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Cryptanalysis of Lattice Based Post-Quantum Cryptosystems

Preliminary Literature Review

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1 List of Definitions, Notation, Abbreviations and Acronyms

CCA chosen ciphertext attack	4
CCA2 adaptive chosen ciphertext attack	4
KEM key encapsulation mechanism	4
LWE learning with errors	4
LWR learning with rounding	4
NIST National Institute of Standards and Technology	4
PKE public-key encryption	2
IND indistinguishability of ciphertexts	4
CVP closest vector problem	5
SVP shortest vector problem	5

Note: All security properties discussed will be for public-key encryption (PKE) schemes.

Def 1.1 (Indistinguishable).

$$(pk, sk) \leftarrow \Sigma.KeyGen$$

An adversary \mathcal{A} produces two messages $m_0, m_1 \in \mathcal{M}$ (of equal length). We choose a bit $b \in \{0, 1\}$.

$$c = Enc_{pk}(m_b)$$

Give the adversary (c, pk) , and allow them to generate a bit $a \in \{0, 1\}$. If $a = b$, then the adversary has succeeded.

We say an encryption scheme Σ is indistinguishable if the following holds:

$$\mathbb{P}(\mathcal{A} \text{ succeeds}) = \frac{1}{2} + \varepsilon \quad \text{where } \varepsilon \text{ is negligible.}$$

Intuitively, an encryption scheme has indistinguishability if an adversary is given a challenge ciphertext c , they cannot tell if it is from m_0 or m_1 .

Def 1.2 (IND-CPA or CPA security). “Indistinguishability of ciphertexts under chosen plaintext attack” for an encryption scheme Σ .

$$(pk, sk) \leftarrow \Sigma.KeyGen$$

The adversary \mathcal{A} is given pk and outputs two messages m_0, m_1 (of equal length), and is also given a challenge ciphertext:

$$c = Enc_{pk}(m_b) \quad \text{for a chosen } b \in \{0, 1\}.$$

\mathcal{A} now generates a bit $a \in \{0, 1\}$, and if $a = b$ then \mathcal{A} has succeeded. The encryption scheme has CPA security if:

$$\mathbb{P}(\mathcal{A} \text{ succeeds}) = \frac{1}{2} + \varepsilon \quad \text{where } \varepsilon \text{ is negligible.}$$

This can intuitively be thought of as if an attacker is given access to the public key (therefore able to encrypt plaintext's of their choice **but not decrypt**), then if given the encryption of one of two plaintexts, the attacker has negligible advantage over guessing.

Def 1.3 (IND-CCA or CCA security). “Indistinguishability of ciphertexts under a chosen ciphertext attack” for an encryption scheme Σ .

$$(pk, sk) \leftarrow \Sigma.KeyGen$$

The adversary \mathcal{A} is given pk and a decryption oracle $\mathcal{O}_{Dec_{sk}}$, and outputs m_0, m_1 (of equal length). The adversary is only able to query this oracle up until it receives the challenge ciphertext,

$$c = Enc_{pk}(m_b) \quad \text{for a chosen } b \in \{0, 1\}.$$

\mathcal{A} then generates a bit $a \in \{0, 1\}$, and if $a = b$ then \mathcal{A} has succeeded. The encryption scheme has CCA security if:

$$\mathbb{P}(\mathcal{A} \text{ succeeds}) = \frac{1}{2} + \varepsilon \quad \text{where } \varepsilon \text{ is negligible.}$$

Intuitively, this is if an adversary is able to ask for decryptions before given the challenge, once given the challenge ciphertext they have negligible advantage over guessing.

Def 1.4 (IND-CCA2 or CCA2 security). “Indistinguishability of ciphertexts under an adaptive chosen ciphertext attack” for an encryption scheme Σ .

$$(pk, sk) \leftarrow \Sigma.KeyGen$$

The adversary \mathcal{A} is given pk and a decryption oracle $\mathcal{O}_{Dec_{sk}}$, and outputs m_0, m_1 (of equal length). The adversary then receives the challenge ciphertext,

$$c = Enc_{pk}(m_b) \quad \text{for a chosen } b \in \{0, 1\}.$$

but may continue to query $\mathcal{O}_{Dec_{sk}}$ provided the requested decryption is not of c . \mathcal{A} then generates a bit $a \in \{0, 1\}$, and if $a = b$ then \mathcal{A} has succeeded. The encryption scheme has CCA security if:

$$\mathbb{P}(\mathcal{A} \text{ succeeds}) = \frac{1}{2} + \varepsilon \quad \text{where } \varepsilon \text{ is negligible.}$$

Intuitively, this can be thought of as if an adversary has the ability to decrypt any ciphertext other than the challenge, can they decrypt the challenge.

