# Measure Theory

Lecture Notes

Yingzhen Yang

# Classes of subsets (semi-algebras, algebras and sigma-algebras)

## Definition 1.1

$\mathcal{L} \subseteq \mathcal{P}(\Omega)$ is a semi-algebra if (1) $\Omega \in \mathcal{L}$; (2) $A, B \in \mathcal{L} \Rightarrow A \cap B \in \mathcal{L}$; (3) $A \in \mathcal{L} \Rightarrow A^c = \bigcup_{i=1}^n E_i, \{E_i\}_{i=1}^n \in \mathcal{L}$.

## Definition 1.2

$\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is an algebra if (1) $\Omega \in \mathcal{A}$; (2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$; (3) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.

## Definition 1.3

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is an $\sigma$-algebra if (1) $\Omega \in \mathcal{F}$; (2) $\{A_j\} \subseteq \mathcal{F} \Rightarrow \bigcap_{j \geq 1} A_j \in \mathcal{F}$; (3) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

- $\mathcal{A}(\mathcal{L})$ is an algebra, and $\forall \mathcal{L} \subseteq \mathcal{B}, \mathcal{B}$ is an algebra, $\mathcal{A}(\mathcal{L}) \subseteq \mathcal{B}$. $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$.

# Classes of subsets (semi-algebras, algebras and sigma-algebras)

**Lemma 1.1**

$\mathcal{L} \subseteq \mathcal{P}(\Omega)$ is a semi-algebra, $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$. Then $A \in \mathcal{A}(\mathcal{L}) \iff \exists \{E_j\}_{j=1}^n \subseteq \mathcal{L}, A = \bigcup_{j=1}^n E_j$.

-

# Classes of subsets (semi-algebras, algebras and sigma-algebras)

**Definition 1.4**

-

### Definition 1.5

-

# Caratheodory theorem

- $\mu\colon \mathcal{L} \to \mathbb{R}^+ \cup \{+\infty\}$ is a measure on the semi-algebra $\mathcal{L}$, which can be extended to $\nu\colon \mathcal{A} \to \mathbb{R}^+ \cup \{+\infty\}$ in a unique way where $\mathcal{A}$ is the algebra generated by $\mathcal{L}$. If $\mu$ is $\sigma$-additive, $\nu$ is also $\sigma$-additive. The goal is to extend $\nu$ to $\pi\colon \sigma(\mathcal{L}) \to \mathbb{R}^+ \cup \{+\infty\}$ where $\sigma(\mathcal{L})$ is the $\sigma$-algebra generated by $\mathcal{L}$ ($\sigma(\mathcal{A}) = \sigma(\mathcal{L})$).
- We will construct an outer measure $\pi^*\colon \mathcal{P}(\Omega) \to \mathbb{R}^+ \cup \{+\infty\}$ such that $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ is an $\sigma$-algebra, $\mathcal{A} \subseteq \mathcal{M}$, $\pi^*|_{\mathcal{M}}$ is $\sigma$-additive, and $\pi^*|_{\mathcal{A}} = \nu$.

# Caratheodory theorem

- Step 1. Construct $\pi^*: \mathcal{P}(\Omega) \to \mathbb{R}^+ \cup \{+\infty\}$. Let $A \subseteq \Omega$, $\pi^*(A) = \inf_{\{A_i\}_{i\geq 1} \subseteq \mathcal{A}, A \subseteq \bigcup_{i\geq 1} A_i} \sum_{i\geq 1} \nu(A_i)$.

**Definition 1.6**

(Outer measure) $\mu: \mathcal{L} \to \mathbb{R}^+ \cup \{+\infty\}$ for $\mathcal{L} \subseteq \mathcal{P}(\Omega), \emptyset \in \mathcal{L}$ is an outer measure if (1) $\mu(\emptyset) = 0$; (2) $E \subseteq F, E, F \in \mathcal{L} \Rightarrow \mu(E) \leq \mu(F)$. (3) $E \subseteq \bigcup_{i\geq 1} E_i \Rightarrow \mu(E) \leq \sum_{i\geq 1} \mu(E_i)$.

**Claim 1.1**

$\pi^*$ is an outer measure.

- The above claim can be proof by checking the three properties of an outer measure.

## Caratheodory theorem

- Step 2. Define $\mathcal{M} = \{A \subseteq \Omega \colon \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c), \forall E \subseteq \Omega\}$. We will show that $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a $\sigma$-algebra.
- Let $A \in \mathcal{A}$. $\forall \varepsilon > 0$, $\exists \{E_i\} \subseteq \mathcal{A}$ such that $\varepsilon + \pi^*(E) \geq \sum_{i \geq 1} \nu(E_i)$.
  Note that $\{E_i \cap A\}$ and $\{E_i \cap A^c\}$ are cover of $E \cap A$ and $E \cap A^c$ by elements of $\mathcal{A}$, and $\sum_{i \geq 1} \nu(E_i) = \sum_{i \geq 1} \nu(E_i \cap A) + \sum_{i \geq 1} \nu(E_i \cap A^c)$, it follows that $\varepsilon + \pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c) \Rightarrow \pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c)$.
- We have that $\Omega \in \mathcal{M}$, and $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$. Now we need to show $\{A_j\} \subseteq \mathcal{M} \Rightarrow \bigcup_{j \geq 1} A_j \in \mathcal{M}$. To this end, we first show $A, B \in \mathcal{M} \Rightarrow A \cup B \in \mathcal{M}$. We have $\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) = \pi^*(E \cap A) + \pi^*(E \cap A^c \cap B) + \pi^*(E \cap A^c \cap B^c) \geq \pi^*(E \cap (A \cup B)) + \pi^*(E \cap (A \cup B)^c)$.

## Caratheodory theorem

- We showed that $\mathcal{M}$ is closed by finite union. Now we need to show $\{A_j\} \subseteq \mathcal{M} \Rightarrow A = \bigcup_{j\geq 1} A_j \in \mathcal{M}$. $\forall E \subseteq \Omega$, that is, we need to show $\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c)$. We already have $\pi^*(E) \geq \pi^*(E \cap \bigcup_{j\in[n]} A_j) + \pi^*(E \cap (\bigcup_{j\in[n]} A_j)^c) \geq \pi^*(E \cap \bigcup_{j\in[n]} A_j) + \pi^*(E \cap A^c)$. Let $F_n = A_n \setminus (\bigcup_{j=1}^{n-1} A_j), \forall n \geq 2$.

  Claim: $\pi^*(E \cap (\bigcup_{j=1}^{n} A_j)) = \sum_{j=1}^{n} \pi^*(E \cap F_j)$.
- By the above claim, $\pi^*(E) \geq \pi^*(E \cap \bigcup_{j\in[n]} A_j) + \pi^*(E \cap A^c) = \sum_{j=1}^{n} \pi^*(E \cap F_j) + \pi^*(E \cap A^c)$ which holds $\forall n \geq 1$. It follows that $\pi^*(E) \geq \sum_{j=1}^{\infty} \pi^*(E \cap F_j) + \pi^*(E \cap A^c) \geq \pi^*(E \cap \bigcup_{j\geq 1} A_j) + \pi^*(E \cap A^c) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \Rightarrow A \in \mathcal{M}$. It follows that $\mathcal{M}$ is a $\sigma$-algebra. Therefore, $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.

# Caratheodory theorem

- Step 3. $\pi^*: \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ is $\sigma$-additive, and $\pi^*|_{\mathcal{A}} = \nu, \pi^*(A) = \nu(A), \forall A \in \mathcal{A}$.
- To prove $\pi^*|_{\mathcal{A}} = \nu$, we have $\pi^*(A) \leq \nu(A), \forall A \in \mathcal{A}$ by the definition of outer measure. Let $F_n = E_n \setminus \bigcup_{j=[n-1]} E_j, \forall n \geq 2$. For any $\{E_j\}_{j\geq 1} \subseteq \mathcal{A}, A \subseteq \bigcup_{j\geq 1} E_j$, we have $\sum\limits_{j\geq 1} \nu(E_j) \geq \sum\limits_{j\geq 1} \nu(F_j \cap A) = \nu(A)$. Therefore, $\pi^*|_{\mathcal{A}} = \nu$.
- To prove $\pi^*$ is $\sigma$-additive, let $\{A_j\}_{j\geq 1} \subseteq \mathcal{M}$, and $A_i \cap A_j = \emptyset, i \neq j$. We have $\pi^*(\bigcup_{j\geq 1} A_j) \leq \sum\limits_{j\geq 1} \pi^*(A_j)$. In step 2, we proved that $\pi^*(\bigcup_{j\geq 1} A_j) \geq \sum\limits_{j\in[n]} \pi^*(A_j), \forall n \geq 1 \Rightarrow \pi^*(\bigcup_{j\geq 1} A_j) \geq \sum\limits_{j\geq 1} \pi^*(A_j)$.

## Caratheodory theorem

- Step 4. Uniqueness. Assume $\mu_1, \mu_2 \colon \sigma(\mathcal{A}) \to \mathbb{R}^+ \cup \{+\infty\}$, and $\mu_1|_{\mathcal{A}} = \mu_2|_{\mathcal{A}} = \nu$. We assume $\mu_1$ is $\sigma$-finite, that is, $\exists \{E_j\}_{j\geq 1} \subseteq \mathcal{A}, \mathbb{E}_j \uparrow \Omega, \mu_1(E_j) < \infty, \forall j \geq 1$. We needs to show $\mu_1 = \mu_2$.
- The proof relies on the monotone class.

**Definition 1.7**

$\mathcal{G} \subseteq \mathcal{P}(\omega)$. $\mathcal{G}$ is a monotone class if (1) $\{A_j\} \subseteq \mathcal{G}, A_j \subseteq A_{j+1} \Rightarrow \bigcup_{j\geq 1} A_j \in \mathcal{G}$; (2) $\{B_j\} \subseteq \mathcal{G}, B_{j+1} \subseteq B_j \Rightarrow \bigcap_{j\geq 1} B_j \in \mathcal{G}$.

**Claim 1.2**

$\mathcal{G}_\alpha, \alpha \in I$ are monotone classes, $\mathcal{G}_\alpha \subseteq \mathcal{P}(\Omega)$. Then $\bigcap_{\alpha\in I} \mathcal{G}_\alpha$ is a monotone class.

- $\mathcal{G}(\mathcal{L}) = \bigcap_{\alpha\in I} \mathcal{G}_\alpha$ where $\mathcal{G}_\alpha, \alpha \in I$ are all the monotone classes which contain $\mathcal{L}$. Monotone class lemma: $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$ if $\mathcal{A}$ is an algebra, where $\mathcal{M}(\mathcal{A})$ is the monotone class generated by $\mathcal{A}$.

## Caratheodory theorem

- To prove the uniqueness, let $B_n = \{E \in \sigma(\mathcal{A})\colon \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\}, \forall n \geq 1$. Then $\mathcal{A} \subseteq B_n, \forall n \geq 1$. Also, $B_n$ is a monotone class, $\forall n \geq 1$. Let $\{A_j\} \subseteq B_n$, $A_j \uparrow A$, $A = \bigcup_{j\geq 1} A_j$. Then $\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n)$. Because $\mu_1, \mu_2$ are $\sigma$-additive on $\sigma(\mathcal{A})$, they are continuous from below, so $\mu_1(A \cap E_n) = \mu_2(A \cap E_n)$. The same result holds for $\{A_j\} \subseteq B_n \downarrow A$. So $B_n$ is a monotone class, and $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A}) \subseteq B_n$. Because $B_n \subseteq \sigma(\mathcal{A})$, $B_n = \sigma(\mathcal{A})$. Now $\forall A \in \sigma(\mathcal{A})$, $A \in B_n, \forall n \geq 1 \Rightarrow \mu_1(A \cap E_n) = \mu_2(A \cap E_n), \forall n \geq 1$. Because $\mu_1, \mu_2$ are $\sigma$-additive on $\sigma(\mathcal{A})$, they are continuous from below, so $\mu_1(A) = \mu_2(A)$.

# Complete measures

## Definition 1.8

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$, $\mu \colon \mathcal{F} \to \mathbb{R}^+ \cup \{+\infty\}$. $(\mu, \mathcal{F})$ is complete if $A \in \mathcal{F}, \mu(A) = 0, E \subseteq A \Rightarrow E \in \mathcal{F}, \mu(E) = 0$.

- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a $\sigma$-algebra, $\mu \colon \mathcal{F} \to \bar{\mathbb{R}}^+$ is a measure. Goal: $\mathcal{F} \subseteq \bar{\mathcal{F}}$, $\bar{\mu} \colon \bar{\mathcal{F}} \to \mathbb{R}^+ \cup \{+\infty\}$, such that $\bar{\mu}|_{\mathcal{F}} = \mu$, and $(\bar{\mu}, \bar{\mathcal{F}})$ is complete.
- $\bar{\mathcal{F}} = \{A \cup N \colon A \in \mathcal{F}, N \subseteq E \in \mathcal{F}, \mu(E) = 0\}$.

## Claim 1.3

$\bar{\mathcal{F}}$ is a $\sigma$-algebra.

