\| = \sup_{\|\ell\|=1} |\ell(x)| = \ell_0(x)$ for all $x \in$ NLS $\mathbf{X}$ and some $\ell_0 \in \mathbf{X}'$. Apply this result to $\ell \in \mathbf{X} = C'[-1,1]$, and assume that $C[-1,1]$ is reflexive. Then $\|\ell\| = \sup_{\|L\|=1, L\in C''[-1,1]} |L(\ell)| = \sup_{\|L_x\|=1, L_x\in C''[-1,1]} |L_x(\ell)| = \ell(x)$ for some $x \in C[-1,1]$.
Consider $\ell(g) = \int_{-1}^{0} g(t)\mathrm{d}t - \int_0^1 g(t)\mathrm{d}t$. Then $|l(g)| \leq 2\|g\| \Rightarrow \|\ell\| \leq 2$. In addition, one can construct g_ε with $\|g_\varepsilon\| = 1$ and $|\ell(g_\varepsilon)| \geq 2(1-\varepsilon)$ for any $\varepsilon > 0$. So we have $\|\ell\| = 2$. On the other hand, $\forall x \in C[-1,1]$, it can be verified that $\ell(x) < 2 = \|\ell\|$. The contradiction shows that $C[-1,1]$ is not reflexive. □

Reflexive Space

Theorem 1.9

With NLS $\mathbf{X}$, then $\mathbf{X}'$ is separable $\Rightarrow$ $\mathbf{X}$ is separable.

Proof.

There exists $\{\ell_n\}_{n\geq 1}$ dense in $\mathbf{X}'$, so that there exists $\{z_n\}_{n\geq 1}$ such that $\|z_n\| = 1$ and $\ell_n(z_n) \geq \frac{1}{2}\|\ell_n\|$, $\forall n \geq 1$. We prove that CLS$(\{z_n\}_{n\geq 1}) = \mathbf{X}$.
Otherwise, let $\mathbf{X} \neq \mathbf{Y} =$ CLS$(\{z_n\}_{n\geq 1})$. Then there exists $x \notin \mathbf{Y}$. Define the subspace $\{\alpha x + y \colon \alpha \in \mathbb{R}, y \in \mathbf{Y}\}$. Then there exists $\ell \in \mathbf{X}'$ such that $\ell(z_n) = 0, \forall n \geq 1$, $\ell(x) \neq 0$. By divding ℓ by its norm, we can assume $\|\ell\| = 1$. As a result, there exists ℓ_n such that $\|\ell_n - \ell\| \leq \varepsilon$. Then $\|\ell_n\| \geq 1 - \varepsilon$. Also, $\|\ell_n\| \leq 2\ell_n(z_n)$ and $|\ell_n(z_n) - \ell(z_n)| \leq \|\ell_n - \ell\|\|z_n\| \leq \varepsilon \Rightarrow |\ell_n(z_n)| \leq \varepsilon$. This contradiction shows that $\mathbf{X} =$ CLS$(\{z_n\}_{n\geq 1}) = \overline{\left\{\sum_{j=1}^{N} \alpha_j z_{n_j} : N \geq 1, \{\alpha_j\} \subseteq \mathbb{R}\right\}}$. By restricting $\{\alpha_j\}$ to $\{\widehat{\alpha}_j\}$ where $\widehat{\alpha}_j \in \mathbb{Q}$, $\mathbf{X}$ is separable. □

Reflecxive Space

Theorem 1.10

If NLS $\mathbf{X}$ is reflexive, $\mathbf{Y} \subseteq \mathbf{X}$ is a closed linear space of $\mathbf{X}$. Then $\mathbf{Y}$ is reflexive.

Proof.

Let $m \in \mathbf{Y}'$. By the Hahn-Banach Theorem, m can be extended to $\widehat{m} \in \mathbf{X}'$. For $L \in \mathbf{Y}''$, define $L_0 \in \mathbf{X}''$ such that $L_0(\widehat{m}) = L(\widehat{m}_{\mathbf{Y}})$ where $\widehat{m} \in \mathbf{X}'$ and $\widehat{m}_{\mathbf{Y}}$ is the restriction of $\widehat{m}$ on $\mathbf{Y}$. Since $\mathbf{X}$ is reflexive, there exists $z \in \mathbf{X}$ such that $L_0(\widehat{m}) = \widehat{m}(z)$.
We now prove that $z \in \mathbf{Y}$. Otherwise, construct a subspace $A_0 = \{\alpha z + y \colon \alpha \in \mathbb{R}, y \in \mathbf{Y}\}$, and define the BLS f on this subspace such that $f(y) = 0, \forall y \in \mathbf{Y}$ and $f(z) \neq 0$. By the Hahn-Banach Theorem, f is extended to $F \colon \mathbf{X} \to \mathbb{R}$ such that $F(w) = f(w), \forall w \in A_0$. Now $L_0(F) = L(F_{\mathbf{Y}}) = 0$. On the other hand, $L_0(F) = F(z) \neq 0$. The contradiction shows that $z \in \mathbf{Y}$.
Because $\widehat{m}$ is an extension of m from $\mathbf{Y}$ to $\mathbf{X}$, so $\widehat{m}(z) = m(z)$. As a result, $L(m) = L_0(\widehat{m}) = \widehat{m}(z) = m(z)$ holds for all $L \in \mathbf{Y}''$ and all $m \in \mathbf{Y}'$. $\square$

The Dual of $C[a, b]$

- $\mathbf{X}$ is a MS (Metric Space) with $\mathcal{B}(\mathbf{X})$ (Borel σ-algebra of $\mathbf{X}$).
- Finite Signed measure on $\mathbf{X}$, i.e. $\mu \colon \mathcal{B}(\mathbf{X}) \to [-\infty, \infty]$. $\mu(\emptyset) = 0$ and μ is σ-additive. $\mathcal{M}_{\mathbf{X}}$: all finite signed measures on $\mathbf{X}$
- $\mathbf{X} = [a, b]$, $\rho \in \mathrm{BV}(\mathbf{X})$.
  $\mathcal{S} = \{(c, d], [a, d'] \colon a \leq c < d \leq b, a \leq d' \leq b\}$. v is a measure on $\mathcal{S}$ defined by $v(c, d] = \rho(d) - \rho(c)$, $v[a, d'] = \rho(d')$. Because $\mathcal{S}$ is a semi-algebra, by the Caratheodory Theorem, v is extended to $\mathcal{B}(\mathbf{X})$, i.e. $v \colon \mathcal{B}(\mathbf{X}) \to \mathbb{R}$ with $v[a, t] = \rho(t)$.
- With $v \in \mathcal{M}_{\mathbf{X}}$, we can define $\rho(t) = v[a, t]$ so that $\rho \in \mathrm{BV}(\mathbf{X})$. Note that $v(c, d] = \rho(d) - \rho(c)$ and $\|\rho\| = \|v\|$.

The Dual of $C[a, b]$

Theorem 1.11

(Riesz representation theorem) Let $\mathbf{X} = [a, b]$. Then $C'[a, b] = \mathrm{BV}(\mathbf{X}) = \mathcal{M}_{\mathbf{X}}$.

Proof.

Let $\ell \in C'[a, b]$. Define NLS $\mathbf{Y} = B[a, b] = \left\{h : [a, b] \to \mathbb{R} \mid \sup_{x \in \mathbf{X}} h(x) < \infty\right\}$. By the Hahn-Banach Theorem, ℓ is extended to $L\colon B[a, b] \to \mathbb{R}$ such that $\|L\| = \|\ell\|, L(f) = \ell(f), \forall f \in C[a, b]$. Define $L(\mathbb{1}_{\{[a,t]\}}) = \rho(t)$ and $\rho\colon [a, b] \to \mathbb{R}$. For each partition $\pi = \{t_0 = a < t_1 < t_2 < \ldots < t_N = b\}$

$$
\begin{aligned}
&\sum_{j=0}^{N-1} |\rho(t_{j+1}) - \rho(t_j)| = \sum_{j=0}^{N-1} s_j \rho(t_{j+1}) - \rho(t_j) \\
&= \sum_{j=0}^{N-1} L(s_j \left(\mathbb{1}_{\{[a,t_{j+1}]\}} - \mathbb{1}_{\{[a,t_j]\}}\right)) = L(\sum_{j=0}^{N-1} s_j \mathbb{1}_{\{(t_j,t_{j+1}]\}}) \triangleq L(u) \\
&\leq \|L\| \|u\| \leq \|L\| = \|\ell\|,
\end{aligned}
$$

because $\|u\| \leq 1$. Therefore, $\|\rho\| \leq \|\ell\|$, and $\rho \in \mathrm{BV}(\mathbf{X})$. □

The Dual of $C[a,b]$

Claim 1.4

$\ell(f) = \int_{\mathbf{X}} f(t) \mathrm{d}\rho(t)$.

Proof.

Approximate the $f \in C[a,b]$ with combination of indicator functions. Define $h_\pi(t) = f(a)\mathbb{1}_{\{[a,t_1]\}} + \sum_{j=1}^{N-1} f(t_j)\mathbb{1}_{\{(t_j,t_{j+1}]\}}$. Then $\lim_{\|\pi\|\to 0} h_\pi = f$. Note that $h_\pi(t) = f(a)\mathbb{1}_{\{a\}} + \sum_{j=0}^{N-1} f(t_j)\mathbb{1}_{\{(t_j,t_{j+1}]\}}$.

$$\ell(f) = L(f) = \lim_{\|\pi\|\to 0} L(h_\pi) = \lim_{\|\pi\|\to 0}\left(f(a)\rho(a) + \sum_{j=0}^{N-1} f(t_j)\mathbb{1}_{\{(t_j,t_{j+1}]\}}\right)$$

$$= \lim_{\|\pi\|\to 0}\left(f(a)\rho(a) + \sum_{j=0}^{N-1} f(t_j)\left(\rho(t_{j+1}) - \rho(t_j)\right)\right) = \int_{\mathbf{X}} f(t)\mathrm{d}\rho(t)$$

$$= \int_{\mathbf{X}} f(t)\mathrm{d}v.$$

The Dual of $C[a,b]$

- It remains to show that $\|\ell\| \le \|\rho\|$. For $f \in C[a,b]$,

$$|\ell(f)| = \left| \int_{\mathbf{X}} f \mathrm{d}\rho(t) \right| \le \|f\|_\infty \|\rho\|,$$

  so that $\|\ell\| \le \|\rho\|$.

Claim 1.5

Let $\rho \in \mathrm{BV}(\mathbf{X})$, $\ell_\rho(f) = \int_{\mathbf{X}} f(t)\mathrm{d}\rho(t)$ for $f \in C[a,b]$. Then $\ell_\rho \in C'[a,b]$ and $\|\ell_\rho\| = \|\rho\|$.

Proof.

By previous arguments, $\ell_\rho \in C'[a,b]$ and $\|\ell_\rho\| \le \|\rho\|$. By the Hahn-Banach Theorem, ℓ_ρ is extened to $F_\rho \colon \mathbf{Y} \to \mathbb{R}$. Define $\lambda(t) = F_\rho(\mathbb{1}_{\{[a,t]\}})$. Then $\|\lambda\| \le \|F_\rho\| = \|\ell_\rho\|$. Then $\ell_\rho(f) = F_\rho(f) = \int_{\mathbf{X}} f(t)\mathrm{d}\lambda(t) = \int_{\mathbf{X}} f(t)\mathrm{d}\rho(t)$ for any $f \in C[a,b]$. So that $\lambda = \rho$ and $\|\rho\| = \|\lambda\| \le \|\ell_\rho\|$. $\square$

The Dual of $C[a,b]$

Claim 1.6

Let $h \in L^1[a,b]$, and $\ell(f) = \int_a^b fh\mathrm{d}t$. Then $\ell \in C'[a,b]$ and $\|\ell\| = \|h\|_1$.

Proof.

It can be verified that $|\ell(f)| \le \|f\| \|h\|_1$, so that $\ell \in C'[a,b]$ and $\|\ell\| \le \|h\|_1$. Let $\rho(t) = L(\mathbb{1}_{\{[a,t]\}})$, then by the Riesz representation theorem, $\ell(f) = \int_a^b f\mathrm{d}\rho$. This indicates that $\int_a^b f\mathrm{d}\rho = \int_a^b fh\mathrm{d}t$ for all $f \in C[a,b]$. Then ρ is absolutely continuous w.r.t. the Lebesgue mesure $\mathrm{d}t$ and its Radon-Nikodym derivative is h, i.e. $\rho(t) = \int_a^t h(s)\mathrm{d}s$. So that $\|\rho\| = \|h\|_1$. Because $\|\ell\| = \|\rho\|$, we have $\|\ell\| = \|h\|_1$. □

An Application of the H-B Theorem

Lemma 1.6

Let $\mathbf{X} = C[a,b]$, and $\delta_t \in \mathbf{X}'$ for some $t \in [a,b]$ indicates $\delta_t(u) = u(t), \forall u \in \mathbf{X}$. Fix $\ell \in \mathbf{X}'$, suppose $\exists u \in \mathbf{X}$ such that $\ell(u) = \|\ell\| \|u\|$. Assume that $\sup_{t\in[a,b]} u(t)$ is attained at $t_1 < t_2 \ldots < t_N$. Then there exist $\{\alpha_j\}_{j=1}^N$, $\ell = \sum_{j=1}^{N} \alpha_j \delta_{t_j}$, and $\|\ell\| = \sum_{j=1}^{N} |\alpha_j|$.

Proof.

Because $\ell \in C'[a,b]$, there exists FSM such that $\ell(v) = \int_a^b v(t)\mathrm{d}\mu(dt) = \int_a^b v(t)\mathrm{d}\rho(t)$. $\square$

Existence of Green Function

Definition 1.13

(Green Function) $\mathbf{D} \subseteq \mathbb{R}^2$: Domain, $x_0 \in \mathbf{D}$, $G_{x_0} : \bar{\mathbf{D}} \to \mathbb{R}$ is a Green Function (GF) if
(a) $G_{x_0}(y) = K_{x_0}(y) - \overline{\ln|y - x_0|}$. (note that $\triangle \ln|y - x_0| = 0, y \neq x_0$)
(b) $K_{x_0}(y) = \ln|y - x_0|, y \in \partial\mathbf{D}$.
(c) $K_{x_0} \in C(\bar{\mathbf{D}}) \cap C^2(\mathbf{D})$, K_{x_0} is harmonic on $\mathbf{D}$.

Theorem 1.12

$\mathbf{D} \subseteq \mathbb{R}^2$ is bounded domain, $\mathbf{B} = \partial\mathbf{D}$. Assume $\forall x_0 \in \mathbf{D}$, $\exists G_{x_0} \in C^2(\mathbf{D}) \cap C^1(\bar{\mathbf{D}})$ which is a GF. Let $f \in C(\mathbf{B})$, $m(z)$ with $\|m(z)\|_2 = 1$ is the unit normal vector perpendicular to the tangent plane at $z \in B$. Define $U(x_0) = \int_{\mathbf{B}} f(z) \frac{\partial G_x(z)}{\partial m} \sigma(\mathrm{d}z)$. Then U is the solution to the Laplace equation

$$\begin{cases} \triangle U = 0 \text{ on } \mathbf{D}, \\ U = f \text{ on } \partial\mathbf{D}. \end{cases}$$

Existence of Green Function

Theorem 1.13

Assume $\mathbf{D} \subseteq \mathbb{R}^2$ is a bounded domain, $\mathbf{B} = \partial\mathbf{D}$ is C^1. Then $\forall x_0 \in D, \exists$ GF function G_{x_0}.

Proof.

Let $\mathbf{X} = C(B)$ equipped with $\|\|_\infty$ norm.
$\mathbf{Y} \subseteq \mathbf{X}, \mathbf{Y} = \{f \in \mathbf{X} \colon \exists$sol to $\triangle U = 0$ on $\mathbf{D}, U = f$ on $\mathbf{B}\}$. First.
$\mathbf{Y} \neq \emptyset, f =$ const $, U =$ const , and $U \in C(\bar{\mathbf{D}}) \cap C^2(\mathbf{D})$ □

Closed Convex Subsets of a Hilbert Space

Theorem 1.14

$\mathbf{X}$ is a Hilbert Space (HS). $\mathbf{K}$ is a closed convex subset of $\mathbf{X}$.
$x \in \mathbf{X}, d = d(x, \mathbf{K}) = \inf\{\|x-y\|, y \in \mathbf{K}\}$. Then $!\exists z \in \mathbf{K}$,
$\|z-x\| = d(x, \mathbf{K})$.

Proof.

First, construct $\{z_n\}_{n\geq 1}$ s.t. $\|x - z_n\| \leq d + \frac{1}{n}$. Then by
$\|z_n - z_m\|^2 + \|z_n + z_m - 2x\|^2 = 2\|z_n - x\|^2 + 2\|z_m - x\|^2 \Rightarrow$
$\|z_n - z_m\| \overset{n,m\to\infty}{\to} 0$ and $\{z_n\}_{n\geq 1}$ is a Cauchy sequence. It follows that there exisits $z \in \mathbf{X}$ such that $\lim_n z_n = z$, and $d \leq \|x - z\| \leq d + \frac{1}{n}$ for any $n \geq 1$, so $\|z - x\| = d$.
For uniqueness, let $\|z_1 - x\| = \|z_2 - x\| = d$. By
$\|z_1 - z_2\|^2 + \|z_1 + z_2 - 2x\|^2 = 2\|z_1 - x\|^2 + 2\|z_2 - x\|^2 = 4d^2$ and
$\|z_1 + z_2 - 2x\|^2 \geq 4d^2$, we have
$\|z_1 - z_2\|^2 \leq 0 \Rightarrow \|z_1 - z_2\| = 0, z_1 = z_2$. □

Closed Convex Subsets of a Hilbert Space

Definition 1.14

$Y \subseteq \mathbf{X}$ is a linear subspace of $\mathbf{X}$, $Y^{\perp} := \{x \in \mathbf{X} \colon \langle x, y \rangle = 0, \forall y \in Y\}$.

Proposition 1.6

$\mathbf{X}$ is a H.S., $Y \subseteq \mathbf{X}$ is a closed linear subspace of $\mathbf{X}$. Then

(1) $Y^{\perp}$ is a closed linear space.

(2) $\mathbf{X} = Y \bigoplus Y^{\perp}$, that is, $\forall x \in \mathbf{X}$, $\exists y \in Y, y^{\perp} \in Y^{\perp}$ s.t. $x = y + y^{\perp}$.

(3) $\left(Y^{\perp}\right)^{\perp} = Y$.

Closed Convex Subsets of a Hilbert Space

Proof.

(1) It can be verifed by checking the definition of lienar and closed space.

(2) Let $y_0 = \inf_{y \in Y} \|x - y\|$, that is, y_0 is the orthogonal projection of x onto Y. Consider $F(t) = \|x - y_0 + ty\|^2$ for $y \in Y$, then $F(t)$ achieves minimum at $t = 0 \Rightarrow$ Re $\langle x - y_0, y\rangle = 0$ for all $y \in Y$. By considering $\|x - y_0 + ity\|^2$ we have Im $\langle x - y_0, y\rangle = 0$ for all $y \in Y$. Therefore, $\langle x - y_0, y\rangle = 0$ for all $y \in Y \Rightarrow x - y_0 \in Y^{\perp}$. Therefore, $x = y_0 + x - y_0$ with $y_0 \in Y, x - y_0 \in Y^{\perp}$. Noting that $Y \cap Y^{\perp} = \{0\}$, so that $\mathbf{X} = Y \bigoplus Y^{\perp}$.

(3) First, $\left(Y^{\perp}\right)^{\perp} \subseteq Y$. To see this, let $z \in \left(Y^{\perp}\right)^{\perp}$, then $\langle z, w\rangle = 0, \forall w \in Y^{\perp}$. By Part (2), $z = u + v, u \in Y, v \in Y^{\perp} \Rightarrow v = 0, z = u \in Y$, so that $\left(Y^{\perp}\right)^{\perp} \subseteq Y$. By the definiition, $Y \subseteq \left(Y^{\perp}\right)^{\perp}$. Therefore, $\left(Y^{\perp}\right)^{\perp} = Y$.

□

Bounded Linear Functional on HS

Theorem 1.15

$\mathbf{X}$ is a HS, $\ell\colon \mathbf{X} \to \mathbb{K}$ is a BLF. Then $\exists x \in \mathbf{X}, \ell(y) = \langle y, x\rangle$.

Lemma 1.7

$\mathbf{X}$ is HS.

(1) ℓ is a BLF, $\ell \neq 0$, $N_\ell = \{x \in \mathbf{X} \colon \ell(x) = 0\}$. N_ℓ has Co-Dim 1. $\exists w \in \mathbf{X}, \mathbf{X} = \{\alpha w \colon \alpha \in \mathbb{R}\} \bigoplus N_\ell$.

(2) ℓ, m are BLFs, $N_\ell = N_m$. Then $\exists c \in \mathbb{K}, \ell = cm$.

Proof.

(1) $\exists w \in \mathbf{X}, \ell(w) \neq 0$. $\forall x \in \mathbf{X}, x = \frac{\ell(x)}{\ell(w)} w + \underbrace{\left(x - \frac{\ell(x)}{\ell(w)} w\right)}_{\in N_\ell}$. Let $z \in \{\alpha w \colon \alpha \in \mathbb{R}\} \cap N_\ell$, then $z = \alpha w, \ell(z) = 0 \Rightarrow \alpha \ell(w) = 0 \Rightarrow \alpha = 0$, so $z = 0$. Therefore, $\{\alpha w \colon \alpha \in \mathbb{R}\} \cap N_\ell = \{0\}$, and it follows that $\mathbf{X} = \{\alpha w \colon \alpha \in \mathbb{R}\} \bigoplus N_\ell$.

□

Bounded Linear Functional on HS

Proof Cont'd.

(2) $\ell = m = 0$ if $\ell = 0$. If $\ell \neq 0$, $\exists w \in \mathbf{X}, \ell(w) \neq 0$, and $\mathbf{X} = \{\alpha w \colon \alpha \in \mathbb{R}\} \bigoplus N_\ell$ by Part (1). Now $\forall x \in \mathbf{X}, x = \alpha w + n, n \in N_\ell = N_m$.
$\ell(x) = \alpha \ell(w) = \alpha \frac{\ell(w)}{m(w)} m(w) = \frac{\ell(w)}{m(w)} m(\alpha w + n) = \frac{\ell(w)}{m(w)} m(x)$
$(m(w) \neq 0)$. Setting $c = \frac{\ell(w)}{m(w)}$ we have $\ell = cm$.

□

Lemma 1.8

$\mathbf{X}$ is HS, ℓ is a BLF, then N_ℓ is closed.

Proof.

Let $\{x_n\} \subseteq \mathbf{X}, \mathbf{x}_n \to x$, then $\ell(x) = \lim_{n \to \infty} \ell(x_n) = 0 \Rightarrow x \in N_\ell$.

