

Contents

1	Norms	2
2	Functional Analysis Theorems	3
3	Fourier Transformation	5
4	Important Inequalities	6
4.1	Sobolev Embedding	6
5	Basic Analysis, Measure Theory	7

Here everything can be written down that helps with understanding the Non-Linear Wave Equations Lecture and solving the exercises.

1 Norms

Definition 1 (Sobolev space). *The following is called Sobolev space:*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k : \exists D^\alpha u \in L^p(\Omega)\} \quad (1)$$

with the Sobolev Norm, if $p < \infty$:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \quad (2)$$

and if $p = \infty$:

$$\|u\|_{W^{k,p}(\Omega)} = \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}. \quad (3)$$

Notation:

$$H^k(\Omega) := W^{k,2}(\Omega) \quad (4)$$

$W^{k,p}(\Omega)$ is a Banach space. $C^{k,p}(\Omega)$ is a dense subset of $W^{k,p}(\Omega) \subset L^p(\Omega)$. $H^k(\Omega)$ is a Hilbertspace.

Definition 2 (fractional Sobolev space; source: exercise session 1). *For $s \in \mathbb{R}$ (?) $H^s(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm*

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} \quad (5)$$

where $(1 + |\xi|^2)^{\frac{s}{2}}$ corresponds to a derivative in physical space.

Definition 3 (homogeneous Sobolev norm; source: mathoverflow(!)). *If the following quantity is bounded, then we have*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \quad (6)$$

Definition 4 (\dot{H}^1 -norm; source: notes p 16 (!)).

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \quad (7)$$

Definition 5 (mixed spaces; source: lecture notes p.39). *For $1 \leq r < \infty$, $1 \leq q < \infty$ we define $L_t^q L_x^r$ to be the space of (equivalence class of) measurable functions on $\mathbb{R}_t \times \mathbb{R}_x^n$ such that the norm*

$$\|u\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} dt \left(\int_{\mathbb{R}^n} |u(t, x)|^r dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}} \quad (8)$$

Theorem 6 (properties of mixed spaces; source: lecture notes p.39). *The following facts hold for the mixed spaces $L_t^q L_x^r$:*

1. *The $L_t^q L_x^r$ are Banach spaces.*
2. *Smooth functions of compact support $C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$ are dense in $L_t^q L_x^r$.*
3. *The dual of $L_t^q L_x^r$ can be identified with $L_t^{q'} L_x^{r'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.*

2 Functional Analysis Theorems

Definition 7 (adjoint operator). *The adjoint of a (bounded, linear) operator $T : X \rightarrow Y$ is defined by*

$$T^* : Y \rightarrow X, \langle Tx, y \rangle = \langle x, T^*y \rangle. \quad (9)$$

*We have that $T^{**} = T$ and $\|T\| = \|T^*\|$. (letzte Aussage überprüfen)*

Theorem 8 (Gronwall's Lemma). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative continuous function and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative integrable function such that*

$$f(t) \leq A + \int_0^t f(s)g(s)ds \quad (10)$$

holds for some $A \geq 0$ for every $t \in [0, T]$. Then

$$f(t) \leq A \exp\left(\int_0^t g(s)ds\right) \quad (11)$$

holds for every $t \in [0, T]$.

Theorem 9 (Hahn-Banach). *Let ϕ be a bounded linear functional on a subspace M of a normed (real) vectorspace X . Then there exists a bounded linear functional Φ on X which is an extension of ϕ to X and has the same norm, i.e.*

$$\|\Phi\|_X = \|\phi\|_M, \quad (12)$$

where

$$\|\Phi\|_X = \sup_{x \in X, \|x\|=1} |\Phi(x)|, \|\phi\|_M = \sup_{x \in M, \|x\|=1} |\phi(x)|. \quad (13)$$

Theorem 10 (Arzelà-Ascoli theorem, source: Wiki). *Consider a sequence of functions (f_n) defined on a compact interval. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence (f_{n_k}) that converges uniformly.*

Definition 11 (uniformly equicontinuous). *A sequence of functions (f_n) is said to be uniformly equicontinuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$|f_n(x) - f_n(y)| < \varepsilon \quad (14)$$

whenever $|x - y| < \delta$ for all functions f_n in the sequence.