## Proof.

We have $\Omega \in \bar{\mathcal{F}}$. If $A \in \bar{\mathcal{F}}, A = E \cup N, E \in \mathcal{F}, N \subseteq H \in \mathcal{F}, \mu(H) = 0$. Then $A^c = ((E \cup N)^c \cap H) \cup ((E \cup N)^c \cap H^c) = ((E \cup N)^c \cap H) \cup (E^c \cap H^c) \in \bar{\mathcal{F}}$. Let $\{A_j\}_{j \geq 1} \subseteq \bar{\mathcal{F}}$, then $A_j = E_j \cup N_j, E_j \in \mathcal{F}, N_j \subseteq H_j \in \mathcal{F}, \mu(H_j) = 0$. Then $\bigcup_{j \geq 1} A_j = \left(\bigcup_{j \geq 1} E_j\right) \cup \left(\bigcup_{j \geq 1} N_j\right) \in \bar{\mathcal{F}}$. $\square$

## Complete measures

- Define $\bar{\mu}(A \cup N) = \mu(A)$. $\bar{\mu}$ is well defined: if $A \cup N = B \cup M$, then $\mu(A) = \bar{\mu}(A \cup N) = \bar{\mu}(B \cup M) \leq \bar{\mu}(B) + \bar{\mu}(M) = \mu(B)$. Similarly, $\mu(B) \leq \mu(A) \Rightarrow \mu(A) = \mu(B)$.
- Also, $\bar{\mu}(A) = \mu(A), \forall A \in \mathcal{F}$, so $\bar{\mu}|_{\mathcal{F}} = \mu$.
- $\bar{\mu}$ is $\sigma$-additive. To see this, let $A = \bigcup_{j\geq 1} A_j \in \bar{\mathcal{F}}, A_i \cap A_j = \emptyset, A_j \in \bar{\mathcal{F}}, A_j = E_j \cup N_j, E_j \in \mathcal{F}, N_j \subseteq H_j, H_j \in \mathcal{F}, \mu(H_j) = 0$. Then $A = \bigcup_{j\geq 1} E_j + \bigcup_{j\geq 1} N_j, \bar{\mu}(A) = \mu(\bigcup_{j\geq 1} E_j) = \sum\limits_{j\geq 1} \mu(E_j) = \sum\limits_{j\geq 1} \bar{\mu}(A_j)$.
- $(\bar{\mu}, \bar{\mathcal{F}})$ is complete, or $\bar{\mathcal{F}}$ is $\bar{\mu}$-complete. Let $A \subseteq E \in \bar{\mathcal{F}}, \bar{\mu}(E) = 0$. We need to show $A \in \bar{\mathcal{F}}$ with $\bar{\mu}(A) = 0$. We have $E = B \cup N, B \in \mathcal{F}, N \subseteq H \in \mathcal{F}, \mu(B) = \mu(H) = 0$. Then $A = \emptyset \cup A \in \bar{\mathcal{F}}$ (note that $A \subseteq B \cup H$), and $\bar{\mu}(A) = \mu(\emptyset) = 0$.

## Complete measures

- $\mu\colon \mathcal{F} \to \mathbb{R}^+ \cup \{+\infty\}$, which is extended to $\bar{\mu}\colon \bar{\mathcal{F}}_\mu \to \mathbb{R}^+ \cup \{+\infty\}$. Such extension is unique. Let $\nu\colon \bar{\mathcal{F}}_\mu \to \mathbb{R}^+ \cup \{+\infty\}, \nu(A) = \bar{\mu}(A), \forall A \in \mathcal{F}$. Then $\bar{\mu}(B) = \nu(B), \forall B \in \bar{\mathcal{F}}_\mu$.
- To see this, we have $B = E \cup N, E \in \mathcal{F}, N \subseteq H \in \mathcal{F}, \mu(H) = \nu(H) = 0$. Then $\bar{\mu}(B) = \mu(E) = \nu(E) \leq \nu(B)$. Also, $\nu(B) \leq \nu(E) + \nu(H) = \nu(E) = \mu(E) = \bar{\mu}(B)$. So $\nu(B) = \bar{\mu}(B), \forall B \in \bar{\mathcal{F}}$.
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a $\sigma$-algebra, $\mu\colon \mathcal{F} \to \mathbb{R}^+ \cup \{+\infty\}$, $\mathcal{F}$ is $\mu$-complete if $A \subseteq E \in \mathcal{F}, \mu(E) = 0 \Rightarrow A \in \mathcal{F}, \mu(A) = 0$.

**Claim 1.4**

Let $\mathcal{M}$ be the Lebesgue measurable sets, and $\pi^*\colon \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ is the outer measure. Then $\mathcal{M}$ is $\pi^*$-complete.

# Complete measures

**Proof.**

We need to prove $A \subseteq B \in \mathcal{M}, \pi^*(B) = 0 \Rightarrow A \in \mathcal{M}, \pi^*(A) = 0$. To this end, we need to show $\forall E \subseteq \Omega, \pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c)$. Because $\pi^*(E \cap A) \leq \pi^*(B) = 0, \pi^*(E \cap A^c) \leq \pi^*(E)$, we have $\forall E \subseteq \Omega, \pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c) \Rightarrow A \in \mathcal{M}, \pi^*(A) = 0$. □

# Approximation theorems

- $\pi^*(A) < \infty, A \in \mathcal{M}$, $\mathcal{M}$ is the measurable sets. $\mathcal{F}$ is the $\sigma$-algebra generated by the algebra $\mathcal{A}$ (refer to the Caratheodory theorem), $F \in \mathcal{F}, A \subseteq F$. Goal: $\pi^*(A) = \pi^*(F)$.

## Theorem 1.1

Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be an algebra, $\mathcal{F} = \sigma(\mathcal{A})$ is the $\sigma$-algebra generated by $\mathcal{A}$. $\mu\colon \mathcal{F} \to \bar{\mathbb{R}}^+$. $A \in \mathcal{F}, \mu(A) < \infty$. Then $\forall \varepsilon > 0, \exists E \in \mathcal{A}, \mu(E \setminus A) + \mu(A \setminus E) < \varepsilon$.

## Proof.

$A \in \mathcal{F}, \mu(A) < \infty, \mu(A) = \pi^*(A) = \inf_{\{A_i\} \subseteq \mathcal{A}, A \subseteq \bigcup_{i \geq 1} A_i} \sum_{i \geq 1} \nu(A_i)$.

$\forall \varepsilon > 0, \exists \{A_i\} \subseteq \mathcal{A}, A \subseteq \bigcup_{i \geq 1} A_i, \pi^*(A) \leq \sum_{i \geq 1} \nu(A_i) \leq \pi^*(A) + \varepsilon \Rightarrow \exists n_0, \sum_{i \geq n_0} \nu(A_i) \leq \varepsilon$.

Let $E = \bigcup_{i \in [n_0]} A_i$. Then $\pi^*(E \setminus A) \leq \pi^*(\bigcup_{i \geq 1} A_i \setminus A) \leq \pi^*(\bigcup_{i \geq 1} A_i) - \pi^*(A) \leq \varepsilon$.

Also, $\pi^*(A \setminus E) = \pi^*(A \setminus \bigcup_{i \in [n_0]} A_i) \leq \pi^*(\bigcup_{i \geq 1} A_i \setminus \bigcup_{i \in [n_0]} A_i) = \pi^*(\bigcup_{i > n_0} A_i) \leq \sum_{i > n_0} \pi^*(A_i) \leq \varepsilon$. □

## Approximation theorems

### Remark 1.1

$\Omega$ is $\sigma$-finite $(\mu)$ $(\Omega = \bigcup_{i\geq 1} E_i, E_i \in \mathcal{A}, \mu(E_i) < \infty)$. $\bar{\mu}\colon \bar{\mathcal{F}} \to \bar{\mathbb{R}}^+$ is the completion of $(\mu, \mathcal{F})$. Then the above theorem also holds for $A \in \bar{\mathcal{F}}$, that is, if $\bar{\mu}(A) < \infty, \exists E \in \mathcal{A}, \bar{\mu}(E \setminus A) + \bar{\mu}(A \setminus E) < \varepsilon$.

- $\Omega$ is a topological space, let $\mathcal{B}$ be the Borel sets of $\Omega$ (smallest $\sigma$-algebra containing all the open sets of $\Omega$).
  $\mathcal{B} \subseteq \mathcal{F}, \mu\colon \mathcal{F} \to \mathbb{R}^+ \cup \{+\infty\}$. $\mu$ is regular if $\forall A \in \mathcal{F}$, $\exists F \subseteq A \subseteq G$ $F$ closed, $\exists G$ open, $\mu(G \setminus F) \leq \varepsilon$.

### Remark 1.2

$\mathcal{B} \subseteq \mathcal{F}$, $\mu$ is regular, then $\mathcal{F} \subseteq \bar{\mathcal{B}}_\mu$. To see this,
$\forall A \in \mathcal{F}, \forall n \geq 1, \exists F_n, G_n$ such that $F_n \subseteq A \subseteq G_n, \mu(G_n \setminus F_n) \leq \frac{1}{n}$. Let
$F = \bigcup_{n\geq 1} F_n, G = \bigcap_{n\geq 1} G_n, F \in \mathcal{B}, G \in \mathcal{B}$. Then
$\mu(G \setminus F) \leq \mu(G_n \setminus F_n) \leq \frac{1}{n}, \forall n \geq 1 \Rightarrow \mu(G \setminus F) = 0$. Now
$A = F \cup (A \setminus F), F \in \mathcal{B}, A \setminus F \subseteq G \setminus F \in \mathcal{B}$, so $A \in \bar{\mathcal{B}}$.

# Approximation theorems

## Theorem 1.2

$\mu$ is Lebesgue measure, $\mu\colon \mathcal{L} \to \mathbb{R}^+ \cup \{+\infty\}$ where $\mathcal{L}$ is the Lebesgue $\sigma$-algebra. Then $\mu$ is regular ($\forall \varepsilon > 0, \forall A \in \mathcal{L}, \exists F \subseteq A \subseteq G$, $F$ closed, $G$ open, $\mu(G \setminus F) \leq \varepsilon$)

## Proof.

We need to find $G$ open, $A \subseteq G$, $\mu(G \setminus A) \leq \varepsilon$. Let $E_n = [-n, n], A_n = A \cap E_n$. Then $\mu(A_n) < \infty, \forall \varepsilon > 0, \exists \left\{B_{n,k}\right\}, B_{n,k} \in \mathcal{A}, A_n \subseteq \bigcup_{k\geq 1} B_{n,k}$, such that $\mu(A_n) \leq \sum_{k\geq 1} \nu(B_{n,k}) \leq \mu(A_n) + \frac{\varepsilon}{2^n}$. Since $B_{n,k} \in \mathcal{A}, B_{n,k} = \bigcup_{j\in[\ell_{n,k}]} I_{n,k,j}, I_{n,k,j} = (a_{n,k,j}, b_{n,k,j}] \subset J_{n,k,j} = (a_{n,k,j}, c_{n,k,j}), c_{n,k,j} = b_{n,k,j} + \delta_{n,k,j}$. Then $B_{n,k} \subseteq G_{n,k} = \bigcup_{j\in[\ell_{n,k}]} J_{n,k,j}$.
$\mu(G_{n,k}) \leq \sum_{j\in[\ell_{n,k}]} \mu(I_{n,k,j}) + \delta_{n,k,j} = \mu(B_{n,k}) + \frac{\varepsilon}{2^n 2^k}$, $\sum_{j\in[\ell_{n,k}]} \delta_{n,k,j} \leq \frac{\varepsilon}{2^n 2^k}$ (note that $\{I_{n,k,j}\}$ are disjoint).
We have $A_n \subseteq \bigcup_{k\geq 1} G_{n,k} \triangleq G_n$, and $G_n$ is open. $\mu(G_n) \leq \sum_{k\geq 1} \mu(B_{n,k}) + \frac{\varepsilon}{2^n} \leq \mu(A_n) + \frac{2\varepsilon}{2^n}$.

Let $G = \bigcup_{n\geq 1} G_n$ which is open, since $A = \bigcup_{n\geq 1} A_n$, $\mu(G \setminus A) \leq \sum_{n\geq 1} \mu(G_n \setminus A_n) \leq 2\varepsilon$.

We also have $A^c \subseteq H$, $H$ is open, such that $\mu(H \setminus A^c) \leq \varepsilon$. Now take $F = H^c$, then $F \subseteq A, \mu(A \setminus F) = \mu(A \cap F^c) = \mu(H \setminus A^c) \leq \varepsilon$. $\square$

## Approximation theorems

**Remark 1.3**

$\forall A \in \mathcal{L}, \exists R \in F_\delta, \exists S \in G_\delta$ ($F_\delta$ denotes countable union of closed sets, $G_\delta$ denotes countable intersection of open sets), such that $R \subseteq A \subseteq S, \mu(S \setminus R) = 0$.

- Integration $(\Omega, \mathcal{F}, \mu)$, $f\colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$. We aim to design the integral $I$ such that (1) $I(\alpha f + g) = \alpha I(f) + I(g)$; (2) $I(f) \geq 0, \forall f \geq 0$; (3) $f_n \uparrow f \Rightarrow I(f_n) \to I(f)$.

## Measure on a countable product of spaces

- $\{\Omega_j, \mathcal{F}_j, \mu_j\}_{j\geq 1}$, $\mu_j(\Omega_j) = 1$. $\Omega^{(n)} = \prod_{m\geq n} \Omega_m$. Let $\mathcal{L} = \left\{E_1 \times E_2 \ldots \times E_n \times \Omega^{(n+1)} \mid n \geq 1, E_i \in \mathcal{F}_i, i \in [n]\right\}$. For $E = E_1 \times E_2 \ldots \times E_n \times \Omega^{(n+1)} \in \mathcal{L}$, define $\mu(E) = \prod_{i=1}^{n} \mu_i(E_i)$.

**Claim 1.5**

$\mathcal{L}$ is a semi-algebra, and $\mu$ is additive on $\mathcal{L}$.

**Proof.**

It can be verified that: 1) $\Omega \in \mathcal{L}$; 2) $A, B \in \mathcal{L} \Rightarrow A \cap B \in \mathcal{L}$; 3) If $A \in \mathcal{L}$, then $A^c$ is a finite disjoint union of elements of $\mathcal{L}$. $\square$

- Let $\mathcal{A}$ be the algebra generated by $\mathcal{L}$. Each element of $\mathcal{A}$ is a finite disjoint union of elements of $\mathcal{L}$. $\mu$ is extended from $\mathcal{L}$ to $\mathcal{A}$ (such extension is unique).