□

Bounded Linear Functional on HS

Theorem 1.16

$\mathbf{X}$ is a HS, $\ell\colon \mathbf{X} \to \mathbb{K}$ is a BLF. Then $\exists x \in \mathbf{X}, \ell(y) = \langle y, x\rangle$.

Proof.

Let $\ell \neq 0$, otherwise we can set $y = 0$. Then $\exists w \in \mathbf{X}, \ell(w) \neq 0$, and $\mathbf{X} = \{\alpha w\colon \alpha \in \mathbb{R}\} \bigoplus N_\ell$, and N_ℓ is a CLS. It follows that $\mathbf{X} = N_\ell \bigoplus N_\ell^\perp$. By the claim below, $\dim(N_\ell^\perp) = 1$, so $\exists z$ s.t. $N_\ell^\perp = \{\alpha z\colon \alpha \in \mathbb{K}\}$, and $\mathbf{X} = N_\ell \bigoplus \{\alpha z\colon \alpha \in \mathbb{K}\}$. Consider BLF $m(x) = \langle x, z\rangle$, then $N_m = \left(N_\ell^\perp\right)^\perp = N_\ell$ (since N_ℓ is CLS). Therefore, $\ell(x) = cm(x) = c\langle x, z\rangle = \langle x, \bar{c}z\rangle$ by the previous lemma. □

Bounded Linear Functional on HS

Claim 1.7

$\dim(N_\ell^\perp) = 1$.

Proof.

First of all, $\dim(N_\ell \neq \{0\}$. Otherwise, $\mathbf{X} = N_\ell$ contradicting with the case that $\ell \neq 0$. Then $\exists z \neq 0, z \in N_\ell$. Let $z_1, z_2 \in N_\ell^\perp, z_1, z_2 \neq 0$. Then $z_1 = \alpha_{11} w + \alpha_{12} n_1, z_2 = \alpha_{21} w + \alpha_{22} n_2 \Rightarrow \alpha_{21} z_1 - \alpha_{11} z_2 = \alpha_{21}\alpha_{12} n_1 - \alpha_{11}\alpha_{22} n_2 \in N_\ell \Rightarrow \alpha_{21} z_1 - \alpha_{11} z_2 \in N_\ell \cap N_\ell^\perp = \{0\}$. Because $\alpha_{21}, \alpha_{11} \neq 0$ (otherwise $z_1 = 0$ or $z_2 = 0$), $z_1 = \frac{\alpha_{11}}{\alpha_{21}} z_2$. This proves that $\dim(N_\ell^\perp) = 1$. □

Bounded Linear Functional on HS

Theorem 1.17 (Lax-Milgram)

$\mathbf{X}$ is HS, $B\colon \mathbf{X}\times\mathbf{X}\to\mathbb{K}$. Suppose the following conditions hold.

(1) $B(\cdot,x)$ is linear, $B(x,\cdot)$ is sesqui-linear:
$\forall x,y_1,y_2\in\mathbf{X}, B(x,\alpha y_1+y_2)=\bar{\alpha}B(x,y_1)+B(x,y_2)$.

(2) $\exists c_1,\forall x,y\in\mathbf{X}, |B(x,y)|\le c_1\|x\|\|y\|$.

(3) $\exists c_0,\forall x\in\mathbf{X}, B(x,x)\ge c_0\|x\|^2$.

Then $\forall \ell\colon \mathbf{X}\to\mathbb{K}$ which is a BLF, $\exists x\in\mathbf{X}$ s.t. $\ell(y)=B(y,x),\forall y\in\mathbf{X}$.

Proof.

First , $B(\cdot,x)$ is a BLF, so there exists $T(x)\in\mathbf{X}$ s.t. $B(y,x)=\langle y,T(x)\rangle$. We have

(1) $T\colon \mathbf{X}\to\mathbf{X}$ is linear, $T(\alpha x_1+x_2)=\alpha T(x_1)+T(x_2)$ (this can be checked by the definition of T : $B(y,\alpha x_1+x_2)=\langle y,T(\alpha x_1+x_2)\rangle=\bar{\alpha}B(y,x_1)+B(y,x_2)=\langle y,\alpha T(x_1)\rangle+\langle y,T(x_2)\rangle$ holds for any $y\in\mathbf{X}\Rightarrow T(\alpha x_1+x_2)=\alpha T(x_1)+T(x_2)$.)

Bounded Linear Functional on HS

Cont'd.

(2) $A = \{T(x) \colon x \in \mathbf{X}\}$ is a CLS of $\mathbf{X}$. To see this, let $\{y_n\} \subseteq A, y_n \to y$. We have following property for T: $\exists c_1, c_2$, s.t. $c_1\|x\|^2 \leq \|T(x)\|^2 \leq c_1\|x\|^2$. Because $B(y,x) = \langle y, T(x)\rangle$, let $y = x$ we have
$c_0\|x\|^2 \leq B(x,x) = \langle x, T(x)\rangle \leq \|x\| \|T(x)\| \Rightarrow c_0\|x\| \leq \|T(x)\|$.
Let $y = T(x)$ we have $\|T(x)\|^2 \leq c_1\|T(x)\| \|x\|$.

Using this property, with $y_n = T(x_n)$, $\{x_n\}$ is a Cauchy sequence because $\{y_n\}$ is a Cauchy sequence. It follows that $x_n \to x \in \mathbf{X}$ by the completeness of $\mathbf{X}$, and $\|y_n - T(x)\| = \|T(x_n - x)\| \to 0$. Let $y = T(x) \in \mathbf{X}$, then $y_n \to y$, so it is proved that A is closed.

Now we prove that $A = \mathbf{X}$. Suppose $A \neq \mathbf{X}$, then $\mathbf{X} = A \bigoplus A^{\perp}$. We proves that $A^{\perp} = \{0\}$. Let $z \in A^{\perp}$, so
$\langle z, y\rangle = 0 = \langle z, T(x)\rangle = B(z,x), \forall x \in \mathbf{X}$. Setting $x = z$, we have $0 = B(z,z) \geq c_0\|z\|^2 \Rightarrow z = 0$. Therefore, $\mathbf{X} = A$. Let ℓ be a BLF, then $\exists y \in \mathbf{X}, \ell(z) = \langle z, y\rangle, \forall z \in \mathbf{X}$. Since $y = T(x)$ for some $x \in \mathbf{X}$, $\ell(z) = \langle z, T(x)\rangle = B(z,x)$.

Orthonormal Sets and Closed Linear Spans

Definition 1.15

$\{x_\theta : \theta \in I\}$ $(x_\theta \in \mathbf{X})$. Linear Span of $\{x_\theta : \theta \in I\}$ is the smallest linear set containing $\{x_\theta : \theta \in I\}$.

LS $\{x_\theta : \theta \in I\} = \left\{ \sum_{j=1}^{m} \alpha_j \mathbf{x}_{\theta_j}, M \geq 1, \alpha_j \in \mathbb{K}, \theta_j \in I, j \in [M] \right\}$.

CLS $\{x_\theta : \theta \in I\}$ is the smallest closed linear set containing $\{x_\theta : \theta \in I\}$.

Proposition 1.7

$\mathbf{X}$ is a HS, and

CLS $\{x_\theta : \theta \in I\} = \overline{\left\{ \sum_{j=1}^{M} \alpha_j x_{\theta_j} : M \geq 1, \alpha_j \subseteq \mathbb{K}, \theta_j \in I, j \in [M] \right\}}$.

Then

$z \in$ CLS $\{x_\theta : \theta \in I\} \iff \forall x \in \mathbf{X}, \langle x, x_\theta \rangle = 0, \forall \theta \in I \Rightarrow \langle z, x \rangle = 0.$

Orthonormal Sets and Closed Linear Spans

Proof.

Let $Y = \text{CLS}\{x_\theta : \theta \in I\}$. $z \in Y \Rightarrow z = \lim_{p\to\infty} \sum_{j=1}^{M_p} \alpha_{j,p} x_{\theta_{j,p}}$. Since

$\langle x, x_\theta \rangle = 0, \forall \theta \in I$, $\langle z, x \rangle = \lim_{p\to\infty} \sum_{j=1}^{M_p} \alpha_{j,p} \left\langle x_{\theta_{j,p}}, x \right\rangle = 0.$

Because Y is CLS, $\mathbf{X} = Y \bigoplus Y^\perp$. $z = u + v, u \in Y, v \in Y^\perp$. Suppose $z \neq 0$ (otherwise the conclusion is trivially true), and $z \notin Y$. It follows that $v \neq 0, 0 \neq \|v\|^2 = \langle v, v \rangle = \langle z - u, v \rangle = \langle z, v \rangle$. On the other hand, $\langle x_\theta, v \rangle = 0, \forall \theta \in I$ because $x_\theta \in Y, \forall \theta \in I$, so $\langle z, v \rangle = 0$. This contradiction shows that $z \in Y$. □

Definition 1.16

$\{x_\theta : \theta \in I\}$ is an Orthonormal Family if (1) $\|x_\theta\| = 1$, (2) $\langle x_\theta, x_{\theta'} \rangle = 0$ if $\theta \neq \theta'$.

Definition 1.17

$\{x_\theta : \theta \in I\}$ is a basis of $\mathbf{X}$ if (1) it is an Orthonormal Family, (2) $\text{CLS}\{x_\theta : \theta \in I\} = \mathbf{X}$.

Orthonormal Sets and Closed Linear Spans

Lemma 1.9

$\{x_\theta : \theta \in I\}$ is an orthonormal set, $x \in \mathbf{X}$, $\alpha_\theta = \langle x, x_\theta \rangle$.

(1) $\{\theta : \alpha_\theta \neq 0\}$ is at most countable.

(2) $\sum_{\theta \in I} |\alpha_\theta|^2 \leq \|x\|^2$.

Proof.

Let $J = \{\theta_k : k \geq 1\} \subseteq I$ be countable. With $\alpha_j = \left\langle x, x_{\theta_j} \right\rangle$ we have

$$\left\| \sum_{j=1}^{M} \alpha_j x_{\theta_j} - x \right\|^2 = \sum_{j=1}^{M} |\alpha_j|^2 - 2\text{Re} \left\langle \sum_{j=1}^{M} \alpha_j x_{\theta_j}, x \right\rangle + \|x\|^2$$

$$= \sum_{j=1}^{M} |\alpha_j|^2 - 2\text{Re} \sum_{j=1}^{M} \alpha_j \bar{\alpha}_j + \|x\|^2$$

$$= \|x\|^2 - \sum_{j=1}^{M} |\alpha_j|^2 \geq 0,$$

Orthonormal Sets and Closed Linear Spans

Proof Cont'd.

which means that $\sum_{j\in J'} |\alpha_j|^2 \le \|x\|^2$ holds for any finite set J'. It follows that $\sum_{j\ge 1} |\alpha_j|^2 \le \|x\|^2$, and for $J \subseteq I$ which is countable, $\sum_{\theta\in J} |\alpha_\theta|^2 \le \|x\|^2$.

Define $J_m = \left\{\theta \in I\colon\ |\alpha_\theta| \ge \frac{1}{m}\right\}$. $\sum_{j\in J_m} |\alpha_j|^2 \le \|x\|^2 \Rightarrow J_m$ is finite. It follows that $\{\theta\colon\ \alpha_\theta \ne 0\} = \bigcup_{m\ge 1} J_m$ is countable.

Let $I(x) = \{\theta\colon\ \alpha_\theta \ne 0\}$ which is countable. Then $\sum_{\theta\in I} |\alpha_\theta|^2 = \sum_{\theta\in I(x)} |\alpha_\theta|^2 \le \|x\|^2$. □

Proposition 1.8

$\{x_\theta\colon \theta \in I\}$ is an orthonormal set, then

$$\mathrm{CLS}\,\{x_\theta\colon \theta \in I\} = \left\{\sum_{j\ge 1} \alpha_j x_{\theta_j} : \alpha_j \in \mathbb{K}, \sum_{j\ge 1} |\alpha_j|^2 < \infty, \theta_j \in I\right\}.$$

Orthonormal Sets and Closed Linear Spans

Proof.

Let $A = \left\{ \sum_{j \geq 1} \alpha_j x_{\theta_j} : \alpha_j \in \mathbb{K}, \sum_{j \geq 1} |\alpha_j|^2 < \infty, \theta_j \in I \right\}$. We will prove that A is linear and closed.

Let $\{z_n\} \subseteq A$, $z_n \to z$, $z_n = \sum_{j \geq 1} \alpha_j^n x_{\theta_j}^n$. Let

$J_n = \left\{ \alpha_j^n : j \geq 1 \right\}$, $J = \bigcup_{n \geq 1} J_n = \left\{ \widehat{\theta}_k : k \geq 1 \right\}$, then $z_n = \sum_{k \geq 1} \beta_k^n \widehat{\theta}_k$.

Define the map $T : \left\{ \sum_{k \geq 1} \beta_k \widehat{\theta}_k : \sum_{k \geq 1} |\beta_k|^2 < \infty \right\} \to \ell^2 =$

$\left\{ \{\alpha_j : j \geq 1\} : \sum_{j \geq 1} |\alpha_j|^2 < \infty \right\}$, $T(\sum_{k \geq 1} \beta_k \widehat{\theta}_k) = \{\beta_k : k \geq 1\}$. Also, let $z' = \sum_{k \geq 1} \beta_k \widehat{\theta}_k$, then

$\left\| T(z') \right\|^2 = \sum_{k \geq 1} |\beta_k|^2 = \left\| z' \right\|^2$, so T is an isometry.

Because $z_n \to z$, $\{z_n\}$ is a Cauchy sequence, and $\{T(z_n)\}$ is also a Cauchy sequence because T is an isometry. Because $\{T(z_n)\} \subseteq \ell^2$ and ℓ^2 is complete, $T(z_n) \to \{\beta_k : k \geq 1\}$ with $\sum_{k \geq 1} |\beta_k|^2 < \infty$.

Let $w = \sum_{k \geq 1} \beta_k \widehat{\theta}_k$. Because $\|z_n - w\| = \|T(z_n - w)\| \to 0$, $z_n \to w$ and

$w \in \left\{ \sum_{k \geq 1} \beta_k \widehat{\theta}_k : \sum_{k \geq 1} |\beta_k|^2 < \infty \right\} \subseteq A$. Recall that $z_n \to z$, so $z = w \in A$, so A is closed. □

Orthonormal Sets and Closed Linear Spans

Proof Cont'd.

By checking the definition, A is also linear. Therefore, CLS $\{x_\theta : \theta \in I\} \subseteq A$. To prove $A \subseteq$ CLS $\{x_\theta : \theta \in I\}$, let $x \in A$, then $x = \sum_{j\geq 1} \alpha_j x_{\theta_j}$, $\sum_{j\geq 1} |\alpha_j|^2 < \infty$. Let $x_n = \sum_{j=1}^{n} \alpha_j x_{\theta_j}$, then $x_n \in$ CLS $\{x_\theta : \theta \in I\}$. Since $\sum_{j\geq 1} |\alpha_j|^2 < \infty$, $x_n \to x$, it follows that $x \in$ CLS $\{x_\theta : \theta \in I\}$. □

Remark 1.2

$\mathbf{X}$ is a HS, $\{x_\theta : \theta \in I\}$ is an orthonormal set, $x = \sum_{j\geq 1} \alpha_j x_{\theta_j}$ with $\sum_{j\geq 1} |\alpha_j|^2 < \infty$. Then $\|x\|^2 = \sum_{j\geq 1} |\alpha_j|^2$. Also, $\alpha_j = \langle x, x_{\theta_j} \rangle$. To see this, let $x_n = \sum_{j=1}^{n} \alpha_j x_{\theta_j}$, then $\lim_{n\to\infty} \langle x_n, x_{\theta_j} \rangle = \alpha_j = \langle x, x_{\theta_j} \rangle$.

Orthonormal Bases

Theorem 1.18

All HS contains orthonormal basis.

Proof.

$\mathbf{X}$ is HS, $\Omega = \left\{ \underbrace{\{x_\theta : \theta \in I\}}_{\text{orthonormal set}} \right\}$. For $A, B \in \Omega$, $A < B$ if $A \subseteq B$. Then $<$ is a partial order. If $\Lambda = \{A_\beta : \beta \in J\} \subseteq \Omega$ is a totally ordered subset of Ω, for $A, B \in \Lambda$, either $A \subseteq B$ or $B \subseteq A$. Let $\tilde{A} = \bigcup_{\beta \in J} A_\beta$. Then $\tilde{A}$ is an upper bound for Λ. By Zorn's Lemma, there exists a maximal element $\{x_\theta : \theta \in I\}$ in Ω. Let CLS $\{x_\theta : \theta \in I\} = Y$. Suppose $\mathbf{X} \neq Y$, then $\mathbf{X} = Y \bigoplus Y^\perp$, $\exists y \in Y^\perp, y \neq 0$. Let $z = \frac{y}{\|y\|}$. Then $\{x_\theta : \theta \in I\} \cup \{z\}$ is an orthonormal set, contradicting with the fact that $\{x_\theta : \theta \in I\}$ is a maximal element in Ω. Therefore, $\mathbf{X} =$ CLS $\{x_\theta : \theta \in I\}$, and $\{x_\theta : \theta \in I\}$ is an orthonormal basis of $\mathbf{X}$. □

Lemma 1.10 (Gram-Schmidt)

$\mathbf{X}$ is HS. $\{x_p : p \geq 1\}$ are linearly independent (either finite or countable). Then $\exists \{y_p : p \geq 1\}$ which is an orthonormal set such that (1) $\forall M \geq 1$, span $\{x_i : i \in [M]\} =$ span $\{y_j : j \in [M]\}$; (2) Cardinality of $\{x_i : i \in [M]\}$ is the same as cardinality of $\{y_j : j \in [M]\}$.

Orthonormal Bases

Proof.

By induction, first consider $p = 1$ and set $y_1 = \frac{x_1}{\|x_1\|}$. For $M \geq 1$ with span $\{x_i : i \in [M]\}$ = span $\{y_j : j \in [M]\}$ and span $\{y_j : j \in [M]\}$ is an orthonormal set. set

$$\widehat{y}_{M+1} = x_{M+1} - \sum_{j=1}^{M} \langle x_{M+1}, y_j \rangle y_j, y_{M+1} = \frac{\widehat{y}_{M+1}}{\left\| \widehat{y}_{M+1} \right\|}.$$

Then we have $\left\langle \widehat{y}_{M+1}, y_j \right\rangle = 0$. Now we prove that span $\{x_i : i \in [M+1]\}$ = span $\{y_j : j \in [M+1]\}$. To see this, let $x \in$ span $\{x_i : i \in [M+1]\}$, then

$$x = \sum_{j=1}^{M+1} \alpha_j x_j \alpha_{M+1} x_{M+1} + \sum_{j=1}^{M} \alpha_j x_j = \alpha_{M+1} x_{M+1} + \sum_{j=1}^{M} \beta_j y_j \in \text{span} \left\{ y_j : j \in [M], \widehat{y}_{M+1} \right\}.$$

It follows that span $\{x_i : i \in [M+1]\} \subseteq$ span $\left\{ y_j : j \in [M], \widehat{y}_{M+1} \right\}$. Because span $\left\{ y_j : j \in [M], \widehat{y}_{M+1} \right\} \subseteq$ span $\{x_i : i \in [M+1]\}$, we have span $\{x_i : i \in [M+1]\}$ = span $\left\{ y_j : j \in [M], \widehat{y}_{M+1} \right\}$ = span $\{y_j : j \in [M+1]\}$. Also, span $\{y_j : j \in [M+1]\}$ is an orthonormal set by induction and construction of y_{M+1}.
By the above argument, for finite set $\{x_i : i \in [M]\}$, the finite orthonormal set $\{y_j : j \in [M]\}$ is constructed. For countably infinite set $\{x_p : p \geq 1\}$, the orthonormal set $\{y_p : p \geq 1\}$ is also countably infinite. Therefore, Cardinality of $\{x_i : i \geq 1\}$ is the same as cardinality of $\{y_j : j \geq 1\}$. □

Orthonormal Bases

Lemma 1.11

$\mathbf{X}$ is HS, $\{x_\theta : \theta \in I\}$ and $\{y_\beta : \beta \in J\}$ are two orthonormal basis. Then Card $\{x_\theta : \theta \in I\}$ = Card $\{y_\beta : \beta \in J\}$.

Proof.

When $\{x_\theta : \theta \in I\}$ or $\{y_\beta : \beta \in J\}$ is finite, we can use the Gram-Schmidt lemma to show that the other is also finite with the same number of elements. Now we suppose one of them is infinite.
For $\theta \in I$, let $J_\theta = \{\beta \in J : \langle y_\beta, x_\theta\rangle \neq 0\}$. Then $J_\theta \subseteq J$ is at most countable. We will show that $J \subseteq \bigcup_{\theta \in I} J_\theta$, so Card $\{x_\theta : \theta \in I\} \leq$ Card $\{y_\beta : \beta \in J\}$. To see this, let $\beta \in J$. Because $\{x_\theta : \theta \in I\}$ is an orthonormal basis, so
$y_\beta \in$ CLS $\{x_\theta : \theta \in I\} \Rightarrow y_\beta = \sum_{k \geq 1} \alpha_k x_{\theta_k}, \alpha_k = \langle y_\beta, x_{\theta_k}\rangle$. Since $y_\beta \neq 0$, there exists at least one k such that $\alpha_k = \langle y_\beta, x_{\theta_k}\rangle \neq 0 \Rightarrow y_\beta \in J_{\theta_k}$. Therefore, $J \subseteq \bigcup_{\theta \in I} J_\theta$, and it follows that Card $\{x_\theta : \theta \in I\} \leq$ Card $\{y_\beta : \beta \in J\}$ (noting that each J_θ is at most countably infinite, and countably infinite set has the least cardinality among infinite sets). By swapping the roles of I, J, we have Card $\{y_\beta : \beta \in J\} \leq$ Card $\{x_\theta : \theta \in I\}$. It follows that Card $\{x_\theta : \theta \in I\}$ = Card $\{y_\beta : \beta \in J\}$. □

Orthonormal Bases

By the above lemma, the cardinality of every orthonormal basis of a HS is the same.

Definition 1.18

$\mathbf{X}$ is HS, then $\dim(\mathbf{X})$ is defined as the cardinality of any orthonormal basis of $\mathbf{X}$.

