Contents

Here everything can be written down that helps with understanding the Non-Linear Wave Equations Lecture and solving the exercises.

1 Norms

Definition 1 (Sobolev space)**.** *The following is called Sobolev space:*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \| \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k : \exists D^\alpha u \in L^p(\Omega)\} \tag{1}$$

with the Sobolev Norm, if $p < \infty$:

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}} \tag{2}$$

and if $p = \infty$:

$$\|u\|_{W^{k,p}(\Omega)} = \max_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}. \tag{3}$$

Notation:

$$H^k(\Omega) := W^{k,2}(\Omega) \tag{4}$$

$W^{k,p}(\Omega)$ is a Banach space. $C^{k,p}(\Omega)$ is a dense subset of $W^{k,p}(\Omega) \subset L^p(\Omega)$. $H^k(\Omega)$ is a Hilbertspace.

Definition 2 (fractional Sobolev space; source: exercise session 1)**.** *For $s \in \mathbb{R}$ (?) $H^s(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm*

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} \tag{5}$$

where $(1+|\xi|^2)^{\frac{s}{2}}$ corresponds to a derivative in physical space.

Definition 3 (homogeneous Sobolev norm; source: mathoverflow(!!))**.** *If the following quantity is bounded, then we have*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \tag{6}$$

Definition 4 ($\dot{H}^1$-norm; source: notes p 16 (!!))**.**

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\nabla \phi|^2 dx\Big)^{\frac{1}{2}} \tag{7}$$

2 Functional Analysis Theorems

Definition 5 (adjoint operator). *The adjoint of a (bounded, linear) operator $T : X \to Y$ is defined by*

$$T^* : Y \to X, \langle Tx, y\rangle = \langle x, T^* y\rangle. \tag{8}$$

We have that $T^{**} = T$ and $\|T\| = \|T^*\|$. (letzte Aussage überprüfen)

Theorem 6 (Gronwall´s Lemma). *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and $g : \mathbb{R} \to \mathbb{R}$ be a non-negative integrable function such that*

$$f(t) \leq A + \int_0^t f(s)g(s)ds \tag{9}$$

holds for some $A \geq 0$ for every $t \in [0, T]$. Then

$$f(t) \leq A \exp(\int_0^t g(s)ds) \tag{10}$$

holds for every $t \in [0, T]$.

Theorem 7 (Hahn-Banach). *Let ϕ be a bounded linear functional on a subspace M of a normed (real) vectorspace X. Then there exists a bounded linear functional Φ on X which is an extension of ϕ to X and has the same norm, i.e.*

$$\|\Phi\|_X = \|\phi\|_M, \tag{11}$$

where

$$\|\Phi\|_X = \sup_{x\in X, \|x\|=1} |\Phi(x)|, \|\phi\|_M = \sup_{x\in M, \|x\|=1} |\phi(x)|. \tag{12}$$

Theorem 8 (Arzelà-Ascoli theorem, source: Wiki). *Consider a sequence of functions (f_n) defined on a compact interval. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence (f_{n_k}) that converges uniformly.*

Definition 9 (uniformly equicontinuous). *A sequence of functions (f_n) is said to be uniformly equicontinuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$|f_n(x) - f_n(y)| < \varepsilon \tag{13}$$

whenever $|x - y| < \delta$ for all functions f_n in the sequence.

This is the version of Arzelà-Ascoli used on example sheet 6, problem*, in a specific norm:

Theorem 10 (Arzelà-Ascoli theorem, source: solution to sheet 5)**.** *Suppose there is a sequence (of solutions)* $(\phi^{(i)})$ *in* $C^{k-1}([0,T]\times\mathbb{R}^n)$ *that converges to* ϕ *in a* $C^{k-1}([0,T]\times\mathbb{R}^n)$*-sense and is uniformly bounded, meaning* $\|\partial^k\phi\|_{L^\infty}\leq A$ *(A not dependent on i).*
Then, there exists a subsequence $(\phi^{(i_\lambda)})$ *that converges in* $C^k([0,T]\times\mathbb{R}^n)$*.*