# Measure on a countable product of spaces

Let $\mathcal{L}^{(t)}, \mathcal{A}^{(t)}, \mu^{(t)}$ be the semi-algebra, algebra and measure for $\Omega^{(t)}$. Let $A(x_1) \in \mathcal{L}^{(2)}$ be the section of $A$ at $x_1 \in \Omega_1$ for $A \in \mathcal{L}$.

## Claim 1.6

Let $A \in \mathcal{L}$. Then $\mu^{(2)}(A(x))$ for $x \in \Omega_1$ is $\mathcal{F}_1$-measurable, and $\mu(A) = \int \mu^{(2)}(A(x)) \mathrm{d}\mu^{(1)}(x)$.

## Proof.

Let $A = E_1 \times E_2 \ldots \times E_n \times \Omega^{(n+1)}$, then $\mu^{(2)}(A(x)) = \mathbb{1}_{\{x \in E_1\}} \prod_{j=2}^{n} \mu_i(E_j)$ and $\mu(A) = \int \mu^{(2)}(A(x)) \mathrm{d}\mu^{(1)}(x)$. $\square$

## Remark 1.4

This claim can be extended for $A \in \mathcal{A}, A = \bigcup_{j=1}^{n} A^{(j)}$ with $\left\{A^{(j)}\right\}$ in $\mathcal{L}$.

# Measure on a countable product of spaces

## Theorem 1.3

$\mu$ on $\mathcal{A}$ is continuous from above at $\emptyset$.

## Proof.

Let $\left\{A^{(n)}\right\}_{n\geq 1} \subseteq \mathcal{A}$ and $A^{(n)} \downarrow \emptyset$, we will prove that $\lim_n \mu(A^{(n)}) = 0$. To this end, we will prove that if there exisits $\varepsilon > 0$ such that $\mu(A^{(n)}) \geq \varepsilon$ for all $n \geq 1$, and $A^{(n)} \downarrow$, then $\bigcap_{n\geq 1} A^{(n)} \neq \emptyset$. Define $B^{(n)} = \left\{x \in \Omega_1 \mid \mu^{(2)}(A^{(n)}(x)) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{F}_1$. Then $B^{(n+1)} \subseteq B^{(n)}$. Then

$$\varepsilon \leq \mu(A^{(n)}) = \int \mu^{(2)}(A^{(n)}(x))\mathrm{d}\mu^{(1)}(x) \leq \int_{B^{(n)}} \mu^{(2)}(A^{(n)}(x))\mathrm{d}\mu^{(1)}(x)$$

$$+ \int_{\Omega_1 \setminus B^{(n)}} \mu^{(2)}(A^{(n)}(x))\mathrm{d}\mu^{(1)}(x) \leq \varepsilon/2 \left(1 - \mu^{(1)}(B^{(n)})\right) + \mu^{(1)}(B^{(n)})$$

$$\Rightarrow \mu^{(1)}(B^{(n)}) \geq \varepsilon/2$$

It follows that $\bigcap_{n\geq 1} B^{(n)} \neq \emptyset$. $\square$

# Measure on a countable product of spaces

## Cont'd.

We have $A^{(n)} \subseteq \Omega$, $A^{(n+1)} \subseteq A^{(n)}$, $\mu(A^{(n)}) \geq \varepsilon$, and proved that there exists $x_1 \in \bigcap_{n\geq 1} B^{(n)}$, such that $A^{(n)}(x_1) \subseteq \Omega^{(2)}$, $A^{(n+1)}(x_1) \subseteq A^{(n)}(x_1)$, $\mu(A^{(n)}(x_1)) \geq \varepsilon/2$ for all $n \geq 1$. We can iterate this process, at step $k$, we have $(x_1, x_2, \ldots, x_k) \in \prod_{i=1}^{k} \Omega_k$, $A^{(n)}(x_1, x_2, \ldots, x_k) \subseteq \Omega^{(k+1)}$, $A^{(n+1)}(x_1, x_2, \ldots, x_k) \subseteq A^{(n)}(x_1, x_2, \ldots, x_k)$, $\mu(A^{(n)}(x_1, x_2, \ldots, x_k)) \geq \varepsilon/2^k$ for all $n \geq 1$. It follows that $A^{(n)}(x_1, x_2, \ldots, x_k) \neq \emptyset$. For any $n \geq 1$, noting that $A^{(n)} \in \mathcal{A}$, $(x_1, x_2, \ldots) \in A^{(n)}$, so that $(x_1, x_2, \ldots) \in \bigcap_{n\geq 1} A^{(n)} \Rightarrow \bigcap_{n\geq 1} A^{(n)} \neq \emptyset$. □

- We proved that $\mu$ on $\mathcal{A}$ is continuous from above at $\emptyset$. It follows that $\mu$ is $\sigma$-additive, so by the Carathéodory's extension theorem $\mu$ is extended from $\mathcal{A}$ to $\sigma(\mathcal{A}) = \sigma(\mathcal{L})$.

# Fubini Theorem

## Theorem 1.4

(Our goal) Let $\{\Omega_j, \mathcal{F}_j, \mu_j\}_{j=1}^2$ be measurable spaces, and $\{\Omega_j, \mu_j\}_{j=1}^2$ are $\sigma$-finite. $\mu = \mu_1 \times \mu_2, \mathcal{F} = \mathcal{F}_1 \bigotimes \mathcal{F}_2$. $f\colon \Omega_1 \times \Omega_2 \to \bar{\mathbb{R}}, f \geq 0$.

$$\int_{\Omega_1} \left[ \int_{\Omega_2} f_x(y) \mathrm{d}\mu_2(y) \right] \mathrm{d}\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f \mathrm{d}\mu.$$

## Claim 1.7

Let $f\colon \Omega \to \bar{\mathbb{R}}$ and $\mathcal{F}$-measurable. Then $\forall x \in \Omega_1$, $f_x\colon \Omega_2 \to \bar{\mathbb{R}}$ with $f_x(y) = f(x,y)$ is $\mathcal{F}_2$-measurable, that is, $f_x \in \mathcal{F}_2$.

## Proof.

Let $\bar{\mathcal{B}}$ be the $\sigma$-algebrea of the extended Borel sets. We will show $f_x^{-1}(B) \in \mathcal{F}_2$ for all $B \in \bar{\mathcal{B}}$, or $E_x \in \mathcal{F}_2$ for $E = f^{-1}(B)$. Because $E \in \mathcal{F}$, it follows that $E_x \in \mathcal{F}_2$ for $x \in \Omega_1$ and $E \in \mathcal{F}$. $\square$

# Monotone Class Lemma

## Lemma 1.2

(Monotone class lemma) Let $\mathcal{A}$ be an algebra, and $\mathcal{M}(\mathcal{A})$ be the monotone class generated by $\mathcal{A}$, $\sigma(\mathcal{A})$ be the $\sigma$-algebra generated by $\mathcal{A}$. Then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$.

## sketch.

It is clear that a $\sigma$-algebra is a monotone class, so $\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. To prove the converse inclusion $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, it suffices to prove that $\mathcal{M}(\mathcal{A})$ is a $\sigma$-algebra, and it suffices to prove that $\mathcal{M}(\mathcal{A})$ is an algebra because algebra + monotone $\Rightarrow$ $\sigma$-algebra.
To this end, for $E \in \mathcal{A}$, define $\mathcal{H}(E) = \{A\colon E \setminus A \in \mathcal{M}(\mathcal{A}), A \setminus E \in \mathcal{M}(\mathcal{A}), A \cap E \in \mathcal{M}(\mathcal{A})\}$. it is clear that $\mathcal{A} \subseteq \mathcal{H}(E)$ and $\mathcal{H}(E)$ is a monotone class, so $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{H}(E)$.
Now for $E \in \mathcal{M}(\mathcal{A})$, define $\mathcal{H}'(E) = \{A\colon E \setminus A \in \mathcal{M}(\mathcal{A}), A \setminus E \in \mathcal{M}(\mathcal{A}), A \cap E \in \mathcal{M}(\mathcal{A})\}$. By the above argument, $\mathcal{A} \subseteq \mathcal{H}'(E)$. $\mathcal{H}'(E)$ is also a monotone class, so $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{H}'(E)$. It follows that $\mathcal{M}(\mathcal{A})$ is an algebra, so it is also a $\sigma$-algebra $\Rightarrow \sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

□

# Monotone class lemma

## Proof.

$\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. We need to prove that $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, and we will show that $\mathcal{M}(\mathcal{A})$ is an algebra first. For $E \subseteq \mathcal{M}(\mathcal{A})$, define $\mathcal{G}(E) = \{F \in \mathcal{M}(\mathcal{A}) \colon E \setminus F, E \cap F, F \setminus E \in \mathcal{M}(\mathcal{A})\}$.
Claim: $E \in \mathcal{A} \Rightarrow \mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}(E)$. To show the claim, note that $\forall H \in \mathcal{A}$, $E \setminus F, E \cap F, F \setminus E \in \mathcal{A} \Rightarrow \mathcal{A} \subseteq \mathcal{G}(E)$. It follows that $\mathcal{A} \subseteq \mathcal{G}(E)$. Let $H_k \uparrow H$, $H_k \in \mathcal{G}(E)$, then $E \setminus H_k \in \mathcal{M}(\mathcal{A}) \downarrow E \setminus H$, so $E \setminus H \in \mathcal{M}(\mathcal{A})$ because $\mathcal{M}(\mathcal{A})$ is a monotone class. Similary, $E \cap H_k \in \mathcal{M}(\mathcal{A}) \uparrow E \cap H \in \mathcal{M}(\mathcal{A})$, and $H_k \setminus E \in \mathcal{M}(\mathcal{A}) \uparrow H \setminus E \in \mathcal{M}(\mathcal{A})$. It follows that $H \in \mathcal{G}(E)$ because $H \in \mathcal{M}(\mathcal{A})$, and $\mathcal{G}(E)$ is a monotone class (similar argument applies to $H_k \downarrow H$). Therefore, $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}(E)$.
Claim: $E \in \mathcal{M}(\mathcal{A}) \Rightarrow \mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}(E)$. We need to prove that (1) $\mathcal{G}(E)$ is a monotone class (2) $\mathcal{A} \subseteq \mathcal{G}(E)$. (1) can be approved by the same argument in the above claim. To prove (2), let $H \in \mathcal{A}$. By the above claim, we have $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}(H)$. Since $E \in \mathcal{M}(\mathcal{A}) \Rightarrow E \in \mathcal{G}(H) \Rightarrow E \setminus H, E \cap H, H \setminus E \in \mathcal{M}(\mathcal{A}) \Rightarrow H \in \mathcal{G}(E)$. It follows that $\mathcal{A} \subseteq \mathcal{G}(E)$.
Now we show $\mathcal{M}(\mathcal{A})$ is an algebra. (1) $\Omega \in \mathcal{M}(\mathcal{A})$ holds because $\mathcal{A}$ is an algebra. (2) $\forall E \in \mathcal{M}(\mathcal{A})$, then by the above claim ,$E \in \mathcal{G}(\Omega) \Rightarrow E^c \in \mathcal{M}(\mathcal{A})$. (3) $\forall E \in \mathcal{M}(\mathcal{A}), \forall F \in \mathcal{M}(\mathcal{A}) \Rightarrow E \in \mathcal{G}(F) \Rightarrow E \cap F \in \mathcal{M}(\mathcal{A})$.
Finally we show $\mathcal{M}(\mathcal{A})$ is a $\sigma$-algebra. Let $A_j \in \mathcal{M}(\mathcal{A})$, then $B_n = \bigcap_{j=1}^{n} A_n \in \mathcal{M}(\mathcal{A})$ because $\mathcal{M}(\mathcal{A})$ is an algebra. Because $\mathcal{M}(\mathcal{A})$ is a monotone class, $B_n \uparrow \bigcup_{j\geq 1} A_j \Rightarrow \bigcup_{j \geq 1} A_j \in \mathcal{M}(\mathcal{A})$.
So $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

□

# Absolute continuity of the Lebesgue integral

## Theorem 1.5

Assume $f$ is Lebesgue integrable. Then $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\int_A |f| \, \mathrm{d}\mu \leq \varepsilon$ if $\mu(A) \leq \delta$.