Orthonormal Bases

Lemma 1.12

Let $\{x_\theta : \theta \in I\}$ be an orthonormal set, $\mathbf{X}$ is a HS, $x \in \mathbf{X}$, $\alpha_\theta = \langle x, x_\theta \rangle$.
Then $\sum\limits_{\theta \in I} |\alpha_\theta|^2 \leq \|x\|^2$ (Bessel inequality).
If $\{x_\theta : \theta \in I\}$ is an orthonormal basis, then

$$\mathbf{X} = \mathsf{CLS}\,\{x_\theta : \theta \in I\} = \left\{ \sum_{j \geq 1} \alpha_j x_{\theta_j}, \theta_j \in I, \alpha_j \in \mathbb{K}, \sum_{j \geq 1} \left|\alpha_{\theta_j}\right|^2 < \infty \right\}.$$

If $x \in \mathbf{X}$, then $x = \sum\limits_{j \geq 1} \alpha_j x_{\theta_j} = \lim\limits_{M \to \infty} \sum\limits_{j=1}^{M} \alpha_j x_{\theta_j}$. It follows that

(1) $\alpha_j = \left\langle x, x_{\theta_j} \right\rangle$ (by noting that
$\left\langle x, x_{\theta_j} \right\rangle = \lim\limits_{M \to \infty} \left\langle x_n, x_{\theta_j} \right\rangle, x_n = \sum\limits_{j=1}^{M} \alpha_j x_{\theta_j}$).

(2) $\langle x, x_\theta \rangle = 0$ for $\theta \notin \{\theta_j : j \geq 1\}$.

(3) $\|x\|^2 = \sum\limits_{j \geq 1} \left|\alpha_{\theta_j}\right|^2 = \sum\limits_{\theta \in I} |\alpha_\theta|^2$ (PARSEVAL identity),
$x = \sum\limits_{\theta \in I} \langle x, x_\theta \rangle\, x_\theta$.

Orthonormal Bases

Proof.

Because $x \in \mathbf{X}$ and $\{x_\theta : \theta \in I\}$ is an orthonormal basis of $\mathbf{X}$, there exists $\{\alpha_j : j \geq 1\}, \{\theta_j : j \geq 1\}$ such that $x = \lim_{M \to \infty} \sum_{j=1}^{m} \alpha_j x_{\theta_j}$. Define $z_m = \sum_{j=1}^{m} \alpha_j x_{\theta_j}$, then $x = \lim_{m \to \infty} z_m$.

(1) $\left\langle x, x_{\theta_j} \right\rangle = \lim_{m \to \infty} \left\langle z_m, x_{\theta_j} \right\rangle = \lim_{m \to \infty} \left\langle \sum_{k=1}^{m} \alpha_k x_{\theta_k}, x_{\theta_j} \right\rangle = \alpha_j$.

(2) $\langle x, x_\theta \rangle = \lim_{m \to \infty} \langle z_m, x_\theta \rangle = \lim_{m \to \infty} \left\langle \sum_{k=1}^{m} \alpha_k x_{\theta_k}, x_\theta \right\rangle = 0$, if $\theta \notin \{\theta_j : j \geq 1\}$.

(3) $\|x\|^2 = \lim_{m \to \infty} \|z_m\|^2 = \lim_{m \to \infty} \left\| \sum_{j=1}^{m} \alpha_j x_{\theta_j} \right\|^2 = \lim_{m \to \infty} \sum_{j=1}^{m} |\alpha_j|^2 = \sum_{j \geq 1} |\alpha_j|^2 = \sum_{j \geq 1} \left| \left\langle x, x_{\theta_j} \right\rangle \right|^2 = \sum_{\theta \in I} |\langle x, x_\theta \rangle|^2$. Also, $x = \sum_{\theta \in I} \langle x, x_\theta \rangle x_\theta$.

$\square$

A Quadratic Variational Problem

The Dirichlet Principle

- $G \subseteq \mathbb{R}^d$ is an open bounded set, and $G \neq \emptyset$.

$$\begin{cases} -\triangle U & = f \text{ on } G, \\ U & = g \text{ on } \partial G. \end{cases} \quad \text{(DP1)}$$

Here $\triangle U(x) = \sum_{j=1}^{d} \left(\partial_{x_j}^2 U \right)(x)$,
$f\colon G \to \mathbb{R}, g\colon \partial G \to \mathbb{R}, f \in C(G), g \in C(\partial G)$. We look for $U \in C^2(\bar{G})$ which satisfies the above equations.

- Let $w \in C_0^\infty(G)$, where $C_0^\infty(G)$ is the set of functions with compact support contained on G and infinitely differentiable. Multiplying both sides of the first equation by w and take the integral, we have (also using integral by parts)

$$-\int_G (\triangle U)(x) w(x) \mathrm{d}x = \int_G f(x) w(x) \mathrm{d}x$$

$$\Rightarrow \sum_{j=1}^{d} \int_G \left(\partial_{x_j} U \right)(x) \left(\partial_{x_j} w \right)(x) \mathrm{d}x = \int_G f(x) w(x) \mathrm{d}x, \forall w \in C_0^\infty(G) \text{ (DP2)}$$

The Dirichlet Principle

- Define $H(U) = \frac{1}{2} \sum_{j=1}^{d} \int_G \left(\left(\partial_{x_j} U \right) (x) \right)^2 \mathrm{dx}$. Consider

$$\inf \left\{ H(V) - \int_G f(x)V(x)\mathrm{d}x \colon V \in C^1(\bar{G}), V = g \text{ on } \partial G \right\} \quad \text{(DP3)}.$$

Assume $U \in C^2(\bar{G})$ is a solution to (DP3), then $(\triangle U)(x) = f(x)$ for $x \in G$.

Lemma 1.13

Let $H(U) = \frac{1}{2} \int_G \|\nabla U\|_2^2 \mathrm{d}x$, and $U \in C^2(\bar{G})$ is a solution to (DP3). Then $(\triangle U)(x) = f(x)$ for $x \in G$.

The Dirichlet Principle

Proof.

Let $w \in\in C_0^\infty(G)$, then
$T(\varepsilon) = H(U + \varepsilon w) - \int_G f(x)\,(U + \varepsilon w)\,(x)\mathrm{d}x \geq H(U) - \int_G f(x)U(x)\mathrm{d}x$ for $|\varepsilon| \leq 1, \varepsilon \in \mathbb{R}$.
We have $T(\varepsilon) = \frac{1}{2}\int_G \|\nabla U\|_2^2\mathrm{d}x + \varepsilon \int_G \nabla U \cdot \nabla w\mathrm{d}x + \frac{\varepsilon^2}{2}\int_G \|\nabla w\|_2^2\mathrm{d}x - \int_G f(x)U(x)\mathrm{d}x - \varepsilon \int_G f(x)w(x)\mathrm{d}x$. It follows that
$T'(\varepsilon) = \int_G \nabla U \cdot \nabla w\mathrm{d}x + \varepsilon \int_G \|\nabla w\|_2^2\mathrm{d}x - \int_G f(x)w(x)\mathrm{d}x$. Because $T'(0) = 0$, we have $\int_G \nabla U \cdot \nabla w\mathrm{d}x = \int_G f(x)w(x)\mathrm{d}x$. By integral by parts, $\int_G -\triangle U w\mathrm{d}x = \int_G f(x)w(x)\mathrm{d}x$ holds $\forall w \in\in C_0^\infty(G)$, that is, $\int_G (\triangle U + f)\,w\mathrm{d}x = 0, \forall w \in\in C_0^\infty(G)$. It follows that $\triangle U + f = 0$ a.s. on G. □

Generalized Derivatives and Sobolev Spaces

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Generalized Derivatives and Sobolev Spaces

Proposition 1.9

$W_2^1(G), \langle , \rangle_{1,2}$ is HS.

Proof.

Let $\{U_n : n \geq 1\}$ be a Cauchy sequence, $\|U\|_{1,2}^2 = \|U\|^2 + \sum_{j=1}^{N} \left\|\partial_{x_j} U\right\|^2$. Then $\exists U \in L^2(G)$ s.t. $U_n \overset{L^2(G)}{\to} U$, and $\exists v_j \in L^2(G)$ s.t. $\partial_{x_j} U \overset{L^2(G)}{\to} v_j$.

Claim 1.8

U has general derivative $\partial_{x_j} U = v_j$.

Proof.

$\forall f \in C_0^\infty(G)$, we have $\int_G \left(\partial_{x_j} f\right) U_n \mathrm{d}x = -\int_G f \partial_{x_j} U_n \mathrm{d}x$. Let $n \to \infty$, then $\int_G \left(\partial_{x_j} f\right) U \mathrm{d}x = -\int_G f v_j \mathrm{d}x \Rightarrow \partial_{x_j} U = v_j$. □

By the above claim, $\partial_{x_j} U \overset{L^2(G)}{\to} v_j = \partial_{x_j} U$. Therefore, $\|U_n - U\|_{1,2}^2 \overset{n\to\infty}{\to} = 0$ for some $U \in W_2^1(G)$. It follows that $W_2^1(G), \langle , \rangle_{1,2}$ is HS. □

Generalized Derivatives and Sobolev Spaces

Definition 1.19

$\overset{o}{W}{}_2^1(G) = \overline{C_0^\infty(G)}$ and the closure is in the space of $W_2^1(G)$.

Claim 1.9

$\overset{o}{W}{}_2^1(G)$ is HS.

Observation 1.1

Y is a linear subspace of a HS $\mathbf{X}$, then $\bar{Y}$ is HS.

$\overset{o}{W}{}_2^1(G) = \{U \in W_2^1(G) \colon U = 0$ at $\partial G\}$.

Lemma 1.14

$(d = 1)$ $U \in \overset{o}{W}{}_2^1(a, b) \Rightarrow \exists v \in C([a, b])$, $u = v$ a.s., $v(a) = v(b) = 0$.

Generalized Derivatives and Sobolev Spaces

Generalized Derivatives and Sobolev Spaces

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Uniform boundedness principle

Theorem 1.19

$\mathbf{X}$ is a complete Metric Space (MS), then $\mathbf{X}$ Baire: $\{E_n : n \geq 1\}$ where each E_n is open and dense in $\mathbf{X}$, then $E = \bigcap_{n\geq 1} E_n$ is dense in $\mathbf{X}$.

Proof.

□

Uniform Boundedness Principle

Theorem 1.20

$\mathbf{X}$ is a complete MS, $f_\alpha : \mathbf{X} \to \mathbb{R}$ is continuous for $\alpha \in I$. $\forall x \in \mathbf{X}, \sup_{\alpha \in I} f_\alpha(x) \leq M(x) < \infty$. Then $\exists G$ which is open and $c < \infty$ s.t. $\sup_{\alpha \in I} f_\alpha(x) \leq c, \forall x \in G$.

Weak Convergence

- $\mathbf{X}$ is a Normed Linear Space (NLS).

Definition 1.20

$\{x_n : n \geq 1\} \subseteq \mathbf{X}$, $x_n \rightharpoonup x$ (weak convergence) if $\forall \ell \in \mathbf{X}', \ell(x_n) \to \ell(x)$.

- Observation 1 If $x_n \to x, \|x_n - x\| \to 0$, then $\forall \ell \in \mathbf{X}', \ell(x_n) \to \ell(x)$. It means that strong convergence indicates weak convergence.
- Observation 2 The converse is not true. Let $\ell^2 = \left\{ \{a_j : j \geq 1\} : a_j \in \mathbb{R}, \sum\limits_{j \geq 1} |a_j|^2 < \infty \right\}$ which is a HS. Let $m \in \left(\ell^2\right)' \Rightarrow \exists b \in \ell^2, m(a) = \langle b, a \rangle = \sum\limits_{j \geq 1} a_j b_j$. Let $\{x_n : n \geq 1\} \subseteq \ell^2, x_n = \{0, 0, \ldots, 1, 0, 0, \ldots, 0\}$ (only the n-the element is 1). Then $x_n \rightharpoonup 0$, but $x_n \nrightarrow 0$ ($\|x_n\| = 1$).
- $\mathbf{X} = C[0,1], \|x\| = \sup\{|x(t)| : t \in [0,1]\}$.

- Cont'd

$$x_n(t) = \begin{cases} nt & t \in [0, \frac{1}{n}] \\ 2 - nt & t \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & t \in [\frac{2}{n}, 1] \end{cases}.$$

Then $x_n \nrightarrow 0$ ($\|x_n\| = 1$), Assume that $x_n \not\rightharpoonup 0$. Then $\exists \ell \in \mathbf{X}', \ell(x_n) \nrightarrow 0$. $\exists \delta > 0, n_k, |\ell(x_{n_k})| \geq \delta, \forall k$. Suppose $\ell(x_{n_k}) \geq \delta$ for infinitely many k's. Without loss of generality, $\ell(x_{n_k}) \geq \delta, \forall k$, and $n_{k+1} > 2n_k$. Let $y_M = \sum_{k=1}^{M} x_{n_k}$. We will prove that $\|y_M\| \leq 3$ for all $M \geq 1$.

Note that $n_{k+1} \geq 2n_k \Rightarrow n_k < \frac{1}{2^{R-k}} n_R$ for $R > k$. When $t \in [0, \frac{1}{n_M}], y_M(t) = \sum_{k=1}^{M} x_{n_k} = \sum_{k=1}^{M} n_k t \leq \frac{1}{n_M} \sum_{k=1}^{M} \frac{1}{2^{M-k}} n_M \leq 2$. When $t \in [\frac{1}{n_M}, \frac{1}{n_{M-1}}], y_M(t) = \sum_{k=1}^{M} x_{n_k} \leq 1 + \sum_{k=1}^{M-1} n_k t \leq 1 + \frac{1}{n_{M-1}} \sum_{k=1}^{M-1} \frac{1}{2^{M-1-k}} n_{M-1} \leq 3$.
$\|y_M\| \leq 3$ can also be proved for $t \in [\frac{1}{n_{M-1}}, \frac{1}{n_{M-2}}], \ldots$.

Because $|\ell(x_{n_k})| \geq \delta, \forall k$, it follows that $\ell(y_M) \geq M\delta$. On the other size, $|\ell(y_M)| \leq \|\ell\| \|y_M\| \leq 3\|\ell\|$. This contradiction shows that $x_n \rightharpoonup 0$.

Uniform Boundedness of Weak Converging Sequences

Lemma 1.15

$\mathbf{X}$ is a NLS, $\{x_n : n \geq 1\}$ is a sequence s.t.

(1) $\|x_n\| \leq c, \forall n \geq 1$,

(2) $\ell(x_n) \to \ell(x), \forall \ell \in A \subseteq \mathbf{X}'$, where A is dense in $\mathbf{X}'$,

then $x_n \rightharpoonup x$.

Proof.

$\forall \varepsilon > 0, \exists \ell \in A$, s.t. $\|m - \ell\| \leq \varepsilon$. We have

$$|m(x_n - x)| \leq \underbrace{|(m-\ell)\,(x_n - x)|}_{\leq \|m-\ell\|\,(c+\|x\|\,)} + \underbrace{|\ell(x_n - x)|}_{\to 0}$$

$$\Rightarrow \limsup_{n\to\infty} |m(x_n - x)| = 0.$$

It follows that $m(x_n) \to m(x), \forall m \in \mathbf{X}'$, and $x_n \rightharpoonup x$. $\square$

Uniform Boundedness of Weak Converging Sequences

Theorem 1.21

$\mathbf{X}$ is a Banach Space (BS). $f_\alpha : \mathbf{X} \to \mathbb{R}$, (1) f_α is continuous, (2) $f_\alpha(x+y) \leq f_\alpha(x) + f_\alpha(y)$, (3) $f_\alpha(\beta x) = |\beta| \, f_\alpha(x)$. Moreover, $\forall x \in \mathbf{X}, \exists M(x) < \infty, \sup_{\alpha \in I} |f_\alpha(x)| \leq M(x)$. Then $\exists c < \infty, \sup_{\alpha \in I} |f_\alpha(x)| \leq c\|x\|, \forall x \in \mathbf{X}$.

Proof.

By the previous theorem, there exists open set $G \subseteq \mathbf{X}$ and $M < \infty$ s.t. $\sup_{\alpha \in I} |f_\alpha(x)| \leq M, \forall x \in G$. Let $B(z,r) \subseteq G$, then $\sup_{\alpha \in I} |f_\alpha(z+y)| \leq M, \forall y \in B(0,r)$. Let $\|y\| = \frac{r}{2}$. Then
$f_\alpha(y) \leq f_\alpha(y+z) + f_\alpha(-z) = f_\alpha(y+z) + f_\alpha(z) \leq 2M, f_\alpha(y) \geq f_\alpha(y+z) - f_\alpha(z) \geq -2M$.
Then $\forall x \in \mathbf{X}, |f_\alpha(x)| = \left| f_\alpha(\frac{rx}{2\|x\|} \frac{2\|x\|}{r}) \right| \leq \frac{4M\|x\|}{r}$. □

Corollary 1.2

$\mathbf{X}$ is a BS, $\ell_\alpha \in \mathbf{X}', \alpha \in I$.
$\forall x \in \mathbf{X}, \exists M(x) < \infty, \sup_{\alpha \in I} |\ell_\alpha(x)| \leq M(x)$. Then $\exists c < \infty, \sup_{\alpha \in I} \|\ell_\alpha\| \leq c$.

Uniform Boundedness of Weak Converging Sequences

Proof.

Let $f_\alpha(x) = |\ell_\alpha(x)|$. Then $f_\alpha(x)$ satisfies the conditions in the above theorem, $\exists c < \infty, \sup_{\alpha\in I} |f_\alpha(x)| \le c\|x\|, \forall x \in \mathbf{X} \Rightarrow \sup_{\alpha\in I} |f_\alpha(x)| \le c\|x\|$, that is, $\sup_{\alpha\in I} \|\ell_\alpha(x)\| \le c$. □

Corollary 1.3

$\mathbf{X}$ is a NLS, $\{x_\alpha : \alpha \in I\} \subseteq \mathbf{X}$.
$\forall \ell \in \mathbf{X}', \exists c(\ell) < \infty, \sup_{\alpha\in I} |\ell(x_\alpha)| \le c(\ell)$. Then
$\exists c < \infty, \sup_{\alpha\in I} \|x_\alpha\| \le c$.

Proof.

Note that $\mathbf{X}'$ is complete so it is a BS, and $L_x : \mathbf{X} \to \mathbf{X}'', L_x(\ell) = \ell(x), \|L_x\| = \|x\|$. Then by the above theorem, $\exists c < \infty, \sup_{\alpha\in I} \|L_{x_\alpha}\| \le c \Rightarrow \sup_{\alpha\in I} \|x_\alpha\| \le c$. □

Corollary 1.4

$\mathbf{X}$ is a NLS, $\{x_n : n \ge 1\} \subseteq \mathbf{X}$, $x_n \rightharpoonup x$. Then $\sup_{n\ge 1} \|x_n\| \le \infty$.

Uniform Boundedness of Weak Converging Sequences

Proof.

$\forall \ell \in \mathbf{X}', \ell(x_n) \to \ell(x) \Rightarrow \exists c(\ell) < \infty, \sup_{n\geq 1} L_{x_n}(\ell) \leq c(\ell)$. It follows that $\exists c_0 < \infty, \sup_{n\geq 1} \|x_n\| \leq c_0$. □

Proposition 1.10

$\mathbf{X}$ is a NLS, $\{x_n : n \geq 1\} \subseteq \mathbf{X}$, $x_n \rightharpoonup x \Rightarrow \|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

Proof.

By the HB theorem, $\exists \ell \in \mathbf{X}', \ell(x) = \|x\|, \|\ell\| = 1$. Then $x_n \rightharpoonup x \Rightarrow \ell(x) = \lim_{n\to\infty} \ell(x_n) = \liminf_{n\to\infty} \ell(x_n) \leq \liminf_{n\to\infty} \|\ell\| \|x_n\| = \liminf_{n\to\infty} \|x_n\|$. □

Weak Sequentially Compactness

Definition 1.21

$\mathbf{X}$ is a BS, $C \subseteq \mathbf{X}$ is Weak Sequentially Compact (WSC) subset if $\forall \{x_n : n \geq 1\} \subseteq C$, $\exists \{n_k : k \geq 1\}, \exists x \in C$ s.t. $x_{n_k} \rightharpoonup x$.

Observation 1.2

C is WSC $\Rightarrow C$ is bounded, that is, $\exists c_0 < \infty, \|x\| \leq c_0, \forall x \in C$. To see this, suppose $\{x_n : n \geq 1\}$ with $\|x_n\| \geq n, \forall n \geq 1$. Then $\exists x \in C, c_0 < \infty, x_{n_k} \rightharpoonup x \Rightarrow \|x_{n_k}\| \leq c_0, \forall k$. The contradiction shows that C is bounded.

Theorem 1.22

$\mathbf{X}$ is a BS which is reflexive ($\mathbf{X}'' = \mathbf{X}$), then $B[0,1] = \{x \in \mathbf{X} : \|x\| \leq 1\}$ is WSC.

Remark 1.3

Because dual spaces are allways complete, a reflexivie NLS must be a BS, that is why $\mathbf{X}$ is a BS in the conditions of the above theorem.

Weak Sequentially Compactness

Proof.

We consider the case that $\mathbf{X}$ is separable. Then $\mathbf{X}'' = \mathbf{X}$ is also separable, and $\mathbf{X}'$ is separable. Let $D = \{m_j : j \geq 1\} \subseteq \mathbf{X}'$ be dense in $\mathbf{X}'$. Let $\{x_n : n \geq 1\} \subseteq B[0,1]$, and we will prove that $x_{n_k} \rightharpoonup x$.

Because $|m_1(x_n)| \leq \|m_1\| \|x_n\| = \|m_1\|$ is bounded, $\exists \{n_k\}$ s.t. $m_1(x_{n_k}) \to A(m_1)$, and $|A(m_1)| \leq \|m_1\| \|x_{n_k}\| = \|m_1\|$. Similarly, $|m_2(x_{n_k})| \leq \|m_2\| \|x_{n_k}\| = \|m_2\|$ is bounded, $\exists \left\{n'_k\right\}$ s.t. $m_1(x_{n'_k}) \to A(m_2)$, and $|A(m_2)| \leq \|m_2\| \left\|x_{n'_k}\right\| = \|m_2\|$. Iteratively applying this process, $\exists n_k$ s.t. $m_j(x_{n_k}) \to A(m_j), |A(m_j)| \leq \|m_j\|, \forall j \geq 1$. We now prove that $x_{n_k} \rightharpoonup x$ for some $x \in B[0,1]$.

Claim 1.10

$\forall m \in \mathbf{X}'$, $m(x_{n_k}) \to A(m), |A(m)| \leq \|m\|$.

Proof.

$\forall \varepsilon > 0, \exists m_j \in D, \|m_j - m\| \leq \frac{\varepsilon}{3}$.

$|m(x_{n_k}) - m(x_{n_l})| \leq |m(x_{n_k}) - m_j(x_{n_k})| + |m(x_{n_l}) - m_j(x_{n_l})| + |m_j(x_{n_k}) - m_j(x_{n_l})| \leq \frac{2\varepsilon}{3} + |m_j(x_{n_k}) - m_j(x_{n_l})| \leq \varepsilon$ for $k, l \geq k_0$, because $m_j(x_{n_k})$ converges so it is a Cauchy sequence. It follows that $\{m(x_{n_k}) : k \geq 1\}$ is a Cauchy sequence and it converges, so we can define $A(m) = \lim_{k\to\infty} m(x_{n_k})$, and $|A(m)| \leq \limsup_{k\to\infty} \|m\| \|x_{n_k}\| = \|m\|$. □

□

Weak Sequentially Compactness

Cont'd.