This is the version of Arzelà-Ascoli used on example sheet 6, problem*, in a specific norm:

Theorem 12 (Arzelà-Ascoli theorem, source: solution to sheet 5). *Suppose there is a sequence (of solutions) $(\phi^{(i)})$ in $C^{k-1}([0, T] \times \mathbb{R}^n)$ that converges to ϕ in a $C^{k-1}([0, T] \times \mathbb{R}^n)$ -sense and is uniformly bounded, meaning $\|\partial^k \phi\|_{L^\infty} \leq A$ (A not dependent on i).*

Then, there exists a subsequence $(\phi^{(i_\lambda)})$ that converges in $C^k([0, T] \times \mathbb{R}^n)$.

Theorem 13 (Banach-Alaoglu, basic version). *Let (u_k) be a bounded sequence in a Hilbert space H , i.e. $\|u_k\|_H \leq C$. Then there exists a subsequence which converges weakly in H .*

Definition 14 (Weak Convergence; source: Wiki). *A sequence of elements (x_n) in a Hilbert space H is said to converge weakly to $x \in H$ if*

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle \quad (15)$$

for all $y \in H$, $\langle \cdot, \cdot \rangle$ denoting the inner product of H .

Notation: $x_n \rightharpoonup x$

Theorem 15 (Riesz Representation thm).

Theorem 16 (Banach's Fixed Point Thm; source: Wiki). *Let $(X, \|\cdot\|)$ be a non-empty complete metric space and $T : X \rightarrow X$ a contraction mapping. Then T admits a unique fixed point x^* in X , so $T(x^*) = x^*$.*

Definition 17 (contraction mapping). *A map $T : x \rightarrow X$ is called a contraction mapping if there exists a $q \in [0, 1)$ such that*

$$\|T(x) - T(y)\| \leq q\|x - y\| \quad (16)$$

for all $x, y \in X$.

3 Fourier Transformation

Open Question: What is the advantage of Fourier representation?

Definition 18 (Schwartz space). *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a vectorspace given by*

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \forall \alpha, \beta \in \mathbb{N}_0^n, \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty\} \quad (17)$$

equipped with a countable family of semi-norms

$$\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|. \quad (18)$$

You can think of the Schwartz space as functions that decay faster than any polynomial.

Definition 19 (Fourier transform). *Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform by*

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx. \quad (19)$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{+2\pi i x \cdot \xi} dx. \quad (20)$$

Theorem 20 (properties of Fourier transform and norms; source: exercise session 1). *For $f \in \mathcal{S}(\mathbb{R}^n)$ the following holds:*

1. $\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$
2. $\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$

Theorem 21 (properties of the Fourier transform; source: exercise session 1). *For $f \in \mathcal{S}(\mathbb{R}^n)$ the following properties hold:*

1. $\mathcal{F}(\partial_i f(x)) = 2\pi i \xi_i \mathcal{F}(f)(\xi)$
2. $\mathcal{F}(x_i f(x)) = -\frac{1}{2\pi i} \partial_{\xi_i} \mathcal{F}(f)(\xi)$

4 Important Inequalities

Theorem 22 (The interpolation inequality). *We have*

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C(s_1, s_2, s_n) \|f\|_{H^{s_1}(\mathbb{R}^n)}^{\theta_1} \|f\|_{H^{s_2}(\mathbb{R}^n)}^{\theta_2}, \quad (21)$$

where $0 \leq s_1 \leq s \leq s_2$, $\theta_1 + \theta_2 = 1$ and $\theta_1 s_1 + \theta_2 s_2 = s$.