## Proof.

By Dominated Convergence Theorem (DCT),

$$\lim_{\lambda \to \infty} \int_{\{|f| > \lambda\}} f \mathrm{d}\mu = 0.$$

Note that

$$\int_A |f| \, \mathrm{d}\mu = \int_{A \cap \{|f| > \lambda\}} |f| \, \mathrm{d}\mu + \int_{A \cap \{|f| > \lambda\}^c} |f| \, \mathrm{d}\mu$$

$$\leq \int_{\{|f| > \lambda\}} |f| \, \mathrm{d}\mu + \lambda \mu(A) \leq \int_{\{|f| > \lambda\}} |f| \, \mathrm{d}\mu + \lambda \delta,$$

the conclusion is proved. $\square$

# Fubini Theorem

## Theorem 1.6

Let $\{\Omega_j, \mathcal{F}_j, \mu_j\}_{j=1}^2$ be measurable spaces, and $\{\Omega_j, \mu_j\}_{j=1}^2$ are $\sigma$-finite. $\mu = \mu_1 \times \mu_2, E \in \mathcal{F} = \mathcal{F}_1 \bigotimes \mathcal{F}_2$. $f\colon \Omega_1 \times \Omega_2 \to \bar{\mathbb{R}}, f \geq 0$. Then $x \to \mu_2(E_x)$ is $\mathcal{F}_1$-measurable, and $y \to \mu_2(E^y)$ is $\mathcal{F}_2$-measurable. Moreover,

$$\int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \int_{\Omega_2} \mu_2(E^y) \mathrm{d}\mu_2 = \mu(E).$$

## Proof.

Assume that $\mu_1(\Omega_1) < \infty, \mu_2(\Omega_2) < \infty$. $(i)$ If $E \in \mathcal{L}, E = A \times B$, then $\mu_2(E_x) = \mathbb{I}_{\{x \in A\}} \mu_2(B) \in \mathcal{F}_1$. $(ii)$ If $E \in \mathcal{A}$ and $\mathcal{A}$ is the algera generated by $\mathcal{L}$, then $E = \bigcup_{j=1}^n E_j = \bigcup_{j=1}^n A_j \times B_j$, and $\mu_2(E_x) = \sum_{j=1}^n \mathbb{I}_{\{x \in A_j\}} \mu_2(B_j) \in \mathcal{F}_1$. $(iii)$ We then define $\mathcal{G} = \{E \in \mathcal{F}\colon \mu_2(E_x) \in \mathcal{F}_1\}$. Then $\mathcal{L} \subseteq \mathcal{G}$. We now prove that $\mathcal{G}$ is a monotone class, so that $\mathcal{G}$ includes the $\sigma$-algebrea generated by $\mathcal{A}$. To this end, let $\{E^n\} \subseteq \mathcal{G}$ and $E^n \uparrow E$. It follows that $\mu_2(E) = \lim_n \mu_2(E_x^n)$. Because each $\mu_2(E_x^n) \in \mathcal{F}_1$ for $n \geq 1$, we have $\mu_2(E_x) \in \mathcal{F}_1 \Rightarrow E \in \mathcal{G}$. Now let $\{E^n\} \subseteq \mathcal{G}$ and $E^n \downarrow E$. It follows that $\mu_2(E) = \lim_n \mu_2(E_x^n)$ becuase $\mu_2(\Omega_2) < \infty$, and $\mu_2(E_x) \in \mathcal{F}_1 \Rightarrow E \in \mathcal{G}$.

As a result, $\mathcal{G}$ is a monotone class and it incudes the $\sigma$-algebra generated by $\mathcal{A}$ which is $\mathcal{F}$. On the other hand, $\mathcal{G} \subseteq \mathcal{F}$. So that $\mathcal{G} = \mathcal{F}$, and every $E \in \mathcal{F}$ satisfies $\mu_2(E_x) \in \mathcal{F}_1$. □

# Fubini Theorem

## Cont'd.

Now due to the $\sigma$-finite measurability, let $\{A_n\}, \{B_n\}$ be measurable sets in $\Omega_1$ and $\Omega_2$ with $\mu_1(A_n) < \infty, \mu_2(B_n) < \infty, A_n \in \mathcal{F}_1, B_n \in \mathcal{F}_2$ for all $n \geq 1$ and $\Omega_1 = \bigcup_{n\geq 1} A_n, \Omega_2 = \bigcup_{n\geq 1} B_n, A_n \uparrow, B_n \uparrow$. Then $\bigcup_{n\geq 1} F_n = \Omega_1 \times \Omega_2$ with $F_n = A_n \times B_n$ for $n \geq 1$. Using the argument above, $\mu_2\left((E \cap F_n)_x\right) \in \mathcal{F}_1$ for all $n \geq 1$. Because $(E \cap F_n)_x \uparrow E_x$, we have $\mu_2(E_x) = \lim_n \mu_2\left((E \cap F_n)_x\right) \in \mathcal{F}_1$.

Now we prove $\int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \mu(E)$. Assume that $\mu_1(\Omega_1) < \infty, \mu_2(\Omega_2) < \infty$. $(i)$ If $E \in \mathcal{L}, E = A \times B$. Then $\int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \int_{\Omega_1} \mathbb{I}_{\{x\in A\}} \mu_2(B) \mathrm{d}\mu_1 = \mu(E)$. $(ii)$ If $E \in \mathcal{A}$, then $\int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \sum_{j=1}^{n} \int_{\Omega_1} \mathbb{I}_{\{x\in A_j\}} \mu_2(B_j) \mathrm{d}\mu_1 = \sum_{j=1}^{n} \mu_1(A_j)\mu_2(B_j) = \mu(E)$. $(iii)$ We define $\mathcal{G} = \left\{ E \in \mathcal{F} \colon \int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \mu(E) \right\}$. Let $E_n \in \mathcal{G} \uparrow E$. Then by Monotone Convergence Theorem (MCT),
$\lim_n \int_{\Omega_1} \mu_2((E_n)_x) \mathrm{d}\mu_1 = \int_{\Omega_1} \lim_n \mu_2((E_n)_x) \mathrm{d}\mu_1 = \int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \lim_n \mu(E_n) = \mu(E)$.
Let $E_n \in \mathcal{G} \downarrow E$. Using Dominated Convergence Theorem (DCT), we still have
$\lim_n \int_{\Omega_1} \mu_2((E_n)_x) \mathrm{d}\mu_1 = \int_{\Omega_1} \lim_n \mu_2((E_n)_x) \mathrm{d}\mu_1 = \int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \lim_n \mu(E_n) = \mu(E)$
because $\mu_2(\Omega_2) < \infty$. Therefore, $\mathcal{G}$ is a monotone class which includes $\mathcal{A}$, and it follows that $\mathcal{G} = \mathcal{F}$.
Now let
$\Omega_1 = \bigcup_{n\geq 1} A_n, \Omega_2 = \bigcup_{n\geq 1} B_n, \mu_1(A_n) < \infty, \mu_2(B_n) < \infty, A_n \in \mathcal{F}_1, B_n \in \mathcal{F}_2, A_n \uparrow, B_n \uparrow$.
Then $\bigcup_{n\geq 1} F_n = \Omega_1 \times \Omega_2$ with $F_n = A_n \times B_n$ for $n \geq 1$. Using the argument above,
$\int_{\Omega_1} \mu_2((E \cap F_n)_x) \mathrm{d}\mu_1 = \mu(E \cap F_n)$ for all $n \geq 1$ and all $E \in \mathcal{F}$. By MCT,
$\int_{\Omega_1} \mu_2(E_x) \mathrm{d}\mu_1 = \mu(E)$. □

# Fubini Theorem

## Theorem 1.7

(Tonelli Theorem) Let $\{\Omega_j, \mathcal{F}_j, \mu_j\}_{j=1}^2$ be measurable spaces, and $\{\Omega_j, \mu_j\}_{j=1}^2$ are $\sigma$-finite. $\mu = \mu_1 \times \mu_2, \mathcal{F} = \mathcal{F}_1 \bigotimes \mathcal{F}_2$. $f \colon \Omega_1 \times \Omega_2 \to \bar{\mathbb{R}}, f \geq 0$.

$$\int_{\Omega_1} \left[ \int_{\Omega_2} f_x(y) \mathrm{d}\mu_2(y) \right] \mathrm{d}\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f \mathrm{d}\mu = \int_{\Omega_2} \left[ \int_{\Omega_1} f_y(x) \mathrm{d}\mu_1(x) \right] \mathrm{d}\mu_2(y).$$

## Proof.

$(i)$ Let $f = c \mathbb{I}_{\{E\}}$ with $c \geq 0, E \in \mathcal{F}$, then $f_x = c \mathbb{I}_{\{E_x\}}$, $\int_{\Omega_2} f_x \mathrm{d}\mu_2(y) = c\mu_2(E_x) \in \mathcal{F}_1$, and $\int_{\Omega_1} \int_{\Omega_2} f_x \mathrm{d}\mu_2(y) \mathrm{d}\mu_1(x) = \int_{\Omega_1} c\mu_2(E_x) \mathrm{d}\mu_1(x) = c\mu(E) = \int_{\Omega_1 \times \Omega_2} f \mathrm{d}\mu$. $(ii)$ Let $f = \sum_{j=1}^{n} c_j \mathbb{I}_{\{E_j\}}$ with $c_j \geq 0, E_j \in \mathcal{F}$ for all $j \in [n]$. By part $(i)$ and linearity of integral we have $\int_{\Omega_1} \left[ \int_{\Omega_2} f_x(y) \mathrm{d}\mu_2(y) \right] \mathrm{d}\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f \mathrm{d}\mu$. $(iii)$ Now let $f \geq 0$, and $\left\{ f^{(j)} \right\}_{j \geq 1}$ be a sequence of simple functions and $f^{(j)} \uparrow f$. Then $f_x^{(j)} \uparrow f_x$, so by MCT, $\int f_x^{(j)} \mathrm{d}\mu_2 \uparrow \int f_x \mathrm{d}\mu_2$. By applying MCT again, $\int \left[ \int f_x^{(j)} \mathrm{d}\mu_2 \right] \mathrm{d}\mu_1 \uparrow \int \left[ \int f_x \mathrm{d}\mu_2 \right] \mathrm{d}\mu_1$. On the other hand, by part $(ii)$ and MCT, $\int \left[ \int f_x^{(j)} \mathrm{d}\mu_2 \right] \mathrm{d}\mu_1 = \int f^{(j)} \mathrm{d}\mu \uparrow \int f \mathrm{d}\mu$. As a result, $\int \left[ \int f_x \mathrm{d}\mu_2 \right] \mathrm{d}\mu_1 = \int f \mathrm{d}\mu$. $\square$

# Fubini Theorem

## Theorem 1.8

(Fubini Theorem) Let $\{\Omega_j, \mathcal{F}_j, \mu_j\}_{j=1}^2$ be measurable spaces, and $\{\Omega_j, \mu_j\}_{j=1}^2$ are $\sigma$-finite. $\mu = \mu_1 \times \mu_2, \mathcal{F} = \mathcal{F}_1 \bigotimes \mathcal{F}_2$. $f\colon \Omega_1 \times \Omega_2 \to \bar{\mathbb{R}}$, and $f$ is integrable.

$$\int_{\Omega_1} \left[ \int_{\Omega_2} f_x(y) \mathrm{d}\mu_2(y) \right] \mathrm{d}\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f \mathrm{d}\mu = \int_{\Omega_2} \left[ \int_{\Omega_1} f_y(x) \mathrm{d}\mu_1(x) \right] \mathrm{d}\mu_2(y).$$

## Proof.

$f = f^+ - f^-$. Then $\int f^+ \mathrm{d}\mu < \infty$ and $\int f_x^+ \mathrm{d}\mu_2$ is $\mathcal{F}_1$-integrable. Define $E = \left\{x\colon \int f_x^+ \mathrm{d}\mu_2 < \infty\right\}$, and $g^+(x) = \int f_x^+ \mathrm{d}\mu_2$ if $x \in E$, and $g^+(x) = 0$ otherwise. Then $g^+(x) = \int f_x^+ \mathrm{d}\mu_2 \mathbb{I}_{\{E\}}$. Since $\int f_x^+ \mathrm{d}\mu_2$ is measurable, $E \in \mathcal{F}_1$ and $g^+$ is $\mathcal{F}_1$-measurable. Define $g^-(x) = \int f_x^- \mathrm{d}\mu_2 \mathbb{I}_{\{\int f^- \mathrm{d}\mu_2 < \infty\}}$, then $g^-$ is also measurable. We can then define $g = g^+ - g^-$ because $g^+(x) < \infty, g^-(x) < \infty$ for all $x \in \Omega_1$.

$$\int f \mathrm{d}\mu = \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu \overset{\text{Tonelli}}{=} \int \left[ \int f_x^+ \mathrm{d}\mu_2 \right] \mathrm{d}\mu_1 - \int \left[ \int f_x^- \mathrm{d}\mu_2 \right] \mathrm{d}\mu_1$$

$$= \int g^+(x) \mathrm{d}\mu_1(x) - \int g^-(x) \mathrm{d}\mu_1(x) = \int \left( g^+(x) - g^-(x) \right) \mathrm{d}\mu_1(x) = \int g(x) \mathrm{d}\mu_1(x)$$

# Fubini Theorem

## Remark 1.5

Let $\{\Omega_j, \mathcal{F}_j, \mu_j\}_{j=1}^2$ be measurable spaces, and $\{\Omega_j, \mu_j\}_{j=1}^2$ are $\sigma$-finite. $\mu = \mu_1 \times \mu_2, \mathcal{F} = \mathcal{F}_1 \bigotimes \mathcal{F}_2$, and $f$ is $\mathcal{F}$-measurable. If $\int \left[\int |f_x| \, \mathrm{d}\mu_2\right] \mathrm{d}\mu_1 < \infty$, then the conclusion of the Fubnini Theorem holds. This is because $f^+, f^- \leq |f|$, so $\int \left[\int f_x^+ \mathrm{d}\mu_2\right] \mathrm{d}\mu_1 \leq \int \left[\int |f_x| \, \mathrm{d}\mu_2\right] \mathrm{d}\mu_1 < \infty$, and for the same reason $\int \left[\int f_x^- \mathrm{d}\mu_2\right] \mathrm{d}\mu_1 < \infty$. By Tonelli Theorem, $\int f \mathrm{d}\mu = \int \left[\int f_x^+ \mathrm{d}\mu_2\right] \mathrm{d}\mu_1 - \int \left[\int f_x^- \mathrm{d}\mu_2\right] \mathrm{d}\mu_1$ so that $f$ is integrable, and Fubini Theorem holds.

# Hahn Decomposition Theorem

## Hahn decomposition theorem

From Wikipedia, the free encyclopedia

In mathematics, the **Hahn decomposition theorem**, named after the Austrian mathematician Hans Hahn, states that for any measurable space $(X, \Sigma)$ and any signed measure $\mu$ defined on the $\sigma$-algebra $\Sigma$, there exist two $\Sigma$-measurable sets, $P$ and $N$, of $X$ such that:

1. $P \cup N = X$ and $P \cap N = \varnothing$.
2. For every $E \in \Sigma$ such that $E \subseteq P$, one has $\mu(E) \geq 0$, i.e., $P$ is a positive set for $\mu$.
3. For every $E \in \Sigma$ such that $E \subseteq N$, one has $\mu(E) \leq 0$, i.e., $N$ is a negative set for $\mu$.