Claim 1.11

$A\colon \mathbf{X}' \to \mathbb{K}, A \in \mathbf{X}''$.

Proof.

First, we can check by definition that $A(m_1 + m_2) = A(m_1) + A(m_2), A(\alpha m) = \alpha A(m)$. By the above claim, $|A(m)| \le \|m\|, \forall m \in \mathbf{X}'$, it follows that $A \in \mathbf{X}''$. Because $\mathbf{X}$ is reflexive, $\exists x \in \mathbf{X}, A(m) = m(x), \|A\| = \|x\| \le 1$, so $x \in B[0,1]$. By the above claim again, $m(x_{n_k}) \to A(m) = m(x), \forall m \in \mathbf{X}'$. It follows that $x_{n_k} \rightharpoonup x$. □

Now suppose $\mathbf{X}$ is not separable. Let $\{x_n : n \ge 1\} \subseteq B[0,1]$, and we will prove that $x_{n_k} \rightharpoonup x$. Let $Y = \text{CLS}\{x_n : n \ge 1\}$, then Y is separable. Y is reflexive a closed subspace of the reflexive space $\mathbf{X}$. By applying the first part of the proof, $\exists n_k, x \in Y, \|x\| \le 1$, s.t. $\forall m \in Y', m(x_{n_k}) \to m(x)$. $\forall \ell \in \mathbf{X}'$, we have $\ell_Y : Y \to \mathbb{K}$ which the restriction of ℓ on Y: $\forall z \in Y, \ell_Y(z) = \ell(z)$. Then $\ell(x_{n_k}) = \ell_Y(x_{n_k}) \to \ell_Y(x) = \ell(x), \forall \ell \in \mathbf{X}' \Rightarrow x_{n_k} \rightharpoonup x$. □

Remark 1.4

$\mathbf{X}$ is a BS and it is reflexive. Then $B[0,1]$ is WSC, but $B[0,1]$ is not compact with respect to the strong topology.

Weak* Topology

- M is a BS, and $\mathbf{X} = M'$. $M \subseteq M''$.
  $m \in M, L_m(x) = x(m), \forall x \in \mathbf{X}$. Let $\{x_n : n \geq 1\} \subseteq \mathbf{X}$, $x_n \rightharpoonup x$ if
  $\forall \ell \in M'', \ell(x_n) \to \ell(x)$.

Definition 1.22

M is a BS, and $\mathbf{X} = M'$. $\{x_n : n \geq 1\} \subseteq \mathbf{X}$, $x_n \xrightarrow{w*} x$ if
$\forall m \in M, x_n(m) \to x(m)$.

Observation 1.3

(1) $w*$ convergence is weaker than weak convergence due to the fact that $M \subseteq M''$. With weak convergence, $\ell(x_n) \to \ell(x), \forall \ell \in \mathbf{X}' = M''$. Because $L_m \in \mathbf{X}', \forall m \in M$, it follows that $L_m(x_n) = x_n(m) \to x(m) = L_m(x)$.

(2) M is reflexive ($M'' = M$), then $x_n \xrightarrow{w*} x \Rightarrow x_n \rightharpoonup x$.

Weak* Topology

Example 1.4

Signed measures $\mathbf{X}$ on $[-1,1]$ with Borel σ-algebra $\mathcal{B}$.
$x \in \mathbf{X}, \|x\| = x^+([-1,1]) + x^-([-1,1]), x = x^+ - x^-$. We say that $x_n \to x$ if $\forall f \in C[-1,1], \int_{[-1,1]} f \mathrm{d}x_n \to \int_{[-1,1]} f \mathrm{d}f$. Let $M = C[-1,1]$ be a BS equipped with supremum norm, $f \in M, \|f\|_\infty$. Then $M' = \mathbf{X}$.
$\int_{[-1,1]} f \mathrm{d}x_n \to \int_{[-1,1]} f \mathrm{d}f$ is equivalent to $x_n(f) \to x(f), \forall f \in M$, or in other words, $x_n \stackrel{w*}{\to} x$.

Example 1.5

We will have an example where $x_n \stackrel{w*}{\to} x$, but $x_n \not\to x$.
$M'' = \mathbf{X}' = L^\infty[-1,1]$, which is the space of bounded functions. Let $x_n(\mathrm{d}t)$ be a measure which is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}t$ with density $x_n(t)$, that is, $x_n(\mathrm{d}t) = x_n(t)\mathrm{d}t$. Let

$$x_n(t) = \begin{cases} n & t \in [-\frac{1}{2n}, \frac{1}{2n}] \\ 0 & \text{otherwise} \end{cases}.$$

Weak* Topology

Example 1.6 (Cont'd)

Let $\delta_y(A) = 1$ if $y \in A$, and 0 otherwise. Then $x_n \overset{w*}{\to} \delta_0$. To see this, $\forall f \in M$,
$x_n(f) = \int_{[-1,1]} f \mathrm{d}x_n(t) = n \int_{[-\frac{1}{2n}, \frac{1}{2n}]} \mathrm{d}t \overset{n\to\infty}{\to} f(0) = \int_{[-1,1]} f \mathrm{d}\delta_0(t)$. Therefore, $x_n \overset{w*}{\to} \delta_0$.
To show that, we need to show that $\exists f \in M'' = L^\infty[-1,1]$ such that
$x_n(f) = \int_{[-1,1]} f \mathrm{d}x_n(t) = n \int_{[-\frac{1}{2n}, \frac{1}{2n}]} \mathrm{d}t \overset{n\to\infty}{\nrightarrow} f(0) = \int_{[-1,1]} f \mathrm{d}\delta_0(t)$. It is easy to find such f which is not continuous at 0.

Proposition 1.11

M is a BS, $\mathbf{X} = M'$. Let $\{x_n : n \geq 1\} \subseteq \mathbf{X}$ and $x_n \overset{w*}{\to} x$. Then $\exists c_0 < \infty, \sup_n \|x_n\| \leq c_0$.

Proof.

$x_n \overset{w*}{\to} x \Rightarrow \forall m \in \mathbf{X}, \exists c(m) < \infty$ s.t. $\sup_n |x_n(m)| \leq c(m)$. By the previous theorem on the unique boundedness principle, $\exists c_0 < \infty$ s.t. $\sup_n \|x_n\| \leq c_0$. □

Weak* Topology

Remark 1.5

M is a BS, $\mathbf{X} = M'$, $\{x_n : n \geq 1\} \subseteq \mathbf{X}, x\ in \mathbf{X}$ and $x_n \overset{w*}{\to} x$. It is proved that $\sup_n \|x_n\|$ is bounded. Furtheremore, we have $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$. To see this. $\forall \varepsilon > 0, \exists m \in M, \|m\| = 1$ s.t. $|x(m)| \geq \|x\| - \varepsilon$. By the definition of weak* convergence,
$|x(m)| = \lim_{n\to\infty} |x_n(m)| \leq \liminf_{n\to\infty} \|x_n\| \|m\| = \liminf_{n\to\infty} \|x_n\|$. Therefore,
$\|x\| - \varepsilon \leq \liminf_{n\to\infty} \|x_n\|, \forall \varepsilon > 0 \Rightarrow \|x\| \leq \liminf_{n\to\infty} \|x_n\|$.
Another proof: by the definition of weak* convergence, $\forall m \in M, |x(m)| = \lim_{n\to\infty} |x_n(m)| \leq \liminf_{n\to\infty} \|x_n\| \|m\| \Rightarrow \|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

Definition 1.23

M is a BS, $\mathbf{X} = M'$. $C \subseteq \mathbf{X}$ is Weak* Sequentially Compact (W*SC) if $\forall \{x_n : n \geq 1\} \subseteq C, \exists \{n_k\}, \exists x \in C$ s.t. $x_{n_k} \overset{w*}{\to} x$.

Weak* Topology

Theorem 1.23

M is a BS which is separable, $\mathbf{X} = M'$. Then $B[0,1] \subseteq \mathbf{X}$ is W*SC.

Proof.

Let $\{x_n : n \geq 1\} \subseteq B[0,1]$, and $D = \{m_j : j \geq 1\}$ be dense in M. Because $|x_n(m_1)| \leq \|x_n\| \|m_1\| \leq \|m_1\|$, $\{x_n(m_1)\}$ is bounded, so that $\exists \{n_k\}$ s.t. $x_{n_k}(m_1) \to A(m_1)$. Iteratively applying this process, $\exists \{n_k\}$ s.t. $x_{n_k}(m_j) \to A(m_j), \forall j \geq 1$.

Claim 1.12

$x_{n_k}(m) \to A(m), \forall m \in M$, where $A \in \mathbf{X}$, $\|A\| \leq 1$.

Proof.

$\forall m \in M, \exists m_j \in D$ s.t. $\|m_j - m\| \leq \frac{\varepsilon}{3}$. We have $|x_{n_k}(m) - x_{n_l}(m)| \leq |x_{n_k}(m) - x_{n_k}(m_j)| + |x_{n_l}(m) - x_{n_l}(m_j)| + |x_{n_k}(m_j) - x_{n_l}(m_j)| \leq \frac{2\varepsilon}{3} + |x_{n_k}(m_j) - x_{n_l}(m_j)| \leq \varepsilon$ when $k, l \geq k_0$ for some k_0. It follows that $\{x_{n_k}(m) : k \geq 1\}$ is a Cauchy sequence, and we can define $A(m)$ be the limit of $x_{n_k}(m)$, that is, $\lim_{k\to\infty} x_{n_k}(m) = A(m)$.

By checking the definition, A is linear: $A(\alpha m_1 + m_2) = \alpha A(m_1) + A(m_2), \alpha \in \mathbb{K}, m_1, m_2 \in M$. In addition,
$\forall m \in M, |A(m)| = \lim_{k\to\infty} |x_{n_k}(m)| \leq \liminf_{k\to\infty} \|x_{n_k}\| \|m\| \Rightarrow \|A\| \leq \liminf_{k\to\infty} \|x_{n_k}\| \leq 1$.
Therefore, $A \in \mathbf{X}$, $\|A\| \leq 1$. □

By the above claim, $A \in \mathbf{X}$, $A \in B[0,1]$, and $x_{n_k} \overset{w*}{\to} A \Rightarrow B[0,1] \subseteq \mathbf{X}$ is W*SC. □

Applications of Weak Convergence

- $\mathbf{X} = C[-1,1]$, $f \in \mathbf{X}\colon [-1,1] \to \mathbb{R}$, $\|f\|_\infty = \sup\{|f(x)| : x \in [-1,1]\}$. $\mathbf{X}' = M$ is the set of finite signed measures defined on the Borel σ-filed (σ-algebra) $\mathcal{B}$. By the Hahn Decomposition theorem in measure theory, $\mu \in M$ can be decomposed as $\mu = \mu^+ - \mu^-$, $\|\mu\| = \mu^+([-1,1]) + \mu^-([-1,1])$, which is the total variation of this measure (also the total variation of the function $\phi(t) = \mu([-\infty, t])$).
- Let $\{f_n : n \geq 1\} \subseteq \mathbf{X}$, $\mu_n(\mathrm{d}t) = f_n(t)\mathrm{d}t$ is a measure absolutely continuous w.r.t. the Lebesgue measure $\mathrm{d}t$ with density function $f_n(t)$. We will show the conditions under which $\mu_n(\mathrm{d}t) \stackrel{w*}{\to} \delta_0(\mathrm{d}t)$ holds, where $\delta_y(A) = 1$ for $y \in A$, and 0 otherwise. Note that $\mu_n(\mathrm{d}t) \stackrel{w*}{\to} \delta_0(\mathrm{d}t) \iff \int_{[-1,1]} g\mu_n(\mathrm{d}t) \to \int_{[-1,1]} g\delta_0(\mathrm{d}t), \forall g \in \mathbf{X}$.

Applications of Weak Convergence

- $\forall g \in \mathbf{X}, \int_{[-1,1]} g f_n \mathrm{d}t \to g(0) \iff$
  (1) $\int_{[-1,1]} f_n(t)\mathrm{d}t \to 1$
  (2) $\forall g \in C^\infty([-1,1]), 0 \notin$ supp(g), $\int_{[-1,1]} f_n(t)g(t)\mathrm{d}t \to 0$, where supp$(g) := \overline{\{x\colon g(x) \neq 0\}}$.
  (3) $\exists c_0 < \infty, \int_{[-1,1]} |f_n(t)|\,\mathrm{d}t \le c_0$

Proof.

$\Rightarrow$:

(1) Set $g = 1$

(2) Noting that g can be chosen such that $g(0) = 0$

(3) Because $\mu_n \overset{w*}{\to} \delta_0$, $\exists c_0, \sup_n \|\mu_n\| \le c_0$.
$\|\mu_n\| = \mu_n^+([-1,1]) + \mu_n^-([-1,1])$, where
$\mu_n^+(\mathrm{d}t) = f_n^+\mathrm{d}t, \mu_n^-(\mathrm{d}t) = f_n^-\mathrm{d}t$. It follows that
$\|\mu_n\| = \int_{[-1,1]} f_n^+\mathrm{d}t + \int_{[-1,1]} f_n^-\mathrm{d}t = \int_{[-1,1]} |f_n|\,\mathrm{d}t \le c_0$.

$\Leftarrow$: It suffices to prove $\int_{[-1,1]} g f_n \mathrm{d}t \to g(0), \forall g \in \mathbf{X}, g(0) = 0$. To see this, let $h \in \mathbf{X}, g(t) = h(t) - h(0)$. Then $\int_{[-1,1]} f_n(t)g(t)\mathrm{d}t \to g(0) = 0 \Rightarrow \int_{[-1,1]} f_n(t)h(t)\mathrm{d}t \to h(0)\int_{[-1,1]} f_n(t)\mathrm{d}t$. Noting that

Applications of Weak Convergence

Cont'd.

We now prove that conditions (1)-(3) guarantee that $\int_{[-1,1]} g f_n \mathrm{d}t \to g(0), \forall g \in \mathbf{X}, \exists \delta > 0, g(t) = 0,$ for $|t| \leq \delta$. To see this, we can construct a function ϕ which is (1) smooth, (2) $\phi(t) \geq 0$, (3) $\phi(t) = 0$ for $|t| > 1$, (4) $\int_{\mathbb{R}} \phi(t)\mathrm{d}t = 1$. Let $\phi_\varepsilon(t) = \frac{1}{\varepsilon}\phi(\frac{t}{\varepsilon}), g_\varepsilon = \phi_\varepsilon \star g$, that is. $g_\varepsilon(t) = \int_{\mathbb{R}} \phi_\varepsilon(s)g(t-s)\mathrm{d}s = \int_{\mathbb{R}} \phi_\varepsilon(t-s)g(s)\mathrm{d}s$. Therefore, $g_\varepsilon \in C^\infty([-1,1]), g_\varepsilon(t) = 0$ for $|t| \leq \delta - \varepsilon$ with $\varepsilon < \delta$. By condition (2), $\int_{[-1,1]} g_\varepsilon f_n \mathrm{d}t \to 0$.

Due to the continuity of g, $\forall \eta > 0, \exists a > 0$, s.t. $|g(t) - g(s)| \leq \eta$ when $|t-s| \leq a$. We have $|g_\varepsilon(t) - g(t)| = \left|\int_{\mathbb{R}} \phi_\varepsilon(s)g(t-s)\mathrm{d}s - g(t)\right| \leq \int_{[-\varepsilon,\varepsilon]} \phi_\varepsilon(s)\left|g(t-s) - g(t)\right|\mathrm{d}s \leq \eta$ when $\varepsilon \leq a$, and it follows that $\|g_\varepsilon - g\|_\infty \leq \eta$. Then, when $\varepsilon < \min\{\delta, a\}$, $\left|\int_{[-1,1]} g(t)f_n(t)\mathrm{d}t - \int_{[-1,1]} g_\varepsilon(t)f_n(t)\mathrm{d}t\right| \leq \int_{[-1,1]} |g(t) - g_\varepsilon(t)|\,\mathrm{d}t \leq c_0\eta$. This result combined with $\int_{[-1,1]} g_\varepsilon f_n \mathrm{d}t \to 0$ shows that $\int_{[-1,1]} g f_n \mathrm{d}t \to 0$, $\forall g \in \mathbf{X}, \exists \delta > 0, g(t) = 0,$ for $|t| \leq \delta$. □

Applications of Weak Convergence

- $f \in C(R), f(t+2\pi) = f(t), \forall t \in \mathbb{R}, a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\mathrm{d}t, n \in \mathbb{Z}$.
- $f(\theta) = \sum_j a_j e^{ij\theta}, f_n(\theta) = \sum_{j=-n}^{n} a_j e^{ij\theta}$. It is expected that $f_n(\theta) \to f(\theta)$. We will prove that $\exists f$, s.t. $f_n(0) \nrightarrow f(0)$, that is, $\sum_{j=-n}^{n} a_j \nrightarrow f(0)$.
- $\sum_{j=-n}^{n} a_j = \frac{1}{2\pi}\sum_{j=-n}^{n}\int_{-\pi}^{\pi} f(t)e^{-ijt}\mathrm{d}t = \int_{-\pi}^{\pi} f(t)g_n(t)\mathrm{d}t$, where $g_n(t) = \frac{1}{2\pi}\sum_{j=-n}^{n} e^{-ijt}$. We will show that $\int_{-\pi}^{\pi} |g_n(t)|\,\mathrm{d}t \to \infty$, so condition (3) in the previous result is not satisfied. Therefore, $\exists f \in C(R)$ s.t. $\int_{-\pi}^{\pi} f(t)g_n(t)\mathrm{d}t \nrightarrow f(0)$.

Applications of Weak Convergence

- By direct computation, $g_n(t) = \frac{1}{2\pi}\frac{\sin\left((n+\frac{1}{2})t\right)}{\sin(\frac{t}{2})}$, and $\int_{-\pi}^{\pi} \frac{\left|\sin\left((n+\frac{1}{2})t\right)\right|}{\left|\sin(\frac{t}{2})\right|}\mathrm{d}t$. It can be proved that $\left|\frac{\sin x}{x}\right| \leq c_0 =$ const. $\quad, |x| \leq \pi$. It follows that $\int_{-\pi}^{\pi} \frac{\left|\sin\left((n+\frac{1}{2})t\right)\right|}{\left|\sin(\frac{t}{2})\right|}\mathrm{d}t \geq 2c_0 \int_{-\pi}^{\pi} \left|\sin\left(n+\frac{1}{2}\right)t\right| \frac{1}{|t|}\mathrm{d}t = 2c_0 \int_{-(n+\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \left|\frac{\sin\theta}{\theta}\right| \mathrm{d}\theta$. By removing small intervals around $n\pi, n \in \mathbb{Z}$, $|\sin\theta|$ is bounded from below by a constant, it can be proved that $\int_{-(n+\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \left|\frac{\sin\theta}{\theta}\right| \mathrm{d}\theta \geq c_1 \log n \to \infty$.

Bounded Linear Operators

- $\mathbf{X}, \mathbf{Y}$ are two BS, $M\colon \mathbf{X} \to \mathbf{Y}$ which is linear if $M(\alpha x + y) = \alpha M(x) + M(y), \forall \alpha \in \mathbb{K}, x, y \in \mathbf{X}$.

Definition 1.24

$\mathbf{X}, \mathbf{Y}$ are two BS, $M\colon \mathbf{X} \to \mathbf{Y}$ is continuous if $\forall \{x_n : n \geq 1\} \subseteq \mathbf{X}, x_n \to x \Rightarrow M(x_n) \to M(x)$.

Definition 1.25

$\mathbf{X}, \mathbf{Y}$ are two BS, $M\colon \mathbf{X} \to \mathbf{Y}$ is bounded if $\exists c_0 < \infty, \forall x \in \mathbf{X}, \|Mx\|_{\mathbf{Y}} \leq c_0 \|x\|_{\mathbf{X}}$.

Lemma 1.16

$\mathbf{X}, \mathbf{Y}$ are two BS, $M\colon \mathbf{X} \to \mathbf{Y}$ is bounded $\iff$ M is continuous.

Bounded Linear Operators

Proof.

$\Rightarrow$: Let $\{x_n : n \geq 1\} \subseteq \mathbf{X}, x_n \to x$, then $\|M(x_n - x)\|_{\mathbf{Y}} \leq c_0 \|x_n - x\|_{\mathbf{X}} \to 0$.
$\Leftarrow$: suppose M is not bounded, so $\exists \{x_n : n \geq 1\} \subseteq \mathbf{X}$ s.t. $\|Mx_n\|_{\mathbf{Y}} \geq n\|x\|_{\mathbf{X}}$. Define $y_n = \frac{x_n}{\|x_n\|_{\mathbf{X}}}$, then $\|My_n\|_{\mathbf{Y}} \geq n$. However, by the boundedness of M, $\|My_n\|_{\mathbf{Y}} \leq c_0 \|y_n\|_{\mathbf{X}} = c_0$. The contradiction shows that M is bounded. □

- $\mathbf{X}, \mathbf{Y}$ are NLS, $M : \mathbf{X} \to \mathbf{Y}$ is bounded, we will construct $M_0 : \bar{\mathbf{X}} \to \bar{\mathbf{Y}}$ which is bounded, and $\bar{\mathbf{X}}$ is the completion of $\mathbf{X}$, $\bar{\mathbf{Y}}$ is the completion of $\mathbf{Y}$.
  $\bar{\mathbf{X}} = \{\{x_n\} : \{x_n\} \subseteq \mathbf{X}$ which is a Cauchy sequence$\}$. Two Cauchy sequences are equivalent, denoted by $[x_n] \sim [\tilde{x}_n]$, if $x_n - \tilde{x}_n \to 0$.
  $\|[x_n]\| := \lim_{n\to\infty} \|x_n\|$.
- $M_0 : \bar{\mathbf{X}} \to \bar{\mathbf{Y}}$, $M_0([x_n]) = [Mx_n]$. Now we show that M_0 is well-defined. First, $\mathbf{X} \subseteq \bar{\mathbf{X}}$ by letting $[x] = \{x, x, x, \ldots\}$. If $[x_n]$ is a Cauchy sequence, by the boundedness of M, $M[x_n]$ is also a Cauchy sequence. IF $[x_n] = [y_n]$, then $\|M(x_n - y_n)\| \leq c_0 \|x_n - y_n\| \to 0$, so $M_0([x_n]) = [Mx_n] = [My_n] = M_0([y_n])$. $M_0([x]) = [Mx]$, so M_0 is an extension of M from $\mathbf{X}$ to $\bar{\mathbf{X}}$.