Note: We shall use \mathcal{B} to denote the set of 8-bit unsigned integers (or bytes), ie. the set $\{0, \dots, 255\}$

2 Introduction

In this PLR, we shall discuss two lattice-based cryptosystems submitted as part of the National Institute of Standards and Technology (NIST) post-quantum call for proposals round 3. These are Saber[1], an indistinguishability of ciphertexts (IND)-chosen ciphertext attack (CCA) Mod-learning with rounding (LWR) based key encapsulation mechanism (KEM) and KYBER[4], an IND-adaptive chosen ciphertext attack (CCA2) Mod-learning with errors (LWE) based KEM. In this preliminary literature review we will lay the foundation for my full dissertation, ensuring that both myself and the reader fully understand the cryptosystems being discussed.

This will be broken down into:

1. Algorithm specification of KYBER.PKE
2. Demonstration of security properties
3. Transformation of KYBER.PKE to KYBER.KEM
4. Algorithm specification of Saber.PKE
5. Demonstration of security properties
6. Transformation of Saber.PKE to Saber.KEM

This will lay the foundation that will allow us to conduct cryptanalysis on both cryptosystems.

3 Understanding the LWE/LWR problem

KYBER.PKE is a module LWE based encryption scheme, and Saber.PKE a module LWR based encryption scheme; both relying on the hardness of the of the LWE/LWR problem(s) - both believed to be hard for both classical and quantum computers. Below is an informal mod q set-up for the LWE problem for a prime q :

1. Let us chose an n dimensional vector $\mathbf{s} \in \mathbb{F}_q^n$. This is our secret.
2. Let us randomly and uniformly generate an $m \times n$ matrix \mathbf{A} over \mathbb{F}_q from elements in \mathbb{F}_q .
3. Let us generate an m dimensional vector, \mathbf{e} , s.t. $\mathbf{e}_i \sim \chi \forall i \in 1, \dots, m$ independently for the distribution χ on \mathbb{F}_q centred on 0.
4. Let $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$ over \mathbb{F}_q

Now, given (\mathbf{A}, \mathbf{b}) , find \mathbf{s} . This problem can be reduced to solving the lattice problems shortest vector problem (SVP) or closest vector problem (CVP). First we consider the q -ary lattice

$$\mathcal{L}_{Im(\mathbf{A})} = \{y \in \mathbb{Z}^m | y = \mathbf{A}z \mod q \text{ for some } z \in \mathbb{Z}^n\}$$

which is generated by the column vectors of our matrix $\mathbf{A} \mod q$. The LWR problem instead uses deterministic rounding instead of adding the small random error $\mathbf{e}[2]$.

4 Algorithm specification of Kyber.PKE

KYBER.PKE is defined over the ring $R \equiv \mathbb{Z}/(X^n + 1)$ and $R_q \equiv \mathbb{Z}_q[X]/(X^n + 1)$ where $n = 2^{n'-1}$ s.t. $X^n + 1$ is the $2^{n'}$ th cyclotomic polynomial[4]. In relation to our earlier explained LWE problem, R is \mathbb{F} and R_q is \mathbb{F}_q . We begin, by generating our matrix \mathbf{A} using the following algorithm:

Algorithm 1 Generate keys.

```

 $N := 0$ 
 $(\rho, \sigma) := G(d)$  ▷ Where  $G$  is a hash function s.t.  $G : \mathcal{B}^* \rightarrow \mathcal{B}^{32} \times \mathcal{B}^{32}$ 
for  $i \leftarrow 0, k - 1$  do
  for  $j \leftarrow 0, k - 1$  do
     $\mathbf{A} := \text{Parse}(\text{XOF}(\rho, j, i))$  ▷ This essentially generates a random  $k \times k$  matrix over  $R_q$  as  $\rho$  is pseudorandom from the hash function  $G()$ .
  end for
end for
for  $i \leftarrow 0, k - 1$  do
   $\mathbf{s} := \text{CBD}(\text{PRF}(\sigma, N))$  ▷ Where CBD is a function outputting a polynomial in  $R_q$  with the coefficients distributed central-binomially.  $\text{PRF}$  is a pseudorandom function,  $\text{PRF} : \mathcal{B}^{32} \times \mathcal{B} \rightarrow \mathcal{B}^*$ .
   $N := N + 1$ 
end for
for  $i \leftarrow 0, k - 1$  do
   $\mathbf{e} := \text{CBD}(\text{PRF}(\sigma, N))$ 
   $N := N + 1$ 
end for
 $\mathbf{s} := \text{NTT}(\mathbf{s})$  ▷ Where NTT is a bijection mapping  $f \in R_q$  to a polynomial with the coefficient vector.
 $\mathbf{e} := \text{NTT}(\mathbf{e})$ 
 $\mathbf{b} := \mathbf{A}\mathbf{s} + \mathbf{e}$ 
return  $\mathbf{A}, \mathbf{s}, \mathbf{b}, \mathbf{e}$ 

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From this, we have our public and private keys, $pk := (\mathbf{b} \bmod q) \parallel \rho$ and $sk := \mathbf{s} \bmod q$ (both encoded).

Algorithm 2 Encryption

Input: $pk, m \in \mathcal{B}^{32}$

first we must extract \mathbf{A} and \mathbf{b} from pk .

$\rho := pk + 12 \cdot k \cdot \frac{n}{8}$ \triangleright ρ was simply appended to the end of \mathbf{b} so we can extract it simply. As we now have ρ we can re-construct \mathbf{A} like we did in key generation.

for $i \leftarrow 0, k-1$ **do**

for $j \leftarrow 0, k-1$ **do**

$\mathbf{A}^T := \text{Parse}(\text{XOF}(\rho, i, j))$

end for

end for

for $i \leftarrow 0, k-1$ **do**

$\mathbf{r} := \text{CBD}(\text{PRF}(r, N))$

\triangleright Where $r \in \mathcal{B}^{32}$ is a random coin.

$N := N + 1$

end for

for $i \leftarrow 0, k-1$ **do**

$\mathbf{e}_1 := \text{CBD}(\text{PRF}(r, N))$

$N := N + 1$

end for

$e_2 := \text{CBD}(\text{PRF}(r, N))$

$\mathbf{r} := \text{NTT}((r))$

$\mathbf{u} := \text{NTT}^{-1}(\mathbf{A}^T \circ \mathbf{r}) + \mathbf{e}_1$

$v := \text{NTT}^{-1}(\mathbf{b}^T \circ \mathbf{r}) + e_2 + m$

return $(\mathbf{u} \| v)$

It is important that the ciphertext composes of two parts, only one of which is dependent on the message, so that the receiver has enough information in order to decrypt correctly.

Algorithm 3 Decryption

Input: sk, c

$m := v - \text{NTT}^{-1}(\mathbf{s}^T \circ \text{NTT}(\mathbf{u}))$

return m

It may not be readily obvious why it is this decryption works:

$$\begin{aligned} v - \text{NTT}^{-1}(\mathbf{s}^T \circ \text{NTT}(\mathbf{u})) &= \text{NTT}^{-1}(\mathbf{b}^T \circ \mathbf{r} + e_2 + m) - \text{NTT}^{-1}(\mathbf{s}^T \circ \text{NTT}(\mathbf{u})) \\ &= \text{NTT}^{-1}(\mathbf{b}^T \circ \mathbf{r} + e_2 + m) - \text{NTT}^{-1}(\mathbf{s}^T \circ \mathbf{A}^T \circ \mathbf{r} + \mathbf{e}_1) \\ &= \text{NTT}^{-1}(\mathbf{b}^T \circ \mathbf{r} + e_2 + m - \mathbf{s}^T \circ \mathbf{A}^T \circ \mathbf{r} + \mathbf{e}_1) \\ &= \text{NTT}^{-1}((\mathbf{A} \circ \mathbf{s})^T \circ \mathbf{r} - (\mathbf{s}^T \circ \mathbf{A}^T) \circ \mathbf{r} + \mathbf{e}^T + \mathbf{e}_1 + e_2 + m) \\ &= \text{NTT}^{-1}(m + \mathbf{e}^T + \mathbf{e}_1 + e_2) \end{aligned}$$

Therefore, we decrypt each bit as 0 if it is closer to 0 than $\lfloor \frac{q}{2} \bmod q \rfloor$, otherwise we decrypt as 1.¹

¹I'm not 100% sure on the last step as I have only seen an LWE setup encrypting a single bit and so you can disregard errors by seeing if it is approx. 0 or $\frac{q}{2}$.

5 Applying our security notions to MLWE and Kyber.PKE

It can be seen that the security of KYBER.PKE can be reduced to the hard problem of MLWE, and in both our security notions, we rely on the statement