Moreover, this decomposition is essentially unique, meaning that for any other pair $(P', N')$ of $\Sigma$-measurable subsets of $X$ fulfilling the three conditions above, the symmetric differences $P \triangle P'$ and $N \triangle N'$ are $\mu$-null sets in the strong sense that every $\Sigma$-measurable subset of them has zero measure. The pair $(P, N)$ is then called a *Hahn decomposition* of the signed measure $\mu$.

# Hahn Decomposition Theorem

## Proof of the Hahn decomposition theorem [ edit ]

**Preparation:** Assume that $\mu$ does not take the value $-\infty$ (otherwise decompose according to $-\mu$). As mentioned above, a negative set is a set $A \in \Sigma$ such that $\mu(B) \leq 0$ for every $\Sigma$-measurable subset $B \subseteq A$.

**Claim:** Suppose that $D \in \Sigma$ satisfies $\mu(D) \leq 0$. Then there is a negative set $A \subseteq D$ such that $\mu(A) \leq \mu(D)$.

**Proof of the claim:** Define $A_0 := D$. Inductively assume for $n \in \mathbb{N}_0$ that $A_n \subseteq D$ has been constructed. Let

$$t_n := \sup(\{\mu(B) \mid B \in \Sigma \text{ and } B \subseteq A_n\})$$

denote the supremum of $\mu(B)$ over all the $\Sigma$-measurable subsets $B$ of $A_n$. This supremum might *a priori* be infinite. As the empty set $\varnothing$ is a possible candidate for $B$ in the definition of $t_n$, and as $\mu(\varnothing) = 0$, we have $t_n \geq 0$. By the definition of $t_n$, there then exists a $\Sigma$-measurable subset $B_n \subseteq A_n$ satisfying

$$\mu(B_n) \geq \min\left(1, \frac{t_n}{2}\right).$$

Set $A_{n+1} := A_n \setminus B_n$ to finish the induction step. Finally, define

$$A := D \Big\backslash \bigcup_{n=0}^{\infty} B_n.$$

As the sets $(B_n)_{n=0}^{\infty}$ are disjoint subsets of $D$, it follows from the sigma additivity of the signed measure $\mu$ that

$$\mu(A) = \mu(D) - \sum_{n=0}^{\infty} \mu(B_n) \leq \mu(D) - \sum_{n=0}^{\infty} \min\left(1, \frac{t_n}{2}\right).$$

This shows that $\mu(A) \leq \mu(D)$. Assume $A$ were not a negative set. This means that there would exist a $\Sigma$-measurable subset $B \subseteq A$ that satisfies $\mu(B) > 0$. Then $t_n \geq \mu(B)$ for every $n \in \mathbb{N}_0$, so the series on the right would have to diverge to $+\infty$, implying that $\mu(A) = -\infty$, which is not allowed. Therefore, $A$ must be a negative set.

**Construction of the decomposition:** Set $N_0 = \varnothing$. Inductively, given $N_n$, define

$$s_n := \inf(\{\mu(D) \mid D \in \Sigma \text{ and } D \subseteq X \setminus N_n\}).$$

as the infimum of $\mu(D)$ over all the $\Sigma$-measurable subsets $D$ of $X \setminus N_n$. This infimum might *a priori* be $-\infty$. As $\varnothing$ is a possible candidate for $D$ in the definition of $s_n$, and as $\mu(\varnothing) = 0$, we have $s_n \leq 0$. Hence, there exists a $\Sigma$-measurable subset $D_n \subseteq X \setminus N_n$ such that

$$\mu(D_n) \leq \max\left(\frac{s_n}{2}, -1\right) \leq 0.$$

By the claim above, there is a negative set $A_n \subseteq D_n$ such that $\mu(A_n) \leq \mu(D_n)$. Set $N_{n+1} := N_n \cup A_n$ to finish the induction step. Finally, define

$$N := \bigcup_{n=0}^{\infty} A_n.$$

As the sets $(A_n)_{n=0}^{\infty}$ are disjoint, we have for every $\Sigma$-measurable subset $B \subseteq N$ that

$$\mu(B) = \sum_{n=0}^{\infty} \mu(B \cap A_n)$$

by the sigma additivity of $\mu$. In particular, this shows that $N$ is a negative set. Next, define $P := X \setminus N$. If $P$ were not a positive set, there would exist a $\Sigma$-measurable subset $D \subseteq P$ with $\mu(D) < 0$. Then $s_n \leq \mu(D)$ for all $n \in \mathbb{N}_0$ and

$$\mu(N) = \sum_{n=0}^{\infty} \mu(A_n) \leq \sum_{n=0}^{\infty} \max\left(\frac{s_n}{2}, -1\right) = -\infty,$$

which is not allowed for $\mu$. Therefore, $P$ is a positive set.

**Proof of the uniqueness statement:** Suppose that $(N', P')$ is another Hahn decomposition of $X$. Then $P \cap N'$ is a positive set and also a negative set. Therefore, every measurable subset of it has measure zero. The same applies to $N \cap P'$. As

$$P \triangle P' = N \triangle N' = (P \cap N') \cup (N \cap P'),$$

this completes the proof. Q.E.D.

# Radon-Nikodym Theorem

- $(\Omega, \mathcal{F}, \mu)$, $\mu$ is $\sigma$-finite, $\mu\colon \mathcal{F} \to \bar{\mathbb{R}}^+$. $\nu$ is also $\sigma$-finite. $\nu\colon \mathcal{F} \to (-\infty, +\infty]$.

## Definition 1.9

$\nu \ll \mu$ ($\nu$ is absolutely continuous w.r.t. $\mu$) if $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

## Example 1.1

$f$ is an integrable function, and $\nu(A) = \int_A f \mathrm{d}\mu$, then $\nu \ll \mu$.

## Definition 1.10

$\nu \perp \mu$ ($\nu$ is singular w.r.t. $\mu$) if $\exists A \in \mathcal{F}$, $\mu(A) = 0, \nu(A^c) = 0$ (with $\nu\colon \mathcal{F} \to \mathbb{R}^+$, $\nu(E) = 0, \forall E \subseteq A^c$).

## Example 1.2

$\nu = \sum_j c_j \delta_{q_j}$ where $q_j \in \mathbb{Q}$ and $\sum_j c_j < \infty$, where $\delta$ is a Dirac measure with $\delta_x(A) = 1$ if $x \in A$, and $0$ otherwise. Let $\lambda$ be the Lebesgue measure, then $\nu(\mathbb{Q}^c) = 0, \lambda(\mathbb{Q}) = 0$.

# Radon-Nikodym Theorem

## Theorem 1.9

$(\Omega, \mathcal{F}, \mu)$, $\mu$ is $\sigma$-finite, $\mu\colon \mathcal{F} \to \bar{\mathbb{R}}^+$. $\nu$ is also $\sigma$-finite. $\nu\colon \mathcal{F} \to (-\infty, +\infty]$. Then (1) $\exists \nu_1, \nu_2$, $\nu = \nu_1 + \nu_2, \nu_1 \ll \mu, \nu_2 \perp \mu$; (2) Decomposition is unique, and $\exists f$ which is $\mathcal{F}$-measurable, and $\nu_1(A) = \int_A f \mathrm{d}\mu$ for all $A \in \mathcal{F}$.

## Proof.

$(i)$ We assume that (1) $\nu\colon \mathcal{F} \to \bar{\mathbb{R}}^+ = [0, \infty]$, and (2) $\nu, \mu$ are finite. Define $\mathcal{H} = \left\{f\colon f \geq 0, \int_A f \mathrm{d}\mu \leq \nu(A), \forall A \in \mathcal{F}\right\}$, and $\alpha = \sup_{f \in \mathcal{H}} \int_\Omega f \mathrm{d}\mu \leq \nu(\Omega) < \infty$. We will find $g \in \mathcal{H}$ s.t. $\int_\Omega g \mathrm{d}\mu = \alpha$. We will then let $\nu_1(A) = \int_A g \mathrm{d}\mu$ and $\nu_2 = \nu - \nu_1$, and drop the assumptions (1)-(2).

We have a function sequence $\{f_n\}_{n \geq 1}$ s.t. $\alpha - \frac{1}{n} \leq \int_\Omega f_n \mathrm{d}\mu \leq \alpha$. Let $g_n = \max\{f_1, f_2, \ldots, f_n\} \uparrow$. For $k \in [n]$, let $E_{n,k} = \{x\colon g_n(x) = f_k(x)\}$, then

$$\int_A g_n \mathrm{d}\mu = \sum_{k=1}^{n} \int_{A \cap E_{n,k}} f_k \mathrm{d}\mu \leq \sum_{k=1}^{n} \nu(A \cap E_{n,k}) = \nu(A).$$ So $g_n \in \mathcal{H}$. Since $g_n \uparrow$, let $g_n \uparrow g$.

Because $\int_A g_n \mathrm{d}\mu \leq \nu(A)$, by MCT, $\int_A g \mathrm{d}\mu \leq \nu(A) \Rightarrow g \in \mathcal{H}$.

$\int_\Omega g \mathrm{d}\mu \geq \int_\Omega g_n \mathrm{d}\mu \geq \int_\Omega f_n \mathrm{d}\mu \geq \alpha - \frac{1}{n}$ for all $n \geq 1$ and $\int_\Omega g \mathrm{d}\mu \leq \alpha$, so $\int_\Omega g \mathrm{d}\mu = \alpha$. Define $\nu_1(A) = \int_A g \mathrm{d}\mu$ and $\nu_2(A) = \nu(A) - \nu_1(A)$, $\forall A \in \mathcal{F}$. $\square$

# Radon-Nikodym Theorem

## Cont'd.

Define $\sigma_n = \nu_2 - \frac{1}{n}\mu$. $\sigma_n$ is a signed measure, so $\exists P_n, N_n$, $P_n = N_n^c, E \subseteq P_n \Rightarrow \sigma_n(E) \geq 0, E \subseteq N_n \Rightarrow \sigma_n(E) \leq 0$. Then $g + \frac{1}{n}\mathbb{I}_{\{P_n\}} \in \mathcal{H}$, because $\int_A g + \frac{1}{n}\mathbb{I}_{\{P_n\}} \mathrm{d}\mu = \nu_1(A) + \frac{1}{n}\mu(P_n \cap A) \leq \nu_1(A) + \nu_2(P_n \cap A) \leq \nu_1(A) + \nu_2(A) = \nu(A)$.
We have $\int_\Omega g + \frac{1}{n}\mathbb{I}_{\{P_n\}} \mathrm{d}\mu = \alpha + \frac{1}{n}\mu(P_n) \leq \alpha$, so $\mu(P_n) = 0$. Let $P = \bigcup_{n\geq 1} P_n$, $N = P^c = \bigcap_{n \geq 1} N_n$, then $\mu(P) = 0$. $\nu_2(N) \leq \nu_2(N_n) \leq \frac{1}{n}\mu(N_n) \leq \frac{1}{n}\mu(\Omega)$ for all $n \geq 1$, so $\nu_2(N) = 0$.
Now we remove assumption (2), and let $\mu, \nu$ be $\sigma$-finite.
$\exists E_n \uparrow \Omega, \bigcup_{n\geq 1} E_n = \Omega, \mu(E_n) < \infty, \exists F_n \uparrow \Omega, \bigcup_{n \geq 1} F_n = \Omega, \nu(F_n) < \infty$. Then $G_n = E_n \cap F_n \uparrow \Omega, \mu(G_n) < \infty, \nu(G_n) < \infty$. Let $H_k = G_k \setminus G_{k-1}, \bigcup_{j\geq 1} H_j = \Omega, \mu(H_j) < \infty, \nu(H_j) < \infty$, define $\mu_j(A) = \mu(H_j \cap A), \nu_j(A) = \nu(H_j \cap A)$. By the previous argument, $\nu_j = \nu_j^1 + \nu_j^2, \nu_j^1 \ll \mu_j, \nu_j^1 \perp \mu_j$. Let $\nu^1 = \sum_j \nu_j^1, \nu^2 = \sum_j \nu_j^2$, then $\nu^1 \ll \mu, \nu^2 \perp \mu$.

Now we remove assumption (1), and let $\nu\colon \mathcal{F} \to (-\infty, +\infty]$. By the Hahn Decompsition Theorem, $\nu = \theta_1 - \theta_2$ with $\theta_1(A) = \nu(P \cap A), \theta_2(A) = -\nu(P^c \cap A), \forall A \in \mathcal{F}$ where $P$ is by the Hahn decomposition theorem, $\theta_j : \mathcal{F} \to [0, \infty]$. By the previous argument, $\theta_1 = \theta_1^1 + \theta_1^2, \theta_1^1 \ll \mu, \theta_1^2 \perp \mu$. $\theta_2 = \theta_2^1 + \theta_2^2, \theta_2^1 \ll \mu, \theta_2^2 \perp \mu$. Then $\theta_1^1(A) = \int_A f_1^1 \mathrm{d}\mu, \theta_2^1(A) = \int_A f_2^1 \mathrm{d}\mu, \forall A \in \mathcal{F}$. Let $\nu_1 = \theta_1^1 - \theta_2^1$, then
$\nu_1(A) = \nu_1(A \cap P) + \nu_1(A \cap P^c) = \theta_1(A \cap P) - \theta_2(A \cap P^c) = \int_A (f_1^1 \mathbb{I}_{\{P\}} - f_2^1 \mathbb{I}_{\{P^c\}}) \mathrm{d}\mu \ll \mu$. □

# Radon-Nikodym Theorem

## Cont'd.

Note that $f^1 \triangleq f_1^1 \mathbb{I}_{\{P\}} - f_2^1 \mathbb{I}_{\{P^c\}}$ is well defined. Also, $\exists A_1, A_2 \in \mathcal{F}$ s.t. $\mu(A_1) = \mu(A_2) = 0, \theta_1^2(A_1^c) = \theta_2^2(A_2^c) = 0$. Let $A = A_1 \cup A_2$ and $\nu_2 = \theta_1^2 - \theta_2^2$, it follows that $\mu(A) = 0, \theta_1^2(A^c) = \theta_2^2(A^c) = 0 \Rightarrow \nu_2(A) = 0$. In this way, $\nu = \nu_1 + \nu_2$ with $\nu_1 \ll \mu, \nu_1(A) = \int_A f^1 \mathrm{d}\mu$, and $\nu_2 \perp \mu$.