Bounded Linear Operators

- For $[x_n], [z_n]$, we have $\alpha[x_n] + [z_n] = [\alpha x_n + z_n]$, so $M_0(\alpha[x_n] + [z_n]) = M_0([\alpha x_n + z_n]) = [M(\alpha x_n + z_n)] = [\alpha M x_n + M z_n] = \alpha[M x_n] + [M z_n] = \alpha M_0([x_n]) + M_0([z_n])$.
- M_0 is bounded. Let $[x_n] \in \bar{\mathbf{X}}$, $\|M_0([x_n])\| = \|[M x_n]\| = \lim\limits_{n\to\infty} \|M x_n\| \leq c_0 \lim\limits_{n\to\infty} \|x_n\| = c_0 \|[x_n]\|$.
- From now on, we always assume $\mathbf{X}, \mathbf{Y}$ are BS because if they are not, they can be extended to BS $\bar{\mathbf{X}}, \bar{\mathbf{Y}}$ by the above process.
- $\mathbf{X}, \mathbf{Y}$ are BS, $M\colon \mathbf{X} \to \mathbf{Y}$, M is a map or an operator. Let M be a Bounded Linear Operator (BLS). $\|M\| = \sup_{x \in \mathbf{X}, \neq 0} \frac{\|Mx\|}{\|x\|} = \sup_{x \in \mathbf{X}, \|x\|=1} \|Mx\|$.
- M is a BLO $\iff \|M\| < \infty$. Define $\mathcal{L}(\mathbf{X}, \mathbf{Y}) := \{M\colon M\colon \mathbf{X} \to \mathbf{Y}, M \text{ is a BLO}\}$.

Bounded Linear Operators

- Properties of $\|M\|, M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$:
  1. $\|M\| \geq 0, \|M\| = 0 \Rightarrow M = 0,$
  2. $\|\alpha M\| = |\alpha| \, \|M\|,$
  3. $\|M + N\| \leq \|M\| + \|N\|.$

Proposition 1.12

$\mathbf{X}, \mathbf{Y}$ are BS, then $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ is a BS.

Proof.

By checking the definition, $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ is a linear space. Let $\{M_n : n \geq 1\}$ be a Cauchy sequence in $\mathcal{L}(\mathbf{X}, \mathbf{Y})$, we will prove that there exists $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ s.t. $\lim\limits_{n \to \infty} M_n = M$. To see this,
$\forall x \in \mathbf{X}, \{Mx_n : n \geq 1\}$ is a Cauchy sequence. This is because
$\|M_n x - M_m x\| \leq \|M_n - Mm\| \|x\| \leq \varepsilon \|x\|$, and $\|M_n - Mm\| \leq \varepsilon, \forall n, m \geq n_0$ since $\{M_n : n \geq 1\}$ is a Cauchy sequence.
Therefore, M is defined by $Mx := \lim\limits_{n \to \infty} M_n x, \forall x \in \mathbf{X}$.
$\forall x \in \mathbf{X}, \|x\| = 1, \|(M_n - M)x\| \leq \|(M_n - M_n)x\| + \varepsilon$ due to $M_n x \to Mx$, and $\|(M_n - M_n)x\| \leq \varepsilon \|x\|, \forall n, m \geq n_0$ because $\{M_n : n \geq 1\}$ is a Cauchy sequence. It follows that $\lim\limits_{n \to \infty} \sup_{x \in \mathbf{X}, \|x\| = 1} \|(M_n - M)x\| = 0 \Rightarrow \lim\limits_{n \to \infty} \|M_n - M\| = 0$. Therefore, $M_n \to M$ in operator norm, and $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ is complete. $\square$

Bounded Linear Operators

Remark 1.6

$\mathbf{X}, \mathbf{Y}$ are BS, $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$. $N = \{x \in \mathbf{X} \colon Mx = 0\}$ is the null space of M. By checking the definition, N is a CLS of $\mathbf{X}$. $x \sim y$ if $x - y \in N$.$M_0 \colon \mathbf{X} \mid_N \to \mathbf{Y}, M_0([x]) = Mx$, $\|[x]\| = \inf\{\|y\| : y \in [x]\}$.

Lemma 1.17

M_0 is a BLO.

Proof.

(1) M_0 is well defined. Let $[x] = [y]$, then $x - y \in N$, and $M_0([x]) = Mx = My = M_0([y])$. (2) M_0 is linear. Note that $\alpha[x] + [y] = [\alpha x + y]$, and $M_0(\alpha[x] + [y]) = M_0([\alpha x + y]) = M(\alpha x + y) = \alpha Mx + My = \alpha M_0([x]) + M_0([y])$. (3) M_0 is bounded. $\forall \varepsilon > 0, \exists y \in [x]$ s.t. $\|y\| \leq \|[x]\| + \varepsilon$.
$\|M_0([x])\| = \|M_0[y]\| = \|My\| \leq \|M\|\|y\| \leq \|M\| \left(\|[x]\| + \varepsilon\right)$. Because this inequality holds for any $\varepsilon > 0$, $\|M_0([x])\| \leq \|M\|\|[x]\| \Rightarrow \|M_0\| \leq \|M\|$. Furthermore. $\|M\| \leq \|M_0\|$. To see this, $\forall x \in \mathbf{X}$, $\|Mx\| = \|M_0([x])\| \leq \|M_0\|\|[x]\| \leq \|M_0\|\|x\| \Rightarrow \|M\| \leq \|M_0\|$. Therefore, $\|M_0\| = \|M\|$. □

Bounded Linear Operators

Remark 1.7

- M_0 is injective. Let $[x] = [y]$, then $x - y \in N$, $M_0([x]) = Mx = My = M_0([y])$.
- rangeM_0 = range$M$. Let $y \in$ rangeM_0, then $\exists [x], M_0([x]) = Mx = y \Rightarrow y \in$ range$M$. Let $y \in$ range$M$, then $\exists x \in \mathbf{X}, Mx = M_0([x]) = y \Rightarrow y \in$ rangeM_0.

Transpose of Linear Operators

- $\mathbf{X}, \mathbf{Y}$ are BS, $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$. $m \in \mathbf{X}', m \colon \mathbf{X} \to \mathbb{K}$, $\ell \in \mathbf{Y}', \ell \colon \mathbf{Y} \to \mathbb{K}$. Then $\ell M \colon \mathbf{X} \to \mathbb{K}, (\ell M)(x) = \ell(M(x))$. it can be verified that $\ell M \in \mathbf{X}'$.
- $M' \colon \mathbf{Y}' \to \mathbf{X}', M'\ell = \ell M$. $\forall \ell \in \mathbf{Y}'$, $M'\ell = \sup_{x \in \mathbf{X}, \|x\|=1} \|M'\ell(x)\| = \sup_{x \in \mathbf{X}, \|x\|=1} \|\ell M(x)\| \leq \sup_{x \in \mathbf{X}, \|x\|=1} \|\ell\| \|Mx\| \leq \sup_{x \in \mathbf{X}, \|x\|=1} \|M\| \|x\| = \|M\|$. Therefore, $\|M'\ell\| \leq \|M\| \|\ell\| \Rightarrow \|M'\| \leq \|M\|$.
- We use the convention $m(x) = \langle x, m \rangle, m \in \mathbf{X}'$. Then $(M'\ell)(x) = \langle x, M'\ell \rangle = \ell(Mx) = \langle Mx, \ell \rangle$, so $\langle x, M'\ell \rangle = \langle Mx, \ell \rangle, \forall x \in \mathbf{X}, \ell \in \mathbf{Y}'$.

Lemma 1.18

$\|M'\| = \|M\|$

Transpose of Linear Operators

Proof.

Because $\left\|M'\right\| \leq \|M\|$, we only need to prove that $\|M\| \leq \left\|M'\right\|$. Note that $\forall y \in \mathbf{Y}$, by the HB theorem, $\|y\| = \sup_{\ell \in \mathbf{Y}', \|\ell\|=1} |\ell(y)|$. It follows that $\|M\| = \sup_{x \in \mathbf{X}, \|x\|=1} \|Mx\| = \sup_{x \in \mathbf{X}, \|x\|=1} \sup_{\ell \in \mathbf{Y}', \|\ell\|=1} \|\ell(Mx)\| = \sup_{x \in \mathbf{X}, \|x\|=1} \sup_{\ell \in \mathbf{Y}', \|\ell\|=1} \left\|(M'\ell)(x)\right\| \leq \sup_{x \in \mathbf{X}, \|x\|=1} \sup_{\ell \in \mathbf{Y}', \|\ell\|=1} \left\|M'\ell\right\| \|x\| = \sup_{\ell \in \mathbf{Y}', \|\ell\|=1} \left\|M'\ell\right\| = \left\|M'\right\|$. $\square$

Remark 1.8

$M_1, M_2 \in \mathcal{L}(\mathbf{X}, \mathbf{Y}), \alpha \in \mathbb{K}$, $(\alpha M_1 + M_2)' = \alpha M_1' + M_2'$. To see this, $\forall \ell \in \mathbf{Y}'$, $(\alpha M_1 + M_2)' \ell = \ell \left(\alpha M_1 + M_2\right) = \alpha \ell M_1 + \ell M_2 = \alpha M_1' \ell + M_2' \ell = \left(\alpha M_1' + M_2'\right) \ell$.

Definition 1.26

$N_M = \{x \in \mathbf{X} \colon Mx = 0\} \subseteq \mathbf{X}$, $R(M) = \{Mx \colon x \in \mathbf{X}\} \subseteq \mathbf{Y}$, $N_{M'} = \{\ell \in \mathbf{Y}' \colon M'\ell = 0\}$, $R(M') = \{\ell \in \mathbf{Y}' \colon M'\ell\}$. For $A \subseteq \mathbf{Y}$, define $A^\perp := \{\ell \in \mathbf{Y}' \colon \ell(x) = 0, \forall x \in A\} \subseteq \mathbf{Y}'$.

Transpose of Linear Operators

Lemma 1.19

$R_M^{\perp} = N_{M'}$

Proof.

We first prove that $R_M^{\perp} \subseteq N_{M'}$. $\forall \ell \in R_M^{\perp}, \ell(Mx) = 0, \forall x \in \mathbf{X} \Rightarrow (M'\ell)(x) = 0, \forall x \in bX$, therefore, $M'\ell = 0$, and $\ell \in N_{M'}$.
We then prove that $N_{M'} \subseteq R_M^{\perp}$. $\forall \ell \in N_{M'}$, $M'\ell = 0$, so $(M'\ell)(x) = 0, \forall x \in \mathbf{X} \Rightarrow \ell(Mx) = 0, \forall x \in \mathbf{X}$. It follows that $\ell(x) = 0, \forall x \in R(M) \Rightarrow \ell \in R_M^{\perp}$. □

Definition 1.27

$(R_{M'})^{\perp} = \{x \in \mathbf{X} \colon m(x) = 0, \forall m \in R_{M'}\} \subseteq \mathbf{X}$.

Lemma 1.20

$N_M = (R_{M'})^{\perp}$

Transpose of Linear Operators

Proof.

We first prove that $N_M \subseteq (R_{M'})^\perp$. $\forall x \in N_M$, $Mx = 0 \Rightarrow \ell(Mx) = 0, \forall \ell \in \mathbf{Y}'$. It follows that $(M'\ell)(x) = 0, \forall \ell \in \mathbf{Y}'$, or $m(x) = 0, \forall m \in R_{M'} \Rightarrow x \in (R_{M'})^\perp$.
We then prove that $(R_{M'})^\perp \subseteq N_M$. $\forall x \in (R_{M'})^\perp$, $m(x) = 0, \forall m \in R_{M'}$. It follows that $(M'\ell)(x) = 0, \forall \ell \in \mathbf{Y}' \Rightarrow \ell(Mx) = 0, \forall \ell \in \mathbf{Y}'$. By choosing $\ell \in \mathbf{Y}'$ such that $\ell(Mx) = \|Mx\| = 0$ (by the HB theorem) $\Rightarrow Mx = 0$, so $x \in N_M$. □

- $\mathbf{X}$ is a HS, $M : \mathbf{X} \to \mathbf{X}$, $\ell \in \mathbf{X}'$, then $\exists y \in \mathbf{X}, \ell(x) = \langle x, y \rangle = \ell_y(x)$. Let $M^* : \mathbf{X} \to \mathbf{X}, M^* y = M'\ell_y \in \mathbf{X}'$. Then $!\exists z \in \mathbf{X}, M'\ell_y = \ell_z$, and $M^* y = z$.

Claim 1.13

$\forall x, y \in \mathbf{X}, \langle Mx, y \rangle = \langle x, M^* y \rangle$

Proof.

$\prod x M^* y = (M^* \ell_y)(x) = \ell_y(Mx) = \langle Mx, y \rangle$. □

Strong and Weak Convergence of Operators

- $\mathbf{X}, \mathbf{Y}$ are BS, $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$. The uniform topology is with respect to the operator norm $\|M\| = \sup_{x \in \mathbf{X}, \|x\|=1} \|Mx\|$.
- Strong topology $x \in \mathbf{X}, \Gamma_x : \mathcal{L}(\mathbf{X}, \mathbf{Y}) \to \mathbf{Y}, \gamma_x(M) = Mx$. (1) $\forall x \in \mathbf{X}$, Γ_x is continuous; (2) the weakest topology
- Weak topology
  $x \in \mathbf{X}, \ell \in \mathbf{Y}', \Gamma_{x,\ell} : \mathcal{L}(\mathbf{X}, \mathbf{Y}) \to \mathbb{K}, \Gamma_{x,\ell}(M) = \langle Mx, \ell \rangle$. The weak topology is the topology with respect to which $\forall x \in \mathbf{X}, \forall \ell \in \mathbf{Y}', \Gamma_{x,\ell}$ is continuous. It is the weakest toplogy w.r.t. which $\forall x \in \mathbf{X}, \forall \ell \in \mathbf{Y}', \Gamma_{x,\ell}$ is continuous.

Definition 1.28

$\{M_n : n \geq 1\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $\{M_n : n \geq 1\}$ strongly converges if $\forall x \in \mathbf{X}, \{M_n x : n \geq 1\}$ converges strongly in $\|\cdot\|_{\mathbf{Y}}$.

Lemma 1.21

$\{M_n : n \geq 1\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $\forall x \in \mathbf{X}, \{M_n x : n \geq 1\}$ converges strongly in $\|\cdot\|_{\mathbf{Y}}$. Then $\exists M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ s.t. $M_n \xrightarrow{s} M$.

Strong and Weak Convergence of Operators

Proof.

Define the map $A\colon \mathbf{X} \to \mathbf{Y}$, $Ax = \lim\limits_{n\to\infty} M_n x$. We will prove that (1) A is linear; (2) M is bounded: $\exists c_0 < \infty, \|Mx\| \le c_0\|x\|$. (1) can be proved by checking the definition of A. For (2), define $f_n(x) = \|M_n x\|$. Then f_n is (1) sub-additive; (2) positive homogeneous: $f_n(\alpha x) = |\alpha|\, f_n(x)$; (3) continuous. Also, $\forall x \in \mathbf{X}, \exists c(x) < \infty, \sup_n \|M_n x\| \le c(x) \Rightarrow \sup_n |f_n(x)| \le c(x)$. By the Principle of Uniform Boundedness (PUB), $\exists c_0 < \infty$,
$\sup_n |f_n(x)| = \sup_n \|M_n x\| \le c_0\|x\|, \forall x \in \mathbf{X}$. It follows that
$\|Ax\| = \lim\limits_{n\to\infty} \|M_n x\| \le \limsup\limits_{n\to\infty} \|M_n\| \le c_0\|x\|$.

Therefore, $M_n \xrightarrow{s} M$ with $M = A$. □

Definition 1.29

$\{M_n\colon n \ge 1\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $M_n \xrightarrow{w} W$ if $\forall x \in \mathbf{X}, M_n x \rightharpoonup Mx$.

Lemma 1.22

$\{M_n\colon n \ge 1\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $\forall x \in \mathbf{X}, \{M_n x\colon n \ge 1\}$ converges weakly. Then $\exists M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ s.t. $M_n \xrightarrow{w} M$.

Strong and Weak Convergence of Operators

Proof.

Define the map $A\colon \mathbf{X} \to \mathbf{Y}$, $Ax = w \lim_{n\to\infty} M_n x$. We will prove that (1) A is linear; (2) M is bounded: $\exists c_0 < \infty, \|Mx\| \le c_0\|x\|$. (1) can be proved by checking the definition of A. For (2), define $f_n(x) = \|M_n x\|$. Then f_n is (1) sub-additive; (2) positive homogeneous: $f_n(\alpha x) = |\alpha|\, f_n(x)$; (3) continuous. Also, $\forall x \in \mathbf{X}, \exists c(x) < \infty, \sup_n \|M_n x\| \le c(x) \Rightarrow \sup_n |f_n(x)| \le c(x)$. By the Principle of Uniform Boundedness (PUB), $\exists c_0 < \infty$, $\sup_n |f_n(x)| = \sup_n \|M_n x\| \le c_0\|x\|, \forall x \in \mathbf{X}$. Because $Ax = w \lim_{n\to\infty} M_n x$, $\|Ax\| \le \liminf_{n\to\infty} \|M_n x\| \le c_0\|x\|$.

Therefore, $M_n \xrightarrow{w} M$ with $M = A$. □

Lemma 1.23

$\mathbf{X}$ is reflexive, $M_n \rightharpoonup M$ (that is, $M_n \xrightarrow{w} M$). Then $M_n' \rightharpoonup M'$.

Proof.

$\forall x \in \mathbf{X}, M_n x \rightharpoonup Mx \Rightarrow \forall \ell \in bY', \langle M_n x, \ell\rangle \to \langle Mx, \ell\rangle$. It follows that $\forall x \in \mathbf{X}, \forall \ell \in \mathbf{Y}', \left\langle x, M_n'\ell \right\rangle \to \left\langle x, M'\ell \right\rangle$. Because $\mathbf{X} = \mathbf{X}''$ (reflexive), $L(M_n'\ell) \to L(M'\ell), \forall L \in \mathbf{X}'' \Rightarrow M_n' \rightharpoonup M'$. □

Strong and Weak Convergence of Operators

- $\ell^2 = \left\{ \{a_j : j \geq 1\} : \sum_{j\geq 1} |a_j|^2 < \infty \right\}$, $\forall a \in \ell^2, \|a\|^2 = \sum_{j\geq 1} |a_j|^2$. ℓ^2 is reflexive. Define $M_n : \ell^2 \to \ell^2, M_n(\{a_j : j \geq 1\}) = \{a_n, 0, 0, \ldots\}$. Then $\|M_n\| \leq 1$. We will prove that $M_n \xrightarrow{s} 0$, but $M_n' \stackrel{s}{\not\to} 0$.
- Because $\|M_n x\|^2 = \|\{x_n, 0, 0, \ldots\}\|^2 = x_n^2 \to 0$, $M_n \xrightarrow{s} 0$.
- Now we show that $M_n' \stackrel{s}{\not\to} 0$. Let $\mathbf{X} = \ell^2$.
  $\forall x \in \mathbf{X}, \forall \ell \in \mathbf{X}', \ell = \ell_y$,
  $\langle M_n x, \ell\rangle = \langle M_n x, y\rangle = \langle x, M_n'\ell\rangle = \langle x, M_n^* y\rangle$. Let
  $y = \{y_1, y_2, \ldots\}$, then $\langle M_n x, y\rangle = x_n y_1 = \sum_{j\geq 1} x_j z_j, z = M_n^* y$.
  Because this equality holds for all $x, y \in \mathbf{X}$, we must have
  $z_n = y_1, z_k = 0, \forall k \neq n$. If $M_n' \xrightarrow{s} 0$, $M_n^* \to 0$. However,
  $|M_n^* y| = |y_1|$, and $M_n^* \to 0 \Rightarrow M_n^* y \to 0$. This contradiction shows
  that $M_n' \stackrel{s}{\not\to} 0$.

Strong and Weak Convergence of Operators

Theorem 1.24

$\{M_n : n \geq 1\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$. If (1) $\exists c_0 < \infty, \|M_n\| \leq c_0$; (2) $\exists D \subseteq \mathbf{X}$ which is dense in $\mathbf{X}$, and $\forall x \in \mathbf{D}, \{M_n x : n \geq 1\}$ converges strongly, then $\{M_n : n \geq 1\}$ converges strongly, and $\exists M \in \mathcal{L}(\mathbf{X}, \mathbf{Y}), M_n \xrightarrow{s} M$.

Proof.

By the previous lemma, it suffices to show that $\forall x \in \mathbf{X}, \{M_n x : n \geq 1\}$ converges strongly, or equivalent, it is a Cauchy sequence. $\exists y \in D, \|x - y\| \leq \varepsilon$.
$\|M_n x - M_m x\| \leq \|M_n x - M_n y\| + \|M_m x - M_m y\| + \|M_n y - M_m y\| \leq$
$\|M_n\| \|x - y\| + \|M_m\| \|x - y\| + \|M_n y - M_m y\| \leq 2c_0\varepsilon + \|M_n y - M_m y\| \leq (2c_0 + 1)\varepsilon$
when $n, m \geq n_0$. It follows that $\forall x \in \mathbf{X}, \{M_n x : n \geq 1\}$ is a Cauchy sequence so it converges strongly. □

Principal of Uniform Boundedness for Maps and Compositions

Theorem 1.25 (PUB)

$\{M_\alpha : \alpha \in I\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$. $\forall x \in \mathbf{X}, \forall \ell \in \mathbf{Y}', |\langle M_\alpha x, \ell \rangle| \leq c(x, \ell), \forall \alpha \in I$. Then $\exists c_0 < \infty, \|M_\alpha\| \leq c_0, \forall \alpha \in I$ (or $\sup_{\alpha \in I} \|M_\alpha\| \leq c_0$).

Proof.

$\forall x \in \mathbf{X}, y_\alpha := M_\alpha x$. $\forall \ell \in \mathbf{Y}', \exists c(\ell)$, s.t. $|\langle y_\alpha, \ell \rangle| \leq c(x, \ell), \forall \alpha \in I$. It follows that $\exists c_1(x) < \infty$ s.t. $\|y_\alpha\| \leq c_1(x), \forall \alpha \in I$. Therefore, $\forall x \in \mathbf{X}, \exists c_1(x), \|M_\alpha x\| \leq c_1(x), \forall \alpha \in I$. Define $f_\alpha(x) := \|M_\alpha x\|$, then f_α is sub-additive, positive-homogeneous and continuous, and $|f_\alpha(x)| \leq c_1(x), \forall \alpha \in I, \forall x \in \mathbf{X}$. By applying PUB to $\{f_\alpha : \alpha \in I\}$, $\exists c_0 < \infty, |f_\alpha(x)| \leq c_0 \|x\|, \forall x \in \mathbf{X}, \forall \alpha \in I$. It follows that $\sup_{\alpha \in I} \|M_\alpha\| \leq c_0$. □

Remark 1.9

$\{M_n : n \geq 1\} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$, if $M_n \rightharpoonup M$, that is, $M_n x \rightharpoonup Mx, \forall x \in \mathbf{X}$, then $\forall x \in \mathbf{X}, \forall \ell \in \mathbf{Y}', \langle M_n x, \ell \rangle \to \langle Mx, \ell \rangle$. It follows that $\exists c(x, \ell), |\langle M_n x, \ell \rangle| \leq c(x, \ell), \forall n \geq 1$. By the above theorem. $\exists c_0 < \infty, \sup_n \|M_n\| \leq c_0$.