Theorem 23 (Gagliardo-Nirenberg interpolation estimate; source: sheet 5). *Let $\phi \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For $0 < |\alpha| < s$ we have*

$$\|D^\alpha \phi\|_{L^{\frac{2s}{|\alpha|}}(\mathbb{R}^n)} \leq C(|\alpha|, s, n) (\|\phi\|_{L^\infty(\mathbb{R}^n)})^{1-\frac{|\alpha|}{s}} (\|\phi\|_{\dot{H}^s(\mathbb{R}^n)})^{\frac{|\alpha|}{s}}. \quad (22)$$

4.1 Sobolev Embedding

Theorem 24 (Sobolev embedding thm; source: sheet 1). *There exists a constant $C = C(n, s) > 0$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ we have*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{H^s(\mathbb{R}^n)}. \quad (23)$$

Theorem 25 (source: sheet 1). *If s is a non-negative integer, there exists a constant $C = C(s)$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ we have*

$$\frac{1}{C} \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}. \quad (24)$$

Theorem 26 (source: solution to sheet 3). *Let $f \in \mathcal{S}(\mathbb{R}^n)$. If $s > k + \frac{n}{2}$, then*

$$\|f\|_{C_b^k(\mathbb{R}^n, \mathbb{C})} \leq C \|f\|_{H^s}, \quad (25)$$

where the b means bounded.

Theorem 27 (source: sheet 6). *For $0 \leq s < \frac{n}{2}$ there exists a constant $C = C(s, n)$, such that*

$$\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq C \|\phi\|_{\dot{H}^s(\mathbb{R}^n)}. \quad (26)$$

Theorem 28 (second Sobolev embedding thm; source: Wiki). *If $n < pk$ and $r + \alpha = k - \frac{n}{p}$ with $\alpha \in (0, 1)$, then one has the embedding*

$$W^{k,p}(\mathbb{R}^n) \subset C^{r,\alpha}(\mathbb{R}^n). \quad (27)$$

5 Basic Analysis, Measure Theory

Theorem 29 (Young's Inequality). *Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any non-negative a and b it holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (28)$$

Theorem 30 (Young's Inequality Special case). *Let $x, y \in \mathbb{R}^n$, $\delta > 0$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$|xy| \leq \delta |x|^2 + \frac{1}{4\delta} |y|^2. \quad (29)$$

Theorem 31 (Hölder's Inequality; source: Wiki). *Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, if $f \in L^p$ and $g \in L^q$:*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}. \quad (30)$$

Theorem 32 (Lebesgue Monotone Convergence; source: math3ma.com). *Suppose $(f_n : X \rightarrow [0, \infty))$ being a monotonically increasing sequence of measurable functions on a measurable set X such that $f_n \rightarrow f$ pointwise almost everywhere, then*

$$\lim_{n \rightarrow \infty} \int_X f_n = \int_X f. \quad (31)$$

Theorem 33 (Fatou's Lemma; source: Wiki). *Given a measure space $(\Omega, \mathcal{F}, \mu)$ and a set $X \in \mathcal{F}$, let $(f_n : X \rightarrow [0, \infty])$ be a sequence of measurable functions. Define the function $F_X \rightarrow [0, \infty]$ by setting $f(x) = \liminf_{n \rightarrow \infty} f_n(x)$ for every $x \in X$.*

Then f is measurable, and also

$$\int_X f d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu \quad (32)$$

where the integrals may be infinite.

Theorem 34 (Change of Variables Formula; source: Wiki). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $\Phi : \Omega \rightarrow \Phi(\Omega) \subset \mathbb{R}^n$ a diffeomorphism. A function f is integrable on $\Phi(\Omega)$ iff the function $x \mapsto f(\Phi(x)) |Det(D\Phi(x))|$ is integrable on Ω . Then we have*

$$\int_{\Phi(\Omega)} f(y) dy = \int_{\Omega} f(\Phi(x)) |\det(D\Phi(x))| dx. \quad (33)$$

Theorem 35 (Picard-Lindelöf). *Let $G \subset \mathbb{R} \times \mathbb{R}^n$ be open and $f : G \rightarrow \mathbb{R}^n$ a continuous function satisfying a Lipschitz property locally. Then for every $(a, c) \in G$ there exists a $\epsilon > 0$ and a solution*

$$\phi : [a - \epsilon, a + \epsilon] \rightarrow \mathbb{R}^n \tag{34}$$

for the ODE $y' = f(x, y)$ with initial data $\phi(a) = c$.

Theorem 36 (Chain Rule).

$$\frac{\partial}{\partial x} F(\phi(x)) = \frac{\partial F}{\partial \phi} \frac{\partial \phi}{\partial x} \tag{35}$$