Theorem 11 (Banach-Alaoglu, basic version)**.** *Let* (u_k) *be a bounded sequence in a Hilbert space* H*, i.e.* $\|u_k\|_H\leq C$*. Then there exists a subsequence which converges weakly in* H*.*

Definition 12 (Weak Convergence; source: Wiki)**.** *A sequence of elements* (x_n) *in a Hilbert space* H *is said to converge weakly to* $x\in H$ *if*

$$\langle x_n, y\rangle \to \langle x, y\rangle \tag{14}$$

for all $y\in H$*,* $\langle\cdot,\cdot\rangle$ *denoting the inner product of* H*.*
Notation: $x_n \rightharpoonup x$

Theorem 13 (Riesz Representation thm)**.**

3 Fourier Transformation

Open Question: What is the advantage of Fourier representation?

Definition 14 (Schwartz space)**.** *The Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ *is a vectorspace given by*

$$\mathcal{S}(\mathbb{R}^n) = \{f\in C^\infty(\mathbb{R}^n)|\forall\alpha,\beta\in\mathbb{N}_0^n, \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta f(x)|<\infty\} \tag{15}$$

equipped with a countable family of semi-norms

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta f(x)|. \tag{16}$$

You can think of the Schwartz space as functions that decay faster than any polynomial.

Definition 15 (Fourier transform)**.** *Given a function* $f\in\mathcal{S}(\mathbb{R}^n)$*, we define the Fourier transform by*

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx. \tag{17}$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}(f)(\xi) := \int_{\mathbb{R}^n} f(x)e^{+2\pi i x\cdot\xi}dx. \tag{18}$$

Theorem 16 (properties of Fourier transform and norms; source: exercise session 1)**.** *For $f \in \mathcal{S}(\mathbb{R}^n)$ the following holds:*

1. $\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$
2. $\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$

Theorem 17 (properties of the Fourier transform; source: exercise session 1)**.** *For $f \in \mathcal{S}(\mathbb{R}^n)$ the following properties hold:*

1. $\mathcal{F}(\partial_i f(x)) = 2\pi i \xi_i \mathcal{F}(f)(\xi)$
2. $\mathcal{F}(x_i f(x)) = -\frac{1}{2\pi i}\partial_{\xi_i}\mathcal{F}(f)(\xi)$

4 Important Inequalities

Theorem 18 (The interpolation inequality)**.** *We have*

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C(s_1, s_2, s_n)\|f\|^{\theta_1}_{H^{s_1}(\mathbb{R}^n)}\|f\|^{\theta_2}_{H^{s_2}(\mathbb{R}^n)}, \tag{19}$$

where $0 \leq s_1 \leq s \leq s_2$, $\theta_1 + \theta_2 = 1$ and $\theta_1 s_1 + \theta_2 s_2 = s$.

Theorem 19 (Gargliardo-Nirenberg interpolation estimate; source: sheet 5)**.** *Let $\phi \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For $0 < |\alpha| < s$ we have*

$$\|D^\alpha \phi\|_{L^{\frac{2s}{|\alpha|}}(\mathbb{R}^n)} \leq C(|\alpha|, s, n)(\|\phi\|_{L^\infty(\mathbb{R}^n)})^{1-\frac{|\alpha|}{s}}(\|\phi\|_{\dot{H}^s(\mathbb{R}^n)})^{\frac{|\alpha|}{s}}. \tag{20}$$

4.1 Sobolev Embedding

Theorem 20 (Sobolev embedding thm; source: sheet 1)**.** *There exists a constant $C = C(n, s) > 0$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ we have*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \tag{21}$$

Theorem 21 (source: sheet 1)**.** *If s is a non-negative integer, there exists a constant $C = C(s)$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ we have*

$$\frac{1}{C}\sum_{|\alpha|\leq s}\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C\sum_{|\alpha|\leq s}\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}. \tag{22}$$

Theorem 22 (source: solution to sheet 3)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. If $s > k + \frac{n}{2}$, then*

$$\|f\|_{C^k_b(\mathbb{R}^n,\mathbb{C})} \leq C\|f\|_{H^s}, \tag{23}$$

where the b means bounded.