Principal of Uniform Boundedness for Maps and Compositions

- $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are BS, $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y}), N \in \mathcal{L}(\mathbf{Y}, \mathbf{Z})$. Then $NM\colon \mathbf{X} \to \mathbf{Z}$. Then $NM \in \mathcal{L}(\mathbf{X}, \mathbf{Z}), \|NM\| \le \|N\|\|M\|$. To see this,$\forall x \in \mathbf{X}, \|NMx\| \le \|N\|\|Mx\| \le \|N\|\|M\|\|x\| \Rightarrow NM$ is bounded, $\|NM\| \le \|N\|\|M\|$.
- $M'\colon \mathbf{Y}' \to \mathbf{X}', N'\colon \mathbf{Z}' \to \mathbf{Y}'$. Then $(NM)' = M'N'\colon \mathbf{Z}' \to \mathbf{X}'$. To see this, $\forall \ell \in \mathbf{Z}', \forall x \in \mathbf{X}$,
  $\langle x, (NM)'\ell\rangle = \langle NMx, \ell\rangle = \langle Mx, N'\ell\rangle = \langle x, M'N'\ell\rangle$

Open Map Principle

Theorem 1.26

$M : \mathbf{X} \to \mathbf{Y}$, $\mathbf{X}, \mathbf{Y}$ are BS. M is surjective (onto): $\forall y \in \mathbf{Y}, \exists x \in \mathbf{X}, Mx = y$. Then $\exists r > 0, B(0,r) \subseteq MB(0,1)$.

Definition 1.30 (Baire Principle)

$\mathbf{S}$ is a topological space, $\mathbf{S}$ satisfies Bair if $\forall \{G_n : n \geq 1, G_n \text{ is open and dense in } \mathbf{S}\}, \bigcap_{n\geq 1} G_n$ is dense in $\mathbf{S}$.

Theorem 1.27

$\mathbf{S}$ is a complete metric space, then $\mathbf{S}$ satisfies Baire.

Remark 1.10

$\mathbf{S}$ satisfies Bair, and $\{F_n : n \geq 1, F_n \text{ is closed}\}$, $\bigcup_{n\geq 1} F_n = \mathbf{S}$. Then $\exists m \geq 1, \exists$open set $G, G \subseteq F_m$. To see this, if $\forall n \geq 1, F_n$ does not contain any open set in $\mathbf{S}$. Then $G_n = F_n^c$ is open and dense in $\mathbf{X}$. By Baire, $G = \bigcap_{n\geq 1} G_n$ is dense in $\mathbf{S}$, so $G \neq \emptyset$, $G^c = \bigcup_{n\geq 1} F_n \subset \mathbf{S}$, contradicting with $\bigcup_{n\geq 1} F_n = \mathbf{S}$.

Open Map Principle

Theorem 1.28

$M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $\mathbf{X}, \mathbf{Y}$ are BS. M is surjective (onto):
$\forall y \in \mathbf{Y}, \exists x \in \mathbf{X}, Mx = y$. Then $\exists r > 0, B(0,r) \subseteq MB(0,1)$.

Proof.

Step 1: $\exists m \geq 1$, open $G \subseteq \mathbf{Y}, G \subseteq \overline{MB(0,m)}$.
Proof: $\mathbf{Y}$ is a complete metric space. Because M is surjective,
$\mathbf{Y} \subseteq \bigcup_{n\geq 1} MB(0.m) \subseteq \bigcup_{n\geq 1} \overline{MB(0.m)}$. By the Baire and the above remark,
$\exists m \geq 1, \exists$open set $G, G \subseteq \overline{MB(0,m)}$.
Step 2: $\exists m \geq 1, \exists r > 0, B(0,r) \subseteq \overline{MB(0,m)}$.
Proof: by Step 1, $\exists y \in \mathbf{Y}, \exists r > 0$ s.t. $B(y,r) \subseteq \overline{MB(0,m)}$.
$\exists x \in \mathbf{X}, y = Mx \Rightarrow B(Mx,r) \subseteq \overline{MB(0,m)} \Rightarrow B(0,r) \subseteq \overline{B(-x,m)}$. Note that
$B(-x,m) \subseteq B(0, \|x\| + m) \subseteq B(0, \|x\| + m)$. Choose $m' \geq \|x\| + m$, then
$B(-x,m) \subseteq B(0,m') \Rightarrow \overline{MB(-x,m)} \subseteq \overline{MB(0,m')}$. It follows that $B(0,r) \subseteq \overline{MB(0,m')}$. Set
$m' \to m$, we have Step 2 proved.
Step 3: $\exists s > 0, B(0,s) \subseteq \overline{MB(0,1)}$.
Proof: By Step 2, $B(0,r/\lambda) \subseteq \overline{MB(0,m/\lambda)}, \forall \lambda > 0$. Set $\lambda = m \Rightarrow B(0,s) \subseteq \overline{MB(0,1)}$ with
$s = r/m$. Furthermore, $\forall k \geq 1, B(0, s/2^k) \subseteq \overline{MB(0, 1/2^k)}$. □

Open Map Principle

Cont'd.

Step 4: $B(0,s) \subseteq MB(0,2)$
Proof: $\forall y \in B(0,s), \exists x_0 \in B(0,1), \|y - Mx_0\| < s/2$. Because $B(0,s/2) \subseteq \overline{MB(0,1/2)}, \exists x_1 \in B(0,1/2), \|y - Mx_0 - Mx_1\| < s/2^2$. Iteratively applying this process, $\forall k \geq 1, \exists x_t \in B(0,1/2^t), t \in [k-1], \left\|y - M(\sum_{t=0}^{k-1} x_t)\right\| < s/2^k$. Let $s_k = \sum_{t=0}^{k-1} x_t$, then $\{s_k\}$ is a Cauchy sequence, so $s_k \to x \in \mathbf{X}$, $\|x\| \leq \sum_{t\geq 0} \|x_t\| < 2$. Also, $\|y - Mx\| \leq \|y - Ms_k\| + \|Ms_k - Mx\| \leq s/2^k + \|Ms_k - Mx\| \Rightarrow \|y - Mx\| = 0, y = Mx \in MB(0,2)$.
Therefore, $B(0,s) \subseteq MB(0,2)$.
By Step 4, $B(0,s/2) \subseteq MB(0,1)$. Set $r = s/2$, $B(0,r) \subseteq MB(0,1)$. □

Theorem 1.29 (Open Map Theorem)

$M \in \mathcal{L}(\mathbf{X},\mathbf{Y})$, $\mathbf{X},\mathbf{Y}$ are BS, M is surjective. Then M maps open sets to open sets.

Open Map Principle

Cont'd.

$G \subseteq \mathbf{X}$ is an open set, we will prove that MG is open in $\mathbf{Y}$. $\forall y \in MG, \exists x \in \mathbf{X}, y = Mx$.
$\exists \varepsilon > 0, B(x, \varepsilon) \subseteq G$. Because $B(0, r) \subseteq MB(0, 1)$,
$B(0, r\varepsilon) \subseteq MB(0, \varepsilon) \Rightarrow y + B(0, r\varepsilon) = B(y, r\varepsilon) \subseteq Mx + MB(0, \varepsilon) = MB(x, \varepsilon) \subseteq MG$.
Therefore, MG is open. $\square$

Theorem 1.30

$M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $\mathbf{X}, \mathbf{Y}$ are BS, M is a bijection: $\forall y \in \mathbf{Y}, \exists x \in \mathbf{X}, y = Mx$ (surjective) and $Mx = My \Rightarrow x = y$ (injective). Then M^{-1} is bounded ($M^{-1} \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$).

Proof.

Because M is a bijection, M^{-1} exists and it is a linear map.
$\exists r > 0, B(0, r) \subseteq MB(0, 1)$.. $\forall y \in \mathbf{Y}, \|y\| = \frac{r}{2} \Rightarrow \exists x \in B(0, 1) \subseteq \mathbf{X}, y = Mx, x = M^{-1}y$.
$\forall z \in \mathbf{Y}, Z = \frac{z}{\|z\|}\frac{r}{2}$, then $\|Z\| = \frac{r}{2}$. $\exists x \in B(0, 1) \subseteq \mathbf{X}, Z = Mx \Rightarrow x = M^{-1}Z =$
$\frac{r}{2\|z\|}M^{-1}z \Rightarrow M^{-1}z = \frac{2\|z\|}{r}x, \left\|M^{-1}z\right\| \leq \frac{2\|z\|}{r}\|x\| \leq \frac{2\|z\|}{r} \Rightarrow \left\|M^{-1}\right\| \leq \frac{2\|z\|}{r}$. $\square$

Closed Graph Theorem

Definition 1.31

$M\colon \mathbf{X} \to \mathbf{Y}$, $\mathbf{X}, \mathbf{Y}$ are BS, $G = \{(x, Mx)\colon x \in \mathbf{X}\}$.
$\|(x, Mx)\| = \|x\| + \|Mx\|$ is the norm on G.

Definition 1.32

M is a closed operator if G is closed. That is, $\forall\, \{(x_n, Mx_n)\} \subseteq G$, $x_n \to x, Mx_n \to y \Rightarrow (x, y) \in G, y = Mx$. If $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ (then M is bounded), then M is a closed operator.

Remark 1.11

M is a closed operator $\Rightarrow$ G is a complete NLS, that is, G is BS.

Proof.

To see this, let $\{(x_n, Mx_n)\}$ is a Cauchy sequence, because $\|x_n - x_m\| \le \|(x_n, Mx_n) - (x_m, Mx_m)\|$, $\{x_n : n \ge 1\}$ is also a Cauchy sequence, so that $\exists x \in \mathbf{X}, x_n \to x$. Similarly, $\{Mx_n : n \ge 1\}$ is also a Cauchy sequence $\Rightarrow \exists y \in \mathbf{Y}, Mx_n \to y$. It follows that $(x_n, Mx_n) \to (x, y)$. Because G is close, $(x, y) \in G$ with $y = Mx$. Therefore, G is a complete NLS so G is a BS. $\square$

Closed Graph Theorem

Theorem 1.31 (Closed Graph Theorem)

$M: \mathbf{X} \to \mathbf{Y}$ is a Linear Map (LM), $\mathbf{X}, \mathbf{Y}$ are BS, then M is a closed operator $\Rightarrow$ M is bounded (that is, $M \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$).

Proof.

By the above remark, G is a BS. Define $A\colon G \to \mathbf{X}$, $A(x.Mx) = x$. Then A is a bijection: A is surjective, and $A(x_1, Mx_1) = A(x_2, Mx_2) \Rightarrow x_1 = x_2, Mx_1 = Mx_2$. Also, A is bounded: $\|A(x, Mx)\| = \|x\| \leq \|(x.Mx)\| = \|x\| + \|Mx\|$. Therefore, by the previous theorem, A^{-1} is bounded. It follows that $\left\|A^{-1}x\right\| = \|x\| + \|Mx\| \leq \left\|A^{-1}\right\| \|x\| \Rightarrow \|Mx\| \leq \left(\left\|A^{-1}\right\| - 1\right) \|x\|$. $\square$

Definition 1.33

$\mathbf{X}$ is a LS. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are compatible if $x_n \xrightarrow{\|\cdot\|_1} x, x_n \xrightarrow{\|\cdot\|_2} y \Rightarrow x = y$.

Closed Graph Theorem

Corollary 1.5

$\mathbf{X}$ is a LS, $\|\cdot\|_1$ and $\|\cdot\|_2$ are compatible. Then $(\mathbf{X}, \|\cdot\|_1), (\mathbf{X}, \|\cdot\|_2)$ are BS $\Rightarrow \exists 0 < c_1 < c_2 < \infty, \forall x \in \mathbf{X}, c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1$.

Proof.

Define $A\colon (\mathbf{X}, \|\cdot\|_1) \to (\mathbf{X}, \|\cdot\|_2), Ax = x$. Then the graph of A, $G = \{(x, x)\}$, is closed. Then A is bounded $\Rightarrow \exists c_2 < \infty, \forall x \in \mathbf{X}, \|Ax\|_2 = \|x\|_2 \leq c_2\|x\|_1$. Similarly, $\exists c_1 < \infty, \forall x \in \mathbf{X}, \|x\|_1 \leq \frac{1}{c_1}\|x\|_2$. □

- $\mathbf{X}$ is a BS, $\mathbf{X} = \mathbf{A} \bigoplus \mathbf{B}$, $\mathbf{A}, \mathbf{B}$ are closed subspaces of $\mathbf{X}$. $\forall x \in \mathbf{X}, \exists a \in \mathbf{A}, b \in \mathbf{B}$, s.t. $x = a + b$. Moreover. if $x = a_1 + b_1 = a_2 + b_2$ with $a_1, a_2 \in \mathbf{A}, b_1, b_2 \in \mathbf{B}$ $\Rightarrow a_1 = a_2, b_1 = b_2$.
- Define $P_{\mathbf{A}}(x)\colon \mathbf{X} \to \mathbf{A}, P_{\mathbf{A}}(x) := a$. $P_{\mathbf{A}}$ is linear, $P_{\mathbf{A}}b = 0, \forall b \in \mathbf{B}$, and $P_{\mathbf{A}}(a) = a, \forall a \in \mathbf{A}$. $P_{\mathbf{A}}^2 = P_{\mathbf{A}}, P_{\mathbf{A}}P_{\mathbf{B}} = 0, P_{\mathbf{B}}P_{\mathbf{A}} = 0$.

Closed Graph Theorem

Definition 1.34

$M : \mathbf{X} \to \mathbf{X}$ is a projection if $M^2 = M$.

Corollary 1.6

$\mathbf{X}$ is a BS, $\mathbf{X} = \mathbf{A} \bigoplus \mathbf{B}$, $\mathbf{A}, \mathbf{B}$ are closed subspaces of $\mathbf{X}$. Then $P_{\mathbf{A}}$ is bounded.

Proof.

We prove that the graph of $P_{\mathbf{A}}$, $G(P_{\mathbf{A}})$, is closed. Let $(x_n, P_{\mathbf{A}}(x_n)) \to (x, a)$. Then $x_n \to x$, $P_{\mathbf{A}}(x_n) \to a$. Let $x_n = a_n + b_n, a_n \in \mathbf{A}, b_n \in \mathbf{B}$. $x_n \to x$, $P_{\mathbf{A}}(x_n) = a_n \to a$. $\mathbf{A}, \mathbf{B}$ are closed $\Rightarrow a \in \mathbf{A}, \exists b \in \mathbf{B}$ s.t. $b_n \to b$. It follows that $x = a + b$, $P_{\mathbf{A}}(x) = a, (x, a) \in G(P_{\mathbf{A}})$. Therefore, $G(P_{\mathbf{A}})$ is closed $\Rightarrow P_{\mathbf{A}}$ is bounded by the Closed Graph Theorem. $\square$

Examples of Bounded Linear Maps: Integral Operators

- Integral operator: $(\mathbf{S}_j, \mathcal{B}_j, \mu_j)$, $\mu(\mathbf{S}_j) < \infty$, $j = 1, 2$. $p \in [1, \infty], L^p(\mu_j) = \left\{ f \text{ measurable } : \int_{\mathbf{S}_j} |f|^p \, \mathrm{d}\mu_j < \infty \right\}$. When $p = \infty$, $L^\infty = \{ f \text{ measurable } : \|f\|_\infty = \text{ess sup}\, |f| < \infty \}$. $\|f\|_p = \left(\int_{\mathbf{S}_j} |f|^p \, \mathrm{d}\mu_j \right)^{\frac{1}{p}}$.
- $C_b(\mathbf{S}_j)$ is the set of bounded and continuous functions defined on $\mathbf{S}_j$. Because $\mu(\mathbf{S}_j) < \infty$, $C_b(\mathbf{S}_j) \subseteq L^p(\mu_j)$.
- $A \colon L^p(\mu_1) \to L^q(\mu_2)$.
- Function $K \colon \mathbf{S}_1 \times \mathbf{S}_2 \to \mathbb{C}$, define $(Af)(s) = \int_{\mathbf{S}_1} K(t,s) f(t) \mu_1(\mathrm{d}t), f \in L^p(\mu_1), s \in \mathbf{S}_2$.
- Case 1: $A \colon L^1(\mu_1) \to L^\infty(\mu_2), f \in L^1(\mu_1)$.

$$
\begin{aligned}
\|Af\|_\infty &= \sup_{s \in \mathbf{S}_2} |(Af)(s)| = \sup_{s \in \mathbf{S}_2} \left| \int_{\mathbf{S}_1} K(t,s) f(t) \mu_1(\mathrm{d}t) \right| \\
&\leq \underbrace{\sup_{s \in \mathbf{S}_2} \sup_{t \in \mathbf{S}_1} |K(t,s)|}_{c_0} \|f\|_1,
\end{aligned}
$$

that is, $\|A\| \leq c_0$.

Examples of Bounded Linear Maps: Integral Operators

- Case 2: $A\colon L^\infty(\mu_1) \to L^1(\mu_2)$, $f \in L^\infty(\mu_1)$.

$$\|Af\|_1 = \int_{\mathbf{S}_2} |(Af)(s)|\,\mu_2(\mathrm{d}s) = \int_{\mathbf{S}_2} \left|\int_{\mathbf{S}_1} K(t,s)f(t)\mu_1(\mathrm{d}t)\right| \mu_2(\mathrm{d}s)$$

$$\leq \underbrace{\int_{\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)|\,\mu_1(\mathrm{d}t)\mu_2(\mathrm{d}s)}_{c_0} \cdot \sup_{t\in\mathbf{S}_1} |f(t)| = c_0\|f\|_\infty$$

that is, $\|A\| \leq c_0$.

- Case 3: $A\colon L^2(\mu_1) \to L^2(\mu_2)$, $f \in L^2(\mu_1)$.

$$\|Af\|_2^2 = \int_{\mathbf{S}_2} |(Af)(s)|^2\,\mu_2(\mathrm{d}s) = \int_{\mathbf{S}_2} \left|\int_{\mathbf{S}_1} K(t,s)f(t)\mu_1(\mathrm{d}t)\right|^2 \mu_2(\mathrm{d}s)$$

$$\leq \int_{\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)|^2\,\mu_1(\mathrm{d}t) \cdot \int_{\mathbf{S}_1} |f(t)|^2\,\mu_1(\mathrm{d}t)\ \mu_2(\mathrm{d}s)$$

$$= \underbrace{\int_{\mathbf{S}_2}\int_{\mathbf{S}_1} K(t,s)^2\mu_1(\mathrm{d}t)\mu_2(\mathrm{d}s)}_{c_0^2}\,\|f\|_2^2,$$

that is, $\|A\| \leq c_0$.

Examples of Bounded Linear Maps: Integral Operators

- Case 4: $A\colon L^2(\mu_1) \to L^2(\mu_1)$, $f \in L^2(\mu_1)$.
  $\|Af\|_2 = \sup_{h\in L^2(\mu_1), \|h\|_2=1} |\langle Af, h\rangle|$. We have

$$|\langle Af, h\rangle| = \left|\int_{\mathbf{S}_2} (Af)(s)h(s)\mu_2(\mathrm{d}s)\right| = \left|\int_{\mathbf{S}_2}\int_{\mathbf{S}_1} K(t,s)f(t)\mu_1(\mathrm{d}t)\, h(s)\mu_2(\mathrm{d}s)\right|$$

$$\leq \int_{\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)f(t)h(s)|\,\mu_1(\mathrm{d}t)\mu_2(\mathrm{d}s)$$

$$\leq \int_{\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)|\left(\frac{\gamma f^2(t)}{2} + \frac{h^2(s)}{2\gamma}\right)\mu_1(\mathrm{d}t)\mu_2(\mathrm{d}s) \quad (\text{ for any } A > 0)$$

$$= \frac{\gamma}{2}\int_{\mathbf{S}_1}\int_{\mathbf{S}_2} |K(t,s)|\, f^2(t)\mu_2(\mathrm{d}s)\mu_1(\mathrm{d}t) + \frac{1}{2\gamma}\int_{\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)|\, h^2(s)\mu_1(\mathrm{d}t)\mu_2(\mathrm{d}s)$$

$$\leq \frac{\gamma}{2}\|f\|_2^2 \sup_{t\in\mathbf{S}_1}\int_{\mathbf{S}_2} |K(t,s)|\,\mu_2(\mathrm{d}s) + \frac{\|h\|_2^2}{2\gamma}\sup_{s\in\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)|\,\mu_1(\mathrm{d}t) \quad (\|h\|_2 = 1)$$

$$= \frac{\gamma}{2}\|f\|_2^2 \sup_{t\in\mathbf{S}_1}\int_{\mathbf{S}_2} |K(t,s)|\,\mu_2(\mathrm{d}s) + \frac{1}{2\gamma}\sup_{s\in\mathbf{S}_2}\int_{\mathbf{S}_1} |K(t,s)|\,\mu_1(\mathrm{d}t)$$

$$= \frac{\gamma}{2}\|f\|_2^2 c_1 + \frac{1}{2\gamma}c_2$$

Examples of Bounded Linear Maps: Integral Operators

- Now the RHS is minimized with $\gamma = \sqrt{c_2/(c_1\|f\|_2)}$, and when the RHS is minimized, $|\langle Af, h\rangle| \le \sqrt{c_1c_2}\|f\|_2$. It follows that $\|A\| \le \sqrt{c_1c_2}$.

Symmetric Operators

Definition 1.35

$\mathbf{X}$ is a HS with $\mathbb{K}$, $D(A) \subseteq \mathbf{X}$ is a linear subspace of $\mathbf{X}$, $A\colon D(A) \to \mathbf{X}$. A is symmetric if (a) $D(A)$ is dense; (b) $\forall x, y \in D(A)$, $\langle Ax, y\rangle = \langle x, Ay\rangle$.

Definition 1.36

$\lambda \in \mathbb{K}$ is an eigenvalue of A if $\exists x \neq 0, x \in \mathbf{X}, Ax = \lambda x$. x is called the eigenfunction.

Proposition 1.13

$A\colon D(A) \to \mathbf{X}$ is a symmetric operator. Then

(1) $\langle Ax, x\rangle \in \mathbb{R}, \forall x \in D(A)$. To see this, note that $\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$.

(2) $\lambda \in \mathbb{K}$ is an eigenvalue $\Rightarrow \lambda \in \mathbb{R}$. To see this, $\langle Ax, x\rangle = \lambda \|x\|^2 = \langle x, Ax\rangle = \overline{\lambda}\|x\|^2 \Rightarrow \lambda = \overline{\lambda}$.

Symmetric Operators

Proposition 1.14 (Cont'd)

(3) λ_1, λ_2 are eigenvalues of A, $\lambda_1 \neq \lambda_2, Ax_i = \lambda_i x_i, i \in [2], x_1 \neq 0, x_2 \neq 0, \Rightarrow \langle x_1, x_2 \rangle = 0$. To see this, $\langle Ax_1, x_2 \rangle = \langle x_1, Ax_2 \rangle \Rightarrow \lambda_1 \langle x_1, x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$, $(\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle = 0 \Rightarrow \langle x_1, x_2 \rangle = 0$.

