

# Inhaltsverzeichnis

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Here everything can be written down that helps with understanding the Non-Linear Wave Equations Lecture and solving the exercises.

## 1 Norms

**Definition 1** (Sobolev space).

**Definition 2** (fractional Sobolev space; source: exercise session 1). *For  $s \in \mathbb{R}$  (?)  $H^s(\mathbb{R}^n)$  is the completion of  $C_0^\infty(\mathbb{R}^n)$  with respect to the norm*

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} \quad (1)$$

where  $(1 + |\xi|^2)^{\frac{s}{2}}$  corresponds to a derivative in physical space.

## 2 Functional Analysis Theorems

**Theorem 3** (Arzelà-Ascoli theorem, source: Wiki). *Consider a sequence of functions  $(f_n)$  defined on a compact interval. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence  $(f_{n_k})$  that converges uniformly.*

**Definition 4** (uniformly equicontinuous). *A sequence of functions  $(f_n)$  is said to be uniformly equicontinuous if, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that*

$$|f_n(x) - f_n(y)| < \varepsilon \quad (2)$$

whenever  $|x - y| < \delta$  for all functions  $f_n$  in the sequence.

This is the version of Arzelà-Ascoli used on example sheet 6, problem\*, in a specific norm:

**Theorem 5** (Arzelà-Ascoli theorem, source: solution to sheet 5). *Suppose there is a sequence (of solutions)  $(\phi^{(i)})$  in  $C^{k-1}([0, T] \times \mathbb{R}^n)$  that converges to  $\phi$  in a  $C^{k-1}([0, T] \times \mathbb{R}^n)$ -sense and is uniformly bounded, meaning  $\|\partial^k \phi\|_{L^\infty} \leq A$  ( $A$  not dependent on  $i$ ).*

*Then, there exists a subsequence  $(\phi^{(i_\lambda)})$  that converges in  $C^k([0, T] \times \mathbb{R}^n)$ .*

**Theorem 6** (Banach-Alaoglu).

**Theorem 7** (Riesz Representation thm).

### 3 Fourier Transformation

Open Question: What is the advantage of Fourier representation?

**Definition 8** (Schwartz space). *The Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  is a vectorspace given by*

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \forall \alpha, \beta \in \mathbb{N}_0^n, \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty\} \quad (3)$$

*equipped with a countable family of semi-norms*

$$\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|. \quad (4)$$

You can think of the Schwartz space as functions that decay faster than any polynomial.

**Definition 9** (Fourier transform). *Given a function  $f \in \mathcal{S}(\mathbb{R}^n)$ , we define the Fourier transform by*

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx. \quad (5)$$

*The inverse Fourier transform is given by*

$$\mathcal{F}^{-1}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{+2\pi i x \cdot \xi} dx. \quad (6)$$

**Theorem 10** (properties of Fourier transform and norms; source: exercise session 1). *For  $f \in \mathcal{S}(\mathbb{R}^n)$  the following holds:*

1.  $\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$
2.  $\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$

**Theorem 11** (properties of the Fourier transform; source: exercise session 1). *For  $f \in \mathcal{S}(\mathbb{R}^n)$  the following properties hold:*

1.  $\mathcal{F}(\partial_i f(x)) = 2\pi i \xi_i \mathcal{F}(f)(\xi)$
2.  $\mathcal{F}(x_i f(x)) = -\frac{1}{2\pi i} \partial_{\xi_i} \mathcal{F}(f)(\xi)$

## 4 Important Inequalities

### 4.1 Sobolev Embedding

**Theorem 12** (Sobolev embedding thm; source: sheet 1). *There exists a constant  $C = C(n, s) > 0$  such that for every  $f \in H^s(\mathbb{R}^n)$  with  $s > \frac{n}{2}$  we have*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \quad (7)$$

**Theorem 13** (source: sheet 1). *If  $s$  is a non-negative integer, there exists a constant  $C = C(s)$  such that for every  $f \in H^s(\mathbb{R}^n)$  with  $s > \frac{n}{2}$  we have*

$$\frac{1}{C} \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}. \quad (8)$$

**Theorem 14** (second Sobolev embedding thm; source: Wiki). *If  $n < pk$  and  $r + \alpha = k - \frac{n}{p}$  with  $\alpha \in (0, 1)$ , then one has the embedding*

$$W^{k,p}(\mathbb{R}^n) \subset C^{r,\alpha}(\mathbb{R}^n). \quad (9)$$

## 5 Basic Analysis

**Theorem 15** (Young's Inequality). *Let  $p$  and  $q$  be positive real numbers such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, for any non-negative  $a$  and  $b$  it holds*

$$ab = \frac{a^p}{p} + \frac{b^q}{q}. \quad (10)$$