Now we prove the uniqueness of the decomposition. Suppose that $\nu = \nu_1 + \nu_2 = \bar{\nu}_1 + \bar{\nu}_2$. $\exists A, B$ s.t. $\mu(A) = \mu(B) = 0, \nu_2(A^c \cap F) = \bar{\nu}_2(B^c \cap G) = 0, \forall F \in \mathcal{F}, \forall G \in \mathcal{F}$. Let $C = A \cup B$. $\forall E \in \mathcal{F}$, $\mu(E \cap C) = 0 \Rightarrow \nu_1(E \cap C) = \bar{\nu}_1(E \cap C) = 0$. Also, $\nu_2(E \cap C^c) = \bar{\nu}_2(E \cap C^c) = 0$. As a result,

$$\begin{aligned}(\nu_1 - \bar{\nu}_1)(E) &= (\nu_1 - \bar{\nu}_1)(E \cap C) + (\nu_1 - \bar{\nu}_1)(E \cap C^c) = (\nu_1 - \bar{\nu}_1)(E \cap C^c) \\ &= (\bar{\nu}_2 - \nu_2)(E \cap C^c) = 0.\end{aligned}$$

It follows that $\nu_1 = \bar{\nu}_1$, so $\nu_2 = \bar{\nu}_2$, and the decomposition of $\nu$ is unique. □

# Almost sure and almost uniform

- $(\Omega, \mathcal{F}, \mu)$, $\Omega, \mu$ is $\sigma$-finite, $\mathcal{F}$ is $\sigma$-complete. $f\colon \Omega \to \bar{\mathbb{R}}, g\colon \Omega \to \bar{\mathbb{R}}$. If $f$ is $\mathcal{F}$-measurable and $g = f$ a.e., then $g$ is also $\mathcal{F}$-measurable.
- $f \sim g$ if $\mu(f \neq g) = 0$. Let $M = \{f \mid f\colon \Omega \to \bar{\mathbb{R}}, f \in \mathcal{F}\}$, then we have the equivalent classes $\mathcal{M} = M| \sim$.
- $f_n \to f$ a.e.
- uniform convergence a.e.
- almost uniform convergence
- Pointwise convergence $f, f_n\colon E \to \bar{\mathbb{R}}$ if $\forall x \in E, f_n(x) \to f(x)$.
- Almost sure convergence $f, f_n\colon \Omega \to \mathbb{R}$.

**Definition 1.11**

If $f_n$

# Almost sure and almost uniform

## Definition 1.12

ess sup$f$ $= \inf\{a > 0\colon \mu(\{x\colon |f(x)| > a\}) = 0\}$.

- ess sup$f$ $\leq \sup_{x\in\Omega}|f|$. Also, let ess sup$f$ $= c$, then $\mu(\{x\colon |f(x)| > c\}) = 0$. To see this, we have $\forall n \geq 1, \mu(\left\{x\colon |f(x)| > c + \frac{1}{n}\right\}) = 0$. Then $\mu(\{x\colon |f(x)| > c\}) \leq \sum_{n\geq 1} \mu(\left\{x\colon |f(x)| > c + \frac{1}{n}\right\}) = 0$.
- $f \sim g \Rightarrow$ ess sup$f$ $=$ ess sup$g$. To see this, $f = g$ on $E \in \mathcal{F}$ with $\mu(E^c) = 0$. It can be verified that $\mu(\{x\colon |g(x)| > c\}) \leq \mu(\{x\colon |f(x)| > c\})$. Set $c =$ ess sup$f$, then $\mu(\{x\colon |g(x)|\}) > c) \leq \mu(\{x\colon |f(x)| > c\}) = 0$. Therefore, ess sup$g$ $\leq$ ess sup$f$. Switching $f$ and $g$ we have ess sup$f$ $\leq$ ess sup$g$, so that ess sup$f$ $=$ ess sup$g$.

# Almost sure and almost uniform

## Claim 1.8

$d(f,g) = \text{ess sup}\,|f-g|$ is a distance.

## Proof.

First, ess sup $|h| = 0 \Rightarrow h = 0$ a.e. Now we prove that $d(f,h) \leq d(f,g) + d(g,h)$. Set $a = d(f,g), b = d(g,h)$, we have

$$\mu(\{x\colon\ |f(x)-h(x)| > a+b\}) \leq \mu(\{x\colon\ |f(x)-g(x)| > a\}) + \mu(\{x\colon\ |g(x)-h(x)| > b\}) = 0$$

$$\Rightarrow d(f,h) \leq a+b = d(f,g) + d(g,h).$$

□

## Claim 1.9

$f_n \to f$ uniformly a.e. $\Leftrightarrow d(f_n, f) \to 0$.

## Proof.

$\Rightarrow$: $\exists E \in \mathcal{F}, \mu(E^c) = 0$, $f_n \to f$ uniformly on $E$. $\forall \varepsilon > 0$, $\exists n_0$ s.t. $d(f_n, f) \leq \varepsilon, \forall n \geq n_0$. The proves that $d(f_n, f) \to 0$.
$\Leftarrow$: ess sup $|f_n - f| \to 0$ indicates that $\forall \varepsilon > 0$, $\exists n_0$, s.t. ess sup $|f_n - f| \leq \varepsilon, \forall n \geq n_0$. It follows that $\mu(\{x\colon\ |f_n - f| > \varepsilon\}) = 0, \forall n \geq n_0$.

□

# Almost sure and almost uniform

## Cont'd.

$\forall k \geq 1, \exists n_k, \forall n \geq n_k, \mu(\left\{x\colon |f_n - f| > \frac{1}{2^k}\right\}) = 0 \Rightarrow \mu(\bigcup_{n \geq n_k} \left\{x\colon |f_n - f| > \frac{1}{2^k}\right\}) = 0$.
Therefore, $\mu(\bigcup_{k\geq 1} \bigcup_{n \geq n_k} \left\{x\colon |f_n - f| > \frac{1}{2^k}\right\}) = 0$. Let
$E = \bigcap_{k\geq 1} \bigcap_{n \geq n_k} \left\{x\colon |f_n - f| \leq \frac{1}{2^k}\right\}$, then $\mu(E^c) = 0$ and $f_n \to f$ uniformly on $E$. □

## Remark 1.6

Let $\mathcal{L}_\infty = \{f \in \mathcal{M}\colon \text{ess sup}\,|f| < \infty\}$. Then for $f, g \in \mathcal{L}_\infty, \alpha \in \mathbb{R}$, $\alpha f + g \in \mathcal{L}_\infty$. ess sup $|\alpha f| = |\alpha|$ ess sup $|f|$, and ess sup $|f + g| \leq$ ess sup $|f|$ + ess sup $|g|$.

## Example 1.3

Let $f_n(x) = x^n, x \in [0, 1], f(x) = 0$, then $f_n \to f$ a.e., but $f_n$ does not uniformly converge to $f$.

# Almost sure and almost uniform

## Definition 1.13

$f_n, f : \Omega \to \bar{\mathbb{R}}$, $f_n \to f$ almost unformly if $\forall \varepsilon > 0, \exists E_\varepsilon \in \mathcal{F}$ s.t. $\mu(E_\varepsilon^c) \leq \varepsilon$, and $f_n \to f$ uniformly on $E_\varepsilon$.

## Example 1.4

Let $f_n(x) = x^n, x \in [0,1], f(x) = 0$, then $f_n \to f$ a.e.. Let $E_\varepsilon = [0, 1-\varepsilon]$, then $\lambda(E_\varepsilon^c) \leq \varepsilon$, and $f_n$ uniformly converge to $f$ on $E_\varepsilon$.

## Claim 1.10

If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e.

## Proof.

$\forall k \geq 1, \exists E_k$, $f_n \to f$ uniformly on $E_k$ and $\mu(E_k^c) \leq \frac{1}{k}$. Then $f_n \to f$ a.e on $\bigcup_{k\geq 1} E_k$, and $\left(\bigcup_{k\geq 1} E_k\right)^c = \bigcap_{k\geq 1} E_k^c \leq \frac{1}{k}$ for all $k \geq 1$, so $\left(\bigcup_{k\geq 1} E_k\right)^c = \bigcap_{k\geq 1} E_k^c = 0$. □

# Almost sure and almost uniform

## Theorem 1.10

(Egoroff) Let $\mu(\Omega) < \infty$. Then $f_n \to f$ a.e. $\Rightarrow f_n \to f$ almost uniformly.

## Proof.

Let $A = \bigcap_{k\geq 1}\bigcup_{n\geq 1}\bigcap_{\ell\geq n}\left\{x\colon |f_\ell(x) - f(x)| \leq \frac{1}{k}\right\}$. Then $f_n$ does not converge to $f$ on $A^c$, so $\mu(A^c) = 0$. Note that $A^c = \bigcup_{k\geq 1}\bigcap_{n\geq 1}\bigcup_{\ell\geq n}\left\{x\colon |f_\ell(x) - f(x)| > \frac{1}{k}\right\}$. $\forall k \geq 1$, $\mu(\bigcap_{n\geq 1}\bigcup_{\ell\geq n}\left\{x\colon |f_\ell(x) - f(x)|) > \frac{1}{k}\right\}) = 0$. Let $A_n = \bigcup_{\ell\geq n}\left\{x\colon |f_\ell(x) - f(x)| > \frac{1}{k}\right\}$, then $A_{n+1} \subseteq A_n$, $A_n \downarrow \bigcap_{n\geq 1} A_n$. Since $\mu(\Omega) < \infty$, $\lim_n \mu(A_n) = \mu(\bigcap_{n\geq 1} A_n) = 0$. Therefore, $\exists n_{\varepsilon,k}, \forall n \geq n_{\varepsilon,k}$, $\mu(\bigcup_{\ell\geq n_{\varepsilon,k}}\left\{x\colon |f_\ell(x) - f(x)| > \frac{1}{k}\right\}) \leq \frac{\varepsilon}{2^k}$. Let $E_\varepsilon^c = \bigcup_{k\geq 1}\bigcup_{\ell\geq n_{\varepsilon,k}}\left\{x\colon |f_\ell(x) - f(x)| > \frac{1}{k}\right\}$, then $\mu(E_\varepsilon^c \leq \varepsilon)$, and $f_n \to f$ uniformly on $E_\varepsilon$. □

# Convergence in Measure

## Lemma 1.3

Let $\mu(\Omega) < \infty$. Then $f_n \to f$ a.e. $\Rightarrow f_n \overset{\text{m}}{\to} f$.

## Proof.

We need to prove that $\mu\left(\left\{x\colon |f_n(x) - f(x)| > \varepsilon'\right\}\right) \to 0$. That is, $\forall\delta > 0, \exists n_{\varepsilon,\delta}$ s.t. $\mu(\{x\colon |f_n(x) - f(x)| > \varepsilon\}) \leq \delta, \forall n \geq n_{\varepsilon,\delta}$.
By Egoroff Theorem, $f_n \to f$ almost uniformly. $\forall\varepsilon > 0, \exists E_\varepsilon \in \mathcal{F}$ s.t. $\mu(E_\varepsilon^c) \leq \varepsilon$, and $f_n \to f$ uniformly on $E_\varepsilon$. Set $\varepsilon = \delta$, then $\exists n_{\varepsilon',\delta}$, $|f_n(x) - f(x)| \leq \varepsilon', \forall n \geq n_{\varepsilon',\delta}$, and

$$\mu\left(\left\{x\colon |f_n(x) - f(x)| > \varepsilon'\right\}\right) \leq \mu\left(\left\{x\colon |f_n(x) - f(x)| > \varepsilon'\right\} \cap E_\delta\right)$$
$$+ \mu\left(\left\{x\colon |f_n(x) - f(x)| > \varepsilon'\right\} \cap E_\delta^c\right) \leq \mu(E_\delta^c) \leq \delta.$$

$\square$

# From convergence in measure to convergence

## Definition 1.14

(Equicontinuous) Measures $\{\mu_\alpha, \alpha \in I\}$ is equicontinuous at $\emptyset$ if $\forall \{B_k\}_{k\geq 1}, B_k \downarrow \emptyset$, $\forall \varepsilon > 0, \exists k_0$, if $k \geq k_0$, $\sup_{\alpha\in I} \mu_\alpha(B_k) \leq \varepsilon$.

## Definition 1.15

$\{\nu_\alpha, \alpha \in I\}$ uniformly absolute continuous w.r.t. $\mu$ if $\forall \varepsilon > 0, \exists \delta$, $\forall B, \mu(B) \leq \delta \Rightarrow \sup_{\alpha\in I} \nu_\alpha(B) \leq \varepsilon$.

## Example 1.5

$\|f_n\|_{L_p} = 1$ and $f_n \to 0$. $\nu_n(A) = \int_A |f_n|^p \,\mathrm{d}\mu, f_n \in L^p, \nu_n \ll \mu$. Let $B_k = (0, \frac{1}{k})$, then $\nu_n(B_k) = \int_{B_k} |f_n|^p \,\mathrm{d}\mu = 1$ for $n \geq k$, meaning that $\nu_n$ is not equicontinuous at $\emptyset$.