(4) $\{x_j : j \geq 1\}$ is an orthonormal basis of $\mathbf{X}$, x_j is an eigenfunction and $Ax_j = \lambda_j x_j, \forall j \geq 1$. Then if μ is an eigenvalue of A, $\mu = \lambda_j$ for some $j \geq 1$.

To see this, let $y \neq 0$ be the eigenfunction corresponding to μ: $Ay = \mu y$. Suppose $\mu \neq \lambda_j, \forall j \geq 1 \Rightarrow \langle y, x_j \rangle = 0, \forall j \geq 1$. Since $\{x_j : j \geq 1\}$ is an orthonormal basis of $\mathbf{X}$, $y = \sum\limits_{j \geq 1} \theta_j x_j$, $\sum\limits_{j \geq 1} |\theta_j|^2 < \infty$.

Then $\langle y, x_j \rangle = 0, \forall j \geq 1 \Rightarrow \theta_j = 0, \forall j \geq 1$. Therefore, $y = 0$. The contradiction shows that $\mu = \lambda_j$ for some $j \geq 1$.

Alternatively,

$$\langle y, x_j \rangle = 0, \forall j \geq 1 \Rightarrow \|y\|^2 = \lim_{n \to \infty} \left\langle y, \sum_{j=1}^{n} \theta_j x_j \right\rangle = 0 \Rightarrow y = 0.$$

Symmetric Operators

- $\mathbf{X}$ is a HS with $\mathbb{K}$, $D(A) \subseteq \mathbf{X}$ is a linear subspace of $\mathbf{X}$, $A\colon D(A) \to \mathbf{X}$, A is symmetric and bounded. $D(A)$ is dense in $\mathbf{X}$. We can extend A from $D(A)$ to $\mathbf{X}$ while A is still symmetric and bounded.
- $\|A\| = \sup_{x\in\mathbf{X},\|x\|=1} \|Ax\|$.

Proposition 1.15

A is bounded and symmetric, then $\|A\| = \sup_{x\in\mathbf{X},\|x\|=1} |\langle Ax, x\rangle|$.

Proof.

Let $M = \sup_{x\in\mathbf{X},\|x\|=1} |\langle Ax, x\rangle|$. Then $M \le \|Ax\|\,\|x\| \le \|A\|\,\|x\|^2 = \|A\|$.
To prove $\|A\| \le M$, we first define $x_+ = \lambda x + \frac{1}{\lambda}Ax, x_- = \lambda x - \frac{1}{\lambda}Ax, \lambda > 0$. Then $x = \frac{x_+ + x_-}{2\lambda}, Ax = \frac{\lambda}{2}(x_+ - x_-)$.

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \left\langle A^2x, x\right\rangle = \left\langle A\frac{\lambda}{2}(x_+ - x_-), \frac{x_+ + x_-}{2\lambda}\right\rangle$$

$$= \frac{1}{4}\left(\langle Ax_+, x_+\rangle + \langle Ax_+, x_-\rangle - \langle Ax_-, x_+\rangle - \langle Ax_-, x_-\rangle\right).$$

$\square$

Symmetric Operators

Cont'd.

$$
\begin{aligned}
&= \frac{1}{4}\left(\langle Ax_+, x_+\rangle + \langle Ax_+, x_-\rangle - \overline{\langle Ax_+, x_-\rangle} - \langle Ax_-, x_-\rangle\right) \quad (\langle Ax_+, x_-\rangle - \overline{\langle Ax_+, x_-\rangle} = 0)\\
&= \frac{1}{4}\left(\langle Ax_+, x_+\rangle - \langle Ax_-, x_-\rangle\right)\\
&\leq \frac{1}{4}\left(M\|x_+\|^2 + M\|x_-\|^2\right)\\
&\leq \frac{M}{4}\left(\left\langle \lambda x + \frac{1}{\lambda}Ax, \lambda x + \frac{1}{\lambda}Ax\right\rangle + \left\langle \lambda x - \frac{1}{\lambda}Ax, \lambda x - \frac{1}{\lambda}Ax\right\rangle\right)\\
&= \frac{M}{4}\left(2\lambda^2\|x\|^2 + \frac{2}{\lambda^2}\|Ax\|^2\right) = \frac{M}{2}\left(\lambda^2\|x\|^2 + \frac{1}{\lambda^2}\|Ax\|^2\right)\\
&\leq \frac{M}{2}\inf_{\gamma>0}\left(\gamma\|x\|^2 + \frac{1}{\gamma}\|Ax\|^2\right) = M\|x\|\,\|Ax\|.
\end{aligned}
$$

It follows that $\|Ax\|^2 \leq M\|x\|\,\|Ax\| \Rightarrow \|Ax\| \leq M\|x\|$, so that $\|A\| \leq M$. □

Eigenvalues of Compact Symmetric Operators

- $\mathbf{X}, \mathbf{Y}$ are Normed Linear Spaces (NLS). $A\colon M \subseteq \mathbf{X} \to \mathbf{Y}$.

Definition 1.37

A is compact if (1) A is continuous; (b) $\forall \{x_n : n \geq 1\} \subseteq \mathbf{X}$ which is bounded ($\exists c_0 < \infty, \sup_{n\geq 1} \|x_n\| \leq c_0$) $\Rightarrow \{Ax_n : n \geq 1\}$ is relative compact, that is, $\exists \{n_k\}$ s.t. $Ax_{n_k} \to y \in \mathbf{Y}$.

Example 1.7

$-\infty < a < b < \infty, C[a,b] = \mathbf{X} = \mathbf{Y}$. $F\colon [a,b]^2 \times [-M,M] \to \mathbb{R}$, and F is continuous. Define $A\colon \mathbf{X} \to \mathbf{X}, \forall x \in \mathbf{X}, \|x\|_\infty \leq M$,
$(Ax)(t) = \int_a^b F(s,t,x(s))\mathrm{d}s$. Then $Ax \in C[a,b]$ and A is compact.
To see this, due to the uniform continuity of F, $Ax \in C[a,b]$.
$\forall \{x_n : n \geq 1\} \subseteq \mathbf{X}, \sup_n \|x\|_\infty \leq c_0$, it can be proved that $\{Ax_n : n \geq 1\}$ is equicontinuous by the uniform convergence of F. Then by the Arzela-Ascoli theorem, $\exists \{n_k\}$ s.t. $Ax_{n_k} \to y \in C[a,b]$. Also, $\forall \{x_n : n \geq 1\}, x_n \to x$, it can be proved that $Ax_n \to Ax$ by the uniform continuity of F. Therefore, A is compact.

Eigenvalues of Compact Symmetric Operators

- $\mathbf{X}, \mathbf{Y}$ are HS. $A\colon D(A) \to \mathbf{X}$, $D(A) \subseteq \mathbf{X}$ is a linear subspace. Suppose A is symmetric and compact $\Rightarrow$ A is linear and continuous $\Rightarrow$ A is bounded. By symmetricity of A, $D(A)$ is dense in $\mathbf{X}$, so A can be extended from $D(A)$ to $\mathbf{X}$ in a unique way. In the sequel, we let $A\colon \mathbf{X} \to \mathbf{X}$ when A is a symmetric and compact operator.
- $\mathbf{X}$ is a HS. $A\colon \mathbf{X} \to \mathbf{X}$ is compact and symmetric, $\langle Ax, y\rangle = \langle x, Ay\rangle, \forall x, y \in \mathbf{X}$.

Eigenvalues of Compact Symmetric Operators

Theorem 1.32

$\mathbf{X}$ is a separable HS, $A\colon \mathbf{X} \to \mathbf{X}$ is compact and symmetric. Then

(1) $\exists$ orthonormal basis $\{x_j : j \geq 1\}$, x_j is an eigenfunction of A, $\forall j \geq 1$.

(2) Let λ_j be eigenvalue associated with x_j. Then $\lambda_j \neq \lambda_k \Rightarrow \langle x_j, x_k \rangle = 0$.

(3) λ is an eigenvalue of A, then λ has finite multiplicity: $\dim(\{x\colon Ax = \lambda x\}) < \infty$.

(4) $\dim(\mathbf{X}) = \infty \Rightarrow$ there are finite number of nonzero eigenvalues, and $\lim\limits_{j\to\infty} \lambda_j = 0$.

Proof.

We first prove with the assumption (H): $Ax = 0 \Rightarrow x = 0$.
Suppose $\mathbf{X} \neq \{0\}$, then $\exists x \neq 0, \|Ax\| > 0 \Rightarrow \|A\| > 0$.

□

Eigenvalues of Compact Symmetric Operators

Cont'd.

Step 1:

Claim 1.14

$\exists x_1 \in \mathbf{X}, \|x_1\| = 1$ s.t. $Ax_1 = \lambda_1 x_1, |\lambda_1| = \|A\|$.

Proof.

We have $\infty > \|A\| = \sup_{x \in \mathbf{X}, \|x\|=1} |\langle Ax, x\rangle|$. $\exists \left\{ z_n : \|z_n\| = 1, n \geq 1 \right\} \subseteq \mathbf{X}$ s.t. $|\langle Az_n, z_n\rangle| \to \|A\|$. Then $\exists \{n_k\}$ s.t. $\langle Az_{n_k}, n_k\rangle \to \|A\| = \lambda_1$ (or $-\lambda_1$), and we still denote $\{n_k\}$ by n for simplicity. We have $\langle Az_n, z_n\rangle \to \|A\| = \lambda$ or $\langle Az_n, z_n\rangle \to -\|A\| = -\lambda$. Assume that $\langle Az_n, z_n\rangle \to \|A\| = \lambda$. Now we will prove that $\lambda_1 z_n - Az_n \to 0$.

$$\begin{aligned} \|\lambda z_n - Az_n\|^2 &= \lambda^2 \langle z_n, z_n\rangle + \|Az_n\|^2 - 2\lambda \langle Az_n, z_n\rangle \\ &\leq 2\lambda^2 \|z_n\|^2 - 2\lambda \langle Az_n, z_n\rangle = 2\lambda^2 - 2\lambda \langle Az_n, z_n\rangle \to 0. \end{aligned}$$

It follows that $\lambda z_n - Az_n \to 0$. Because $\{z_n : n \geq 1\}$ is bounded and A is a compact operator, $\exists n_k, y \in \mathbf{X}$ s.t. $Az_{n_k} \to y$. It follows that $\lambda z_{n_k} \to y, \lambda > 0 \Rightarrow z_{n_k} \to \frac{y}{\lambda} := x_1$. Because A is compact so that A is bounded, $Az_{n_k} \to Ax_1$,
$\lambda z_{n_k} - Az_{n_k} \to 0 \Rightarrow Ax_1 = \lambda x_1, x_1 \in \mathbf{X}, \|x_1\| = \lim_{k\to\infty} \|z_{n_k}\| = 1$.
When $\langle Az_n, z_n\rangle \to -\|A\| = -\lambda$, by the above argument, $\exists x_1 \in \mathbf{X}, \|x_1\| = 1$ s.t. $Ax_1 = -\lambda x_1$. Therefore, $\exists x_1 \in \mathbf{X}, \|x_1\| = 1, \lambda_1 \in \mathbb{R}$ s.t. $Ax_1 = \lambda_1 x_1$, $\lambda_1 = \|A\|$ or $-\|A\|$. □

Eigenvalues of Compact Symmetric Operators

Cont'd.

Step 2: $L(x_1)$ is the Linear Space spanned by x_1. $L(x_1)^\perp = \{y \in \mathbf{X} \colon \langle y, x_1 \rangle = 0\}$.

Claim 1.15

$A_1 \colon L(x_1)^\perp \to L(x_1)^\perp$, $A_1 y = Ay, \forall y \in L(x_1)^\perp$. Then $A_1 L(x_1)^\perp \subseteq L(x_1)^\perp$. A_1 is compact and symmetric.

Proof.

$\forall y \in L(x_1)^\perp$, $A_1 y = Ay$. Because $\langle A_1 y, x_1 \rangle = \langle Ay, x_1 \rangle = \langle y, Ax_1 \rangle = \overline{\lambda} \langle y, x_1 \rangle = 0$, it follows that $A_1 y \in L(x_1)^\perp \Rightarrow A_1 L(x_1)^\perp \subseteq L(x_1)^\perp$.
A_1 is symmetric because A is symmetric. Let $\{x_n \colon n \geq 1\} \subseteq \mathbf{X}, \sup_{n \geq 1} \|x_n\| \leq c_0$. Because A is compact, $\exists \{n_k\}, \exists y \in \mathbf{X}$ s.t. $A_1 x_{n_k} = Ax_{n_k} \to y$. Because $\{A_1 x_{n_k}\} \subseteq L(x_1)^\perp$ and $L(x_1)^\perp$ is closed, $y \in L(x_1)^\perp$. Also, A_1 is bounded because A is bounded. It follows that A_1 is compact and symmetric. □

Assume that $\dim(\mathbf{X}) = \infty$, then $L(x_1)^\perp \neq \{0\}$ (otherwise $\mathbf{X} = L(x_1)$ so $\mathbf{X}$ has dimension 1). Also, $A_1 x = 0 \Rightarrow Ax = 0 \Rightarrow x = 0$. $L(x_1)^\perp$ is closed so it is complete, ad $L(x_1)^\perp$ is a HS. Therefore, we apply Step 1to A_1 and $L(x_1)^\perp$, $\exists x_2 \in L(x_1)^\perp$, $\|x_2\| = 1$ s.t. $A_1 x_2 = \lambda_2 x_2$,
$0 < |\lambda_2| = \|A_2\| = \sup_{z \in \mathbf{X}, z \in L(x_1)^\perp} \|Az\| \leq \|A\| = |\lambda_1|$.
Iteratively applying the above process, $\exists \{x_n \colon n \geq 1\}$, $\|x_n\| = 1, Ax_n = \lambda_n x_n, \forall n \geq 1$. Also, $\langle x_n, x_k \rangle = 0$ when $k \neq n$. $0 < \ldots < |\lambda_j| \leq |\lambda_{j-1}| \leq \ldots \leq |\lambda_1| = \|A\|$.
Step 3: we will prove $\lambda_n \to 0$. Assume that $|\lambda_n| \geq \varepsilon > 0, \forall n \geq 1$. It follows that $\frac{\|x_n\|}{\lambda_n} \leq \frac{1}{|\lambda_n|}$. □

Eigenvalues of Compact Symmetric Operators

Cont'd.

Since A is compact, $\left\{A\frac{x_n}{\lambda_n}\right\} = \{x_n : n \geq 1\}$ is relative compact $\Rightarrow \exists\, \{n_k\}$ s.t. $\{x_{n_k}\}$ converges. However, $\|x_n - x_m\| = \sqrt{2}$ due to $\langle x_n, x_m\rangle = 0$ for $n \neq m$, so such $\{n_k\}$ cannot exist. Therefore, $|\lambda_n| \to 0 \Rightarrow \lambda_n \to 0$.

Step 4: $\forall x \in \mathbf{X}$, $Ax = \sum_{j\geq 1} \lambda_j \langle x, x_j\rangle x_j$. Let $z_k = x - \sum_{j=1}^{k} \langle x, x_j\rangle x_j$, $m \leq k$, then

$\langle z_k, x_m\rangle = \langle x, x_m\rangle - \sum_{j=1}^{k} \langle x, x_j\rangle \langle x_j, x_m\rangle = 0$. It follows that $z_k \in L(x_1, \ldots, x_k)^{\perp}$. Also,

$\|z_k\| \leq \|x\| + \left\|\sum_{j=1}^{k} \langle x, x_j\rangle x_j\right\| \leq 2\|x\|$. We have $\|Az_k\| = \|A_k z_k\| \leq 2\|A_k\|\,\|x\| =$

$2\,|\lambda_k|\,\|x\| \to 0 \Rightarrow Az_k \to 0 \Rightarrow Ax - A\left(\sum_{j=1}^{k} \langle x, x_j\rangle x_j\right) \to 0 \Rightarrow A\left(\sum_{j=1}^{k} \langle x, x_j\rangle x_j\right) \to Ax$,

that is, $\sum_{j=1}^{k} \lambda_j \langle x, x_j\rangle x_j \to Ax$.

Step 5: Now we prove that $x = \sum_{j\geq 1} \langle x, x_j\rangle x_j$. To see this, let $y = \sum_{j\geq 1} \langle x, x_j\rangle x_j$. Then

$Ay = \sum_{j\geq 1} \lambda_j \langle x, x_j\rangle x_j = Ax \Rightarrow A(y - x) = 0 \Rightarrow x = y = \sum_{j\geq 1} \langle x, x_j\rangle x_j$.

Step 6: λ is an eigenvalue of A, then λ has finite multiplicity: $\dim(\{x : Ax = \lambda x\}) < \infty$.

Eigenvalues of Compact Symmetric Operators

Cont'd.

To see this, $\exists x \neq 0$ s.t. $Ax = \lambda x$. Note that $x = \sum_{j \geq 1} \langle x, x_j \rangle x_j$.

$Ax = \sum_{j \geq 1} \lambda_j \langle x, x_j \rangle x_j = \lambda x = \sum_{j \geq 1} \lambda \langle x, x_j \rangle x_j$. It follows that $\sum_{j \geq 1} (\lambda - \lambda_j) \langle x, x_j \rangle x_j = 0$.

Therefore, $\langle x, x_j \rangle = 0, \forall j$ s.t. $\lambda_j \neq \lambda$. Therefore, $x = \sum_{j \geq 1} \langle x, x_j \rangle x_j = \sum_{j \geq 1:\ \lambda_j = \lambda} \langle x, x_j \rangle x_j$.

Because $\lambda_j \to 0$, there can only be finite number of elements in $\{\lambda_j : j \geq 1, \lambda_j = \lambda\}$. It follows that x lie in the span of finite set $\{x_j : j \geq 1, \lambda_j = \lambda\}$.

Note that the above proof holds with the assumption (H): $Ax = 0 \Rightarrow x = 0$ and $\dim(\mathbf{X}) = \infty$. Now if $\dim(\mathbf{X}) = m < \infty$,then repeating Step 2, $\exists \{x_n : n \in [m]\}$ which is a finite set, $\|x_n\| = 1, Ax_n = \lambda_n x_n, \forall n \in [m]$. Also, $\langle x_n, x_k \rangle = 0$ when $k \neq n$.

$0 < |\lambda_m| \leq |\lambda_{m-1}| \leq \ldots \leq |\lambda_1| = \|A\|$. $\forall x \in \mathbf{X}, x = \sum_{j \in [m]} \langle x, x_j \rangle x_j$.

Now let $N(A) = \{x \in \mathbf{X} \colon Ax = 0\}$. It can be verified by definition that $N(A)$ is a CLS by the linearality and continuity of A. Because $\mathbf{X}$ is separable, $N(A)$ as a CLS of $\mathbf{X}$ is also separable. As a separable HS, $\mathbf{X}$ has a countable orthonormal basis $\{w_j : k \geq 1\}$ (the orthonormal basis can be finite if $\dim(N(A)) < \infty$) s.t. $\|w_j\| = 1, \forall j \geq 1, \langle w_j, w_k \rangle = 0, \forall j \neq k$. $y = \sum_{j \geq 1} \langle y, w_j \rangle w_j$.

Decompose $\mathbf{X}$ by $\mathbf{X} = N(A) \bigoplus N(A)^{\perp}$ because $N(A)$ is CLS. $N(A)^{\perp}$ is also a CLS of $\mathbf{X}$.

We prove that $AN(A)^{\perp} \subseteq N(A)^{\perp}$. To see this, $letx \in N(A)^{\perp}$ and $y \in N(A)$, then $\langle Ax, y \rangle = \langle x, Ay \rangle = 0 \Rightarrow Ax \in N(A)^{\perp}$. Define $A_{\perp} : N(A)^{\perp} \to N(A)^{\perp}$ which is the restriction of A on $N(A)^{\perp}$. Because $N(A)^{\perp}$ is a CLS of the separable HS $\mathbf{X}$, $N(A)^{\perp}$ is a separable HS. $A_{\perp}$ is symmetric and compact because it is the restriction of A on $N(A)^{\perp}$, and $A_{\perp} x = 0 \Rightarrow x = 0$. This is because if $x \in N(A)^{\perp}$ and $A_{\perp} x = Ax = 0 \Rightarrow x \in N(A) \Rightarrow x = 0$. □

Eigenvalues of Compact Symmetric Operators

Cont'd.

When dim$(N(A)^{\perp}) < \infty$, we can apply the previous proof to $A_{\perp}$ on $N(A)^{\perp}$ and construct $\{x_n : n \in [m]\}$ which are eigenfunctions with associated eigenvalues $\{\lambda_n : n \in [m]\}$ with properties in the previous proof. Then $\{x_n : n \in [m]\} \bigcup \{w_j : j \geq 1\}$ is an orthonorma basis of $\mathbf{X}$. We can see that $\{w_j : j \geq 1\}$ are eigenfunctions associated with eigenvalue 0.
When dim$(N(A)^{\perp}) = \infty$. By the previous proof we have $\{x_n : n \geq 1\}$, and $\{x_n : n \geq 1\} \bigcup \{w_j : j \geq 1\}$ are an orthonormal basis of $\mathbf{X}$. $\square$

The Fredholm Alternative

- $\mathbf{X}$ is a separable HS. $A\colon \mathbf{X} \to \mathbf{X}$ is symmetric and compact. It means that A is linear, $\langle Ax, y\rangle = \langle x, Ay\rangle, \forall x, y \in \mathbf{X}$, $\forall \{x_n\colon n \geq 1\} \subseteq \mathbf{X}, \sup_{n\geq 1} \|x_n\| \leq c_0 \Rightarrow \{Ax_n\colon n \geq 1\}$ is relative compact (containing a convergent subsequence).
- $z \in \mathbf{X}, \lambda \in \mathbb{K}$, consider the equation $\lambda x - Ax = z$.
- (Homogeneous Equation) $\lambda x - Ax = 0$. Define $N_\lambda = \{x \in \mathbf{X}\colon \lambda x - Ax = 0\}$. $\lambda - A\colon \mathbf{X} \to \mathbf{X}$ is an operator, and N_λ is the null space of $\lambda - A$. N_λ is a CLS of $\mathbf{X}$.

Theorem 1.33

Fix $\lambda \neq 0, z \in \mathbf{X}$. Then $\lambda x - Ax = z$ (EQz) has solution $\iff$ $z \in N_\lambda^\perp$.

The Fredholm Alternative

Proof.