Theorem 23 (source: sheet 6)**.** *For* $0 \leq s < \frac{n}{2}$ *there exists a constant* $C = C(s,n)$*, such that*

$$\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq C\|\phi\|_{\dot{H}^s(\mathbb{R}^n)}. \tag{24}$$

Theorem 24 (second Sobolev embedding thm; source: Wiki)**.** *If* $n < pk$ *and* $r + \alpha = k - \frac{n}{p}$ *with* $\alpha \in (0,1)$*, then one has the embedding*

$$W^{k,p}(\mathbb{R}^n) \subset C^{r,\alpha}(\mathbb{R}^n). \tag{25}$$

5 Basic Analysis, Measure Theory

Theorem 25 (Young´s Inequality)**.** *Let* p *and* q *be positive real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then, for any non-negative* a *and* b *it holds*

$$ab = \frac{a^p}{p} + \frac{b^q}{q}. \tag{26}$$

Theorem 26 (Young´s Inequality Special case)**.** *Let* $x, y \in \mathbb{R}^n$*,* $\delta > 0$*,* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then we have*

$$|xy| \leq \delta|x|^2 + \frac{1}{4\delta}|y|^2. \tag{27}$$

Theorem 27 (Hölder´s Inequality; source: Wiki)**.** *Let* p *and* q *be positive real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$*.Then, if* $f \in L^p$ *and* $g \in L^q$*:*

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{28}$$

Theorem 28 (Lebesgue Monotone Convergence; source: math3ma.com)**.** *Suppose* $(f_n : X \to [0,\infty))$ *being a monotonically increasing sequence of measurable functions on a measurable set* X *such that* $f_n \to f$ *pointwise almost everywhere, then*

$$\lim_{n\to\infty} \int_X f_n = \int_X f. \tag{29}$$

Theorem 29 (Fatou's Lemma; source: Wiki)**.** *Given a measure space* $(\Omega, \mathcal{F}, \mu)$ *and a set* $X \in \mathcal{F}$*, let* $(f_n : X \to [0,\infty])$ *be a sequence of measurable funtions. Define the function* $F_X \to [0,\infty]$ *by setting* $f(x) = \liminf_{n\to\infty} f_n(x)$ *for every* $x \in X$*.*
Then f *is measurable, and also*

$$\int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu \tag{30}$$

where the integrals may be infinite.

Theorem 30 (Change of Variables Formula; source: Wiki)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $\Phi : \Omega \to \Phi(\Omega) \subset \mathbb{R}^n$ *a diffeomorphism. A function* f *is integrable on* $\Phi(\Omega)$ *iff the function* $x \mapsto f(\Phi(x))|Det(D\Phi(x))|$ *is integrable on* Ω*. Then we have*

$$\int_{\Phi(\Omega)} f(y)dy = \int_{\Omega} f(\Phi(x))|\det(D\Phi(x))|dx. \tag{31}$$

Theorem 31 (Picard-Lindelöf)**.** *Let* $G \subset \mathbb{R} \times \mathbb{R}^n$ *be open and* $f : G \to \mathbb{R}^n$ *a continuous function satisfying a Lipschitz property locally. Then for every* $(a, c) \in G$ *there exists a* $\epsilon > 0$ *and a solution*

$$\phi : [a - \epsilon, a + \epsilon] \to \mathbb{R}^n \tag{32}$$

for the ODE $y' = f(x, y)$ *with initial data* $\phi(a) = c$*.*

Theorem 32 (Chain Rule)**.**

$$\frac{\partial}{\partial x} F(\phi(x)) = \frac{\partial F}{\partial \phi} \frac{\partial \phi}{\partial x} \tag{33}$$