$$\mathbb{P}(\mathcal{A} \text{ succeeds}) = 1/2 + \varepsilon \quad \text{where } \varepsilon \text{ is negligible.}$$

and so we can define what $\mathbb{P}(\mathcal{A} \text{ succeeds})$ is in order to begin to analyse the security of KYBER.PKE, and so we have $\mathbb{P}(\mathcal{A} \text{ succeeds}) =$

$$\mathbb{P}(a = i \mid \begin{matrix} (pk, sk) \leftarrow \text{KYBER.PKE.KeyGen}(), \\ (m_0, m_1, s) \leftarrow \mathcal{A}(pk), \\ c \leftarrow \text{KYBER.PKE.Enc}(pk, m_i), \\ a \leftarrow \mathcal{A}(s, c) \end{matrix}) \quad \text{for } i \in \{0, 1\}.$$

Therefore, the advantage of the adversary (in our earlier definition written as ε), $\text{Adv}_{\text{KYBER.PKE}}^{\text{CPA}}(\mathcal{A}) =$

$$|\mathbb{P}(a = i \mid \begin{matrix} (pk, sk) \leftarrow \text{KYBER.PKE.KeyGen}(), \\ (m_0, m_1, s) \leftarrow \mathcal{A}(pk), \\ c \leftarrow \text{KYBER.PKE.Enc}(pk, m_i), \\ a \leftarrow \mathcal{A}(s, c) \end{matrix}) - \frac{1}{2}|$$

In other words, if $\text{Adv}_{\text{KYBER.PKE}}^{\text{CPA}} \approx 0$ (or the adversary has no advantage), then KYBER.PKE is CCA secure.

We can also do the same thing for an MLWE problem, and define the advantage of our adversary over this problem. The MLWE challenge that our adversary must solve is slightly different to our challenge that we gave them for KYBER.PKE, and instead of distinguishing which message created the challenge ciphertext, they must instead determine if the challenge was generated via the MLWE scheme, or if it was instead generated randomly. Thus, our $\text{Adv}^{\text{MLWE}}(\mathcal{A}) =$

$$|\mathbb{P}(b' = 1 \mid \begin{matrix} \mathbf{A} \leftarrow R_q^{m \times k}, \\ (\mathbf{s}, \mathbf{e}) \leftarrow \mathcal{B}^k \times \mathcal{B}^m, \\ \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}, \\ b' \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{b}) \end{matrix}) - \mathbb{P}(b' = 1 \mid \begin{matrix} \mathbf{A} \leftarrow R_q^{m \times k}, \\ \mathbf{b} \leftarrow R_q^m, \\ b' \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{b}) \end{matrix})|$$

It can be seen that $\text{Adv}_{\text{KYBER.PKE}}^{\text{CPA}}(\mathcal{A})$ can be re-written as:

$$|\mathbb{P}(b' = 1 \mid \begin{matrix} \mathbf{A} \leftarrow R_q^{m \times k}, \\ (\mathbf{s}, \mathbf{e}) \leftarrow \{0, 1\}^k \times \{0, 1\}^m, \\ \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} + m_i, \\ a \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{b}) \end{matrix}) - \frac{1}{2}|$$

by using the definitions of KYBER.PKE's algorithms. Thus, we can show that the IND-CCA security of KYBER.PKE is reducible to that of the MLWE scheme, $\text{Adv}^{\text{MLWE}}(\mathcal{A}) \approx$

$$|\mathbb{P}(b' = 1 \mid \begin{matrix} \mathbf{A} \leftarrow R_q^{m \times k}, \\ (\mathbf{s}, \mathbf{e}) \leftarrow \{0, 1\}^k \times \{0, 1\}^m, \\ \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} + m_i, \\ a \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{b}) \end{matrix}) - \frac{1}{2}|$$

as $\mathbb{P}(b' = 1 \mid \begin{matrix} \mathbf{A} \leftarrow R_q^{m \times k}, \\ \mathbf{b} \leftarrow R_q^m, \\ b' \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{b}) \end{matrix})$ can simply be reduced to the expected value $\frac{1}{2}$.

$$\Rightarrow \text{Adv}_{\text{KYBER.PKE}}^{\text{CPA}}(\mathcal{A}) \leq_T \text{Adv}^{\text{MLWE}}(\mathcal{A})$$

6 conclusion

From this, we have come to the conclusion that provided MLWE is a hard problem, then KYBER.PKE can be shown to be IND-CPA secure. From this, we understand that in order to demonstrate security flaws in the scheme, we must address the underlying problem of MLWE or attack the scheme through insecure or improper algorithmic design (in much the same way GGH[8] was attacked). As such, I believe that it would be relevant and more advantageous to focus my project on attacking the security properties of MLWE, as this may have impacts beyond the proposed scheme (although KYBER will still remain a key focus).

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