Let $x \in \mathbf{X}$, $N(A)$ be the null space of A. Because $\mathbf{X}$ is a separable HS and A is symmetric and compact, $\exists\, \{x_j : j \geq 1\}$ which is an orthonormal basis of $N(A)^{\perp}$ and they are eigenfunctions of A associated with eigenvalues $\{\lambda_j : j \geq 1\}$. Also, $\exists\, \{w_j : j \geq 1\}$ which is an orthonormal basis of $N(A)$. $x = \sum_{j\geq 1} \langle x, x_j \rangle x_j + \sum_{j \geq 1} \langle x, w_j \rangle w_j$. Depending on dim$(N(A))$ and dim$(N(A)^{\perp})$, the two summations, $\sum_{j\geq 1} \langle x, x_j \rangle x_j$ and $\sum_{j\geq 1} \langle x, w_j \rangle w_j$, can be finite summations or infinite summations.

Case 1: $\lambda \neq \lambda_j, \forall j \geq 1$. Then $(\lambda - A)x = \sum_{j\geq 1} (\lambda - \lambda_j) \langle x, x_j \rangle x_j + \sum_{j\geq 1} \lambda \langle x, w_j \rangle w_j$. Because $z \in \mathbf{X}$, $z = \sum_{j\geq 1} \langle z, x_j \rangle x_j + \sum_{j\geq 1} \langle z, w_j \rangle w_j$. Then $(\lambda - A)x = z \Rightarrow (\lambda - \lambda_j) \langle x, x_j \rangle = \langle z, x_j \rangle , \lambda \langle x, w_j \rangle = \langle z, w_j \rangle \Rightarrow \langle x, x_j \rangle = \frac{\langle z, x_j \rangle}{\lambda - \lambda_j}, \langle x, w_j \rangle = \frac{\langle z, w_j \rangle}{\lambda}$. As a result, $x = \sum_{j\geq 1} \langle x, x_j \rangle x_j + \sum_{j\geq 1} \langle x, w_j \rangle w_j = \sum_{j\geq 1} \frac{\langle z, x_j \rangle}{\lambda - \lambda_j} x_j + \sum_{j\geq 1} \frac{\langle z, w_j \rangle}{\lambda} w_j$.

Now we prove that $x \in \mathbf{X}$. To see this, $\sum_{j\geq 1} \frac{|\langle z, w_j \rangle|^2}{\lambda^2} \leq \frac{\|z\|^2}{\lambda^2}$ by Bessel inequality, and $\sum_{j\geq 1} \frac{|\langle z, x_j \rangle|^2}{(\lambda - \lambda_j)^2} \leq \sup_{j\geq 1} \frac{1}{(\lambda - \lambda_j)^2} \cdot \|z\|^2 \leq c_1 \|z\|^2$ because $\lambda_j \to 0$. It follows that $x \in \mathbf{X}$, and it can be checked that such x is a solution to (EQz) by plugging x in EQz. We consider the Homogeneous equaiton $\lambda x - Ax = 0$ (HOM). Again, let $x = \sum_{j\geq 1} \langle x, x_j \rangle x_j + \sum_{j\geq 1} \langle x, w_j \rangle w_j$. □

The Fredholm Alternative

Cont'd.

Then $(\lambda - A)x = \sum_{j\geq 1}(\lambda - \lambda_j)\langle x, x_j\rangle x_j + \sum_{j\geq 1}\lambda\langle x, w_j\rangle w_j = 0 \Rightarrow (\lambda - \lambda_j)\langle x, x_j\rangle = 0, \forall j \geq 1; \langle x, w_j\rangle = 0, \forall j \geq 1$. Because $\lambda \neq \lambda_j, \forall j \geq 1$, it follows that $\langle x, x_j\rangle = 0, \forall j \geq 1$. Therefore, $x = 0$.

Under Case 1, We proved that $\exists!$ solution to (EQz) and (HOM), and $N_\lambda = \{0\}$. Therefore, $N_\lambda^\perp = \mathbf{X}$, and (EQz) has solution $\iff$ $z \in N_\lambda^\perp = \mathbf{X}$.

Case 2: $\lambda = \lambda_{j_0}$ for some $j_0 \geq 1$. Then $N_\lambda = N_{\lambda_{j_0}} = \{x \in \mathbf{X}\colon \lambda_{j_0}x = Ax\}$ is a finite dimensional space, and there exists $\left\{x_{j'} : j' \in [p, q]\right\}$ such that $N_\lambda =$ CLS $\left\{x_{j'} : j' \in [p, q]\right\}$. If $z \in N_\lambda^\perp$, then $\langle z, x_{j'}\rangle = 0, \forall j' \in [p, q]$. Now let $x = \sum_{j\geq 1, j\notin[p,q]} \frac{\langle z, x_j\rangle}{\lambda - \lambda_j} x_j + \sum_{j\geq 1} \frac{\langle z, w_j\rangle}{\lambda} w_j$. In the summation $\sum_{j\geq 1, j\notin[p,q]} \frac{\langle z, x_j\rangle}{\lambda - \lambda_j} x_j$, $\lambda - \lambda_j \neq 0, \forall j \geq 1, j \notin [p, q]$. Then $x \in \mathbf{X}$. To see this, $\sum_{j\geq 1, j\notin[p,q]} \frac{|\langle z, x_j\rangle|^2}{(\lambda - \lambda_j)^2} \leq \sup_{j\notin[p,q]} \frac{1}{(\lambda - \lambda_j)^2} \cdot \|z\|^2 \leq c_1\|z\|^2$ because $\lambda_j \to 0$. Also, $\sum_{j\geq 1} \frac{|\langle z, w_j\rangle|^2}{\lambda^2} \leq \frac{\|z\|^2}{\lambda^2}$. It follows that $x \in \mathbf{X}$. Because $(\lambda - A)x = \sum_{j\geq 1, j\notin[p,q]} \langle z, x_j\rangle x_j + \sum_{j\geq 1}\langle z, w_j\rangle w_j = \sum_{j\geq 1}\langle z, x_j\rangle x_j + \sum_{j\geq 1}\langle z, w_j\rangle w_j = z$, x is a solution to (EQz).

Conversely. let x be a solution to (EQz), then $z = \lambda x - Ax$. $\forall y \in N_\lambda$, we have $Ay = \lambda y$, and $\langle z, y\rangle = \langle \lambda x - Ax, y\rangle = \lambda\langle x, y\rangle - \langle Ax, y\rangle = \lambda\langle x, y\rangle - \langle x, Ay\rangle = \lambda\langle x, y\rangle - \langle x, \lambda y\rangle = 0$ because $\lambda = \lambda_{j_0}$ is real. It follows that $z \in N_\lambda^\perp$. □

The Fredholm Alternative

Remark 1.12

$\lambda \neq 0$, and $\lambda \neq \lambda_j, \forall j \geq 1$, and $z \in \mathbf{X}$, $\lambda x - Ax = z$. In the above proof, we see that $x = \sum_{j\geq 1} \frac{\langle z, x_j\rangle}{\lambda - \lambda_j} x_j + \sum_{j\geq 1} \frac{\langle z, w_j\rangle}{\lambda} w_j$ is a solution. We have

$$
\begin{aligned}
\|x\|^2 &= \sum_{j\geq 1} \frac{|\langle z, x_j\rangle|^2}{(\lambda - \lambda_j)^2} + \sum_{j\geq 1} \frac{|\langle z, w_j\rangle|^2}{\lambda^2} \\
&\leq C_1(\lambda) \left(\sum_{j\geq 1} |\langle z, x_j\rangle|^2 + \sum_{j\geq 1} |\langle z, w_j\rangle|^2 \right) = C_1(\lambda)\|z\|^2
\end{aligned}
$$

for some constant $C_1(\lambda)$. $\lambda - A \colon \mathbf{X} \to \mathbf{X}$ is surjective. It is also injective, because if $\lambda x - Ax = 0$ and $x \neq 0$, then λ is an eigenvalue of A. It follows that $\lambda = \lambda_{j_0}$ for some $j_0 \geq 1$ or $\lambda = 0$, contradicting with the assumption. Therefore, $\lambda - A$ is a bijection, and $(\lambda - A)^{-1} \colon \mathbf{X} \to \mathbf{X}, x = (\lambda - A)^{-1} z$. The above inequality shows that $\|(\lambda - A)^{-1} z\| \leq \sqrt{C_1(\lambda)}\|z\|$, so $(\lambda - A)^{-1}$ is bounded.

The Fredholm Alternative

Corollary 1.7

$\lambda \neq 0$, $z \in \mathbf{X}$, and $\lambda x - Ax = z$ has at most one solution. Then (1) $\exists$ $(\lambda - A)^{-1}$ which is a Bounded Linear Operator (BLO); (2) $x = (\lambda - A)^{-1} z$.

Proof.

We will show that $\lambda \neq \lambda_j, \forall j \geq 1$. Suppose that $\lambda = \lambda_j$ for some $j \geq 1$, and x is a solution to $\lambda x - Ax = z$. Let x_j be eigenfunction associated with λ, then $x + \alpha x_j, \forall \alpha \in \mathbb{K}$ is also a solution to $\lambda x - Ax = z$. This contradiction shows that $\lambda \neq \lambda_j, \forall j \geq 1$, and in this case $\lambda x - Ax = z$ has a unique solution. Then we have shown in the previous remark that $(\lambda - A)^{-1}$ exists, which is a Bounded Linear Operator (BLO), and $x = (\lambda - A)^{-1} z$. □

- Some terminologies: $\mathbf{X}$ is a BS, and $A\colon \mathbf{X} \to \mathbf{X}$ is a BLO, $\mathbb{K} = \mathbb{C}$. (1) λ is an eigenvalue of A is $\exists x \neq 0$ s.t. $Ax = \lambda x$. (2) $\rho(A)$ is the Resolvent set, $\rho(A) = \left\{\lambda \in \mathbb{C}\colon (\lambda - A)^{-1} \text{ is a BLO}\right\}$. (3) The spectrum of A is defined as $\sigma(A) = \mathbb{C} \setminus \rho(A)$.
- If $\lambda \in \rho(A)$, $(\lambda - A)^{-1}$ is called Resolvent of A at λ.

The Fredholm Alternative

- We proved that if $\lambda \neq 0$, and $\lambda \neq \lambda_j, \forall j \geq 1$, then $(\lambda - A)^{-1}$ is a BLO, so $\lambda \in \rho(A)$.
- $\Omega(A) = \{\lambda_j : j \geq 1\}$ be the set of all eigenvalues of A. Then $\mathbb{C} \setminus (\Omega \bigcup \{0\}) \subseteq \rho(A)$.
- Let λ be an eigenvalue of A, then $\exists x \neq 0$, $Ax = \lambda x \Rightarrow \lambda - A$ is not injective, and it follows that $\lambda \in \sigma(A)$, and $\Omega(A) \subseteq \sigma(A)$.

The Fredholm Alternative

Corollary 1.8

Suppose 0 is not an eigenvalue of A. Then (1) $\dim(\mathbf{X}) < \infty \Rightarrow 0 \in \rho(A)$; (2) $\dim(\mathbf{X}) = \infty \Rightarrow 0 \in \sigma(A)$.

Proof.

Let $\dim(\mathbf{X}) < \infty$. Then $Ax = 0 \Rightarrow x = 0$, so A is injective. Consider equation $Ax = z, \forall z \in \mathbf{X}$. Then it has the solution $x = \sum_{j=1}^{n} \frac{\langle z, x_j \rangle}{\lambda_j} x_j$. Therefore, A is surjective so A is a bijection, and we have the inverse of A as A^{-1}. Also, $\|x\|^2 = \left\| A^{-1} z \right\|^2 = \sum_{j=1}^{n} \frac{|\langle z, x_j \rangle|^2}{\lambda_j^2} \leq c\|z\|^2, c = \min_{j \in [n]} \frac{1}{\lambda_j^2}$. It follows that A^{-1} is a BLO, and $0 \in \rho(A)$.
Now let $\dim(\mathbf{X}) = \infty$. We still have A is injective, and $N(A) = \{0\}$. Suppose that A^{-1} exists, we will prove that A^{-1} is not bounded. To see this, let $\{x_j : j \geq 1\}$ be an orthonormal basis of $\mathbf{X}$ (because $N(A) = \{0\}$) which is associated with eigenvalues $\{\lambda_j : j \geq 1\}$ with $\lim_{j \to \infty} \lambda_j = 0$. Then $Ax_j = \lambda_j x_j$. Define $z_j = \lambda_j x_j$, then $x_j = A^{-1} z_j = \frac{z_j}{\lambda_j} \Rightarrow \left\| A^{-1} \right\| \geq \frac{1}{\lambda_j}, \forall j \geq 1$. It follows that $\left\| A^{-1} \right\| \geq \frac{1}{\lambda_j} \to \infty$, so A^{-1} is not bounded. Therefore, $0 \in \sigma(A)$. □

An Application to Integral Operators

- $-\infty < a < b < \infty$, $\mathbf{X} = L^2([a,b])$ which is comprised of all $f\colon [a,b] \to \mathbb{R}$, $\int_a^b f^2(x)\mathrm{d}x < \infty$. $\mathbf{X}$ is a HS with $\langle f, g\rangle = \int_a^b f(x)g(x)\mathrm{d}x$. $\mathbf{X}$ is a separable HS.
- Consider a kernel $K\colon [a,b] \times [a,b] \to \mathbb{R}$ satisfying (H1) K is continuous; (H2) K is symmetric: $K(s,t) = K(t,s)$.
- Define $A\colon \mathbf{X} \to \mathbf{X}$ by $(Ax)(t) = \int_a^b K(t,s)x(s)\mathrm{d}s$.

Lemma 1.24

If assumption (H1) holds, then (1) A is a BLO; (2) A is compact; (3) $Ax \in C([a,b])$. If both (H1) and (H2) hold, then A is a symmetric operator.

An Application to Integral Operators

Proof.

We first show that Ax is well defined.
$\left|\int_a^b K(t,s)x(s)\mathrm{d}s\right| \leq \|K\|_\infty \sqrt{b-a}\sqrt{\int_a^b x^2(s)\mathrm{d}s} = \|K\|_\infty \sqrt{b-a}\|x\|$. We then prove that A is bounded.

$$\|Ax\|^2 = \int_a^b \left(\int_a^b K(t,s)x(s)\mathrm{d}s \right)^2 \mathrm{d}t \leq \int_a^b \int_a^b K^2(t,s)\mathrm{d}s \int_a^b x^2(s)\mathrm{d}s\ \mathrm{d}t$$
$$\leq \int_a^b \int_a^b K^2(t,s)\mathrm{d}s\mathrm{d}t \cdot \|x\|^2 \leq \|K\|_\infty^2 (b-a)^2 \|x\|^2.$$

Therefore, A is a BLO. We now prove that $Ax \in C([a,b]), \forall x \in L^2([a,b])$. To see this,

$$\left|(Ax)(t') - (Ax)(t)\right| = \left|\int_a^b \left(K(t,s) - K(t',s)\right) x(s)\mathrm{d}s\right| \leq \int_a^b \left|K(t,s) - K(t',s)\right| |x(s)|\, \mathrm{d}s$$
$$\leq \int_a^b \varepsilon\, |x(s)|\, \mathrm{d}s \leq \varepsilon \sqrt{(b-a)}\|x\|,$$

where the second last inequality is due to the fact that K as a continuous function on the compact domain $[a,b]^2$ is absolutely continuous. Therefore, $Ax \in C([a,b]), \forall x \in L^2([a,b])$. $\square$

An Application to Integral Operators

Cont'd.

Now we prove that A is compact. $\forall \{x_n : n \geq 1\} \subseteq \mathbf{X}, \sup_{n\geq 1} \|x_n\|_\infty \leq c_0$. We will prove that $\{Ax_n : n \geq 1\}$ is relative compact. Let $y_n = Ax_n \in C([a,b])$.

Claim 1.16

(1) $\{y_n : n \geq 1\}$ is uniformly bounded, that is, $\exists c_1 < \infty$, $\sup_{n\geq 1} \|y_n\|_\infty \leq c_1$.
(2) $\{y_n : n \geq 1\}$ is equicontinuous: $\forall \varepsilon > 0, \exists \delta > 0$, s.t.
$\left|t - t'\right| \leq \delta \Rightarrow \sup_{n\geq 1} \left|y_n(t) - y_n(t')\right| \leq \varepsilon$.

Proof.

(1) $|(Ax)(t)| = \left|\int_a^b K(t,s)x(s)\mathrm{d}s\right| \leq \|K\|_\infty \sqrt{b-a}\sqrt{\int_a^b x^2(s)\mathrm{d}s} = \|K\|_\infty \sqrt{b-a}\|x\| \leq \|K\|_\infty \sqrt{b-a}c_0$.
(2) We have

$$\left|y_n(t) - y_n(t')\right| = \left|\int_a^b K(t,s)x_n(s)\mathrm{d}s - \int_a^b K(t',s)x_n(s)\mathrm{d}s\right|$$

$$\leq \int_a^b \left|K(t,s) - K(t',s)\right| |x_n(s)|\,\mathrm{d}s \leq \int_a^b \varepsilon\,|x(s)|\,\mathrm{d}s \leq \varepsilon\sqrt{(b-a)}\|x\| \leq \varepsilon\sqrt{(b-a)}c_0,$$

An Application to Integral Operators

Cont'd.

Proof.

where the second last inequality is due to the fact that K as a continuous function on the compact domain $[a,b]^2$ is absolutely continuous. It follows that
$\sup_{s\in[a,b]}\left|K(t,s)-K(t',s)\right| \leq \varepsilon, \forall t,t' \in [a,b]$ s.t. $\left|t-t'\right| \leq \delta$. □

Then by the Arzela-Ascoli Theorem and the above claim, $\exists\{n_k\}, y \in \mathbf{X}$ s.t. $y_{n_k} \overset{\|\|_\infty}{\to} y$. It follows that $\{Ax_n : n \geq 1\}$ is relative compact,and A is compact.

Lemma 1.25

If both (H1) and (H2) hold, then A is a symmetric operator.

Proof.

$$\langle Ax, y\rangle = \int_a^b (Ax)(t)y(t)\mathrm{d}t = \int_a^b y(t)\mathrm{d}t \int_a^b K(t,s)x(s)\mathrm{d}s = \int_a^b x(s)\mathrm{d}s \int_a^b K(t,s)y(t)\mathrm{d}t$$
$$= \int_a^b x(s)\mathrm{d}s \int_a^b K(s,t)y(t)\mathrm{d}t = \int_a^b x(s)(Ay)(s)\mathrm{d}s = \langle x, Ay\rangle,$$

it follows that A is symmetric. □

An Application to Integral Operators

- By the above proof, if both (H1) and (H2) hold, A is a symmetric and compact operator on a separable HS $\mathbf{X}$. It follows that $\exists \{x_n : n \geq 1\}$ and $\exists \{w_n : n \geq 1\}$ which form an orthonormal basis of $\mathbf{X}$. $\{x_n : n \geq 1\}$ are eigenfunctions of A associated with nonzero eigenvalues $\{\lambda_n : n \geq 1\}$, and $\{w_n : n \geq 1\}$ are associated with eigenvalue 0. If $\{\lambda_n : n \geq 1\}$ is countably infinite, $\lim\limits_{n\to\infty} \lambda_n = 0$. $\forall n \geq 1, \lambda_n \neq 0, \{x \in \mathbf{X} : (\lambda_n - A)x = 0\}$ has finite dimension.
- Let $\{y_k : k \geq 1\} = \{x_n : n \geq 1\} \bigcup \{w_n : n \geq 1\}$. $\forall x \in \mathbf{X}$, $x = \sum\limits_{j\geq 1} \langle x, y_j \rangle\, y_j$, which means that $x = \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle x, y_j \rangle\, y_j$ in $L^2([a,b])$ sense.

Lemma 1.26

Let $x \in \mathbf{X}$, $x = Az$ with $z \in \mathbf{X}$. Then $\forall \varepsilon, \exists n_0, \forall n, m \geq n_0$, $\sup_{t\in[a,b]} \sum\limits_{k=n}^{m} |\langle x, y_k(t) \rangle\, y_k(t)| \leq \varepsilon$.

An Application to Integral Operators

Proof.

We have

$$\sum_{k=n}^{m} |\langle x, y_k\rangle\, y_k(t)| = \sum_{k=n}^{m} |\langle Az, y_k\rangle\, y_k(t)| = \sum_{k=n}^{m} |\langle z, Ay_k\rangle\, y_k(t)|$$

$$= \sum_{k=n}^{m} |\langle z, y_k\rangle\, \lambda_k y_k(t)| = \sum_{k=n}^{m} |\langle z, y_k(t)\rangle\, (Ay_k)(t)| = \sum_{k=n}^{m} |\langle z, y_k\rangle| \left| \int_a^b K(t,s) y_k(s) \mathrm{d}s \right|$$

$$\overset{\text{Cauchy Inequality}}{\leq} \sqrt{\sum_{k=n}^{m} |\langle z, y_k\rangle|^2} \cdot \sqrt{\sum_{k=n}^{m} \left(\int_a^b K(t,s) y_k(s) \mathrm{d}s \right)^2}$$

$$\leq \sqrt{\sum_{k=n}^{m} |\langle z, y_k\rangle|^2} \sqrt{\sum_{k=n}^{m} |\langle K(t,\cdot), y_k\rangle|^2} \leq \sqrt{\sum_{k=n}^{m} |\langle z, y_k\rangle|^2} \cdot \|K(t,\cdot)\|$$

$$\leq \sqrt{\sum_{k=n}^{m} |\langle z, y_k\rangle|^2} \cdot \sqrt{b-a} \|K\|_\infty .$$

Because $\|z\|^2 = \sum_{k\geq 1} |\langle z, y_k\rangle|^2$, $\forall \varepsilon' > 0$, $\exists n_0, \forall n, m \geq n_0$, $\sum_{k=n}^{m} |\langle z, y_k\rangle|^2 \leq \varepsilon'$. □

An Application to Integral Operators

Cont'd.

It follows that $\forall n, m \geq n_0$, $\sum_{k=n}^{m} |\langle x, y_k(t)\rangle\, y_k(t)| \leq \sqrt{\varepsilon'(b-a)}\|K\|_\infty$. Setting $\varepsilon = \sqrt{\varepsilon'(b-a)}\|K\|_\infty$ proves the lemma. □

- Now consider the function $\lambda x - Ax = z, z \in \mathbf{X}$.

Proposition 1.16 (Fredholm Alternative)

Let $\lambda \neq 0$, and $\Omega = \{\lambda_j : j \geq 1\}$ be the set of eigenvalues of A.

(1) If $\lambda \notin \Omega$, then the previous theorem shows that $\lambda x - Ax = z$ has a unique solution given by $x = (\lambda - A)^{-1} z$.

(2) If $\lambda \in \Omega$, then $\exists$ a solution for $\lambda x - Ax = z \iff z \in N_\lambda^\perp$, where $N_\lambda = \{y \in \mathbf{X} : Ay = \lambda y\}$. Because λ is an eigenvalue of A, N_λ is a finite-dimensional space spanned by a finite number of eigenfunctions of A.

(3) If $z \in C([a,b])$ and x is a solution to $\lambda x - Ax = z$, then $x \in C([a,b])$. This follows from the fact that $Ax \in C([a,b])$ and

Thank you!
Questions?