# From convergence in measure to convergence

## Theorem 1.11

$\{f_n\}_{n\geq 1}, f_n \in L^p$, $\{\nu_n\}_{n\geq 1}$ equicontinuous at $\emptyset$, $\nu_n(A) = \int_A |f_n|^p \, \mathrm{d}\mu$, $f_n \xrightarrow{\mathrm{m}} f$. Then $f_n \xrightarrow{L^p} f$.

To prove this theorem, we have the lemma below.

## Lemma 1.4

(equicontinuity+absolute continuity = uniformly absolute continuity) $\{\nu_\alpha, \alpha \in I\}$ equicontinuous at $\emptyset$, and $\nu_\alpha \ll \mu$. Then $\{\nu_\alpha, \alpha \in I\}$ is uniformly absolute continuous w.r.t. $\mu$.

## Proof.

Suppose $\{\nu_\alpha, \alpha \in I\}$ is not uniformly absolute continuous w.r.t. $\mu$, then $\exists \varepsilon > 0, \forall \delta = \frac{1}{2^n}, \exists B_n$, $\mu(B_n) \leq \frac{1}{2^n}$, and $\exists \alpha_n \in I, \nu_{\alpha_n}(B_n) > \varepsilon$. Let $A_k = \bigcup_{n\geq k} B_k$, then $\mu(A_k) \leq \frac{1}{2^{k-1}}$, and $A_k \downarrow A = \bigcap_{j\geq 1} \bigcup_{n\geq j} B_n$. Then $\mu(A) = 0 \Rightarrow \nu_\alpha(A) = 0$. We have $(A_k \setminus A) \downarrow \emptyset$, and $\nu_\alpha(A_k \setminus A) = \nu_\alpha(A_k) \geq \nu_\alpha(B_k)$. Set such $\alpha$ to $\alpha_n$, then $\nu_{\alpha_n}(A_k \setminus A) = \nu_{\alpha_n}(A_k) \geq \nu_{\alpha_n}(B_k) \geq \varepsilon$. This contradicts the fact that $\sup_{\alpha\in I} \nu_\alpha(A_k \setminus A) \overset{k\to\infty}{\to} 0$. $\square$

## From convergence in measure to convergence

**Proof of the Theorem.**

$\square$

**Corollary 1.1**

$\{f_n\}_{n\geq 1}, f_n \in L^p$, and $\exists h \in L^1, |f_n|^p \leq h \Rightarrow \{\nu_n\}_{n\geq 1}$ equicontinuous at $\emptyset$.

**Proof.**

$\nu_n(A) = \int_A |f_n|^p \, \mathrm{d}\mu \leq \int_A h \mathrm{d}\mu = \mu_h(A)$. Let $B_k \downarrow \emptyset$, then $\sup_n \nu_n(B_k) \leq \mu_h(B_k)$. Because $\mu_h$ is finite and $B_k \downarrow \emptyset$, $\mu_h(B_k) \to 0$, so the conclusion holds. $\square$

## From convergence in measure to convergence

### Corollary 1.2

$\{f_n\}_{n\geq 1}$, $f_n \in L^p$, and $\mu(\Omega) < \infty$ and $\{f_n\}$ is uniformly integrable, i.e. $\lim_{A\to\infty} \sup_n \int_{|f_n|>A} |f_n|^p \,\mathrm{d}\mu = 0$. Let $\nu_n(A) = \int_A |f_n|^p \,\mathrm{d}\mu$, then $\{\nu_n\}$ equicontinuous at $\emptyset$.

### Proof.

Let $B_k \downarrow \emptyset$. Then the conclusion follows from

$$\nu_n(B_k) = \int_{B_k} |f_n|^p \,\mathrm{d}\mu \leq A^p \mu(B_k) + \int_{|f_n|>A} |f_n|^p \,\mathrm{d}\mu.$$

$\square$

# Vitali's covering lemma

## Definition 1.16

$E \subseteq \mathbb{R}$, $\{I_\alpha, \alpha \in \mathcal{F}\}$ are intervals, which form a Vitali covering of $E$ if $\forall \varepsilon > 0, \forall x \in E, \exists I_\alpha, \alpha \in \mathcal{F}$ such that $x \in I_\alpha, 0 < |I_\alpha| < \varepsilon$.

## Lemma 1.5

$E \subseteq \mathbb{R}$, $\mathbf{I} = \{I_\alpha, \alpha \in \mathcal{F}\}$ is Vitali covering of $E$, $\{I_\alpha\}$ are closed intervals. Then $\exists \{I_j\}_{j\geq 1}$, $\{I_j\} \subseteq \mathbf{I}$, $I_i \cap I_j = \emptyset, i \neq j$, $\lambda^*(E \setminus \bigcup_{j\geq 1} I_j) = 0$.

## Proof.

First assume $E \subset (k, k+1)$ and prove $\lambda^*(E \setminus \bigcap_{j\geq 1} I_j) = 0, I_j \subseteq (k, k+1)$. Then let $E_k = E \cap (k, k+1)$, and $\lambda^*(E \setminus \bigcap_{j\geq 1} I_j^{(k)}) = 0$. We have $I_j^{(k)} \cap I_{j'}^{(k')} = \emptyset$ if $(j,k) \neq (j',k')$, and $E \subseteq \mathbb{Z} \cup \bigcup_{k\in\mathbb{Z}, j\geq 1} I_j^{(k)}$. Then $\left\{I_j^{(k)}\right\}_{j\geq 1, k\in\mathbb{Z}}$ is a Vitali covering of $E$.

Now we prove the lemma for $E \subseteq (0,1)$ (with $k=0$). It can be verified that $\bar{\mathbf{I}} = \{I_\alpha, \alpha \in \mathcal{F}, I_\alpha \subseteq (0,1)\}$ is a Vitali covering of $E$. $(i)$ If $E = \emptyset$, nothing needs to be proved. If $E \neq \emptyset, x \in E$, let $s_1 = \sup\{|I_\alpha| : \alpha \in \mathcal{F}, I_\alpha \subseteq (0,1)\}$. Then $0 < s_1 \leq 1$. There exists $I_1$ such that $|I_1| > \frac{s_1}{2}$. $(ii)$ If $E \setminus I_1 = \emptyset$, nothing needs to be proved. Otherwise, $x \in E \setminus I_1$. Let $s_2 = \sup\{|I_\alpha| : \alpha \in \mathcal{F}, I_\alpha \subseteq (0,1), I_\alpha \cap I_1 = \emptyset\}$. It can be verified that $0 < s_2 \leq 1$, and there exists $I_2$ such that $|I_2| > \frac{s_2}{2}$ and $I_2 \cap I_1 = \emptyset$. $(iii)$ Suppose we have $\{I_j\}_{1\leq j\leq N-1}$, and $E \setminus \bigcup_{1\leq j\leq N-1} I_j \neq \emptyset$. □

# Vitali's covering lemma

## Cont'd.

Let $s_N = \sup\{|I_\alpha|, \alpha \in \mathcal{F}, I_\alpha \subseteq (0,1), I_\alpha \cap \left(\bigcup_{1\le j\le N-1} I_j\right) = \emptyset\}$, then $0 < s_N \le 1$. There exists $I_N$ such that $|I_N| > \frac{s_N}{2}$ and $I_N \cap \left(\bigcup_{1\le j\le N-1} I_j\right) = \emptyset$.
Construct interval $K_j$ by extending $I_j$ to both sides with a length of $2|I_j|$. We will prove that $E \setminus \bigcup_{1\le j\le N} I_j \subseteq \bigcup_{1\le j\le N} K_j$. Let $x \in E \setminus \bigcup_{1\le j\le N} I_j$. Then $\exists J \in \mathbf{I}$ such that $x \in J$ and $J \cap \bigcup_{1\le j\le N} I_j = \emptyset$, $|J| \le s_{N+1}$. Note that $\sum\limits_{j\ge 1} |I_j| \le 1$, $\lim_{N\to\infty} \sum\limits_{j\ge N} |I_N| = 0$ and $\lim_{N\to\infty} \sum\limits_{j\ge N} s_N = 0$. Because $|J| \ne 0$, let $M$ be the smallest number such that $M > N$ and $J \cap I_M \ne \emptyset$, $J \cap \bigcup_{1\le j\le M-1} I_j = \emptyset$. We have $|J| \le s_M$. Since $|J| \le s_M \le 2|I_M|$ and $J \cap I_M \ne \emptyset$, $J \subseteq K_M$. Therefore, $E \setminus \bigcup_{1\le j\le N} I_j \subseteq \bigcup_{j>N} K_j$.

$$\lambda^*(E \setminus \bigcup_{1\le j\le N} I_j) \le \lambda^*(\bigcup_{j>N} K_j) \le \sum_{j>N} \lambda^*(K_j) \le \sum_{j>N} 5|I_j|.$$

The conclusion holds by noting that $\lim_{N\to\infty} \sum\limits_{j\ge N} |I_N| = 0$.

□

# Vitali's covering lemma

## Remark 1.7

The requirement that $\mathbf{I} = \{I_\alpha, \alpha \in \mathcal{F}\}$ are closed intervals can be removed. This is because if $\mathbf{I}$ is a Vitali covering of $E$, so is $\{\bar{I}_\alpha, \alpha \in \mathcal{F}\}$.

## Remark 1.8

If $\mathbf{I} = \{I_\alpha, \alpha \in \mathcal{F}\}$ (each $I_\alpha$ is a closed interval) is a Vitali covering of $E \subseteq [-L, L]$, then $\forall \varepsilon > 0, \exists \{I_j\}_{j=1}^N$ such that $\lambda^*(E \setminus \bigcup_{1 \le j \le N} I_j) \le \varepsilon$.

## Proof.

By the proof of Vitali's covering lemma, if $E \in [-L, L]$ (or bounded), then

$$\lambda^*(E \setminus \bigcup_{1 \le j \le N} I_j) \le \lambda^*(\bigcup_{j>N} K_j) \le \sum_{j>N} \lambda^*(K_j) \le \sum_{j>N} 5\,|I_j|\,.$$

The conclusion holds by noting that $\lim_{N\to\infty} \sum_{j \ge N} |I_N| = 0$. □

## Remark 1.9

# Differentiability of functions of bounded variations

- $f\colon [a,b] \to R$ is monotonically increasing. Define
$(D^+ f)(x) = \limsup_{h\downarrow 0} \frac{f(x+h)-f(x)}{h}$,
$(D^- f)(x) = \liminf_{h\downarrow 0} \frac{f(x+h)-f(x)}{h}$,
$(D_+ f)(x) = \limsup_{h\downarrow 0} \frac{f(x)-f(x-h)}{h}$,
$(D_- f)(x) = \liminf_{h\downarrow 0} \frac{f(x)-f(x-h)}{h}$.
- $f$ is monotonically increasing. Then $f$ is differentiable at $x$ if $(D^+ f)(x) = (D^- f)(x) = (D_+ f)(x) = (D_- f)(x)$, and we let $f'(x) = (D^+ f)(x)$, $E = \{x \in (a,b)\colon f \text{ is differentiable at } x\}$.

**Theorem 1.12**

$\lambda^*(E^c) = 0$.

# Differentiability of functions of bounded variations

## Proof.

Note that $E^c \subseteq \left\{x \in (a,b) \colon (D_- f)(x) < (D^+ f)(x)\right\} \cup \ldots$. It is enough to prove that $\lambda^* \left(\left\{x \in (a,b) \colon (D_- f)(x) < (D^+ f)(x)\right\}\right) = 0$.

$E_1 = \bigcup_{s,t\in\mathbb{Q}, s<t} \left\{x \in (a,b) \colon (D_- f)(x) < s < t < (D^+ f)(x)\right\} \triangleq \bigcup_{s,t\in\mathbb{Q}, s<t} E_{s,t}$.

We aim to prove that $\lambda^*(E_{s,t}) = 0$ by Vitali's covering lemma. For any $\varepsilon > 0$, there exists open set $G$ such that $\lambda^*(G) \le \lambda^*(E_{s,t}) + \varepsilon$. $\forall x \in E_{s,t}, \exists h_k \downarrow 0$ such that $f(x) - f(x - h_k) \le s h_k$. Then $\mathbf{I} = \{[x - h_k, x] \colon x \in E_{s,t}, f(x) - f(x - h_k) \le s h_k, [x - h_k, x] \subseteq G\}$ is a Vitali covering of $E$. So that $\exists \{I_j\}_{1\le j\le M} \subseteq \mathbf{I}$, $I_j = [x_j - h_j, x_j], I_j \cap I_k = \emptyset$, $\lambda^*(E_{s,t} \setminus \bigcup_{1\le j\le M} I_j) \le \varepsilon$. We have

$$\sum_{j=1}^{m} f(x_j) - f(x_j - h_j) \le s \sum_{j=1}^{M} |I_j| \le s\lambda^*(G) \le s(\lambda + \varepsilon) \text{ where } \lambda = \lambda^*(E_{s,t}).$$

Let $I_j^o = (x_j - h_j, x_j), B = E_{s,t} \cap \bigcup_{1\le j\le M} I_j^o$, then

$$\begin{aligned}
\lambda^*(E_{s,t}) &\le \lambda^*(B) + \lambda^*(E_{s,t} \setminus \bigcup_{1\le j\le M} I_j^o) \\
&\le \lambda^*(B) + \lambda^*(E_{s,t} \setminus \bigcup_{1\le j\le M} I_j) + \lambda^*(\{x_j - h_k, x_j\}_{1\le j\le M}) = \lambda^*(B) + \varepsilon
\end{aligned}$$

$\forall x \in E_{s,t}, \exists r_k \downarrow 0$ such that $f(x + r_k) - f(x) \ge t r_k$. Then $\mathbf{I}' = \left\{[y, y + r_k] \colon y \in B, f(y + r_k) - f(y) \ge t r_k, [y, y + r_k] \subseteq \text{some} I_j^o\right\}$ is a Vitali covering □

# Differentiability of functions of bounded variations

## Proof.

of $B$ (by making $r_k$ small enough). Then $\mathbf{I}'$ is a Vitali covering of $B$. By Vitali's covering lemma, $\exists\,\{J_k\}_{1\le k\le N} \subseteq \mathbf{I}'$ such that $\lambda^*(B \setminus \bigcup_{1\le k\le N} J_k) \le \varepsilon$. We have

$$\lambda^*(B) \le \lambda^*(B \cap \bigcup_{1\le k\le N} J_k) + \lambda^*(B \setminus \bigcup_{1\le k\le N} J_k) \le \sum_{k=1}^{N} |J_k| + \varepsilon$$

$$\Rightarrow \sum_{k=1}^{N} |J_k| \ge \lambda^*(B) - \varepsilon \ge \lambda^*(E_{s,t}) - 2\varepsilon = \lambda - 2\varepsilon.$$

Moreover,

$$\sum_{j=1}^{M} f(x_j) - f(x_j - h_j) \ge \sum_{k=1}^{N} f(y_k + r_k) - f(y_k) \ge t \sum_{k=1}^{N} |J_k| \ge t(\lambda - 2\varepsilon).$$

Therefore, $s(\lambda + \varepsilon) \ge t(\lambda - 2\varepsilon) \Rightarrow \lambda = 0$. □

# Differentiability of functions of bounded variations

## Lemma 1.6

$f\colon [a,b] \to \mathbb{R}$ monotonically increasing, $f'$ is Lebesgue measurable and $f' \geq 0$ a.e. $\int_a^b f' \mathrm{d}\lambda \leq f(b) - f(a)$.

## Proof.

Define $f_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} \to f'(x)$ a.e. This proves that $f'$ is Lebesgue measurable and $f' \geq 0$ a.e. By Fatou's lemma and letting $f(x) = f(b)$ for $x \geq b$,

$$\int_a^b f' \mathrm{d}\lambda \leq \liminf_{n\to\infty} \int_a^b f_n \mathrm{d}\lambda = \liminf_{n\to\infty} \int_a^b n \left( f(x + \frac{1}{n}) - f(x) \right) \mathrm{d}\lambda$$

$$= \liminf_{n\to\infty} n \left( \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f_n \mathrm{d}\lambda - \int_a^b f_n \mathrm{d}\lambda \right) = \liminf_{n\to\infty} n \left( \int_b^{b+\frac{1}{n}} f_n \mathrm{d}\lambda - \int_a^{a+\frac{1}{n}} f_n \mathrm{d}\lambda \right)$$

$$\leq \liminf_{n\to\infty} n \left( f(b) \cdot \frac{1}{n} - f(a) \cdot \frac{1}{n} \right) \leq f(b) - f(a).$$

□

# Differentiability of functions of bounded variations

- $f\colon [a,b] \to \mathbb{R}$, $\pi = \{a = t_0 < t_1 < \ldots < t_P = b\}$, $V_{f,\pi} = \sum_{i=0}^{P-1} |f(t_{i+1} - f(t_i))|$. $P_{f,\pi} = \sum_{i=0}^{P-1} (f(t_{i+1} - f(t_i)))^+$, $N_{f,\pi} = \sum_{i=0}^{P-1} (f(t_{i+1} - f(t_i)))^-$. It follows that $V_{f,\pi} = P_{f,\pi} + N_{f,\pi}$.
- $V_f = \sup_\pi V_{f,\pi}, P_f = \sup_\pi P_{f,\pi}, N_f = \sup_\pi N_{f,\pi}$.
- $f\colon [a,b] \to \mathbb{R}$ is of bounded variation if $V_f < \infty$.

**Claim 1.11**

$V_f = P_f + N_f, f(b) - f(a) = P_f - N_f$.

# Differentiability of functions of bounded variations

## Proof.

$\exists \pi_1, \pi_2$ such that $P_f - \varepsilon \le P_{f,\pi_1} \le P_f, N_f - \varepsilon \le N_{f,\pi_2} \le N_f$. Let $\pi = \pi_1 \cup \pi_2$, then

$$\begin{aligned} &P_f - \varepsilon \le P_{f,\pi} \le P_f, N_f - \varepsilon \le N_{f,\pi} \le N_f \\ &\Rightarrow P_f - N_f - \varepsilon \le P_{f,\pi} - N_{f,\pi} \le P_f - N_f + \varepsilon. \end{aligned}$$

It follows that $f(b) - f(a) = P_f - N_f$. Moreover, since

$$\begin{aligned} &P_f - \varepsilon \le P_{f,\pi} \le P_f, N_f - \varepsilon \le N_{f,\pi} \le N_f \\ &\Rightarrow P_f + N_f - 2\varepsilon \le P_{f,\pi} + N_{f,\pi} = V_{f,\pi} \le P_f + N_f, \end{aligned}$$

we have $P_f + N_f - 2\varepsilon \le V_{f,\pi} \le P_f + N_f \Rightarrow V_f = P_f + N_f$. □

- $f(x) - f(a) = P_f([a,x]) - N_f([a,x])$. Then $P_f([a,x]), N_f([a,x])$ are both monotonically increasing functions, so $f$ as a function of bounded variation is differentiable a.e.

## Absolutely Continuous Functions

- $f\colon [a,b] \to \mathbb{R}$ monotonically increasing, then $\exists f'(x)$ a.e., and $\int_a^b f' \mathrm{d}\lambda \leq f(b) - f(a)$.
- Let $f\colon [a,b] \to \mathbb{R}$ be integrable, $\int_a^b |f| \, \mathrm{d}\lambda < \infty$. Define $F(x) = C + \int_a^x f \mathrm{d}\lambda = \int \mathbb{1}_{\{[a,x]\}} f \mathrm{d}\lambda$.
- Recall the absolute continuity of Lebesgue integral: $\forall \varepsilon, \exists \delta, \lambda(B) < \delta \Rightarrow \int_B f \mathrm{d}\lambda < \varepsilon$.
- $\{I_1, I_2, \ldots, I_M\}, I_j = [a_j, b_j], I_i \cap I_j = \emptyset, \sum_{j=1}^{M} |I_j| < \delta$, then $\int_{\bigcup_{j=1}^M I_j} |f| \, \mathrm{d}\lambda \leq \varepsilon$. It follows that
$$\sum_{j=1}^{M} |F(b_j) - F(a_j)| = \sum_{j=1}^{M} \left| \int_{a_j}^{b_j} f \mathrm{d}\lambda \right| \leq \sum_{j=1}^{M} \int_{a_j}^{b_j} |f| \, \mathrm{d}\lambda \leq \varepsilon.$$

# Absolutely Continuous Functions

## Definition 1.17

$G\colon [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \varepsilon, \exists \delta$ such that

$$I_j = [a_j, b_j], I_i \cap I_j = \emptyset, \sum_{j=1}^{M} |I_j| \leq \delta, \text{ then } \sum_{j=1}^{M} |G(b_j) - G(a_j)| \leq \varepsilon.$$

## Theorem 1.13

Let $f\colon [a,b] \to \mathbb{R}$ and $f$ is integrable, $\int_a^b |f| \, \mathrm{d}\lambda < \infty$.
$F(x) = C + \int_a^x f \mathrm{d}\lambda$. Then $F$ is differentiable, and $F' = f$ a.e.

# Absolutely Continuous Functions

## Proof.

$f = f^+ - f^-$. Define $F_+(x) = C + \int_a^x f^+ \mathrm{d}\lambda$ and $F_-(x) = C + \int_a^x f^- \mathrm{d}\lambda$. We prove that $F_\pm$ is differentiable a.e. with $F'_\pm = f^\pm$.
Suppose that $f \geq 0$. Then $F$ is monotonically increasing function, and $F$ is differentiable a.e., and $\int_a^x F' \mathrm{d}\lambda \leq F(x) - F(a) = \int_a^x f \mathrm{d}\lambda$. $(i)$ Assume $f$ is bounded, $f \leq K$. Define $f_n(x) = \frac{F(x+1/n)-F(x)}{1/n} \to F'(x)$ a.e., and $f_n \leq K$. Therefore, by Dominated Convergence Theorem,

$$\begin{aligned}
\int_a^x F' \mathrm{d}\lambda &= \lim_n \int_a^x f_n \mathrm{d}\lambda = \lim_n n \int_a^x \left(F(x+1/n) - F(x)\right) \mathrm{d}\lambda \\
&= \lim_n n \left[ \int_{a+1/n}^{x+1/n} F \mathrm{d}\lambda - \int_a^x F \mathrm{d}\lambda \right] = \lim_n n \left[ \int_x^{x+1/n} F \mathrm{d}\lambda - \int_a^{a+1/n} F \mathrm{d}\lambda \right] \\
&\overset{*}{=} F(x) - F(a) = \int_a^x f \mathrm{d}\lambda,
\end{aligned}$$

where $(*)$ is due to the continuity of $F$. $(ii)$ Now consider the general case and let $f_M = f \wedge M$, $F_M(x) = C + \int_a^x f_M \mathrm{d}\lambda \leq C + \int_a^x f \mathrm{d}\lambda = F(x)$. By part $(i)$, $\int_a^x F'_M \mathrm{d}\lambda = \int_a^x f_M \mathrm{d}\lambda$. We also have
$F'_M(x) = \lim_{h\to 0} \frac{F_M(x+h)-F_M(x)}{h} = \lim_{h\to 0} \int_x^{x+h} f_M \mathrm{d}\lambda \leq \lim_{h\to 0} \int_x^{x+h} f \mathrm{d}\lambda = F'(x)$ □

## Cont'd.

By MCT and the above arguments,
$\int_a^x f \mathrm{d}\lambda = \lim_{M\to\infty} \int_a^x f_M \mathrm{d}\lambda = \lim_{M\to\infty} \int_a^x F'_M \mathrm{d}\lambda \le \int_a^x F' \mathrm{d}\lambda$. We already know that $\int_a^x F' \mathrm{d}\lambda \le \int_a^x f \mathrm{d}\lambda$. Therefore, $\int_a^x F' \mathrm{d}\lambda = \int_a^x f \mathrm{d}\lambda$ □

## Theorem 1.14

If $f\colon [a,b] \to \mathbb{R}$ absolutely continuous, then $f$ is differentiable a.e., $f'$ is integrable, and $f(x) = f(a) + \int_a^x f' \mathrm{d}\lambda$.

## Claim 1.12

If $f\colon [a,b] \to \mathbb{R}$ absolutely continuous, then $f$ has bounded variation.

## Proof of the theorem.

Because $f$ has bounded variation, $f = f_1 - f_2$ where $f_1, f_2$ are monotonically increasing which are differentiable a.e. $f' = f_1' - f_2'$. The total variation of $f_1$ is $V_{f_1} = f_1(b) - f_1(a) \le V_f$. We have $\int f_1' \mathrm{d}\lambda \le f_1(b) - f_1(a) \Rightarrow f_1'$ is integrable. By the same argument $f_2'$ is integrable. $f'$ as the difference of two integrable functions is integrable. Now we prove $f(x) = f(a) + \int_a^x f' \mathrm{d}\lambda$.
To this end, define $G(x) = f(a) + \int_a^x f' \mathrm{d}\lambda$. Then $G$ is absolutely continuous. By the first theorem, $G$ is differentiable and $G' = f'$ a.e. Define $F = G - f$ as the difference of two absolutely continuous functions, and $F$ is absolutely continuous. Also, $F' = 0$ a.e. By the lemma below, $F$ is constant. So $G - f = C$ and $C = 0$ by taking $x = a$. □

# Absolutely Continuous Functions

## Lemma 1.7

$F\colon [a,b] \to \mathbb{R}$ is absolutely continuous and $F' = 0$ a.e., then $F$ is constant.

## Proof.

Let $c \in (a,b)$. We will construct a Vitali covering of a subset of $(a,c)$. Define $E = \left\{ x \in (a,c) \colon F'(x) = 0 \right\}$. Because $F' = 0$ a.e., $\lambda([a,c] \setminus E) = 0$. For $x \in E$, since $F'(x) = 0$, $\lim_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} = 0$. As a result, $\exists h_0, \forall h < h_0$, $|F(x+h) - F(x)| \leq \varepsilon h$. We construct a family $\mathbf{I} = \{[x, x+h] \colon \ |F(x+h) - F(x)| \leq \varepsilon h, [x, x+h] \subseteq (a,c), x \in E\}$. Then $\mathbf{I}$ is a Vitali covering of $E$. By the Vitali's lemma, given $\delta > 0$, $\exists \left\{I_j\right\}_{j=1}^{M}, I_i \cap I_j = \emptyset, I_j = [x_j, x_j + h_j]$, such that $\lambda(E \setminus \bigcup_{j=1}^{M} I_j) \leq \delta$. Because $\lambda([a,c] \setminus E) = 0, \lambda(E \setminus \bigcup_{j=1}^{M} I_j) \leq \delta$, we have $\lambda([a,c] \setminus \bigcup_{j=1}^{M} I_j) \leq \delta$. We have

$$
\begin{aligned}
&|F(c) - F(a)| \leq |F(x_1) - F(a)| + |F(x_1 + h_1) - F(x_1)| + |F(x_2) - F(x_1 + h_1)| \\
&+ \ldots + |F(b) - F(x_M + h_M)| = \sum_{j=1}^{M} |F(x_j + h_j) - F(x_j)| + \sum_{j=2}^{M} |F(x_j) - F(x_{j-1} + h_{j-1})| \\
&+ |F(x_1) - F(a)| + |F(b) - F(x_M + h_M)| \leq \varepsilon(c-a) + \varepsilon = \varepsilon(c - a + 1),
\end{aligned}
$$

where the last inequality is due to the absolutely continuity $(\delta, \varepsilon)$ of $F$. $\square$

# Absolutely Continuous Functions

•

## Absolutely Continuous Functions

Thank you!
Questions?